Wenlong Zeng; Jianzhou Liu

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A LOWER BOUND SEQUENCE FOR THE MINIMUM EIGENVALUE OF HADAMARD PRODUCT OF AN M-MATRIX AND ITS INVERSE

Wenlong Zeng, Jianzhou Liu, Xiangtan

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Abstract. We propose a lower bound sequence for the minimum eigenvalue of Hadamard product of an M-matrix and its inverse, in terms of an S-type eigenvalues inclusion set and inequality scaling techniques. In addition, it is proved that the lower bound sequence converges. Several numerical experiments are given to demonstrate that the lower bound sequence is sharper than some existing ones in most cases.

Keywords: lower bound sequence; Hadamard product; M-matrix; doubly stochastic matrix; S-type eigenvalue inclusion set

MSC 2020: 15A18, 15A42

1. Introduction

M-matrices play an important role in various fields of science and engineering. Many problems in biology, physics, mathematics, and social sciences are closely related to the M-matrices, such as economic value model matrices and coefficient matrices of inverse network analysis and linear complementarity problems in optimization.

Denote the set of all $n \times n$ real matrices by $\mathbb{R}^{n\times n}$, and N denotes the set $\{1, 2, \ldots, n\}$. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. If A satisfies $a_{ij} \leqslant 0$, $i, j \in N$, $i \neq j$, then A is a Z-matrix. The set of all $n \times n$ Z-matrices is represented by Z_n. The matrix A is a nonsingular M-matrix if and only if $A \in Z_n$ and A can be written as $A = sI - B$, where B is a nonnegative matrix (i.e., all elements are not negative), and $s > \varrho(B)$.

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The set of all $n \times n$ nonsingular M-matrices is denoted by M_n, see [1]. We denote

$$\tau(A) = \min\{\mathrm{Re}(\lambda)\colon\ \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of the matrix A.

For two real matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{n\times n}$, the Hadamard product of A and B is $A \circ B \equiv (a_{ij}b_{ij})$. It is proved that $A \circ B^{-1}$ is an M-matrix if both A and B are M-matrices, see [5], [6].

The question of whether a real matrix may be symmetrized via multiplication by a positive diagonal matrix, is reduced to the corresponding issue for M-matrices and related to Hadamard products. Fiedler et al. in [4] pointed out a measure $\tau(A \circ A^{-1})$, which is the minimum real eigenvalue of $A \circ A^{-1}$. In addition, they showed that

$$0 < \tau(A \circ A^{-1}) \leqslant 1 \tag{1.1}$$

for a nonsingular M-matrix A. Subsequently, Fiedler and Markham in [5] proved that

$$\tau(A \circ A^{-1}) \geqslant \frac{1}{n}, \tag{1.2}$$

and proposed the following conjecture:

$$\tau(A \circ A^{-1}) \geqslant \frac{2}{n}. \tag{1.3}$$

We introduce some notations that will be very useful in explaining some existing results. Suppose $A = (a_{ij})$, $A^{-1} = B = (b_{ij})$. For any $i, j, k \in N$, $j \neq i$, $t = 0, 1, 2, \ldots$, denote

$$R_i = \sum_{j\neq i} |a_{ij}|, \quad d_i = \frac{R_i}{|a_{ii}|}, \quad s_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k}{|a_{jj}|}, \quad s_i = \max_{j\neq i}\{s_{ij}\},$$

$$r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i} |a_{jk}|}, \quad r_i = \max_{j\neq i}\{r_{ji}\}, \quad m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|},$$

$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| m_{ki}}{|a_{jj}|}, \quad u_i = \max_{j\neq i}\{u_{ij}\}, \quad p_{ji}^{(0)} = \min\{s_{ji}, m_{ji}\},$$

$$p_i^{(t)} = \max_{j\neq i}\{p_{ij}^{(t)}\}, \quad h_i^{(t)} = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(t)} - \sum_{k\neq j,i} |a_{jk}| p_{ki}^{(t)}}\right\},$$

$$v_{ji}^{(t)} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| p_{ki}^{(t)} h_i^{(t)}}{|a_{jj}|}, \quad g_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| f_{ki} l_i}{|a_{jj}|}, \quad g_i = \max_{j\neq i}\{g_{ij}\},$$

$$p_{ji}^{(t+1)} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| v_{ki}^{(t)}}{|a_{jj}|}, \quad q_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}}{|a_{jj}|}, \quad q_i = \max_{j\neq i}\{q_{ij}\},$$

$$f_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| q_{ki}}{|a_{jj}|}, \quad l_{ji} = \frac{|a_{ji}|}{|a_{jj}| f_{ji} - \sum_{k\neq j,i} \| a_{jk}| f_{ki}}, \quad l_i = \max_{j\neq i}\{l_{ji}\}.$$

Li et al. in [8] testified the conjecture (1.3) and obtained the result

$$\tau(A\circ A^{-1})\geqslant \min_{i\in N}\Big\{\frac{a_{ii}-s_iR_i}{1+\sum_{j\neq i}s_{ji}}\Big\}, \tag{1.4}$$

which depends only on the entries of matrix A instead of the dimension of matrix A.

Cheng et al. in [3] improved result (1.4) indicating the following result:

$$\tau(A\circ A^{-1})\geqslant \min_{i\in N}\Big\{\frac{a_{ii}-u_iR_i}{1+\sum_{j\neq i}u_{ji}}\Big\}. \tag{1.5}$$

Zhou et al. in [12] proved that

$$\tau(A\circ A^{-1})\geqslant 1-\varrho^2(J_A), \tag{1.6}$$

where $D_A=\operatorname{diag}(a_{ii})$, $C_A=D_A-A$, $J_A=D_A^{-1}C_A$.

Zhao et al. in [11] improved results (1.4) and (1.5), showing the following conclusion:

$$\tau(A\circ A^{-1})\geqslant \min_{i\in N}\Big\{\frac{a_{ii}-p_i^{(t)}R_i}{1+\sum_{j\neq i}p_{ji}^{(t)}}\Big\},\quad t=1,2,\dots \tag{1.7}$$

Chen in [2] obtained the result

$$\begin{aligned}\tau(A\circ A^{-1})\geqslant \min_{i\neq j}\frac{1}{2}\Big\{&a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\&+4\Big(q_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(q_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Big]^{1/2}\Big\}.\end{aligned} \tag{1.8}$$

Huang et al. in [7] gave three lower bounds as follows, which are difficult to be compared theoretically:

$$\begin{aligned}\tau(A\circ A^{-1})\geqslant \min_{i\neq j}\frac{1}{2}\Big\{&a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\&+4\Big(g_i\sum_{k\neq i}|a_{ki}|b_{ii}\Big)\Big(g_j\sum_{k\neq j}|a_{kj}|b_{jj}\Big)\Big]^{1/2}\Big\},\end{aligned} \tag{1.9}$$

$$\tau(A\circ A^{-1})\geqslant \min_{i\in N}\Big\{\frac{a_{ii}-\sum_{j\neq i}|a_{ji}|g_{ji}}{a_{ii}-\sum_{k\neq i}a_{ki}a_{ik}/a_{kk}}\Big\}, \tag{1.10}$$

$$\begin{aligned}\tau(A\circ A^{-1})\geqslant \min_{i\neq j}\frac{1}{2}\Big\{&a_{ii}b_{ii}+a_{jj}b_{jj}-\Big[(a_{ii}b_{ii}-a_{jj}b_{jj})^2\\&+4\Big(b_{ii}\sum_{k\neq i}|a_{ki}|q_{ki}\Big)\Big(b_{jj}\sum_{k\neq j}|a_{kj}|q_{kj}\Big)\Big]^{1/2}\Big\}.\end{aligned} \tag{1.11}$$

The content of this paper can be listed as follows: In Section 2, some lemmas and notations are presented. In Section 3, we exhibit a lower bound sequence for $\tau(A\circ A^{-1})$ by using S-type eigenvalue inclusion set and a series of inequality scaling techniques. For the obtained lower bound sequence, we prove its convergence and give the calculation algorithm. In Section 4, in order to verify the validity and superiority of our results, numerical experiments are carried out by randomly generating M-matrices.

2. Some lemmas and notations

For convenience, let S denote a nonempty subset of N, $n\geqslant 2$, and let $\overline{S}:=N\setminus S$ denote its complement in N. Then for any given matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$, split each row sum $R_i(A)$ and column sum $C_i(A)$ into two parts, depending on S and $\overline{S}$. In other words, for any $i\in N$, denote

$$R_i^S(A)=\sum_{j\in S\setminus\{i\}}|a_{ij}|,\quad R_i^{\overline{S}}(A)=\sum_{j\in\overline{S}\setminus\{i\}}|a_{ij}|,$$
$$C_i^S(A)=\sum_{j\in S\setminus\{i\}}|a_{ji}|,\quad C_i^{\overline{S}}(A)=\sum_{j\in\overline{S}\setminus\{i\}}|a_{ji}|.$$

Lemma 2.1 ([11]). *If $A=(a_{ij})\in M_n$ is a strictly row diagonally dominant matrix, then $A^{-1}=(b_{ij})$ exists, and*

$$b_{ji}\leqslant\frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|v_{ki}^{(t-1)}}{a_{jj}}b_{ii}=p_{ji}^{(t)}b_{ii}\leqslant p_j^{(t)}b_{ii},\quad j,i\in N,\ j\neq i,\ t=1,2,\ldots$$

Remark 2.1. As t increases, $p_{ij}^{(t)}$ and $p_i^{(t)}$ will decrease gradually, and $0\leqslant p_{ij}^{(t)}\leqslant p_i^{(t)}\leqslant 1$, see [11], Lemma 1. By the monotone bounded theorem we know that $p_{ij}^{(t)}$ and $p_i^{(t)}$ are convergent.

Lemma 2.2 ([11]). *If $A=(a_{ij})\in M_n$ is a strictly row diagonally dominant matrix, then $A^{-1}=(b_{ij})$ exists, and*

$$\frac{1}{a_{ii}}\leqslant b_{ii}\leqslant\frac{1}{a_{ii}-\sum_{j\neq i}|a_{ji}|p_{ji}^{(t)}},\quad i,j\in N,\ t=1,2,\ldots$$

Lemma 2.3 ([11]). *If $A = (a_{ij}) \in M_n$ and $B = A^{-1} = (b_{ij})$ is a doubly stochastic matrix, then*

$$b_{ii} \geqslant \frac{1}{1 + \sum_{j \neq i} p_{ji}^{(t)}}, \quad i, j \in N, \ t = 1, 2, \ldots$$

Lemma 2.4 ([10]). *For a given $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and a nonempty subset S, define the Gershgorin-type discs*

$$\Gamma_i^S(A) := \{z \in C\colon \ |z - a_{ii}| \leqslant C_i^S(A)\} \quad (\text{any } i \in S) \tag{2.1}$$

and the sets

$$V_{i,j}^S(A) := \{z \in C\colon \ (|z - a_{ii}| - C_i^S(A))(|z - a_{jj}| - C_j^{\overline{S}}(A)) \leqslant C_i^{\overline{S}}(A) C_j^S(A)\} \tag{2.2}$$

(any $i \in S$, any $j \in \overline{S}$). Then all the eigenvalues of A lie in the region

$$E^S(A) = \Big(\bigcup_{i \in S} \Gamma_i^S(A)\Big) \cup \Big(\bigcup_{i \in S,\, j \in \overline{S}} V_{i,j}^S(A)\Big), \tag{2.3}$$

which is called the S-type eigenvalue inclusion set.

Remark 2.2. See [10]. Let

$$\begin{aligned} S_i &= \{i\} \quad (i \in N), \\ G(A) &= \bigcup_{i \in N} \{z\colon \ |z - a_{ii}| \leqslant R_i(A)\}, \\ K(A) &= \bigcup_{i \neq j} \{z\colon \ |z - a_{ii}||z - a_{jj}| \leqslant R_i(A) R_j(A)\}. \end{aligned}$$

Then

$$\sigma(A) \subseteq \bigcap_{\emptyset \neq S \subseteq N} E^S(A) \subseteq \bigcap_{i \in N} E^{S_i}(A) \subseteq K(A) \subseteq G(A).$$

This means that we use a more precise eigenvalue inclusion set than the one used in other corresponding papers.

Lemma 2.5. *Let S denote a nonempty subset of N. For a given $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and positive real constants $p_1, p_2, \ldots, p_n$, all the eigenvalues of A lie in the region:*

$$\begin{aligned} &\Big(\bigcup_{i \in S} \Big\{z\colon \ |z - a_{ii}| \leqslant p_i \sum_{k \in S \setminus \{i\}} \frac{|a_{ki}|}{p_k}\Big\}\Big) \\ &\bigcup \Big(\bigcup_{i \in S, j \in \overline{S}} \Big\{z\colon \ \Big(|z - a_{ii}| - p_i \sum_{k \in S \setminus \{i\}} \frac{|a_{ki}|}{p_k}\Big)\Big(|z - a_{jj}| - p_j \sum_{k \in \overline{S} \setminus \{j\}} \frac{|a_{kj}|}{p_k}\Big) \\ &\qquad \leqslant p_i \sum_{k \in \overline{S}} \frac{|a_{ki}|}{p_k} p_j \sum_{k \in S} \frac{|a_{kj}|}{p_k}\Big\}\Big). \end{aligned}$$

P r o o f. Since $D^{-1}AD$ has the same eigenvalues as A when D is nonsingular, we apply Lemma 2.4 to $D^{-1}AD$ and thereby obtain additional eigenvalue inclusion sets for A. A particularly convenient choice is $D = \operatorname{diag}(p_1, p_2, \ldots, p_n)$ with all $p_i > 0$. Applying Lemma 2.4 to $D^{-1}AD = [p_j a_{ij}/p_i]$ we get the result we want. $\square$

3. Main result

In this section, a superior lower bound for $\tau(A \circ A^{-1})$ is given. Moreover, the present authors verify the validity and superiority of the lower bound. Without loss of generality, we assume that A^{-1} is a doubly stochastic matrix. Otherwise, there exist two positive diagonal matrices D_1 and D_2 such that $D_1 A^{-1} D_2$ is doubly stochastic, see [10]. The matrix $B = D_2^{-1} A D_1^{-1}$ is an M-matrix and satisfies $\tau(B \circ B^{-1}) = \tau(A \circ A^{-1})$ in virtue of $B \circ B^{-1} = (D_2^{-1} A D_1^{-1}) \circ (D_1 A^{-1} D_2) = (D_1 D_2^{-1})(A \circ A^{-1})(D_1 D_2^{-1})^{-1}$.

To facilitate the formulation in the following theorems, we introduce the notation

$$H_i^S(t) := a_{ii} b_{ii} - p_i^{(t)} b_{ii} C_i^S(A),$$

sometimes abbreviated as H_i^S.

Theorem 3.1. *Let S denote a nonempty subset of N, $n \geqslant 2$. If $A = (a_{ij}) \in M_n$ and $B = A^{-1} = (b_{ij})$ is a doubly stochastic matrix, then for $t = 1, 2, \ldots,$*

$$\begin{aligned} &\tau(A \circ A^{-1}) \\ &\geqslant \max_{\emptyset \neq S \subseteq N} \min_{i \in S,\, j \in \overline{S}} \frac{1}{2}\{H_i^S + H_j^{\overline{S}} - [(H_i^S - H_j^{\overline{S}})^2 + 4 p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A)]^{1/2}\} \\ &= \Omega_t. \end{aligned}$$

P r o o f. Firstly, we assume that A is irreducible. Since $B = A^{-1}$ is a doubly stochastic matrix, we obtain that $Ae = e$, $A^\top e = e$, and A is a nonsingular M-matrix, so we can conclude that

$$a_{ii} = R_i(A) + 1 = C_i(A) + 1 \quad \text{and} \quad a_{ii} > 1,\ i \in N.$$

We have

$$p_i^{(t)} = \max_{j \neq i} \{p_{ij}^{(t)}\} = \max_{j \neq i} \left\{ \frac{|a_{ij}| + \sum_{k \neq i,j} |a_{ik}| v_{kj}^{(t-1)}}{|a_{ii}|} \right\}, \quad i \in N,\ t = 1, 2, \ldots$$

Since A is an irreducible matrix and due to Remark 2.1, we know that $0 < p_i^{(t)} \leqslant 1$. Let $\tau(A \circ B) = \lambda$. It is not hard to get $0 < \lambda \leqslant 1 \leqslant a_{ii} b_{ii}$, $i \in N$. Thus, for any given positive integer t, by Lemma 2.5, either there exists $i \in S$ such that

$$|\lambda - a_{ii}b_{ii}| \leqslant p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}}, \tag{3.1}$$

or there exists $i \in S$, $j \in \overline{S}$ such that

$$\begin{aligned}
&\Big(|\lambda - a_{ii}b_{ii}| - p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}}\Big)\Big(|\lambda - a_{jj}b_{jj}| - p_j^{(t)} \sum_{k\in \overline{S}\setminus\{j\}} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}}\Big) \\
&\quad\leqslant p_i^{(t)} \sum_{k\in\overline{S}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}} p_j^{(t)} \sum_{k\in S} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}} \\
&\quad\leqslant p_i^{(t)} \sum_{k\in\overline{S}} \frac{|a_{ki}|p_{ki}^{(t)}b_{ii}}{p_k^{(t)}} p_j^{(t)} \sum_{k\in S} \frac{|a_{kj}|p_{kj}^{(t)}b_{jj}}{p_k^{(t)}} \quad \text{(by Lemma 2.1)} \\
&\quad\leqslant p_i^{(t)} b_{ii} \sum_{k\in\overline{S}} |a_{ki}| p_j^{(t)} b_{jj} \sum_{k\in S} |a_{kj}| \\
&\quad= p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A).
\end{aligned} \tag{3.2}$$

By inequality (3.1) we know that

$$\begin{aligned}
\lambda &\geqslant a_{ii}b_{ii} - p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}} \\
&\geqslant a_{ii}b_{ii} - p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|p_{ki}^{(t)}b_{ii}}{p_k^{(t)}} \\
&\geqslant a_{ii}b_{ii} - p_i^{(t)} \sum_{k\in S\setminus\{i\}} |a_{ki}|b_{ii} \\
&= a_{ii}b_{ii} - p_i^{(t)} C_i^S(A) b_{ii} \\
&\geqslant \min_{i\in S}\{a_{ii}b_{ii} - p_i^{(t)} C_i^S(A) b_{ii}\}.
\end{aligned} \tag{3.3}$$

Since $0 < \lambda \leqslant a_{kk}b_{kk}$, $k \in N$, by inequality (3.2), we conclude that

$$\begin{aligned}
&p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A) \\
&\quad\geqslant \Big(|\lambda - a_{ii}b_{ii}| - p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}}\Big)\Big(|\lambda - a_{jj}b_{jj}| - p_j^{(t)} \sum_{k\in \overline{S}\setminus\{j\}} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}}\Big) \\
&\quad= \Big(a_{ii}b_{ii} - \lambda - p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}}\Big)\Big(a_{jj}b_{jj} - \lambda - p_j^{(t)} \sum_{k\in \overline{S}\setminus\{j\}} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}}\Big) \\
&\quad= \Big(\lambda - a_{ii}b_{ii} + p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}}\Big)\Big(\lambda - a_{jj}b_{jj} + p_j^{(t)} \sum_{k\in \overline{S}\setminus\{j\}} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}}\Big).
\end{aligned} \tag{3.4}$$

By Lemma 2.1, similarly to (3.2), it is true that

$$p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}} \leqslant p_i^{(t)} b_{ii} C_i^S(A), \quad p_j^{(t)} \sum_{k\in \overline{S}\setminus\{j\}} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}} \leqslant p_j^{(t)} b_{jj} C_j^{\overline{S}}(A). \tag{3.5}$$

Let

$$\begin{aligned} H_i^S &= a_{ii}b_{ii} - p_i^{(t)} b_{ii} C_i^S(A), & H_j^{\overline{S}} &= a_{jj}b_{jj} - p_j^{(t)} b_{jj} C_j^{\overline{S}}(A), \\ K_i^S &= a_{ii}b_{ii} - p_i^{(t)} \sum_{k\in S\setminus\{i\}} \frac{|a_{ki}|b_{ki}}{p_k^{(t)}}, & K_j^{\overline{S}} &= a_{jj}b_{jj} - p_j^{(t)} \sum_{k\in \overline{S}\setminus\{j\}} \frac{|a_{kj}|b_{kj}}{p_k^{(t)}}. \end{aligned}$$

From (3.5), $K_i^S \geqslant H_i^S \geqslant 0$, $K_j^{\overline{S}} \geqslant H_j^{\overline{S}} \geqslant 0$. Then inequality (3.4) can be simplified as

$$(\lambda - K_i^S)(\lambda - K_j^{\overline{S}}) \leqslant p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A), \tag{3.6}$$

which is a one-variable quadratic inequality. For convenience, denote

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\{K_i^S + K_j^{\overline{S}} - [(K_i^S - K_j^{\overline{S}})^2 + 4p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A)]^{1/2}\}, \\ \lambda_2 &= \frac{1}{2}\{K_i^S + K_j^{\overline{S}} + [(K_i^S - K_j^{\overline{S}})^2 + 4p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A)]^{1/2}\}. \end{aligned}$$

It is not difficult to obtain that (3.6) is equivalent to $(\lambda - \lambda_1)(\lambda - \lambda_2) \leqslant 0$. Due to $\lambda_1 < \lambda_2$, we conclude that

$$\lambda \geqslant \lambda_1 = \frac{1}{2}\{K_i^S + K_j^{\overline{S}} - [(K_i^S - K_j^{\overline{S}})^2 + 4p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A)]^{1/2}\}. \tag{3.7}$$

Let $a = K_i^S - H_i^S$, $b = K_j^{\overline{S}} - H_j^{\overline{S}}$, $c = 4p_i^{(t)} b_{ii} C_i^{\overline{S}}(A) p_j^{(t)} b_{jj} C_j^S(A)$ (a, b and c are all nonnegative), then we will show that

$$K_i^S + K_j^{\overline{S}} - [(K_i^S - K_j^{\overline{S}})^2 + c]^{1/2} \geqslant H_i^S + H_j^{\overline{S}} - [(H_i^S - H_j^{\overline{S}})^2 + c]^{1/2}, \tag{3.8}$$

that is

$$a + b + [(H_i^S - H_j^{\overline{S}})^2 + c]^{1/2} \geqslant [(H_i^S + a - H_j^{\overline{S}} - b)^2 + c]^{1/2}. \tag{3.9}$$

equation (3.9) is equivalent to

$$\begin{aligned} &(a+b)^2 + 2(a+b)[(H_i^S - H_j^{\overline{S}})^2 + c]^{1/2} + (H_i^S - H_j^{\overline{S}})^2 + c \\ &\geqslant (H_i^S - H_j^{\overline{S}} + a - b)^2 + c \\ &= (H_i^S - H_j^{\overline{S}})^2 + 2(H_i^S - H_j^{\overline{S}})(a-b) + (a-b)^2 + c. \end{aligned} \tag{3.10}$$

Then (3.10) can be simplified as

$$(3.11)\quad (a+b)^2+2(a+b)[(H_i^S-H_j^{\overline{S}})^2+c]^{1/2}\geqslant (a-b)^2+2(a-b)2(H_i^S-H_j^{\overline{S}}).$$

Through item by item comparison, it is easy to conclude that (3.11) is true, therefore (3.10), (3.9) and (3.8) all hold. Hence, by (3.7) and (3.8), we obtain that

$$(3.12)\quad \begin{aligned}\lambda &\geqslant \frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2+4p_i^{(t)}b_{ii}C_i^{\overline{S}}(A)p_j^{(t)}b_{jj}C_j^S(A)]^{1/2}\}\\ &\geqslant \min_{i\in S, j\in\overline{S}}\frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2+4p_i^{(t)}b_{ii}C_i^{\overline{S}}(A)p_j^{(t)}b_{jj}C_j^S(A)]^{1/2}\}.\end{aligned}$$

We get two lower bounds (3.3) and (3.12) on λ, but there is concrete evidence to suggest that

$$(3.13)\quad \begin{aligned}&\min_{i\in S}\{a_{ii}b_{ii}-p_i^{(t)}C_i^S(A)b_{ii}\}=\min_{i\in S}\{H_i^S\}\\ &\geqslant \min_{i\in S, j\in\overline{S}}\frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2+4p_i^{(t)}b_{ii}C_i^{\overline{S}}(A)p_j^{(t)}b_{jj}C_j^S(A)]^{1/2}\}.\end{aligned}$$

Then we will prove (3.13). For any given $i\in S$, $j\in\overline{S}$, if $H_i^S\leqslant H_j^{\overline{S}}$ holds, then

$$\begin{aligned}&\frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2+4p_i^{(t)}b_{ii}C_i^{\overline{S}}(A)p_j^{(t)}b_{jj}C_j^S(A)]^{1/2}\}\\ &\qquad\leqslant \frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2]^{1/2}\}=H_i^S.\end{aligned}$$

Otherwise, $H_i^S>H_j^{\overline{S}}$, then

$$\begin{aligned}&\frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2+4p_i^{(t)}b_{ii}C_i^{\overline{S}}(A)p_j^{(t)}b_{jj}C_j^S(A)]^{1/2}\}\\ &\qquad\leqslant \frac{1}{2}\{H_i^S+H_j^{\overline{S}}-[(H_i^S-H_j^{\overline{S}})^2]^{1/2}\}=H_j^{\overline{S}}<H_i^S.\end{aligned}$$

Therefore, (3.13) holds.

Whenever a nonempty subset S is given, there is a corresponding eigenvalue inclusion set. So $\tau(A\circ A^{-1})\geqslant\Omega_t$ is obtained.

If A is reducible, without loss of generality, we may assume that A has the block upper triangular form

$$A=\begin{pmatrix}A_{11} & A_{12} & \dots & A_{1s}\\ & A_{22} & \dots & A_{2s}\\ & & \dots & \dots\\ & & & A_{ss}\end{pmatrix}$$

with irreducible diagonal blocks A_{ii}, $i=1,2,\dots,s$. Obviously, $\tau(A\circ A^{-1})=\min\limits_{1\leqslant i\leqslant s}(A_{ii}\circ A_{ii}^{-1})$. Thus, reducible matrix A is reduced to irreducible diagonal blocks A_{ii}. The result of Theorem 3.1 also holds. $\square$

Theorem 3.2. *The sequence* $\{\Omega_t\}$ *obtained from Theorem* 3.1 *is increasing on* $\{t\colon\ t=1,2,\ldots\}$, *consequently, it is convergent.*

P r o o f. Let

$$G(t)=4p_i^{(t)}b_{ii}C_i^{\overline{S}}(A)p_j^{(t)}b_{jj}C_j^S(A),\qquad (3.14)$$
$$\Gamma(t)=H_i^S(t)+H_j^{\overline{S}}(t)-[(H_i^S(t)-H_j^{\overline{S}}(t))^2+G(t)]^{1/2}.$$

From Remark 2.1, we have $p_{ij}^{(t)}$ and $p_i^{(t)}$ are decreasing on $\{t\colon\ t=1,2,\ldots\}$. Hence, $H_i^S(t)$ is increasing and $G(t)$ is decreasing on $\{t\colon\ t=1,2,\ldots\}$. For any given positive integer t_1 and t_2 $(t_1<t_2)$, we assume that

$$H_i^S(t_2)=H_i^S(t_1)+\varepsilon,\quad H_j^{\overline{S}}(t_2)=H_j^{\overline{S}}(t_1)+\delta,\quad G(t_2)=G(t_1)-\theta,\qquad (3.15)$$

where ε, δ and θ are nonnegative. From (3.14) and (3.15), we obtain that

$$\begin{aligned}\Gamma(t_2)-\Gamma(t_1)&=\varepsilon+\delta+[(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+G(t_1)]^{1/2}\\&\quad-[(H_i^S(t_2)-H_j^{\overline{S}}(t_2))^2+G(t_2)]^{1/2}\\&=\varepsilon+\delta+[(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+G(t_1)]^{1/2}\\&\quad-[(H_i^S(t_1)+\varepsilon-H_j^{\overline{S}}(t_1)-\delta)^2+G(t_1)-\theta]^{1/2}.\end{aligned}$$

We will prove that $\Gamma(t_2)-\Gamma(t_1)\geqslant 0$, i.e.,

$$\begin{aligned}&\varepsilon+\delta+[(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+G(t_1)]^{1/2}\\&\qquad\geqslant[(H_i^S(t_1)+\varepsilon-H_j^{\overline{S}}(t_1)-\delta)^2+G(t_1)-\theta]^{1/2},\end{aligned}\qquad (3.16)$$

which is equivalent to

$$\begin{aligned}&(\varepsilon+\delta)^2+(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+G(t_1)\\&\qquad+2(\varepsilon+\delta)[(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+G(t_1)]^{1/2}\\&\quad\geqslant(H_i^S(t_1)+\varepsilon-H_j^{\overline{S}}(t_1)-\delta)^2+G(t_1)-\theta\\&\quad=(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+(\varepsilon-\delta)^2\\&\qquad+2(H_i^S(t_1)-H_j^{\overline{S}}(t_1))(\varepsilon-\delta)+G(t_1)-\theta.\end{aligned}\qquad (3.17)$$

And (3.17) is equivalent to

$$\begin{aligned}&(\varepsilon+\delta)^2+2(\varepsilon+\delta)[(H_i^S(t_1)-H_j^{\overline{S}}(t_1))^2+G(t_1)]^{1/2}\\&\qquad\geqslant(\varepsilon-\delta)^2+2(H_i^S(t_1)-H_j^{\overline{S}}(t_1))(\varepsilon-\delta)-\theta.\end{aligned}\qquad (3.18)$$

Thanks to ε, δ, θ and $G(t)$ being nonnegative, we have

$$(\varepsilon+\delta)^2 \geqslant (\varepsilon-\delta)^2, \tag{3.19}$$

$$2(\varepsilon+\delta) \geqslant 2(\varepsilon-\delta), \tag{3.20}$$

$$[(H_i^S(t_1) - H_j^{\overline{S}}(t_1))^2 + G(t_1)]^{1/2} \geqslant (H_i^S(t_1) - H_j^{\overline{S}}(t_1)), \tag{3.21}$$

$$\theta \geqslant 0. \tag{3.22}$$

By (3.19), (3.20), (3.21) and (3.22), we conclude that (3.18) is valid, therefore (3.17) and (3.16) also hold. Hence, we obtain that $\Gamma(t_2) - \Gamma(t_1) \geqslant 0$. In other word, as t increases, $\Gamma(t)$ will increase gradually. Finally, we know that Ω_t is increasing on $\{t\colon\ t = 1, 2, \ldots\}$. Any term in the sequence $\{\Omega_t\}$ is a lower bound of $\tau(A \circ A^{-1})$, so Ω_t is convergent in virtue of monotone bounded theorem. $\square$

Next, an algorithm of Ω_t is given as follow:

Algorithm 1: an algorithm of Ω_t

Input: a nonsingular M-matrix $A = (a_{ij})$ whose inverse matrix is a doubly stochastic matrix, and t_0;

Output: Ω_{t_0}

1: compute $R_i = \sum\limits_{j\neq i} |a_{ij}|$;

2: compute $d_i = \dfrac{R_i}{|a_{ii}|}$;

3: compute $s_{ji} = \dfrac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k}{|a_{jj}|}$ and $r_{ji} = \dfrac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i} |a_{jk}|}$;

4: compute $s_i = \max\limits_{j\neq i}\{s_{ij}\}$ and $r_i = \max\limits_{j\neq i}\{r_{ji}\}$;

5: compute $m_{ji} = \dfrac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| r_i}{|a_{jj}|}$;

6: compute $p_{ji}^{(0)} = \min\{s_{ji}, m_{ji}\}$;

7: for $t = 0\colon\ t_0$

8: compute $p_i^{(t)} = \max\limits_{j\neq i}\{p_{ij}^{(t)}\}$ and $h_i^{(t)} = \max\limits_{j\neq i}\left\{\dfrac{|a_{ji}|}{|a_{jj}| p_{ji}^{(t)} - \sum_{k\neq j,i} |a_{jk}| p_{ki}^{(t)}}\right\}$;

9: compute $v_{ji}^{(t)} = \dfrac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| p_{ki}^{(t)} h_i^{(t)}}{|a_{jj}|}$;

10: compute $p_{ji}^{(t+1)} = \dfrac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| v_{ki}^{(t)}}{|a_{jj}|}$;

11: end for

12: compute Ω_{t_0};

13: return that Ω_{t_0}.

For a given positive integer t_0, the corresponding lower bound Ω_{t_0} can be obtained. And t_0 can be chosen as small as possible according to the need of different precision. For example, t_0 is selected so that $|\Omega_{t_0} - \Omega_{t_0-1}| \leqslant 10^{-8}$, which is a condition for cycle termination.

4. Numerical example

In this section, the following examples are given to demonstrate the superiority of our result.

Example 4.1. Let us see the following simple M-matrix given in [7]:

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}.$$

We know that $B = A^{-1} = (b_{ij})$ is a doubly stochastic matrix because $Ae = e$ and $A^{\top}e = e$. By direct calculations with MATLAB R2016a, we get the following results.

Theorem	Lower bound
Theorem 3.1 of [3]	0.6624
Theorem 3.1 of [8]	0.8250
Corollary 2 of [4]	0.8364
Theorem 3.2 of [5]	0.8456
Theorem 2 of [9]	0.8904
Theorem 3 of [9]	0.6874
Theorem 4 of [9]	0.8811

Table 1. Lower bounds for $\tau(A \circ A^{-1})$.

If we apply Theorem 3.1, we get $\tau(A \circ A^{-1}) \geqslant \Omega_t$, see Table 2.

t	1	2	3	4
Ω_t	0.9168	0.9188	0.9190	0.9190

Table 2. The lower bounds of $\tau(A \circ A^{-1})$ by Theorem 3.1.

The numerical example shows that the bound of Theorem 3.1 is greater than these corresponding results.

Example 4.2. The last example is not convincing enough. Next, we construct randomly 120 4×4 M-matrices, whose inverse matrices are doubly stochastic. For convenience, the idea, how the random matrices are constructed, is as follows. Firstly, the diagonal entries of the 3 order principal submatrix are generated randomly. Secondly, the off-diagonal entries of the 3 order principal submatrix are generated randomly in the diagonal constraints, Finally, the remaining entries can be calculated, see Algorithm 2. We compare the new lower bound with Theorems 2–4 of [7], i.e., (1.9), (1.10) and (1.11), see Figure 1.

Algorithm 2: a 4×4 M-matrix A is constructed randomly

Output: a 4×4 M-matrix A, whose inverse matrix is doubly stochastic

1: initialization $A = O_{4\times 4}$;
2: $[a_{11}, a_{22}, a_{33}] = 10\times$ rand$(1,3) + [2,2,2]$;
3: for $i = 1:3$
4:     for $j = 1:3$
5:         if $i \neq j$
6:             $a_{ij} = [2 - \min(a_{ii}, a_{jj})]\times$rand$(1)$;
7:         end if
8:     end for
9: end for
10: for $i = 1:3$
11:     $a_{i4} = 1 - a_{i1} - a_{i2} - a_{i3}$;
12:     if $a_{i4} > 0$
13:         return step 3;
14:     end if
15: end for
16: for $i = 1:3$
17:     $a_{4i} = 1 - a_{1i} - a_{2i} - a_{3i}$;
18:     if $a_{4i} > 0$
19:         return step 3;
20:     end if
21: end for
22: if $A^{-1} \geqslant 0$
23:     return A;
24: else
25:     return step 3;
26: end if

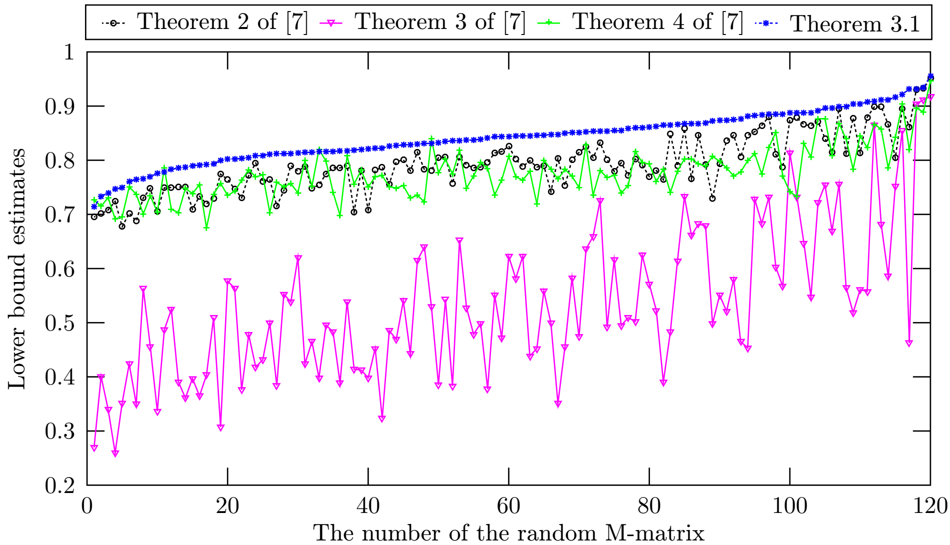

Figure 1. The lower bound estimations derived from Theorems 2–4 of [7] and Theorem 3.1.

In order to observe conveniently, we arrange incrementally the lower bounds obtained from Theorem 3.1. As can be seen in Figure 1, in most cases, the lower bounds obtained from Theorem 3.1 (star symbol) are better than the ones obtained from Theorems 2–4 of [7]. To facilitate our observation, we remove Theorem 3 of [7], see Figure 2.

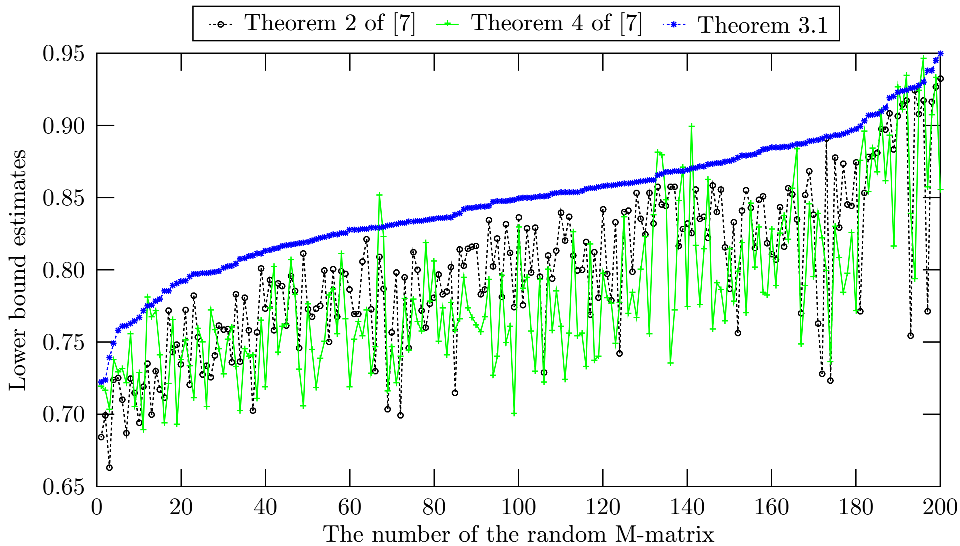

Figure 2. The lower bound estimations derived from Theorems 2–4 of [7] and Theorem 3.1.

In Figure 2, we increase the number of random M-matrices to 200. And the conclusion is consistent with what we have said before, i.e., in most cases, the lower bounds obtained from Theorem 3.1 are better than the ones from Theorems 2 and 4 of [7]. Then we compare the estimate value from Theorem 3.1 with the one from Theorems 2–4 of [7], by subtracting the latter from the former, see Figures 3–5.

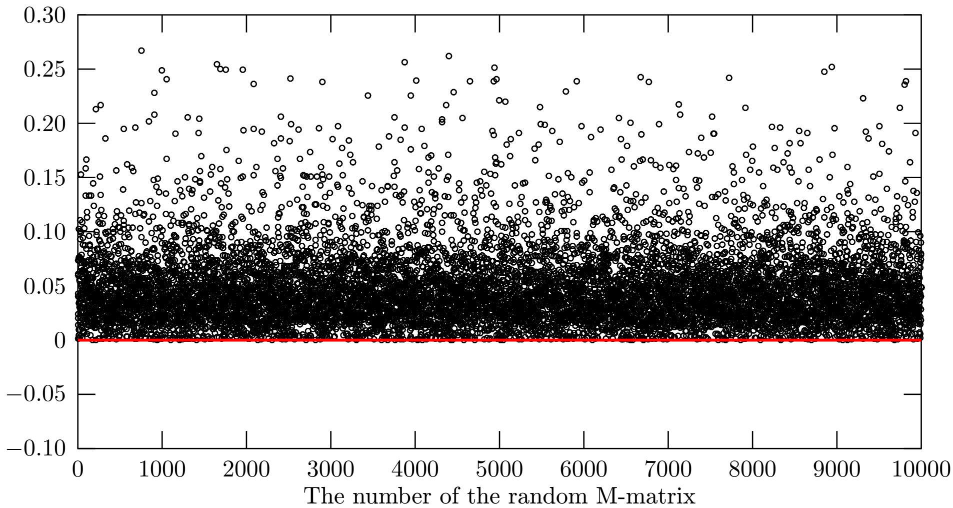

Figure 3. Subtract the lower bounds given by Theorem 2 of [7] from the lower bounds given by Theorem 3.1.

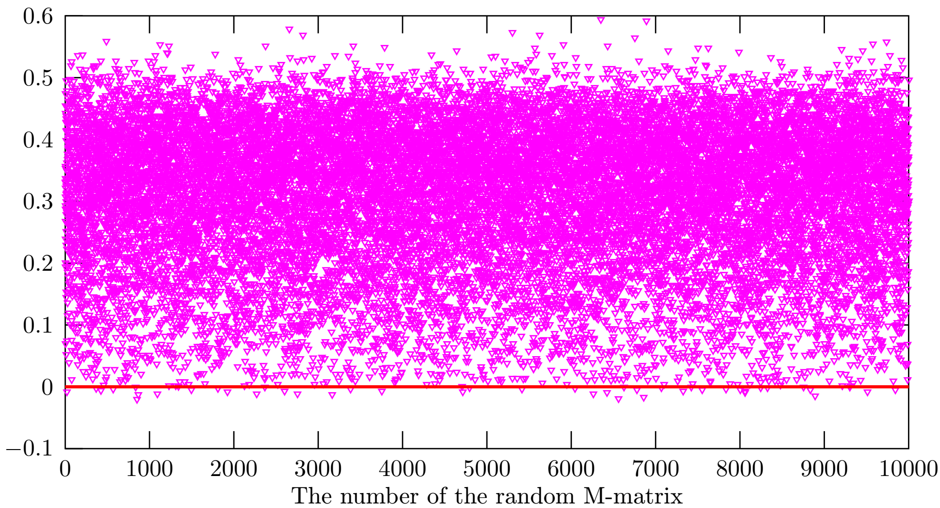

Figure 4. Subtract the lower bounds given by Theorem 3 of [7] from the lower bounds given by Theorem 3.1.

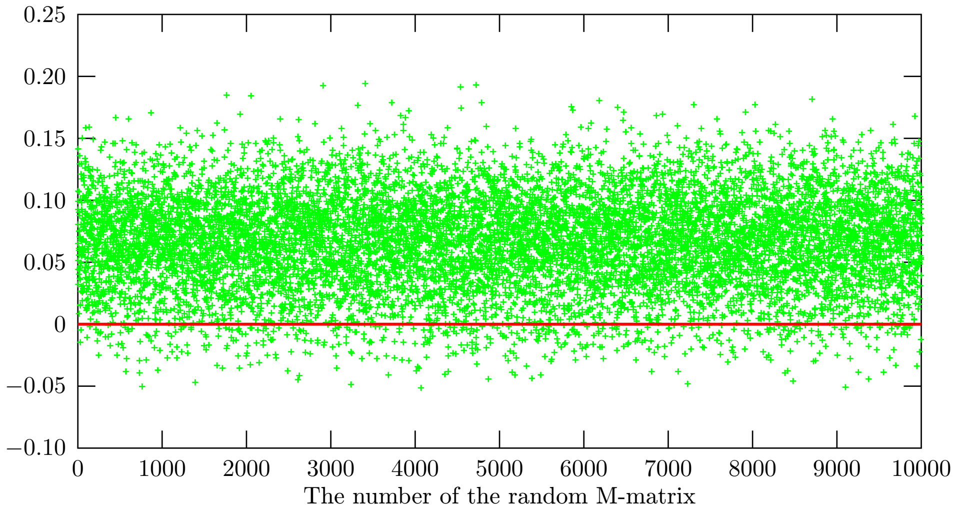

Figure 5. Subtract the lower bounds given by Theorem 4 of [7] from the lower bounds given by Theorem 3.1

In Figures 3–5, we construct randomly 10000 M-matrices, respectively. We can see that most symbols are above the X-axis. It means that the estimated value for lower bound of $\tau(A \circ A^{-1})$ from Theorem 3.1 is larger than the one from Theorems 2–4 of [7]. We count the total number of points above the X-axis, so we can figure out the proportion of points above the X-axis. The result is, that in Figures 2–4, the rates of points above the X-axis are 1.0000, 0.9984, 0.9545, respectively. The superiority of Theorem 3.1 is explained.

5. Conclusion

This paper concerns $\tau(A \circ A^{-1})$, the minimum eigenvalue of Hadamard product of an M-matrix and its inverse, in which A is a nonsingular M-matrix. By using S-type eigenvalue inclusion set and the inequality scaling techniques, the lower bound sequence of $\tau(A \circ A^{-1})$ is obtained, and the convergence of the sequence is proved and the algorithm is given. Finally, we illustrate the superiority and validity of the theorem by numerical experiments with examples presented in other literatures and randomly generated M-matrices. In conclusion, we obtain a superior lower bound sequence for $\tau(A \circ A^{-1})$.

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Authors' addresses: Wenlong Zeng, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, P. R. China, e-mail: `xtuzwl@163.com`; Jianzhou Liu (corresponding author), School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, P. R. China, Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Hunan 411105, P. R. China, e-mail: `liujz@xtu.edu.cn`.