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QUASI-TRACE FUNCTIONS ON LIE ALGEBRAS AND THEIR APPLICATIONS TO 3-LIE ALGEBRAS

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Abstract. We introduce the notion of quasi-trace functions on Lie algebras. As applications we study realizations of 3-dimensional and 4-dimensional 3-Lie algebras. Some comparison results on cohomologies of 3-Lie algebras and Leibniz algebras arising from quasi-trace functions are obtained.

Keywords: quasi-trace function; 3-Lie algebra; Leibniz algebra

MSC 2020: 17B05, 17A42, 17A32, 17B56

1. Introduction

An n-ary groupoid G is a nonempty set with an n-ary operation $f\colon G^n \to G$, see [12]. One may define various $(n-1)$-ary operations on G via f. For example, if G is an n-Lie algebra, then some $(n-1)$-Lie algebras can be defined on G by the method given in [15]. However, in general there seems no apparent construction of n-ary groupoids with some specific properties from $(n-1)$-ary groupoids. For example, there are 3-groups which cannot be derived from any groups, see [12].

This paper is motivated by the construction of 3-Lie algebras from Lie algebras. A vector space L with a 3-ary multilinear skew-symmetric operation $[\cdot,\cdot,\cdot]\colon \otimes^3 L \to L$ is a 3-Lie algebra if

$$\begin{aligned}(1.1)\qquad [\mathrm{x}_1,\mathrm{x}_2,[\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5]] - [\mathrm{x}_3,\mathrm{x}_4,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5]] \\ = [[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3],\mathrm{x}_4,\mathrm{x}_5] + [\mathrm{x}_3,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4],\mathrm{x}_5]\end{aligned}$$

holds for all $\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5 \in L$, see [15]. The identity (1.1) is called the *fundamental identity* (FI for short). Subalgebras and homomorphisms between 3-Lie algebras are

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defined in the obvious way, while an ideal I of a 3-Lie algebra L is a subspace of L satisfying $[I, L, L] \subseteq I$, and the center $Z(L)$ is defined to be the subspace $Z(L) = \{\mathrm{x} \in L\colon\ [\mathrm{x}, \mathrm{y}, \mathrm{z}] = 0 \text{ for all } \mathrm{y}, \mathrm{z} \in L\}$. 3-Lie algebras have a close relation with Nambu mechanics, see [23]. For an extensive review of 3-Lie algebras, see [10].

In [5] a 3-ary operation on the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is introduced to make $\mathfrak{gl}_n(\mathbb{C})$ be a 3-Lie algebra by defining

$$[A, B, C] = \operatorname{tr}(A)[B, C] + \operatorname{tr}(B)[C, A] + \operatorname{tr}(C)[A, B],$$

where tr denotes the trace of square matrices. Note that tr is a linear function on $\mathfrak{gl}_n(\mathbb{C})$ satisfying $\operatorname{tr}([A, B]) = 0$ for any $A, B \in \mathfrak{gl}_n(\mathbb{C})$. This construction was generalized in [4], [6] as follows. Let $\mathfrak{g}$ be a Lie algebra with the bracket $[\cdot, \cdot]$ and $\tau \in \mathfrak{g}^*$ a linear function on $\mathfrak{g}$. Define a 3-ary bracket $[\cdot, \cdot, \cdot]_\tau$ on $\mathfrak{g}$ by

$$(1.2) \qquad [\mathrm{x}, \mathrm{y}, \mathrm{z}]_\tau \triangleq \underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})[\mathrm{y}, \mathrm{z}] \quad \forall\, \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{g}.$$

Hereafter $\underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft}$ denotes the summation over the cyclic permutations of x, y, z:

$$\underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})[\mathrm{y}, \mathrm{z}] = \tau(\mathrm{x})[\mathrm{y}, \mathrm{z}] + \tau(\mathrm{y})[\mathrm{z}, \mathrm{x}] + \tau(\mathrm{z})[\mathrm{x}, \mathrm{y}].$$

Denote the 3-ary groupoid $\mathfrak{g}$ with $[\cdot, \cdot, \cdot]_\tau$ by $\mathfrak{g}_\tau$. If τ is a trace function on $\mathfrak{g}$, that is, $\tau([\mathfrak{g}, \mathfrak{g}]) = 0$, then $\mathfrak{g}_\tau$ is a 3-Lie algebra (see [6], Theorem 3.1 and [4], Theorem 3.3). We denote by $F_{\mathrm{tr}}(\mathfrak{g})$ the set of all trace functions on $\mathfrak{g}$.

The notion of trace functions is closely related to the notion of "subordinate" on Lie subalgebras. Recall that, for a $\tau \in \mathfrak{g}^*$ and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathfrak{h}$ is subordinate to τ if $\tau([\mathfrak{h}, \mathfrak{h}]) = 0$ (see [11], Section 1.12.7). So, $\tau \in F_{\mathrm{tr}}(\mathfrak{g})$ if and only if $\mathfrak{g}$ itself is subordinate to τ.

Trace functions are not enough to induce 3-Lie algebras. For example, as Corollary 3.1 below shows, the unique nonabelian 3-dimensional 3-Lie algebra cannot be induced, using only trace functions, from all except one isoclass of 3-dimensional nonabelian Lie algebras. For other examples see Corollary 4.2.

For any $\tau \in \mathfrak{g}^*$, a sufficient and necessary condition for $\mathfrak{g}_\tau$ to be a 3-Lie algebra is given in Theorem 2.1. Denote the set of those linear functions by $F_{\text{3-Lie}}(\mathfrak{g})$. We show that if $\mathfrak{g}$ is solvable then $\mathfrak{g}_\tau$ ($\tau \in F_{\text{3-Lie}}(\mathfrak{g})$) is also solvable (see Proposition 2.4), and if $\dim \mathfrak{g} \leqslant 3$ then $F_{\text{3-Lie}}(\mathfrak{g}) = \mathfrak{g}^*$, see Lemma 2.1.

Note that $\tau \in \mathfrak{g}^*$ is a trace function if and only if $\ker \tau$ is an ideal of $\mathfrak{g}$. We consider the weaker condition that $\ker \tau$ is just a subalgebra of $\mathfrak{g}$. It is shown that $\ker \tau$ is a subalgebra of $\mathfrak{g}$ if and only if $\underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})\tau([\mathrm{y}, \mathrm{z}]) = 0$ for any x, y, z $\in \mathfrak{g}$, which implies that such a τ also makes $\mathfrak{g}_\tau$ be a 3-Lie algebra. We call $\tau \in \mathfrak{g}^*$ a quasi-trace function on $\mathfrak{g}$ if $\ker\tau$ is a subalgebra of $\mathfrak{g}$. If τ is a quasi-trace function, then $\ker\tau$ is a quasi-ideal in the sense of Amayo, see [1], [2]. Denote the set of all quasi-trace functions on $\mathfrak{g}$ by $F_{\mathrm{qtr}}(\mathfrak{g})$. So we have the inclusions

$$F_{\mathrm{tr}}(\mathfrak{g}) \subseteq F_{\mathrm{qtr}}(\mathfrak{g}) \subseteq F_{\text{3-Lie}}(\mathfrak{g}) \subseteq \mathfrak{g}^*.$$

Quasi-trace functions are related with Leibniz algebras, which we explain briefly as follows. For any $\tau \in \mathfrak{g}^*$ there is a map $\tau^\sharp\colon \wedge^2\mathfrak{g} \to \mathfrak{g}$ given by

$$\tau^\sharp(\mathrm{x}\wedge\mathrm{y}) = \tau(\mathrm{x})\mathrm{y} - \tau(\mathrm{y})\mathrm{x} \quad \forall\,\mathrm{x},\mathrm{y}\in\mathfrak{g}.$$

Following [9], for any vector space L with a 3-ary bracket $[\cdot,\cdot,\cdot]$ we have the following notation: for $\mathrm{X} = \mathrm{x}_1\wedge\mathrm{x}_2$, $\mathrm{Y} = \mathrm{y}_1\wedge\mathrm{y}_2 \in \wedge^2 L$, $\mathrm{x}_3 \in L$, set

$$(1.3)\qquad [\mathrm{X},\mathrm{x}_3] := [\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3] \in L, \quad [\mathrm{X},\mathrm{Y}]_F = [\mathrm{X},\mathrm{y}_1]\wedge\mathrm{y}_2 + \mathrm{y}_1\wedge[\mathrm{X},\mathrm{y}_2] \in \wedge^2 L.$$

Due to Daletskii and Takhtajan (see [9]), if L is a 3-Lie algebra then $\wedge^2 L$ is a Leibniz algebra with the bracket $[\cdot,\cdot]_F$.

It is shown that τ is a quasi-trace function if and only if $\tau^\sharp$ preserves Leibniz brackets, that is, $\tau^\sharp([\mathrm{X},\mathrm{Y}]_F) = [\tau^\sharp(X),\tau^\sharp(\mathrm{Y})]$ for any X, $\mathrm{Y}\in\wedge^2\mathfrak{g}$ (see Lemma 6.2). In this case, $\tau^\sharp$ is a homomorphism of Leibniz algebras from $\wedge^2\mathfrak{g}_\tau$ to $\mathfrak{g}$, where $\mathfrak{g}$ is regarded as a Leibniz algebra.

Based on this observation we consider further the connection between quasi-trace functions and universal enveloping algebras of Lie algebras. In [16], for any 3-Lie algebra L an associative algebra $\mathrm{U}(L)$ is introduced as an analogue of universal enveloping algebras of Lie algebras. For any $\tau \in F_{\text{3-Lie}}(\mathfrak{g})$ we show that $\tau^\sharp$ induces a homomorphism of associative algebras from $\mathrm{U}(\mathfrak{g}_\tau)$ to $\mathrm{U}(\mathfrak{g})$ if and only if τ is a quasi-trace function on $\mathfrak{g}$, where $\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, see Theorem 6.1. This result motivates us to consider some representation theoretic connections between 3-Lie algebras and Lie algebras via quasi-trace functions.

It turns out that, for any quasi-trace function τ on $\mathfrak{g}$, one can construct a representation (V,ϱ_τ) of the 3-Lie algebra $\mathfrak{g}_\tau$ from a representation (V,ϱ) of $\mathfrak{g}$, see Corollary 6.2. Then we consider connections between the Cartan-Eilenberg cohomology $H_\varrho(\mathfrak{g},V)$ and the cohomology $\mathcal{H}^*_{\varrho_\tau}(\mathfrak{g}_\tau,V)$. As partial results we construct 1-cocycles and 2-cocycles for $\mathfrak{g}_\tau$ in V from those of $\mathfrak{g}$ in V. For details, see Propositions 7.1 and 7.2. For a trace function τ, a similar construction of 1-cocycles and 2-cocycles for $\mathfrak{g}_\tau$ from $\mathfrak{g}$ is considered in [3] for the trivial representation and the adjoint representation of $\mathfrak{g}_\tau$. Note that the adjoint representation of $\mathfrak{g}_\tau$ cannot be induced in general from the adjoint representation of $\mathfrak{g}$, since, for example, the algebra homomorphism $\tau^\sharp\colon \mathrm{U}(\mathfrak{g}_\tau) \to \mathrm{U}(\mathfrak{g})$ has the image $\mathrm{U}(\ker\tau)$ (see Corollary 6.1), and hence $\tau^\sharp$ cannot be surjective.

Note that the cohomology $\mathcal{H}^*_\theta(L,V)$ of a 3-Lie algebra L is deduced from the cohomology $HL^*_{l,r}(\wedge^2 L, \mathrm{Hom}(L,V))$ of its associated Leibniz algebra $\wedge^2 L$ given by [21], where (V,θ) is a representation of L and $(\mathrm{Hom}(L,V),l,r)$ is the representation of $\wedge^2 L$ induced from θ. For a brief review see Section 5, where we also give a construction of morphisms from $\mathcal{H}^*_\theta(L,V)$ to $HL^*_{\theta,-\theta}(\wedge^2 L,V)$, see Proposition 5.1. $\mathcal{H}^*_\theta(L,V)$ has been applied to study extensions and deformation of L, see, for example, [14], [19], [22], [24], [25]. Due to [16] there is another cohomology $\mathrm{H}^*(L,V)$ of L defined via the invariant submodule functor. The relation between $\mathcal{H}^*_\theta(L,V)$ and $\mathrm{H}^*(L,V)$ remains open.

For the 3-Lie algebra $\mathfrak{g}_\tau$ (τ being a quasi-trace function on $\mathfrak{g}$) and a representation (V,ϱ) of $\mathfrak{g}$, $\mathrm{H}^*(\mathfrak{g}_\tau,V)$ is related to $H^*_\varrho(\mathfrak{g},V)$ via the algebra homomorphism $\tau^\sharp\colon \mathrm{U}(\mathfrak{g}_\tau)\to\mathrm{U}(\mathfrak{g})$. As an example, if $\mathrm{U}(\mathfrak{g})$ is a projective module of $\mathrm{U}(\mathfrak{g}_\tau)$ via $\tau^\sharp$, then $\mathrm{H}^*(\mathfrak{g}_\tau,V)\cong H^*_\varrho(\mathfrak{g},V)$, see Corollary 6.3. We don't know whether there is a natural morphism from $H^*_\varrho(\mathfrak{g},V)$ to $\mathcal{H}^*_{\varrho_\tau}(\mathfrak{g}_\tau,V)$, but we show that τ induces a morphism from $H^*_\varrho(\mathfrak{g},V)$ to $HL^*_{\varrho_\tau,-\varrho_\tau}(\wedge^2\mathfrak{g}_\tau,V)$, see Proposition 7.3.

We consider only low-dimensional 3-Lie algebras which can be induced from Lie algebras via linear functions. As mentioned earlier, 3-dimensional and 4-dimensional 3-Lie algebras have been studied via some specified Lie algebras and trace functions in [3], [6]. Let $L_{3,1}$ be the unique nonabelian 3-dimensional 3-Lie algebra. We show that for each nonabelian 3-dimensional Lie algebra $\mathfrak{g}$ there is a $\tau\in\mathfrak{g}^*$ such that $L_{3,1}\cong\mathfrak{g}_\tau$. We list all quasi-trace functions on each isoclass of 3-dimensional Lie algebras which induce $L_{3,1}$. For details see Theorem 3.1, Corollaries 3.1 and 3.2 below.

Let $\mathfrak{g}$ be any complex 4-dimensional Lie algebra. We classify all 3-Lie algebras of the form $\mathfrak{g}_\tau$ with τ being a quasi-trace function on $\mathfrak{g}$, see Theorem 4.1. To do this we make a little refinement (see Corollary 4.1) on the classification of complex 4-dimensional 3-Lie algebras given by Filippov, see [15]. We obtain a complete list of all quasi-trace functions for each isoclass of complex 4-dimensional Lie algebras and their induced 3-Lie algebras, see Corollary 4.3. Note that the simple complex 4-dimensional 3-Lie algebra, which is unique up to isomorphism, cannot be realized as $\mathfrak{g}_\tau$ for any Lie algebra $\mathfrak{g}$ and any quasi-trace function τ, since such 3-Lie algebras are always solvable, see Proposition 2.3.

The paper is organized as follows. In Section 2 we introduce the notion of quasi-trace functions on Lie algebras and discuss some basic properties including solvability of 3-Lie algebras induced by linear functions on Lie algebras. In Section 3 we use linear functions, especially quasi-trace functions, to realize 3-dimensional 3-Lie algebras via each isoclass of 3-dimensional Lie algebras. In Section 4 we classify all 4-dimensional 3-Lie algebras of the form $\mathfrak{g}_\tau$, where τ is a quasi-trace function on $\mathfrak{g}$. In Section 5 we review representations and cohomologies of 3-Lie algebras and their associated Leibniz algebras. In Section 6 we show that a linear function on $\mathfrak{g}$ is a quasi-trace function if and only if $\tau^\sharp$ is a homomorphism of Leibniz algebras from $\wedge^2\mathfrak{g}_\tau$ to $\mathfrak{g}$, if and only if $\tau^\sharp$ induces a homomorphism of associative algebras from $\mathrm{U}(\mathfrak{g}_\tau)$ to $\mathrm{U}(\mathfrak{g})$, from which we construct a representation of $\mathfrak{g}_\tau$ from those of $\mathfrak{g}$. In Section 7 we obtain some results on comparison of cohomologies via quasi-trace functions.

Throughout we work on the complex number field $\mathbb{C}$. Notations such as Hom, End, $\oplus$, $\wedge$ are defined over $\mathbb{C}$.

2. Linear functions and their induced 3-Lie algebras

Let $\mathfrak{g}$ be a Lie algebra which may be infinite-dimensional. Let $\tau \in \mathfrak{g}^*$ be a linear function on $\mathfrak{g}$. Then $\operatorname{codim}\ker\tau = \dim\mathfrak{g}/\ker\tau \leqslant 1$. By [8], Lemma 2.1, τ is a representation of $\mathfrak{g}$ on $\mathbb{C}$ if and only if $\tau([\mathfrak{g},\mathfrak{g}]) = 0$, hence if and only if $\ker\tau$ is an ideal of $\mathfrak{g}$. Such linear functions are called trace functions, see [4]. Let $F_{\mathrm{tr}}(\mathfrak{g})$ be the set of trace functions on $\mathfrak{g}$. Then $F_{\mathrm{tr}}(\mathfrak{g})$ is a subspace of $\mathfrak{g}^*$.

Example 2.1. If $\mathfrak{g}$ is a perfect Lie algebra then $F_{\mathrm{tr}}(\mathfrak{g}) = \{0\}$.

Proposition 2.1. *Let $\mathfrak{g}$ be a Lie algebra and $\tau \in \mathfrak{g}^*$. Then $\ker\tau$ is a subalgebra of $\mathfrak{g}$ if and only if τ satisfies*

$$\circlearrowleft_{\mathrm{x,y,z}} \tau(\mathrm{x})\tau([\mathrm{y},\mathrm{z}]) = 0 \quad \forall\, \mathrm{x},\mathrm{y},\mathrm{z} \in \mathfrak{g}. \tag{2.1}$$

Proof. Assume that $\ker\tau$ is a subalgebra of $\mathfrak{g}$. If $\ker\tau = \mathfrak{g}$ then (2.1) follows. Suppose that $\ker\tau \neq \mathfrak{g}$. Then $\operatorname{codim}\ker\tau = 1$, and hence there is a $\mathrm{u} \in \mathfrak{g} \setminus \ker\tau$ such that $\mathrm{x},\mathrm{y},\mathrm{z} \in \mathfrak{g}$ have the form $\mathrm{x} = \mathrm{x}' + a\mathrm{u}$, $\mathrm{y} = \mathrm{y}' + b\mathrm{u}$, $\mathrm{z} = \mathrm{z}' + c\mathrm{u}$, where $\mathrm{x}',\mathrm{y}',\mathrm{z}' \in \ker\tau$ and $a,b,c \in \mathbb{C}$. So $\tau(\mathrm{x}) = a\tau(\mathrm{u})$, $\tau(\mathrm{y}) = b\tau(\mathrm{u})$, $\tau(\mathrm{z}) = c\tau(\mathrm{u})$. Since $\ker\tau$ is a subalgebra of $\mathfrak{g}$, $\tau([\mathrm{y}',\mathrm{z}']) = 0$. Hence

$$\tau(\mathrm{x})\tau([\mathrm{y},\mathrm{z}]) = a\tau(\mathrm{u})\tau([\mathrm{y}'+b\mathrm{u},\mathrm{z}'+c\mathrm{u}]) = ac\tau(\mathrm{u})\tau([\mathrm{y}',\mathrm{u}]) + ab\tau(\mathrm{u})\tau([\mathrm{u},\mathrm{z}']). \tag{2.2}$$

Similarly, we have

$$\tau(\mathrm{y})\tau([\mathrm{z},\mathrm{x}]) = bc\tau(\mathrm{u})\tau([\mathrm{u},\mathrm{x}']) + ab\tau(\mathrm{u})\tau([\mathrm{z}',\mathrm{u}]), \tag{2.3}$$

$$\tau(\mathrm{z})\tau([\mathrm{x},\mathrm{y}]) = bc\tau(\mathrm{u})\tau([\mathrm{x}',\mathrm{u}]) + ac\tau(\mathrm{u})\tau([\mathrm{u},\mathrm{y}']). \tag{2.4}$$

Then (2.1) follows by (2.2), (2.3) and (2.4).

Conversely, assume that (2.1) holds. Fix any $\mathrm{x},\mathrm{y} \in \ker\tau$. It suffices to show that $\tau([\mathrm{x},\mathrm{y}]) = 0$. Without loss of generality we assume that $\ker\tau \neq \mathfrak{g}$. Then there exists an element $\mathrm{z} \in \mathfrak{g}$ such that $\tau(\mathrm{z}) \neq 0$. By $\tau(\mathrm{x}) = \tau(\mathrm{y}) = 0$, $\tau(\mathrm{z}) \neq 0$ and (2.1) it follows that $\tau([\mathrm{x},\mathrm{y}]) = 0$ as required. $\square$

Motivated by Proposition 2.1 and trace functions, we introduce the following definition.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. A linear function $\tau \in \mathfrak{g}^*$ is called a *quasi-trace function* on $\mathfrak{g}$ if τ satisfies (2.1), i.e., τ is a quasi-trace function on $\mathfrak{g}$ if and only if $\ker \tau$ is a subalgebra of $\mathfrak{g}$.

Let $F_{\mathrm{qtr}}(\mathfrak{g})$ be the set of quasi-trace functions on a Lie algebra $\mathfrak{g}$. Note that all trace functions on $\mathfrak{g}$ are quasi-trace functions on $\mathfrak{g}$, that is, $F_{\mathrm{tr}}(\mathfrak{g}) \subseteq F_{\mathrm{qtr}}(\mathfrak{g})$.

Example 2.2. Consider the Lie algebra $\mathfrak{sl}_2$ with a basis $\{e_1, e_2, e_3\}$ such that $[e_1, e_2] = e_3$, $[e_1, e_3] = -2e_1$, $[e_2, e_3] = 2e_2$. Since $\mathfrak{sl}_2 = [\mathfrak{sl}_2, \mathfrak{sl}_2]$ there is no nonzero trace function on $\mathfrak{sl}_2$. Define $\tau \in (\mathfrak{sl}_2)^*$ by $\tau(e_1) = \tau(e_3) = 0$, $\tau(e_2) = 1/2$. Then $\tau \in F_{\mathrm{qtr}}(\mathfrak{sl}_2)$.

Example 2.3. Let $\alpha\colon \mathfrak{g} \to \widetilde{\mathfrak{g}}$ be a homomorphism of Lie algebras. For any $\widetilde{\tau} \in F_{\mathrm{qtr}}(\widetilde{\mathfrak{g}})$ it holds that $\widetilde{\tau}\alpha \in F_{\mathrm{qtr}}(\mathfrak{g})$. Indeed, for any $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{g}$, by a direct computation one obtains

$$\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}} (\widetilde{\tau}\alpha)(\mathrm{x})(\widetilde{\tau}\alpha)([\mathrm{y},\mathrm{z}]) = \circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}} \widetilde{\tau}(\alpha(\mathrm{x}))\widetilde{\tau}([\alpha(\mathrm{y}), \alpha(\mathrm{z})]) = 0,$$

which means that $\widetilde{\tau}\alpha$ satisfies (2.1).

By Proposition 2.1 we have the following result, which is crucial for our further computations.

Corollary 2.1. $\tau \in F_{\mathrm{qtr}}(\mathfrak{g})$ *if and only if*

$$\tau([\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3]_\tau) = 0 \quad \forall\, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3 \in \mathfrak{g}, \tag{2.5}$$

where the 3*-ary bracket* $[\cdot,\cdot,\cdot]_\tau$ *is given by* (1.2).

We give a sufficient and necessary condition on any $\tau \in \mathfrak{g}^*$ such that the 3-ary bracket given by (1.2) makes $\mathfrak{g}_\tau$ a 3-Lie algebra.

Theorem 2.1. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\tau \in \mathfrak{g}^*$. *Then* $\mathfrak{g}_\tau$ *is a* 3*-Lie algebra if and only if for all* $\mathrm{x}_i \in \mathfrak{g}$, *the following identity holds:*

$$\begin{aligned}(\circlearrowleft_{\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5} \tau(\mathrm{x}_3)\tau([\mathrm{x}_4, \mathrm{x}_5]))[\mathrm{x}_1, \mathrm{x}_2] - (\circlearrowleft_{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5} \tau(\mathrm{x}_1)\tau([\mathrm{x}_2, \mathrm{x}_5]))[\mathrm{x}_3, \mathrm{x}_4] \\ = (\circlearrowleft_{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3} \tau(\mathrm{x}_1)\tau([\mathrm{x}_2, \mathrm{x}_3]))[\mathrm{x}_4, \mathrm{x}_5] + (\circlearrowleft_{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4} \tau(\mathrm{x}_1)\tau([\mathrm{x}_2, \mathrm{x}_4]))[\mathrm{x}_5, \mathrm{x}_3].\end{aligned} \tag{2.6}$$

In this case we say $\mathfrak{g}_\tau$ *is a* 3*-Lie algebra induced by* $\mathfrak{g}$ *and* τ.

P r o o f. Since the Lie bracket is skew symmetric, the bracket $[\cdot,\cdot,\cdot]_\tau$ given by (1.2) is also skew symmetric. By (1.1) the FI for $[\cdot,\cdot,\cdot]_\tau$ is

$$\begin{aligned}(2.7)\qquad [\mathrm{x}_1,\mathrm{x}_2,[\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5]_\tau]_\tau-[\mathrm{x}_3,\mathrm{x}_4,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5]_\tau]_\tau \\ = [[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau,\mathrm{x}_4,\mathrm{x}_5]_\tau+[\mathrm{x}_3,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4]_\tau,\mathrm{x}_5]_\tau.\end{aligned}$$

By a direct check we decuce that (2.7) is equivalent to (2.6) due to (1.2) and the Jacobi identity of $\mathfrak{g}$. □

Since (2.1) implies (2.6), by Theorem 2.1 we get the following result, which generalizes Theorem 3.1 in [6].

Corollary 2.2. *Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie algebra. If $\tau\in F_{\mathrm{qtr}}(\mathfrak{g})$, that is, $\tau\in\mathfrak{g}$ is a quasi-trace function on $\mathfrak{g}$, then $\mathfrak{g}_\tau$ is a 3-Lie algebra.*

Remark 2.1. Let $\alpha\colon\ \mathfrak{g}\to\widetilde{\mathfrak{g}}$ be a homomorphism of Lie algebras. For $\tau\in F_{\mathrm{qtr}}(\mathfrak{g})$, $\widetilde{\tau}\in F_{\mathrm{qtr}}(\widetilde{\mathfrak{g}})$, $\alpha\colon\ \mathfrak{g}_\tau\to\widetilde{\mathfrak{g}}_{\widetilde{\tau}}$ need not be a 3-Lie algebra homomorphism.

Example 2.4. Let $\alpha\colon\ \mathfrak{g}\to\mathfrak{g}$ be a Lie algebra endomorphism and $\tau\in F_{\mathrm{qtr}}(\mathfrak{g})$. Then $\tau\alpha\in F_{\mathrm{qtr}}(\mathfrak{g})$ by Example 2.3. One can show that, if $(\mathbf{1}_{\mathfrak{g}}-\alpha^{\mathbf{2}})(\mathfrak{g})\subseteq\ker\tau$ then α is a 3-Lie algebra homomorphism. In particular, if α is an involution of $\mathfrak{g}$, then α is a 3-Lie algebra homomorphism.

Let $F_{\text{3-Lie}}(\mathfrak{g})$ be the set of linear functions on $\mathfrak{g}$ satisfying (2.6), that is, $\tau\in F_{\text{3-Lie}}(\mathfrak{g})$ if and only if $\mathfrak{g}_\tau$ is a 3-Lie algebra. Then

$$(2.8)\qquad F_{\mathrm{tr}}(\mathfrak{g})\subseteq F_{\mathrm{qtr}}(\mathfrak{g})\subseteq F_{\text{3-Lie}}(\mathfrak{g})\subseteq\mathfrak{g}^*.$$

Remark 2.2. Let $\mathfrak{g}$ be a Lie algebra. In general it is difficult to compute $F_{\text{3-Lie}}(\mathfrak{g})$. It might be an interesting question whether a 3-Lie algebra can be induced by $\mathfrak{g}$ and some $\tau\in F_{\text{3-Lie}}(\mathfrak{g})$. Note that different functions in $F_{\text{3-Lie}}(\mathfrak{g})$ may induce isomorphic 3-Lie algebras. We shall discuss 3-dimensional and 4-dimensional 3-Lie algebras in Section 3 and Section 4, respectively.

Example 2.5. If $\mathfrak{g}$ is abelian then $F_{\mathrm{tr}}(\mathfrak{g})=F_{\mathrm{qtr}}(\mathfrak{g})=F_{\text{3-Lie}}(\mathfrak{g})=\mathfrak{g}^*$.

Before we give more examples in Section 3 and Section 4, we present the following examples to show that inclusions in (2.8) may be proper. We use the following notations.

Notation 2.1. *For a Lie algebra $\mathfrak{g}$ with a basis $\{e_i\}_{1\leqslant i\leqslant\dim\mathfrak{g}}$, we denote the corresponding coordinate functions by $\{t_i\}_{1\leqslant i\leqslant\dim\mathfrak{g}}$ and denote the coordinate of $\mathrm{x}\in\mathfrak{g}$ by $(x_i)_{1\leqslant i\leqslant\dim\mathfrak{g}}$.*

Notation 2.2. *In the definition of a Lie algebra or a 3-Lie algebra via the multiplication table of basis elements, omitted brackets are either zero or can be obtained by skew-symmetry.*

Example 2.6. Let $\mathfrak{g}$ be the Lie algebra with a basis $\{e_1, e_2, e_3\}$ and the multiplication table $[e_1, e_2] = e_3$, $[e_1, e_3] = -2e_1$, $[e_2, e_3] = 2e_2$. Then

$$\begin{aligned} F_{\rm tr}(\mathfrak{g}) &= \{0\}, \\ F_{\rm qtr}(\mathfrak{g}) &= \{\tau \in \mathfrak{g}^*\colon\ \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, 4t_1t_2 + t_3^2 = 0\}, \\ F_{\text{3-Lie}}(\mathfrak{g}) &= \mathfrak{g}^*. \end{aligned}$$

Example 2.7. Let $\mathfrak{g}$ be the 3-dimensional Lie algebra with a basis $\{e_1, e_2, e_3\}$ and the multiplication table $[e_1, e_2] = e_2$. Then

$$\begin{aligned} F_{\rm tr}(\mathfrak{g}) &= \{\tau \in \mathfrak{g}^*\colon\ \tau(\mathrm{x}) = t_1x_1 + t_3x_3\}, \\ F_{\rm qtr}(\mathfrak{g}) &= \{\tau \in \mathfrak{g}^*\colon\ \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, t_2t_3 = 0\}, \\ F_{\text{3-Lie}}(\mathfrak{g}) &= \mathfrak{g}^*. \end{aligned}$$

Example 2.8. Let $\mathfrak{g}$ be a 2-dimensional Lie algebra. By Example 2.5 we may assume that $\mathfrak{g}$ has a basis $\{e_1, e_2\}$ with $[e_1, e_2] = e_2$. Then

$$F_{\rm tr}(\mathfrak{g}) = \{\tau \in \mathfrak{g}^*\colon\ \tau(\mathrm{x}) = t_1x_1\}, \quad F_{\rm qtr}(\mathfrak{g}) = \mathfrak{g}^* = F_{\text{3-Lie}}(\mathfrak{g}).$$

It is not accidental that $F_{\text{3-Lie}}(\mathfrak{g}) = \mathfrak{g}^*$ holds in Examples 2.6 and 2.7, since we have the following result which will also be used in Section 3.

Lemma 2.1. *Let* $\mathfrak{g}$ *be a Lie algebra with* $\dim \mathfrak{g} \leqslant 3$. *Then* $F_{\text{3-Lie}}(\mathfrak{g}) = \mathfrak{g}^*$, *that is, for any* $\tau \in \mathfrak{g}^*$, $\mathfrak{g}_\tau$ *is a 3-Lie algebra.*

P r o o f. By Examples 2.5 and 2.8, we may assume that $\dim \mathfrak{g} = 3$. Let $\{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$. By linearity it suffices to check that (2.6) holds for any $\mathrm{x}_i \in \{e_1, e_2, e_3\}$, $1 \leqslant i \leqslant 5$. There are the following two exclusive cases.

Case 1: There exist at least three elements $\mathrm{x}_i, \mathrm{x}_j, \mathrm{x}_k$ which are equal, $1 \leqslant i, j, k \leqslant 5$. Without loss of generality, suppose that $\mathrm{x}_1 = \mathrm{x}_2 = \mathrm{x}_3 = e_1$. Then

$$\begin{aligned} &\Big(\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5}{\circlearrowleft} \tau(\mathrm{x}_1)\tau([\mathrm{x}_2, \mathrm{x}_5])\Big)[\mathrm{x}_3, \mathrm{x}_4] \\ &\qquad = (\tau(e_1)\tau([e_1, \mathrm{x}_5]) + \tau(e_1)\tau([\mathrm{x}_5, e_1]) + \tau(\mathrm{x}_5)\tau([e_1, e_1]))[\mathrm{x}_3, \mathrm{x}_4] = 0, \end{aligned}$$

and hence the left hand side of (2.6) becomes

$$\begin{aligned} &\Big(\underset{\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5}{\circlearrowleft} \tau(\mathrm{x}_3)\tau([\mathrm{x}_4, \mathrm{x}_5])\Big)[\mathrm{x}_1, \mathrm{x}_2] - \Big(\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5}{\circlearrowleft} \tau(\mathrm{x}_1)\tau([\mathrm{x}_2, \mathrm{x}_5])\Big)[\mathrm{x}_3, \mathrm{x}_4] \\ &\qquad = \Big(\underset{\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5}{\circlearrowleft} \tau(\mathrm{x}_3)\tau([\mathrm{x}_4, \mathrm{x}_5])\Big)[e_1, e_1] - 0 = 0. \end{aligned}$$

By a similar computation we get $(\underset{x_1,x_2,x_4}{\circlearrowleft}\tau(x_1)\tau([x_2,x_4]))[x_5,x_3]=0$, and hence the right hand side of (2.6) is

$$(\underset{x_1,x_2,x_3}{\circlearrowleft}\tau(x_1)\tau([x_2,x_3]))[x_4,x_5]+(\underset{x_1,x_2,x_4}{\circlearrowleft}\tau(x_1)\tau([x_2,x_4]))[x_5,x_3]$$
$$=3\tau(e_1)\tau([e_1,e_1])[x_4,x_5]+0=0.$$

So (2.6) holds in this case.

Case 2: There exist at most two elements which are equal. For simplicity, we consider only the subcase $x_1=x_4=e_1$, $x_2=x_5=e_2$, $x_3=e_3$, other subcases are similar. Note that

$$(\underset{x_1,x_2,x_5}{\circlearrowleft}\tau(x_1)\tau([x_2,x_5]))[x_3,x_4]=(\underset{x_1,x_2,x_4}{\circlearrowleft}\tau(x_1)\tau([x_2,x_4]))[x_5,x_3]=0. \tag{2.9}$$

Thus, the left hand side of (2.6) becomes

$$\begin{aligned}(\underset{x_3,x_4,x_5}{\circlearrowleft}\tau(x_3)\tau([x_4,x_5]))[x_1,x_2]&-(\underset{x_1,x_2,x_5}{\circlearrowleft}\tau(x_1)\tau([x_2,x_5]))[x_3,x_4]\\&=(\underset{x_3,x_4,x_5}{\circlearrowleft}\tau(x_3)\tau([x_4,x_5]))[x_1,x_2]-0 \quad \text{(by (2.9))}\\&=(\underset{e_1,e_2,e_3}{\circlearrowleft}\tau(e_1)\tau(e_2,e_3]))[e_1,e_2],\end{aligned} \tag{2.10}$$

while the right hand side of (2.6) is

$$\begin{aligned}(\underset{x_1,x_2,x_3}{\circlearrowleft}\tau(x_1)\tau([x_2,x_3]))[x_4,x_5]&+(\underset{x_1,x_2,x_4}{\circlearrowleft}\tau(x_1)\tau([x_2,x_4]))[x_5,x_3]\\&=(\underset{x_1,x_2,x_3}{\circlearrowleft}\tau(x_1)\tau([x_2,x_3]))[x_4,x_5]+0 \quad \text{(by (2.9))}\\&=(\underset{e_1,e_2,e_3}{\circlearrowleft}\tau(e_1)\tau(e_2,e_3]))[e_1,e_2].\end{aligned} \tag{2.11}$$

So (2.6) holds in this case by (2.10) and (2.11). ☐

Note that Propositions 3.1 and 3.2 in [3] can be generalized to any 3-Lie algebra of the form $\mathfrak{g}_\tau$ as follows. Recall that an ideal I of a 3-Lie algebra L is a subspace of L satisfying $[I,L,L]\subseteq I$. By (1.2) we get the following result.

Proposition 2.2. *Let $\mathfrak{g}$ be a Lie algebra and $\tau\in F_{\text{3-Lie}}(\mathfrak{g})$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ then $\mathfrak{h}_\tau$ is also a subalgebra of $\mathfrak{g}_\tau$. Moreover, if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ then $\mathfrak{h}_\tau$ is an ideal of $\mathfrak{g}_\tau$ if and only if $\mathfrak{h}\subseteq\ker\tau$ or $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{h}$.*

To close this section we consider the solvability of 3-Lie algebras of the form $\mathfrak{g}_\tau$. Nilpotency of $\mathfrak{g}_\tau$ may be treated similarly and we omit the details. Let I be an ideal of a 3-Lie algebra L, see [15]. Put

$$I^{(0)}=I,\quad I^{(n)}=[I^{(n-1)},I^{(n-1)},I^{(n-1)}];\quad I^0=I,\quad I^n=[I^{n-1},I,I]. \tag{2.12}$$

Then I is solvable (or nilpotent) if $I^{(n)}=0$ (or $I^n=0$, respectively) for some $n\geqslant 0$.

The following result generalizes Theorem 3.1 of [3] which states that if $\tau \in F_{\rm tr}(\mathfrak{g})$ then $\mathfrak{g}_\tau$ is solvable. See also Proposition 3.5 in [6].

Proposition 2.3. *Let $\mathfrak{g}$ be a Lie algebra and $\tau \in F_{\rm qtr}(\mathfrak{g})$. Then $\mathfrak{g}_\tau^{(2)} = 0$. In particular, $\mathfrak{g}_\tau$ is solvable.*

Proof. By linearity it suffices to compute $[\mathrm{x},\mathrm{y},\mathrm{z}]_\tau \in \mathfrak{g}_\tau^{(2)}$ for $\mathrm{x} = [\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau$, $\mathrm{y} = [\mathrm{y}_1,\mathrm{y}_2,\mathrm{y}_3]_\tau$, $\mathrm{z} = [\mathrm{z}_1,\mathrm{z}_2,\mathrm{z}_3]_\tau$, and $\mathrm{x}_i,\mathrm{y}_i,\mathrm{z}_i \in \mathfrak{g}$. Since $\tau \in F_{\rm qtr}(\mathfrak{g})$, by (2.5) it follows that $\tau(\mathrm{x}) = \tau(\mathrm{y}) = \tau(\mathrm{z}) = 0$, and hence by (1.2) it follows that $[\mathrm{x},\mathrm{y},\mathrm{z}]_\tau = \tau(\mathrm{x})[\mathrm{y},\mathrm{z}] + \tau(\mathrm{y})[\mathrm{z},\mathrm{x}] + \tau(\mathrm{z})[\mathrm{x},\mathrm{y}] = 0$ as required. □

If $\tau \in F_{3\text{-Lie}}(\mathfrak{g}) \setminus F_{\rm qtr}(\mathfrak{g})$ then (2.5) is not applicable. However, Proposition 2.3 can be generalized as follows. Denote by $[\ker\tau, \ker\tau]$ the linear span of $[\mathrm{x},\mathrm{y}]$, $\mathrm{x},\mathrm{y} \in \ker\tau$. We have the following result.

Proposition 2.4. *Let $\mathfrak{g}$ be a Lie algebra and $0 \neq \tau \in F_{3\text{-Lie}}(\mathfrak{g})$. Then $[\mathfrak{g}_\tau,\mathfrak{g}_\tau,\mathfrak{g}_\tau]_\tau = [\ker\tau,\ker\tau]$. In particular, if $\mathfrak{g}$ is solvable then the 3-Lie algebra $\mathfrak{g}_\tau$ is solvable.*

Proof. Since $\tau \neq 0$, $\operatorname{codim}\ker\tau = 1$. Let $\{f_i\}_{i\in\mathcal{I}}$ be a basis of $\ker\tau$. Choose an $f \in \mathfrak{g} \setminus \ker\tau$. Then $\{f_i\}_{i\in\mathcal{I}} \cup \{f\}$ is a basis of $\mathfrak{g} = \mathfrak{g}_\tau$. Set $t = \tau(f)$. Then $t \neq 0$. Since $f_i \in \ker\tau$ $(i \in \mathcal{I})$ and $f \notin \ker\tau$, by (1.2) it follows that $[\mathfrak{g}_\tau,\mathfrak{g}_\tau,\mathfrak{g}_\tau]_\tau$ is spanned by $[f_i,f_j,f]_\tau = t[f_i,f_j]$, $i,j \in \mathcal{I}$. Note that $[\ker\tau,\ker\tau]$ is spanned by $\{[f_i,f_j]\}$, $i,j \in \mathcal{I}$. Since $t \neq 0$, it follows that $[\mathfrak{g}_\tau,\mathfrak{g}_\tau,\mathfrak{g}_\tau]_\tau = [\ker\tau,\ker\tau]$. □

Remark 2.3. Up to now we have not found an example where $\mathfrak{g}_\tau$ is not solvable for $\tau \in F_{3\text{-Lie}}(\mathfrak{g})$ and $\tau \notin F_{\rm qtr}(\mathfrak{g})$. Note that the converse of the last statement of Proposition 2.4 is not true. For example, let $\mathfrak{g}$ be the Lie algebra with a basis $\{e_1,e_2,e_3\}$ and the multiplication table is given by $[e_1,e_2] = e_3$, $[e_2,e_3] = e_1$, $[e_3,e_1] = e_2$. Define $\tau \in \mathfrak{g}^*$ by $\tau(e_1) = 1$, $\tau(e_2) = \tau(e_3) = 0$. Then $\tau \in F_{3\text{-Lie}}(\mathfrak{g})$ and $(\mathfrak{g}_\tau)^{(1)} = \mathbb{C}e_1$. So $\mathfrak{g}_\tau$ is solvable, while $\mathfrak{g}$ is a simple Lie algebra.

3. Realizations of 3-dimensional 3-Lie algebras

Keep the notation as in last sections, especially Notations 2.1 and 2.2. It is known that there are only two isoclasses of 3-dimensional 3-Lie algebras: the abelian one $L_{3,0}$ and the nonabelian one $L_{3,1}$ (see [15]), where the multiplication table of basis elements of $L_{3,1}$ can be written as $[e_1,e_2,e_3] = e_1$. Since $L_{3,0}$ is abelian it can be induced by either any 3-dimensional Lie algebra with the zero function, or an abelian 3-dimensional Lie algebra with any linear function. In this section we show that $L_{3,1}$ can be realized as $\mathfrak{g}_\tau$, where $\mathfrak{g}$ can be chosen from each isoclass of 3-dimensional nonabelian Lie algebras. Moreover, we give explicitly all linear functions τ such that $L_{3,1} \cong \mathfrak{g}_\tau$.

Throughout this section we always consider 3-dimensional Lie algebras. On the classification of complex 3-dimensional Lie algebras we get the following proposition.

Proposition 3.1 ([7], [13]). *Any complex* 3*-dimensional Lie algebra is isomorphic to one and only one Lie algebra in Table* 1.

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{3,0}$	trivial
$\mathfrak{g}_{3,1}$	$[e_1, e_2] = e_1$
$\mathfrak{g}_{3,2}$	$[e_1, e_3] = e_1 + e_2,\ [e_2, e_3] = e_2$
$\mathfrak{g}_{3,3}$	$[e_1, e_3] = e_1,\ [e_2, e_3] = \alpha e_2,\ \alpha \in \mathbb{C},\ 0 < \lvert\alpha\rvert \leqslant 1$
$\mathfrak{g}_{3,4}$	$[e_1, e_2] = e_3$
$\mathfrak{g}_{3,5}$	$[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1$

Table 1. Classification of complex 3-dimensional Lie algebras.

Remark 3.1. In Section 3.2 of [13], Table 1 is given in terms of derived subalgebras and centers. Let $\mathfrak{g}$ be a 3-dimensional complex Lie algebra. Let $\mathfrak{g}^{(1)}$ and $Z(\mathfrak{g})$ be the derived subalgebra and center of $\mathfrak{g}$, respectively.

(1) If $\dim \mathfrak{g}^{(1)} = 0$ then $\mathfrak{g} \cong \mathfrak{g}_{3,0}$.

(2) Assume that $\dim \mathfrak{g}^{(1)} = 1$. Then $\mathfrak{g} \cong \mathfrak{g}_{3,4}$ if and only if $\mathfrak{g}^{(1)} \subseteq Z(\mathfrak{g})$. In this case $\mathfrak{g}$ is the Heisenberg algebra. $\mathfrak{g} \cong \mathfrak{g}_{3,1}$ if and only if $\mathfrak{g}^{(1)} \nsubseteq Z(\mathfrak{g})$. In this case $\mathfrak{g}$ is the direct sum of a nonabelian Lie algebra and a 1-dimensional Lie algebra.

(3) Assume that $\dim \mathfrak{g}^{(1)} = 2$. Then either $\mathfrak{g} \cong \mathfrak{g}_{3,2}$ or $\mathfrak{g} \cong \mathfrak{g}_{3,3}$, depending on whether there is an $\mathrm{x} \in \mathfrak{g}^{(1)}$ such that $\operatorname{ad} \mathrm{x}$ is diagonalizable.

(4) $\mathfrak{g} \cong \mathfrak{g}_{3,5}$ if and only if $\dim \mathfrak{g}^{(1)} = 3$. In this case $\mathfrak{g}$ is the unique 3-dimensional simple Lie algebra up to isomorphism.

In the proof of Theorem 4.1 of [6] it is shown that $L_{3,1} \cong (\mathfrak{g}_{3,1})_\tau$, where τ is given by $\tau(e_1) = \tau(e_2) = 0$, $\tau(e_3) = 1$, which is a trace function of $\mathfrak{g}_{3,1}$. In fact we have the following theorem.

Theorem 3.1. *Keep the notation as above. For each* $\mathfrak{g}_{3,i}$, $1 \leqslant i \leqslant 5$, *there is* $\tau \in (\mathfrak{g}_{3,i})^*$ *such that* $L_{3,1} \cong (\mathfrak{g}_{3,i})_\tau$. *More precisely:*

(1) $L_{3,1} \cong (\mathfrak{g}_{3,1})_\tau$ *if and only if* $\tau \in S_1 \triangleq \{\tau \in \mathfrak{g}_{3,1}^* \colon \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, t_3 \neq 0\}$.

(2) $L_{3,1} \cong (\mathfrak{g}_{3,2})_\tau$ *if and only if* $\tau \in S_2 \triangleq \{\tau \in \mathfrak{g}_{3,2}^* \colon \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, t_2 \neq 0$ *or* $t_1 \neq t_2\}$.

(3) $L_{3,1} \cong (\mathfrak{g}_{3,3})_\tau$ *if and only if* $\tau \in S_3 \triangleq \{\tau \in \mathfrak{g}_{3,3}^* \colon \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, (t_1, t_2) \neq (0, 0)\}$.

(4) $L_{3,1} \cong (\mathfrak{g}_{3,4})_\tau$ *if and only if* $\tau \in S_4 \triangleq \{\tau \in \mathfrak{g}_{3,4}^* \colon \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, t_3 \neq 0\}$.

(5) $L_{3,1} \cong (\mathfrak{g}_{3,5})_\tau$ *if and only if* $\tau \in S_5 \triangleq \{\tau \in \mathfrak{g}_{3,5}^* \colon \tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3, (t_1, t_2, t_3) \neq (0, 0, 0)\}$.

P r o o f. Since $\dim \mathfrak{g}_{3,i} = 3$, by Lemma 2.1, $F_{3\text{-Lie}}(\mathfrak{g}_{3,i}) = (\mathfrak{g}_{3,i})^*$, which means $(\mathfrak{g}_{3,i})_\tau$ is a 3-Lie algebra for any $\tau \in (\mathfrak{g}_{3,i})^*$. So, due to the classification result on 3-dimensional 3-Lie algebras, it suffices to show that, for each $1 \leqslant i \leqslant 5$, there is a $\tau \in (\mathfrak{g}_{3,i})^*$ such that $(\mathfrak{g}_{3,i})_\tau$ is nonabelian, which is equivalent to $0 \neq [e_1, e_2, e_3]_\tau = \tau(e_1)[e_2, e_3] + \tau(e_2)[e_3, e_1] + \tau(e_3)[e_1, e_2]$. Recall Notation 2.1.

We only show that (1) holds, the other cases are similar and we omit the proof. Suppose that $\tau \in \mathfrak{g}_{3,1}^*$. In view of Table 1, $L_{3,1} \cong (\mathfrak{g}_{3,1})_\tau$ if and only if

$$0 \neq \tau(e_1)[e_2, e_3] + \tau(e_2)[e_3, e_1] + \tau(e_3)[e_1, e_2] = t_3 e_1,$$

which is equivalent to $t_3 \neq 0$. □

For completeness we determine which functions in S_i $(1 \leqslant i \leqslant 5)$ in Theorem 3.1 are trace functions or quasi-trace functions.

Example 3.1. Trace functions on $\mathfrak{g}_{3,i}$ $(1 \leqslant i \leqslant 5)$ are given by Table 2.

Lie algebras	Trace functions
$\mathfrak{g}_{3,1}$	$\tau(\mathrm{x}) = t_2 x_2 + t_3 x_3$
$\mathfrak{g}_{3,2}$	$\tau(\mathrm{x}) = t_3 x_3$
$\mathfrak{g}_{3,3}$	$\tau(\mathrm{x}) = t_3 x_3$
$\mathfrak{g}_{3,4}$	$\tau(\mathrm{x}) = t_1 x_1 + t_2 x_2$
$\mathfrak{g}_{3,5}$	$\tau = 0$

Table 2. Trace functions on 3-dimensional Lie algebras [3].

Example 3.2. Quasi-trace functions on $\mathfrak{g}_{3,i}$ $(1 \leqslant i \leqslant 5)$.

Lie algebras	Quasi-trace functions
$\mathfrak{g}_{3,1}$	$\tau(\mathrm{x}) = t_1 x_1 + t_2 x_2 + t_3 x_3$, $t_1 t_3 = 0$
$\mathfrak{g}_{3,2}$	$\tau(\mathrm{x}) = t_1 x_1 + t_3 x_3$
$\mathfrak{g}_{3,3}$	$\tau(\mathrm{x}) = t_1 x_1 + t_2 x_2 + t_3 x_3$, $(\alpha - 1) t_1 t_2 = 0$
$\mathfrak{g}_{3,4}$	$\tau(\mathrm{x}) = t_1 x_1 + t_2 x_2$
$\mathfrak{g}_{3,5}$	$\tau(\mathrm{x}) = t_1 x_1 + t_2 x_2 + t_3 x_3$, $t_1^2 + t_2^2 + t_3^2 = 0$

Table 3. Quasi-trace functions on 3-dimensional Lie algebras.

P r o o f. We compute $F_{\mathrm{qtr}}(\mathfrak{g}_{3,1})$. Other $F_{\mathrm{qtr}}(\mathfrak{g}_{3,i})$ can be obtained similarly. By Definition 2.1 and linearity, $\tau \in (\mathfrak{g}_{3,1})^*$ is a quasi-trace function if and only if

$$\circlearrowleft_{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3} \tau(\mathrm{x}_1)\tau([\mathrm{x}_2, \mathrm{x}_3]) = 0, \quad \mathrm{x}_i \in \{e_1, e_2, e_3\}. \tag{3.1}$$

Note that (3.1) holds if there are at least two x_i, x_j equal to each other. So $\tau \in F_{\rm qtr}(\mathfrak{g}_{3,1})$ if and only if $\underset{e_1,e_2,e_3}{\circlearrowleft} \tau(e_1)\tau([e_2,e_3]) = 0$. By Table 1 it follows that

$$\tau(e_1)\tau([e_2,e_3]) + \tau(e_2)\tau([e_3,e_1]) + \tau(e_3)\tau([e_1,e_2]) = t_1t_3.$$

So, $\tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3 \in F_{\rm qtr}(\mathfrak{g}_{3,1})$ if and only if $t_1t_3 = 0$. □

By Theorem 3.1 and Example 3.1 we have the following result.

Corollary 3.1. *Let $L_{3,1}$ be the unique (up to isomorphism) nonabelian 3-dimensional 3-Lie algebra.*

(1) *There is no trace function τ on $\mathfrak{g}_{3,i}$ such that $L_{3,1} \cong (\mathfrak{g}_{3,i})_\tau$, $i = 2,3,4,5$.*
(2) *Assume that $L_{3,1} \cong (\mathfrak{g}_{3,1})_\tau$. Then τ is a trace function if and only if $\tau(\mathrm{x}) = t_2x_2 + t_3x_3$, $t_3 \neq 0$.*

So, to obtain $L_{3,1}$ by using only trace functions one has to choose $\mathfrak{g}_{3,1}$ as in the proof of Theorem 4.1 of [6]. By Theorem 3.1 and Example 3.2 we get the following corollary.

Corollary 3.2. *Let $L_{3,1}$ be the unique (up to isomorphism) nonabelian 3-dimensional 3-Lie algebra, S_i the set of functions given by Theorem 3.1.*

(1) *There is no quasi-trace function τ on $\mathfrak{g}_{3,4}$ such that $L_{3,1} \cong (\mathfrak{g}_{3,4})_\tau$.*
(2) *For $1 \leqslant i \leqslant 5$, $i \neq 4$, there are quasi-trace functions τ on $\mathfrak{g}_{3,i}$ such that $L_{3,1} \cong (\mathfrak{g}_{3,i})_\tau$.*

More precisely, such quasi-trace functions are given as follows.

(i) *$\tau \in F_{\rm qtr}(\mathfrak{g}_{3,1}) \cap S_1$ if and only if $\tau(\mathrm{x}) = t_2x_2 + t_3x_3$, $t_3 \neq 0$.*
(ii) *$\tau \in F_{\rm qtr}(\mathfrak{g}_{3,2}) \cap S_2$ if and only if $\tau(\mathrm{x}) = t_1x_1 + t_3x_3$, $t_1 \neq 0$.*
(iii) *$\tau \in F_{\rm qtr}(\mathfrak{g}_{3,3}) \cap S_3$ if and only if $\tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3$ satisfying one of the following conditions:*
   (a) *$\alpha = 1$, $t_1 \neq 0$;*
   (b) *$\alpha = 1$, $t_2 \neq 0$;*
   (c) *$\alpha \neq 1$, $t_1 = 0$, $t_2 \neq 0$;*
   (d) *$\alpha \neq 1$, $t_1 \neq 0$, $t_2 = 0$.*
(iv) *$\tau \in F_{\rm qtr}(\mathfrak{g}_{3,5}) \cap S_5$ if and only if $\tau(\mathrm{x}) = t_1x_1 + t_2x_2 + t_3x_3$, $(t_1,t_2,t_3) \neq (0,0,0)$, $t_1^2 + t_2^2 + t_3^2 = 0$.*

Remark 3.2. The isoclass of type $\mathfrak{g}_{3,3}$ is parametrized by $\alpha \in \mathbb{C}$ with $0 < |\alpha| \leqslant 1$. Though the set S_3 given by Theorem 3.1 is independent of the parameter α, $F_{\rm qtr}(\mathfrak{g}_{3,3}) \cap S_3$ does depend on α.

4. Quasi-trace functions on 4-dimensional Lie algebras and their induced 3-Lie algebras

Recall Notations 2.1 and 2.2. In this section we consider the problem whether a 4-dimensional 3-Lie algebra can be induced by a 4-dimensional Lie algebra via linear functions. This problem has been studied by using trace functions in [3], [6]. Let $\mathfrak{g}$ be a complex 4-dimensional Lie algebra. The main result of this section is that the isoclasses of the 4-dimensional 3-Lie algebras of the form $\mathfrak{g}_\tau$ are determined for quasi-trace functions on $\mathfrak{g}$. Our method depends on the following two facts:

(1) If τ is a nonzero quasi-trace function then $\ker\tau$ is a 3-dimensional subalgebra of $\mathfrak{g}$, while the classification of 3-dimensional Lie algebras is known and given by Proposition 3.1.

(2) All 4-dimensional 3-Lie algebras are classified via their derived subalgebras. We recall Filippov's classification as follows.

Proposition 4.1 ([15], Section 3). *Let L be a complex 4-dimensional 3-Lie algebra. Let $L^{(1)}$ and $Z(L)$ be the derived subalgebra and the center of L, respectively.*

(1) *If $\dim L^{(1)}=0$ then L is abelian, denoted by $L_{4,0}$.*

(2) *Assume that $\dim L^{(1)}=1$.*

  (2.1) *If $L^{(1)}\nsubseteq Z(L)$ then L is given by $[e_1,e_3,e_4]=e_1$, denoted by $L_{4,1}$.*

  (2.2) *If $L^{(1)}\subseteq Z(L)$ then L is given by $[e_2,e_3,e_4]=e_1$, denoted by $L_{4,4}$.*

(3) *If $\dim L^{(1)}=2$ then L is given by either $[e_1,e_2,e_4]=e_3+\alpha e_4$, $[e_1,e_2,e_3]=e_4$ or $[e_1,e_2,e_4]=e_3$, $[e_1,e_2,e_3]=\beta e_4$, where $0\neq\alpha,\beta\in\mathbb{C}$.*

(4) *If $\dim L^{(1)}=3$ then L is given by $[e_2,e_3,e_4]=e_1$, $[e_1,e_3,e_4]=e_2$, $[e_1,e_2,e_4]=e_3$, denoted by $L_{4,5}$.*

(5) *If $\dim L^{(1)}=4$ then L is given by $[e_2,e_3,e_4]=e_1$, $[e_1,e_3,e_4]=e_2$, $[e_1,e_2,e_4]=e_3$, $[e_1,e_2,e_3]=e_4$, denoted by $L_{4,6}$.*

By Lemma 4.1 and Lemma 4.2 below, 3-Lie algebras given by (3) of Proposition 4.1 can be classified further as follows.

Corollary 4.1. *Let L be a complex 4-dimensional 3-Lie algebra with $\dim L^{(1)}=2$. Then L is isomorphic to one and only one of the following algebras:*

(1) $L_{4,2}$: $[e_1,e_2,e_4]=e_3+e_4$, $[e_1,e_2,e_3]=e_4$.

(2) $L_{4,3,\beta}$: $[e_1,e_2,e_4]=e_3$, $[e_1,e_2,e_3]=\beta e_4$, $0<|\beta|\leqslant 1$.

Recall that $n\times n$ matrices A,B are $\mathbb{C}^*$-similar if there exist $0\neq k\in\mathbb{C}$ and an invertible matrix P such that $B=kPAP^{-1}$. The $\mathbb{C}^*$-similar relation is used in classification of 3-dimensional Lie algebras, see [17], page 12.

Lemma 4.1. *Any complex* 2×2 *invertible matrix is* $\mathbb{C}^*$*-similar to one and only one of the matrices* $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\0&\beta\end{smallmatrix}\right)$, $0<|\beta|\leqslant 1$.

P r o o f. Since $\mathbb{C}$ is algebraically closed, by using Jordan canonical forms it follows that any complex 2×2 invertible matrix is $\mathbb{C}^*$-similar to matrices of the forms

$$M_\alpha := \begin{pmatrix}1&\alpha\\0&1\end{pmatrix}, \quad N_\beta := \begin{pmatrix}1&0\\0&\beta\end{pmatrix}, \quad 0\neq\alpha,\beta\in\mathbb{C}. \tag{4.1}$$

Since M_α and N_β are not $\mathbb{C}^*$-similar, it suffices to show the following two claims.

Claim 4.1. For any $0\neq\alpha_1,\alpha_2\in\mathbb{C}$, $\left(\begin{smallmatrix}1&\alpha_1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&\alpha_2\\0&1\end{smallmatrix}\right)$ are $\mathbb{C}^*$-similar.

Claim 4.2. For any $0\neq\beta_1$, $\beta_2\in\mathbb{C}$, $\left(\begin{smallmatrix}1&0\\0&\beta_1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&\beta_2\end{smallmatrix}\right)$ are $\mathbb{C}^*$-similar if and only if either $\beta_1=\beta_2$ or $\beta_1\beta_2=1$.

Claim 4.1 follows by

$$\begin{pmatrix}1&\alpha_2\\0&1\end{pmatrix} = \begin{pmatrix}\alpha_2/\alpha_1&1\\0&1\end{pmatrix}\begin{pmatrix}1&\alpha_1\\0&1\end{pmatrix}\begin{pmatrix}\alpha_2/\alpha_1&1\\0&1\end{pmatrix}^{-1}.$$

The "if" part of Claim 4.2 is clear since for any $0\neq\beta\in\mathbb{C}$,

$$\begin{pmatrix}1&0\\0&1/\beta\end{pmatrix} \sim \begin{pmatrix}1/\beta&0\\0&1\end{pmatrix} = (1/\beta)\begin{pmatrix}1&0\\0&\beta\end{pmatrix}.$$

Conversely, suppose that $\left(\begin{smallmatrix}1&0\\0&\beta_1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&\beta_2\end{smallmatrix}\right)$ are $\mathbb{C}^*$-similar. Then there exist a non-zero number k and an invertible matrix $P=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ such that

$$\begin{pmatrix}1&0\\0&\beta_2\end{pmatrix} = k\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&\beta_1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}, \tag{4.2}$$

which implies that

$$\frac{k(ad-bc\beta_1)}{ad-bc}=1, \quad ab(\beta_1-1)=0, \quad cd(1-\beta_1)=0. \tag{4.3}$$

So, if $\beta_1=1$ then $k=1$, which means that $\left(\begin{smallmatrix}1&0\\0&\beta_1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&\beta_2\end{smallmatrix}\right)$ are similar, and hence $\beta_1=\beta_2$.

If $\beta_1\neq 1$ then $ab=cd=0$ by (4.3). By nonsingularity of P it follows that either $P=\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right)$ or $P=\left(\begin{smallmatrix}0&b\\c&0\end{smallmatrix}\right)$. If $P=\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right)$ then by (4.2) it follows that $\left(\begin{smallmatrix}1&0\\0&\beta_2\end{smallmatrix}\right)=\left(\begin{smallmatrix}k&0\\0&k\beta_1\end{smallmatrix}\right)$, which implies $\beta_1=\beta_2$. If $P=\left(\begin{smallmatrix}0&b\\c&0\end{smallmatrix}\right)$ then by (4.2) it follows that $\left(\begin{smallmatrix}1&0\\0&\beta_2\end{smallmatrix}\right)=\left(\begin{smallmatrix}k\beta_1&0\\0&k\end{smallmatrix}\right)$, which means that $\beta_1\beta_2=1$. □

Assume that L is a complex 4-dimensional 3-Lie algebra and dim $L^{(1)} = 2$. By [15], there is a basis $\{e_1, e_2, e_3, e_4\}$ of L with the multiplication table $[e_1, e_2, e_4] = ae_3 + be_4$, $[e_1, e_2, e_3] = ce_3 + de_4$, where $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an invertible matrix. Moreover, we have the next lemma.

Lemma 4.2 ([15], Section 3)**.** *The* 4*-dimensional* 3*-Lie algebras defined by A and B respectively are isomorphic if and only if A is $\mathbb{C}^*$-similar to B.*

Keep notations $L_{4,i}$ $(i = 0, 1, 4, 5, 6)$ of 4-dimensional 3-Lie algebras given by Proposition 4.1 and $L_{4,2}$, $L_{4,3,\beta}$ given by Corollary 4.1.

Example 4.1. Since $L_{4,6}$ is a simple 3-Lie algebra (see [15], Theorem 4), by Proposition 2.3 for any 4-dimensional Lie algebra $\mathfrak{g}$ there is no quasi-trace function (and hence no trace function) on $\mathfrak{g}$ such that $\mathfrak{g}_\tau \cong L_{4,6}$.

Example 4.2. Let $\mathfrak{g}$ be the 4-dimensional Lie algebra with a basis $\{e_1, e_2, e_3, e_4\}$ and the multiplication table $[e_1, e_3] = e_1$, that is, $\mathfrak{g} \cong \mathfrak{g}_{4,1}$, see Table 4 below. Define $\tau \in \mathfrak{g}^*$ by $\tau(e_1) = \tau(e_4) = 1$, $\tau(e_2) = \tau(e_3) = 0$. By a long but direct check it follows that $\tau \in F_{3\text{-Lie}}(\mathfrak{g})$. By (1.2) the 3-Lie algebra $\mathfrak{g}_\tau$ is given by $[e_1, e_3, e_4]_\tau = e_1$, which means that $\mathfrak{g}_\tau \cong L_{4,1}$. Moreover, since $\tau([e_1, e_3]) = \tau(e_1) \neq 0$, τ is not a trace function on $\mathfrak{g}$.

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_{4,0}$	trivial
$\mathfrak{g}_{4,1}$	$[e_1, e_2] = e_1$
$\mathfrak{g}_{4,2}$	$[e_1, e_2] = e_3$
$\mathfrak{g}_{4,3}$	$[e_1, e_2] = e_2$, $[e_1, e_3] = e_2 + e_3$
$\mathfrak{g}_{4,4}$	$[e_1, e_2] = e_2$, $[e_1, e_3] = \alpha e_3$,$\alpha \in \mathbb{C}$, $0 < \lvert\alpha\rvert \leqslant 1$
$\mathfrak{g}_{4,5}$	$[e_1, e_2] = e_1$, $[e_3, e_4] = e_3$
$\mathfrak{g}_{4,6}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$
$\mathfrak{g}_{4,7}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$
$\mathfrak{g}_{4,8}$	$[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$, $[e_1, e_4] = \alpha e_4$, $\alpha \in \mathbb{C}^*$
$\mathfrak{g}_{4,9}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = \alpha e_2 - \beta e_3 + e_4$, $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ or $\alpha, \beta = 0$
$\mathfrak{g}_{4,10}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = \alpha(e_2 + e_3)$, $\alpha \in \mathbb{C}^*$
$\mathfrak{g}_{4,11}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_2$
$\mathfrak{g}_{4,12}$	$[e_1, e_2] = \frac{1}{3}e_2 + e_3$, $[e_1, e_3] = \frac{1}{3}e_3$, $[e_1, e_4] = \frac{1}{3}e_4$
$\mathfrak{g}_{4,13}$	$[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$, $[e_1, e_4] = 2e_4$, $[e_2, e_3] = e_4$
$\mathfrak{g}_{4,14}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = e_2$, $[e_2, e_3] = e_4$
$\mathfrak{g}_{4,15}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = -\alpha e_2 + e_3$, $[e_1, e_4] = e_4$, $[e_2, e_3] = e_4$, $\alpha \in \mathbb{C}$

Table 4. Classification of complex 4-dimensional Lie algebras, see [7].

Let $\mathfrak{g}$ be a 4-dimensional Lie algebra and $0 \neq \tau \in F_{\rm qtr}(\mathfrak{g})$. Then $\ker \tau$ is a 3-dimensional subalgebra of $\mathfrak{g}$. Keep the notation in Table 1.

Lemma 4.3. *Let $\mathfrak{g}$ be a 4-dimensional Lie algebra and $0 \neq \tau \in F_{\rm qtr}(\mathfrak{g})$.*

(1) $\mathfrak{g}_\tau \cong L_{4,0}$ *if and only if* $\ker \tau \cong \mathfrak{g}_{3,0}$.

(2) $\mathfrak{g}_\tau \cong L_{4,i}$ *if and only if* $\ker \tau \cong \mathfrak{g}_{3,i}$, $i = 1, 4$.

(3) $\mathfrak{g}_\tau \cong L_{4,5}$ *if and only if* $\ker \tau \cong \mathfrak{g}_{3,5}$.

Proof. By Proposition 2.4 we have $(\mathfrak{g}_\tau)^{(1)} = (\ker \tau)^{(1)} \subseteq \ker \tau$, since $\ker \tau$ is a subalgebra of $\mathfrak{g}$.

(1) Since $\mathfrak{g}_\tau \cong L_{4,0}$ if and only if $\mathfrak{g}_\tau$ is abelian, that is, $0 = (\mathfrak{g}_\tau)^{(1)} = (\ker \tau)^{(1)}$, the claim follows by Proposition 3.1.

(2) Choose a basis $\{f_1, f_2, f_3\}$ of $\ker \tau$ and take an $f_4 \in \mathfrak{g} \setminus \ker \tau$. Then $\{f_1, f_2, f_3, f_4\}$ is a basis of $\mathfrak{g}$. Assume that $\mathfrak{g}_\tau \cong L_{4,1}$. Then $\dim(\mathfrak{g}_\tau)^{(1)} = 1$ and $(\mathfrak{g}_\tau)^{(1)} \nsubseteq Z(\mathfrak{g}_\tau)$ by Proposition 4.1. Since $\dim(\ker \tau)^{(1)} = \dim(\mathfrak{g}_\tau)^{(1)} = 1$, to show $\ker \tau \cong \mathfrak{g}_{3,1}$ it suffices to show that $(\ker \tau)^{(1)} \nsubseteq Z(\ker \tau)$ by Remark 3.1. In fact, by $(\mathfrak{g}_\tau)^{(1)} \nsubseteq Z(\mathfrak{g}_\tau)$ there are some $1 \leqslant j, k \leqslant 3$ such that $[f_j, f_k, f_4]_\tau \notin Z(\mathfrak{g}_\tau)$. By choices of f_i and (1.2) it follows that $[f_j, f_k] \notin Z(\mathfrak{g}_\tau)$, that is,

$$[[f_j, f_k], f_l, f_4]_\tau \neq 0 \quad \text{for some } 1 \leqslant l \leqslant 3, \tag{4.4}$$

or equivalently, by (1.2) and choices of f_i,

$$[[f_j, f_k], f_l] \neq 0 \quad \text{for some } 1 \leqslant j, k, l \leqslant 3, \tag{4.5}$$

which means that $(\ker \tau)^{(1)} \nsubseteq Z(\ker \tau)$ as required. Conversely, assume that $\ker \tau \cong \mathfrak{g}_{3,1}$. Then $\dim(\ker \tau)^{(1)} = 1$ and $(\ker \tau)^{(1)} \nsubseteq Z(\ker \tau)$ by Remark 3.1, therefore, $\dim(\mathfrak{g}_\tau)^{(1)} = 1$. By $(\ker \tau)^{(1)} \nsubseteq Z(\ker \tau)$ there are some $1 \leqslant j, k, l \leqslant 3$ such that (4.5) holds, and hence (4.4) holds, which means that $(\mathfrak{g}_\tau)^{(1)} \nsubseteq Z(\mathfrak{g}_\tau)$. So $\mathfrak{g}_\tau \cong L_{4,1}$ as required by Proposition 4.1. Similarly one can show that $\mathfrak{g}_\tau \cong L_{4,4}$ if and only if $\ker \tau \cong \mathfrak{g}_{3,4}$.

(3) By Proposition 4.1, $\mathfrak{g}_\tau \cong L_{4,5}$ if and only if $\dim(\mathfrak{g}_\tau)^{(1)} = 3 = \dim(\ker \tau)^{(1)}$. By Remark 3.1 this is equivalent to $\ker \tau \cong \mathfrak{g}_{3,5}$. □

Now we consider the remaining cases when $\ker \tau \cong \mathfrak{g}_{3,2}$ and $\ker \tau \cong \mathfrak{g}_{3,3}$.

Lemma 4.4. *Let $\mathfrak{g}$ be a complex 4-dimensional Lie algebra and $0 \neq \tau \in F_{\rm qtr}(\mathfrak{g})$.*

(1) *If* $\ker \tau \cong \mathfrak{g}_{3,2}$ *then* $\mathfrak{g}_\tau \cong L_{4,3,(\sqrt{5}-3)/2}$.

(2) *If* $\ker \tau \cong \mathfrak{g}_{3,3}$ *then* $\mathfrak{g}_\tau \cong L_{4,3,-1}$.

P r o o f. (1) By $\ker\tau \cong \mathfrak{g}_{3,2}$ and Table 1 there exists a basis $\{f_2, f_3, f_4\}$ of $\ker\tau$ with the multiplication table given by $[f_3, f_2] = f_3 + f_4$, $[f_4, f_2] = f_4$. Choose an $f_1 \in \mathfrak{g}$ such that $\tau(f_1) = -1$. Then $\{f_1, f_2, f_3, f_4\}$ is a basis of $\mathfrak{g}$ and the multiplication table of $\mathfrak{g}_\tau$ is given by (see (1.2))

$$[f_1, f_2, f_4]_\tau = f_4, \quad [f_1, f_2, f_3]_\tau = f_3 + f_4. \tag{4.6}$$

Then $\dim(\mathfrak{g}_\tau)^{(1)} = 2$, and hence $\mathfrak{g}_\tau$ is isomorphic to either $L_{4,2}$ or $L_{4,3,\beta}$, $0 < |\beta| \leqslant 1$, by Corollary 4.1. Therefore, by Lemma 4.2 it remains to determine the $\mathbb{C}^*$-similar class of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ given by (4.6). Since A has no multiple eigenvalues, neither has kA for any $0 \neq k \in \mathbb{C}$. So, A is $\mathbb{C}^*$-similar to $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ by Lemma 4.1 and $\mathfrak{g}_\tau \cong L_{4,3,\beta}$ for a unique $\beta \in \mathbb{C}$ with $0 < |\beta| \leqslant 1$.

The characteristic polynomials of kA and $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ are given by $\lambda^2 - k\lambda - k^2$, $\lambda^2 - (\beta+1)\lambda + \beta$, respectively. So $k^2 = -\beta$, $k = \beta + 1$, which means that $\beta^2 + 3\beta + 1 = 0$ and hence $\beta = (\sqrt{5} - 3)/2$ by $0 < |\beta| \leqslant 1$.

(2) Assume that $\ker\tau \cong \mathfrak{g}_{3,3}$. By Table 1 there exists a basis $\{f_2, f_3, f_4\}$ of $\ker\tau$ with the multiplication table given by $[f_3, f_2] = f_3$, $[f_4, f_2] = \alpha f_4$. Choose an $f_1 \in \mathfrak{g}$ such that $\tau(f_1) = -1$. Then $\{f_1, f_2, f_3, f_4\}$ is a basis of $\mathfrak{g}$ and the multiplication table of $\mathfrak{g}_\tau$ is $[f_1, f_2, f_4]_\tau = \alpha f_4$, $[f_1, f_2, f_3]_\tau = f_3$, see (1.2). Set $B = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$. Since kB has no multiple eigenvalues for any $0 \neq k \in \mathbb{C}^*$, B is $\mathbb{C}^*$-similar to $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ by Lemma 4.1, and hence $\mathfrak{g}_\tau \cong L_{4,3,\beta}$ for a unique $\beta \in \mathbb{C}$ with $0 < |\beta| \leqslant 1$. The characteristic polynomials of kB and $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$ are given by $\lambda^2 - k^2\alpha$, $\lambda^2 - (\beta+1)\lambda + \beta$, respectively, and hence $\beta = -1$. □

By Lemma 4.3 and Lemma 4.4 we get the main result of this section.

Theorem 4.1. *Let $\mathfrak{g}$ be a complex 4-dimensional Lie algebra and $0 \neq \tau \in F_{\rm qtr}(\mathfrak{g})$. Then we have the following complete and exclusive cases.*

(1) $\mathfrak{g}_\tau \cong L_{4,0}$ *if and only if* $\ker\tau \cong \mathfrak{g}_{3,0}$.
(2) $\mathfrak{g}_\tau \cong L_{4,i}$ *if and only if* $\ker\tau \cong \mathfrak{g}_{3,i}$, $i = 1, 4$.
(3) $\mathfrak{g}_\tau \cong L_{4,3,(\sqrt{5}-3)/2}$ *if and only if* $\ker\tau \cong \mathfrak{g}_{3,2}$.
(4) $\mathfrak{g}_\tau \cong L_{4,3,-1}$ *if and only if* $\ker\tau \cong \mathfrak{g}_{3,3}$.
(5) $\mathfrak{g}_\tau \cong L_{4,5}$ *if and only if* $\ker\tau \cong \mathfrak{g}_{3,5}$.

The following corollary is straightforward.

Corollary 4.2. *Let $\mathfrak{g}$ be a complex 4-dimensional Lie algebra.*

(1) *There is no $\tau \in F_{\rm qtr}(\mathfrak{g})$ such that $\mathfrak{g}_\tau \cong L_{4,2}$ and $\mathfrak{g}_\tau \cong L_{4,6}$.*
(2) *There is no $\tau \in F_{\rm qtr}(\mathfrak{g})$ such that $\mathfrak{g}_\tau \cong L_{4,3,\beta}$ for $\beta \neq \frac{1}{2}(\sqrt{5} - 3), -1$.*

Corollary 4.3. *With the notation given by Table* 4, *all* 4*-dimensional* 3*-Lie algebras induced by quasi-trace functions are given by Table* 5.

P r o o f. We consider $\mathfrak{g}_{4,1}$ only and the other cases are obtained similarly. By Corollary 2.1, $\tau \in F_{\rm qtr}(\mathfrak{g}_{4,1})$ if and only if

$$\begin{aligned}(4.7)\qquad && \tau([e_1,e_2,e_3]_\tau)=0, \quad \tau([e_1,e_2,e_4]_\tau)=0,\\ && \tau([e_1,e_3,e_4]_\tau)=0, \quad \tau([e_2,e_3,e_4]_\tau)=0.\end{aligned}$$

By (1.2), (4.7) is equivalent to $\tau(t_3e_1)=0$, $\tau(t_4e_1)=0$, $\tau(0)=0$, $\tau(0)=0$, i.e., $t_1t_3=t_1t_4=0$. By (1.2) the induced 3-Lie algebra is given by $[e_1,e_2,e_3]_\tau=t_3e_1$, $[e_1,e_2,e_4]_\tau=t_4e_1$, $[e_1,e_3,e_4]_\tau=0$, $[e_1,e_2,e_3]_\tau=0$. □

Remark 4.3. Since different quasi-trace functions may induce isomorphic 3-Lie algebras, it would be better to list isoclasses of 3-Lie algebras of the form $\mathfrak{g}_\tau$ in Table 5. However, by Theorem 4.1, it needs to determine the isoclass of $\ker\tau$, which involves long computations.

5. Cohomology of 3-Lie algebras and Leibniz algebras

In this section we recall representations and cohomologies of 3-Lie algebras and Leibniz algebras for our purpose. Throughout this section L denotes a 3-Lie algebra with a 3-ary bracket $[\cdot,\cdot,\cdot]$. We fix the following notation.

Notation 5.1. *Denote* $\mathrm{x}_1\wedge\mathrm{x}_2\wedge\ldots\wedge\mathrm{x}_n\in\wedge^nL$ *by* $(\mathrm{x}_1,\mathrm{x}_2,\ldots,\mathrm{x}_n)$.

According to Kasymov (see [18]), a representation of L on a vector space V is defined such that $L\oplus V$ is again a 3-Lie algebra with L being a 3-Lie subalgebra and V an abelian ideal, which is equivalent to the following definition.

Definition 5.1 ([18]). A representation of a 3-Lie algebra L on a vector space V is a linear map $\theta\colon \wedge^2L\to\mathrm{End}(V)$ such that for all $\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4\in L$,

$$\begin{aligned}&\theta(\mathrm{x}_1,\mathrm{x}_2)\theta(\mathrm{x}_3,\mathrm{x}_4)-\theta(\mathrm{x}_3,\mathrm{x}_4)\theta(\mathrm{x}_1,\mathrm{x}_2)=\theta([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3],\mathrm{x}_4)+\theta(\mathrm{x}_3,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4]),\\ &\theta(\mathrm{x}_1,[\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4])=\theta(\mathrm{x}_3,\mathrm{x}_4)\theta(\mathrm{x}_1,\mathrm{x}_2)-\theta(\mathrm{x}_2,\mathrm{x}_4)\theta(\mathrm{x}_1,\mathrm{x}_3)+\theta(\mathrm{x}_2,\mathrm{x}_3)\theta(\mathrm{x}_1,\mathrm{x}_4).\end{aligned}$$

A representation θ of L on a vector space V is denoted by (V,θ). For example, we have the adjoint representation (L,ad) of L on itself by (1.1), where the map $\mathrm{ad}\colon\wedge^2L\to\mathrm{End}(L)$ is given by $\mathrm{ad}(\mathrm{x}_1,\mathrm{x}_2)(\mathrm{x}_3)=[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]$.

$\mathfrak{g}_4$	quasi-trace functions	induced 3-Lie algebras
$\mathfrak{g}_{4,0}$	$t_1, t_2, t_3, t_4 \in \mathbb{C}$	abelian 3-Lie algebras
$\mathfrak{g}_{4,1}$	$t_1t_3 = t_1t_4 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_1$, $[e_1, e_2, e_4]_\tau = t_4e_1$
$\mathfrak{g}_{4,2}$	$t_3 = 0$	$[e_1, e_2, e_4]_\tau = t_4e_3$
$\mathfrak{g}_{4,3}$	$t_2 = t_3t_4 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_2$, $[e_1, e_2, e_4]_\tau = t_4e_2$, $[e_1, e_3, e_4]_\tau = t_4(e_2 + e_3)$
$\mathfrak{g}_{4,4}$	$(1 - \alpha)t_2t_3 = t_2t_4 = t_3t_4 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_2$, $[e_1, e_2, e_4]_\tau = t_4e_2$, $[e_1, e_3, e_4]_\tau = \alpha t_4e_3$
$\mathfrak{g}_{4,5}$	$t_1t_3 = t_1t_4 = t_2t_3 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_1$, $[e_1, e_2, e_4]_\tau = t_4e_1$, $[e_1, e_3, e_4]_\tau = t_1e_3$, $[e_2, e_3, e_4]_\tau = t_2e_3$
$\mathfrak{g}_{4,6}$	$t_1^2 + t_2^2 + t_3^2 = 0$, $t_1t_4 = t_2t_4 = t_3t_4 = 0$	$[e_1, e_2, e_3]_\tau = t_1e_1 + t_2e_2 + t_3e_3$, $[e_1, e_2, e_4]_\tau = t_4e_3$, $[e_1, e_3, e_4]_\tau = -t_4e_2$, $[e_2, e_3, e_4]_\tau = t_4e_1$
$\mathfrak{g}_{4,7}$	$t_3 = t_4 = 0$	$[e_1, e_2, e_3]_\tau = -t_2e_4$
$\mathfrak{g}_4$	quasi-trace functions	induced 3-Lie algebras
$\mathfrak{g}_{4,8}$	$(1 - \alpha)t_2t_4 = 0$, $(1 - \alpha)t_3t_4 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_2 - t_2e_3$, $[e_1, e_2, e_4]_\tau = t_4e_2 - \alpha t_2e_4$, $[e_1, e_3, e_4]_\tau = t_4e_3 - \alpha t_3e_4$
$\mathfrak{g}_{4,9}$	$t_2t_4 - t_3^2 = 0$, $\alpha t_2^2 - \beta t_2t_3 + t_3^2 - t_3t_4 = 0$, $\alpha t_2t_3 - \beta t_3^2 + t_3t_4 - t_4^2 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_3 - t_2e_4$, $[e_1, e_2, e_4]_\tau = -\alpha t_2e_2 + (\beta t_2 + t_4)e_3 - t_2e_4$, $[e_1, e_3, e_4]_\tau = -\alpha t_3e_2 + \beta t_3e_3 - (t_3 - t_4)e_4$
$\mathfrak{g}_{4,10}$	$t_2t_4 - t_3^2 = 0$, $\alpha t_2^2 + \alpha t_2t_3 - t_3t_4 = 0$, $\alpha t_2t_3 + \alpha t_3^2 - t_4^2 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_3 - t_2e_4$, $[e_1, e_2, e_4]_\tau = -\alpha t_2e_2 - (\alpha t_2 - t_4)e_3$, $[e_1, e_3, e_4]_\tau = -\alpha t_3(e_2 + e_3) + t_4e_4$
$\mathfrak{g}_{4,11}$	$t_2t_4 - t_3^2 = 0$, $t_2^2 - t_3t_4 = 0$, $t_2t_3 - t_4^2 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_3 - t_2e_4$, $[e_1, e_2, e_4]_\tau = -t_2e_2 - t_4e_3$, $[e_1, e_3, e_4]_\tau = -t_3e_2 + t_4e_4$
$\mathfrak{g}_{4,12}$	$t_3 = 0$	$[e_1, e_2, e_3]_\tau = \frac{1}{3}t_2e_3$, $[e_1, e_2, e_4]_\tau = \frac{1}{3}t_4e_2 + t_4e_3 - \frac{1}{3}t_2e_4$, $[e_1, e_3, e_4]_\tau = \frac{1}{3}t_4e_3$
$\mathfrak{g}_{4,13}$	$t_4 = 0$	$[e_1, e_2, e_3]_\tau = t_3e_2 - t_2e_3 + t_1e_4$, $[e_1, e_2, e_4]_\tau = -2t_2e_4$, $[e_1, e_3, e_4]_\tau = -2t_3e_4$
$\mathfrak{g}_{4,14}$	$t_2^2 - t_3^2 = t_4 = 0$	$[e_1, e_2, e_3]_\tau = -t_2e_2 + t_3e_3 + t_1e_4$
$\mathfrak{g}_{4,15}$	$\alpha t_2^2 - t_2t_3 + t_3^2 = 0$, $t_4 = 0$	$[e_1, e_2, e_3]_\tau = \alpha t_2e_2 - (t_2 - t_3)e_3 + t_1e_4$, $[e_1, e_2, e_4]_\tau = -t_2e_4$, $[e_1, e_3, e_4]_\tau = -t_3e_4$

Table 5. 4-dimensional 3-Lie algebras induced by quasi-trace functions.

Remark 5.1. Unlike Lie algebras, given two representations (V_i, θ_i) of L, in general there is no representation on $\mathrm{Hom}(V_1, V_2)$ induced by θ_i. However, if (V_1, θ_1) is the adjoint representation of L then there is a representation of the Leibniz algebra $\wedge^2 L$ on $\mathrm{Hom}(V_1, V_2)$, see Lemma 5.2 below.

A homomorphism f from (V_1, θ_1) to (V_2, θ_2) can be defined such that f induces a 3-Lie algebra homomorphism $\tilde{f}$ from $L \oplus V_1$ to $L \oplus V_2$ with $\tilde{f}|_L$ being the identity, see [16]. By a direct computation it follows that a linear map $f\colon V_1 \to V_2$ is a homomorphism if and only if

$$(5.1)\qquad \theta_2(\mathrm{x}, \mathrm{y})(f(\mathrm{v})) = f(\theta_1(\mathrm{x}, \mathrm{y})(\mathrm{v})) \quad \forall\, \mathrm{x}, \mathrm{y} \in L, \mathrm{v} \in V_1.$$

So we have the category L-Mod of representations of L. As in the case of Leibniz algebras (see [21]), there is a unital associative algebra $\mathrm{U}(L)$ such that L-Mod is equivalent to the (left) module category $\mathrm{U}(L)$-Mod (see [16], Proposition 4.4), where $\mathrm{U}(L)$ can be (and will be) chosen as the unital associative algebra generated by $\wedge^2 L$ subject to the following defining relations:

$$(5.2)\qquad \mathrm{XY} - \mathrm{YX} = [\mathrm{X}, \mathrm{Y}]_F, \qquad \underset{\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3}{\circlearrowleft} (\mathrm{x}_1 \wedge \mathrm{x}_2)(\mathrm{x}_3 \wedge \mathrm{x}_4) = [\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3] \wedge \mathrm{x}_4,$$

where $\mathrm{X}, \mathrm{Y} \in \wedge^2 L$, $\mathrm{x}_i \in L$, and $[\cdot, \cdot]_F$ is given by (1.3). Indeed, for any representation (V, θ) of L, V becomes a $\mathrm{U}(L)$-module via

$$(5.3)\qquad \mathrm{X}(\mathrm{v}) = \theta(\mathrm{x}_1, \mathrm{x}_2)(\mathrm{v}) \quad \forall\, \mathrm{X} = \mathrm{x}_1 \wedge \mathrm{x}_2 \in \wedge^2 L, \quad \mathrm{v} \in V.$$

Let $\mathrm{H}^*(L, -)$ be the right derived functor of the invariant submodule functor $(-)^L$. Using $\mathrm{U}(L)$ it is shown that (see [16], Proposition 5.2)

$$(5.4)\qquad \mathrm{H}^*(L, V) = \mathrm{Ext}^*_{\mathrm{U}(L)}(\mathbb{C}, V)$$

for any representation (V, θ) of L.

There is another cohomology of L which is induced by that of the Leibniz algebra $\wedge^2 L$ (see (5.11) below). Recall that a Leibniz algebra is a vector space A with a bilinear map $[\cdot, \cdot]\colon A \otimes A \to A$ such that (see [20], [21])

$$(5.5)\qquad [\mathrm{x}, [\mathrm{y}, \mathrm{z}]] = [[\mathrm{x}, \mathrm{y}], \mathrm{z}] + [\mathrm{y}, [\mathrm{x}, \mathrm{z}]], \quad \mathrm{x}, \mathrm{y}, \mathrm{z} \in A.$$

In fact, it is a left Leibniz algebra. In this paper by Leibniz algebras we always mean left Leibniz algebras. By Theorem 2 of [9], for any 3-Lie algebra L, $\wedge^2 L$ becomes a Leibniz algebra with respect to the bracket $[\cdot, \cdot]_F$ given by (1.3), called the *basic Leibniz algebra* of L. We have the following lemma.

Lemma 5.1. *There is a covariant functor F from the category of 3-Lie algebras to the category of Leibniz algebras given by $F(L)=\wedge^2 L$, $F(f)(\mathrm{x}\wedge\mathrm{y})=f(\mathrm{x})\wedge f(\mathrm{y})$, where $f\colon L\to L_1$ is a 3-Lie algebra homomorphism.*

P r o o f. Straightforward. □

A representation of a Leibniz algebra A is a triple (W,l,r), where W is a vector space and $l,r\colon A\to\mathfrak{gl}(W)$ are linear maps satisfying

$$l([\mathrm{a},\mathrm{a}'])=[l(\mathrm{a}),l(\mathrm{a}')],\ r([\mathrm{a},\mathrm{a}'])=[l(\mathrm{a}'),r(\mathrm{a})],\ r(\mathrm{a}')l(\mathrm{a})=-r(\mathrm{a}')r(\mathrm{a}). \tag{5.6}$$

See (1.5) in [21]. The bracket $[\cdot,\cdot]$ on $\mathfrak{gl}(W)$ is the usual commutator. Note that (5.6) is equivalent to $(MLL)'$, $(LML)'$ and $(LLM)'$ given by (1.5) of [21], which define co-representations of right Leibniz algebras.

Set $CL^p(A,W)=\mathrm{Hom}(\otimes^p A,W)$. We get a cochain complex with the coboundary operator $\partial\colon CL^p(A,W)\to CL^{p+1}(A,W)$ given by

$$\begin{aligned}(5.7)\quad &(\partial(\varphi))(\mathrm{a}_1,\dots,\mathrm{a}_{p+1})\\ &=\sum_{j=1}^{p}(-1)^{j+1}l(\mathrm{a}_j)\varphi(\mathrm{a}_1,\dots,\widehat{\mathrm{a}_j},\dots,\mathrm{a}_{p+1})+(-1)^{p+1}r(\mathrm{a}_{p+1})\varphi(\mathrm{a}_1,\dots,\mathrm{a}_p)\\ &\quad+\sum_{1\leqslant j<k\leqslant p+1}(-1)^j\varphi(\mathrm{a}_1,\dots,\widehat{\mathrm{a}_j},\dots,\mathrm{a}_{k-1},[\mathrm{a}_j,\mathrm{a}_k],\mathrm{a}_{k+1},\dots,\mathrm{a}_{p+1}),\end{aligned}$$

where $\varphi\in CL^p(A,W)$, $\mathrm{a}_i\in A$. Here we consider left Leibniz algebras. So (5.7) is the "left" version of (1.8) in [21]. The pth cohomology group is

$$HL^p_{l,r}(A,W)=ZL^p_{l,r}(A,W)/BL^p_{l,r}(A,W),$$

where $ZL^p_{l,r}(A,W)$ (or $BL^p_{l,r}(A,W)$) is the space of p-cocycles (or p- coboundaries, respectively). We have the following result, which is the nongraded version of Proposition 2.2 in [25]. Note that the representations of 3-Lie colour algebras in Definition 2.4 of [25] are a generalization of Definition 5.1.

Lemma 5.2. *Assume that (V,θ) is a representation of a 3-Lie algebra L. Then $(\mathrm{Hom}(L,V),l,r)$ is a representation of the Leibniz algebra $\wedge^2 L$ with $l,r\colon \wedge^2 L\to\mathrm{End}(\mathrm{Hom}(L,V))$ given by*

$$\begin{aligned}(5.8)\quad &(l(\mathrm{x},\mathrm{y})(f))(\mathrm{z})=\theta(\mathrm{x},\mathrm{y})(f(\mathrm{z}))-f([\mathrm{x},\mathrm{y},\mathrm{z}]),\\ &(r(\mathrm{x},\mathrm{y})(f))(\mathrm{z})=f([\mathrm{x},\mathrm{y},\mathrm{z}])-\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\theta(\mathrm{x},\mathrm{y})(f(\mathrm{z})),\end{aligned}$$

respectively, where $\mathrm{x},\mathrm{y},\mathrm{z}\in L$, $f\in\mathrm{Hom}(L,V)$.

Fix any representation (V,θ) of L. For any integer $p \geqslant 1$ we have the canonical isomorphism (can) of vector spaces given by

$$\text{(5.9)}\qquad \begin{aligned} &\mathrm{can}\colon \mathrm{Hom}(\otimes^{p-1}(\wedge^2 L)\otimes L, V) \to \mathrm{Hom}(\otimes^{p-1}(\wedge^2 L), \mathrm{Hom}(L,V)),\\ &\omega \mapsto \widetilde{\omega}\colon\ \widetilde{\omega}(\mathrm{X}_1,\dots,\mathrm{X}_{p-1})(\mathrm{z}) = \omega(\mathrm{X}_1,\dots,\mathrm{X}_{p-1},\mathrm{z}),\quad \mathrm{X}_i\in\wedge^2 L,\ \mathrm{z}\in L,\end{aligned}$$

which induces a map $d_\theta\colon \mathrm{Hom}(\otimes^{p-1}(\wedge^2 L)\otimes L, V) \to \mathrm{Hom}(\otimes^{p}(\wedge^2 L)\otimes L, V)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\otimes^{p-1}(\wedge^2 L)\otimes L, V) & \xrightarrow{\mathrm{can}} & \mathrm{Hom}(\otimes^{p-1}(\wedge^2 L), \mathrm{Hom}(L,V)) \\ \Big\downarrow{\scriptstyle d_\theta} & & \Big\downarrow{\scriptstyle \partial} \\ \mathrm{Hom}(\otimes^{p}(\wedge^2 L)\otimes L, V) & \xrightarrow{\mathrm{can}} & \mathrm{Hom}(\otimes^{p}(\wedge^2 L), \mathrm{Hom}(L,V)) \end{array}$$

commutes. Here ∂ is given by (5.7). Since ∂ is a coboundary operator, so is d_θ. By a direct computation using (5.9) and (5.7) it follows that

$$\text{(5.10)}\qquad \begin{aligned} &(d_\theta(\omega))(\mathrm{X}_1,\dots,\mathrm{X}_p,\mathrm{z})\\ &= \sum_{1\leqslant j<k\leqslant p} (-1)^j \omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_{k-1},[\mathrm{X}_j,\mathrm{X}_k]_F,\mathrm{X}_{k+1},\dots,\mathrm{X}_p,\mathrm{z})\\ &\quad + \sum_{j=1}^{p}(-1)^j\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_p,[\mathrm{X}_j,\mathrm{z}])\\ &\quad + \sum_{j=1}^{p}(-1)^{j+1}\theta(\mathrm{X}_j)\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_p,\mathrm{z})\\ &\quad + (-1)^{p+1}(\theta(\mathrm{y}_p,\mathrm{z})\omega(\mathrm{X}_1,\dots,\mathrm{X}_{p-1},\mathrm{x}_p) + \theta(\mathrm{z},\mathrm{x}_p)\omega(\mathrm{X}_1,\dots,\mathrm{X}_{p-1},\mathrm{y}_p))\end{aligned}$$

for all $\mathrm{X}_i = \mathrm{x}_i\wedge\mathrm{y}_i \in \wedge^2 L$ and $\mathrm{z}\in L$, which is exactly (4) of [19].

For brevity set $\mathcal{C}^{p-1}(L,V) = \mathrm{Hom}(\otimes^{p-1}(\wedge^2 L)\otimes L, V)$. Hence we get a cochain complex $(\oplus_p \mathcal{C}^{p-1}(L,V), d_\theta)$ which is induced from the cochain complex of the Leibniz algebra $\wedge^2 L$. Denote the pth cohomology group by

$$\text{(5.11)}\qquad \mathcal{H}^p_\theta(L,V) = \mathcal{Z}^p_\theta(L,V)/\mathcal{B}^p_\theta(L,V),$$

where $\mathcal{Z}^p_\theta(L,V)$ (or $\mathcal{B}^p_\theta(L,V)$) is the space of $(p+1)$-cocycles (or $(p+1)$-coboundaries, respectively). Therefore, $\mathcal{H}^*_\theta(L,V)$ is deduced from $HL^*_{l,r}(\wedge^2 L, \mathrm{Hom}(L,V))$ via the representation of $\wedge^2 L$ on $\mathrm{Hom}(L,V)$ given by Lemma 5.2.

Lemma 5.3. *Let L be a 3-Lie algebra and (V,θ) a representation of L. Then $(V,\theta,-\theta)$ is a Leibniz algebra representation of $\wedge^2 L$.*

P r o o f. Fix any $\mathrm{x}_i \in L$, $1 \leqslant i \leqslant 4$. By (5.6) it suffices to show that

$$\theta([\mathrm{x}_1 \wedge \mathrm{x}_2, \mathrm{x}_3 \wedge \mathrm{x}_4]_F) = [\theta(\mathrm{x}_1,\mathrm{x}_2), \theta(\mathrm{x}_3,\mathrm{x}_4)],$$

which is equivalent to

$$\theta([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3] \wedge \mathrm{x}_4 + \mathrm{x}_3 \wedge [\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4]) = \theta(\mathrm{x}_1,\mathrm{x}_2)\theta(\mathrm{x}_3,\mathrm{x}_4) - \theta(\mathrm{x}_3,\mathrm{x}_4)\theta(\mathrm{x}_1,\mathrm{x}_2).$$

By Definition 5.1 the result follows. □

Let (V,θ) be a representation of the 3-Lie algebra L. By Lemma 5.3, it is possible to compare $\mathcal{H}^*_\theta(L,V)$ and the cohomology of $\wedge^2 L$ in V. Consider the representation $(V,\ \theta,\ -\theta)$ of the Leibniz algebra $\wedge^2 L$. Fix any $\mathrm{z} \in L$ and $p \geqslant 0$. Then there is a linear inclusion of vector spaces

$$\otimes^p(\wedge^2 L) \hookrightarrow \otimes^p(\wedge^2 L) \otimes L\colon\ \mathrm{X}_1 \otimes \ldots \otimes \mathrm{X}_p \mapsto \mathrm{X}_1 \otimes \ldots \otimes \mathrm{X}_p \otimes \mathrm{z},\ \ \mathrm{X}_i \in \wedge^2 L,$$

which induces a map $f^p = f^p_{\mathrm{z}}\colon\ \mathcal{C}^p(L,V) \to CL^p(\wedge^2 L, V)\colon\ \omega \mapsto \widetilde{\omega}$, where

$$\widetilde{\omega}(X_1,\ldots,X_p) = \omega(X_1,\ldots,X_p,\mathrm{z}), \quad \mathrm{X}_i \in \wedge^2 L,\ 1 \leqslant i \leqslant p. \tag{5.12}$$

Proposition 5.1. *Assume that $\mathrm{z} \in Z(L)$ and $\theta(\mathrm{x} \wedge \mathrm{z}) = 0$ for any $\mathrm{x} \in L$. Then $\{f^p = f^p_{\mathrm{z}}\}_{p\geqslant 0}$ is a cochain map from $\big(\bigoplus_{p\geqslant 0} \mathcal{C}^p(L,V), d_\theta\big)$ to $\big(\bigoplus_{p\geqslant 0} CL^p(\wedge^2 L,V), \partial\big)$, which induces a map $\mathcal{H}^p_\theta(L,V) \to HL^p_{\theta,-\theta}(\wedge^2 L, V)$ given by $[\omega] \mapsto [f^p(\omega)]$, $\omega \in \mathcal{Z}^p_\theta(L,V)$.*

P r o o f. It suffices to show that $\partial \circ f^p = f^{p+1} \circ d_\theta$. Fix any $\mathrm{X}_i \in \wedge^2 L$, $1 \leqslant i \leqslant p+1$. By linearity we may assume that $\mathrm{X}_i = \mathrm{x}_i \wedge \mathrm{y}_i$. By (5.7) and (5.12) we have

$$\begin{aligned}((\partial \circ f^p)(\omega))(\mathrm{X}_1,\ldots,\mathrm{X}_{p+1}) &= (\partial(\widetilde{\omega}))(\mathrm{X}_1,\ldots,\mathrm{X}_{p+1})\\ &= \sum_{j=1}^{p+1}(-1)^{j+1}\theta(\mathrm{X}_j)\omega(\mathrm{X}_1,\ldots,\widehat{\mathrm{X}_j},\ldots,\mathrm{X}_{p+1},\mathrm{z})\\ &\quad + \sum_{1\leqslant j<k\leqslant p+1}(-1)^j\omega(\mathrm{X}_1,\ldots,\widehat{\mathrm{X}_j},\ldots,\mathrm{X}_{k-1},[\mathrm{X}_j,\mathrm{X}_k]_F,\mathrm{X}_{k+1},\ldots,\mathrm{X}_{p+1},\mathrm{z}).\end{aligned}$$

On the other hand, by (5.10) and (5.12) we have

$$
\begin{aligned}
&((f^{p+1}\circ d_\theta)(\omega))(\mathrm{X}_1,\dots,\mathrm{X}_{p+1})\\
&\quad=(d_\theta(\omega))(\mathrm{X}_1,\dots,\mathrm{X}_{p+1},\mathrm{z})\\
&\quad=\sum_{1\leqslant j<k\leqslant p+1}(-1)^j\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_{k-1},[\mathrm{X}_j,\mathrm{X}_k]_F,\mathrm{X}_{k+1},\dots,\mathrm{X}_{p+1},\mathrm{z})\\
&\qquad+\sum_{j=1}^{p+1}(-1)^j\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_{p+1},\underline{[\mathrm{X}_j,\mathrm{z}]})\\
&\qquad+\sum_{j=1}^{p+1}(-1)^{j+1}\theta(\mathrm{X}_j)\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_{p+1},\mathrm{z})\\
&\qquad+(-1)^{p+2}(\underline{\theta(\mathrm{y}_{p+1},\mathrm{z})}\omega(\mathrm{X}_1,\dots,\mathrm{X}_{p-1},\mathrm{x}_{p+1})+\underline{\theta(\mathrm{z},\mathrm{x}_{p+1})}\omega(\mathrm{X}_1,\dots,\mathrm{X}_p,\mathrm{y}_{p+1}))\\
&\quad=\sum_{1\leqslant j<k\leqslant p+1}(-1)^j\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_{k-1},[\mathrm{X}_j,\mathrm{X}_k]_F,\mathrm{X}_{k+1},\dots,\mathrm{X}_{p+1},\mathrm{z})\\
&\qquad+\sum_{j=1}^{p+1}(-1)^{j+1}\theta(\mathrm{X}_j)\omega(\mathrm{X}_1,\dots,\widehat{\mathrm{X}_j},\dots,\mathrm{X}_{p+1},\mathrm{z}),
\end{aligned}
$$

where the underlined terms are zero since $\mathrm{z}\in Z(L)$ and $\theta(\mathrm{x}\wedge\mathrm{z})=0$ for any $\mathrm{x}\in L$ by assumption. So $\partial\circ f_p=f^{p+1}\circ d_\theta$ as required. $\square$

Example 5.1. Let $\theta=\mathrm{ad}$ be the adjoint representation of L. Then the condition $\mathrm{z}\in Z(L)$ implies that $\mathrm{ad}(\mathrm{x}\wedge\mathrm{z})=0$ for any $\mathrm{x}\in L$. So, for any $\mathrm{z}\in Z(L)$, there exists a map $\mathcal{H}^p_\theta(L,L)\to HL^p_{\theta,-\theta}(\wedge^2L,L)$ given by $[\omega]\to[f^p_z(\omega)]$, $\omega\in\mathcal{Z}^p_\theta(L,L)$, where f^p is given by (5.12).

6. Representations of the induced 3-Lie algebras

In this section $\mathfrak{g}$ denotes a Lie algebra with bracket $[\cdot,\cdot]$. Keep notation as in former sections. For any $\tau\in\mathfrak{g}^*$ there is a linear map $\tau^\sharp\colon\wedge^2\mathfrak{g}\to\mathfrak{g}$ given by

$$
\tau^\sharp(\mathrm{x}\wedge\mathrm{y})=\tau(\mathrm{x})\mathrm{y}-\tau(\mathrm{y})\mathrm{x}\quad\forall\,\mathrm{x},\mathrm{y}\in\mathfrak{g}. \tag{6.1}
$$

Lemma 6.1. *Assume that* $0\neq\tau\in\mathfrak{g}^*$. *Then* $\mathrm{im}(\tau^\sharp)=\ker\tau$.

P r o o f. Since $\tau(\tau^\sharp(\mathrm{x}\wedge\mathrm{y}))=0$, $\mathrm{x},\mathrm{y}\in\mathfrak{g}$, $\mathrm{im}(\tau^\sharp)\subseteq\ker\tau$. Fix a basis $\{e_i\}_{i\in\mathcal{I}}$ of $\ker\tau$. Choose any $e\notin\ker\tau$. Then $\{e_i\}_{i\in\mathcal{I}}\cup\{e\}$ is a basis of $\mathfrak{g}$. Since $\tau^\sharp(e_i\wedge e_j)=0$ and $\tau^\sharp(e_i\wedge e)=-\tau(e)e_i\neq0$, $\mathrm{im}(\tau^\sharp)$ is generated by $\{e_i\}_{i\in\mathcal{I}}$. $\square$

Let $\tau \in \mathfrak{g}^*$. Consider the 3-ary bracket $[\cdot,\cdot,\cdot]_\tau$ given by (1.2) on $\mathfrak{g}_\tau = \mathfrak{g}$. On $\wedge^2\mathfrak{g}_\tau$ we have the bracket $[\cdot,\cdot]_F$ with respect to $[\cdot,\cdot,\cdot]_\tau$ given by (1.3).

Lemma 6.2. *$\tau \in \mathfrak{g}^*$ is a quasi-trace function on $\mathfrak{g}$ if and only if the map $\tau^\sharp\colon \wedge^2\mathfrak{g}_\tau \to \mathfrak{g}$ given by* (6.1) *satisfies that $\tau^\sharp([\mathrm{X}_1,\mathrm{X}_2]_F) = [\tau^\sharp(\mathrm{X}_1),\tau^\sharp(\mathrm{X}_2)]$ for any X_1, $\mathrm{X}_2 \in \wedge^2\mathfrak{g}$. In this case, $\tau^\sharp$ is a homomorphism of Leibniz algebras ($\mathfrak{g}$ is regarded as a Leibniz algebra).*

Proof. By linearity we may assume that $\mathrm{X}_i = \mathrm{x}_i \wedge \mathrm{y}_i \in \wedge^2\mathfrak{g}$, $i = 1,2$. By (1.2), (1.3) and (6.1) it follows that

$$\begin{aligned}
\tau^\sharp([\mathrm{X}_1,\mathrm{X}_2]_F) &= \tau^\sharp([\mathrm{x}_1,\mathrm{y}_1,\mathrm{x}_2]_\tau \wedge \mathrm{y}_2 + \mathrm{x}_2 \wedge [\mathrm{x}_1,\mathrm{y}_1,\mathrm{y}_2]_\tau) \\
&= \tau(\mathrm{x}_1)\tau(\mathrm{x}_2)[\mathrm{y}_1,\mathrm{y}_2] - \tau(\mathrm{x}_1)\tau(\mathrm{y}_2)[\mathrm{y}_1,\mathrm{x}_2] - \tau(\mathrm{y}_1)\tau(\mathrm{x}_2)[\mathrm{x}_1,\mathrm{y}_2] \\
&\quad + \tau(\mathrm{y}_1)\tau(\mathrm{y}_2)[\mathrm{x}_1,\mathrm{x}_2] + \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{x}_2]_\tau)\mathrm{y}_2 - \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{y}_2]_\tau)\mathrm{x}_2 \\
&= [\tau(\mathrm{x}_1)\mathrm{y}_1 - \tau(\mathrm{y}_1)\mathrm{x}_1, \tau(\mathrm{x}_2)\mathrm{y}_2 - \tau(\mathrm{y}_2)\mathrm{x}_2] \\
&\quad + \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{x}_2]_\tau)\mathrm{y}_2 - \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{y}_2]_\tau)\mathrm{x}_2 \\
&= [\tau^\sharp(\mathrm{X}_1),\tau^\sharp(\mathrm{X}_2)] + \underline{\tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{x}_2]_\tau)\mathrm{y}_2 - \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{y}_2]_\tau)\mathrm{x}_2}.
\end{aligned} \tag{6.2}$$

So, if $\tau \in F_{\mathrm{qtr}}(\mathfrak{g})$ then $\tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{x}_2]_\tau) = \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{y}_2]_\tau) = 0$ by Corollary 2.1, which means that $\tau^\sharp([\mathrm{X}_1,\mathrm{X}_2]_F) = [\tau^\sharp(\mathrm{X}_1),\tau^\sharp(\mathrm{X}_2)]$ as required. In particular, since $\mathfrak{g}_\tau$ is a 3-Lie algebra, which implies that $\wedge^2\mathfrak{g}_\tau$ is a Leibniz algebra with respect to $[\cdot,\cdot]_F$ by [9], $\tau^\sharp$ is a Leibniz algebra homomorphism.

Conversely, assume that $\tau^\sharp([\mathrm{X}_1,\mathrm{X}_2]_F) = [\tau^\sharp(\mathrm{X}_1),\tau^\sharp(\mathrm{X}_2)]$ for any X_1, $\mathrm{X}_2 \in \wedge^2\mathfrak{g}$. By (6.2) it follows that

$$\tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{x}_2]_\tau)\mathrm{y}_2 - \tau([\mathrm{x}_1,\mathrm{y}_1,\mathrm{y}_2]_\tau)\mathrm{x}_2 = 0 \quad \forall\, \mathrm{x}_i,\mathrm{y}_i \in \mathfrak{g}. \tag{6.3}$$

Fix any $\mathrm{x},\mathrm{y},\mathrm{z} \in \mathfrak{g}$. By Corollary 2.1, to show $\tau \in F_{\mathrm{qtr}}(\mathfrak{g})$ it suffices to show that $\tau([\mathrm{x},\mathrm{y},\mathrm{z}]_\tau) = 0$. If $\dim\mathfrak{g} = 1$ then $\mathfrak{g}$ is abelian and hence τ is always a quasi-trace function by Example 2.5. So we may assume that $\dim\mathfrak{g} \geqslant 2$ and $\mathrm{z} \neq 0$. Then, we can choose $\mathrm{z}' \in \mathfrak{g}$ such that z,z' are linearly independent.

Set $\mathrm{x}_1 = \mathrm{x}$, $\mathrm{y}_1 = \mathrm{y}$, $\mathrm{x}_2 = \mathrm{z}$, $\mathrm{y}_2 = \mathrm{z}'$. Then $\tau([\mathrm{x},\mathrm{y},\mathrm{z}]_\tau)\mathrm{z}' - \tau([\mathrm{x},\mathrm{y},\mathrm{y}']_\tau)\mathrm{z} = 0$ by (6.3). Since z,z' are linearly independent, $\tau([\mathrm{x},\mathrm{y},\mathrm{z}]_\tau) = 0$ as required. $\square$

Let $\tau \in F_{3\text{-Lie}}(\mathfrak{g})$. Recall the associative algebra $\mathrm{U}(\mathfrak{g}_\tau)$ (see (5.2)) associated to $\mathfrak{g}_\tau$. Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

Theorem 6.1. *Let $\tau \in F_{3\text{-Lie}}(\mathfrak{g})$. The map $\tau^\sharp$ given by* (6.1) *induces a homomorphism of associative algebras from $\mathrm{U}(\mathfrak{g}_\tau)$ to $\mathrm{U}(\mathfrak{g})$ if and only if τ is a quasi-trace function on $\mathfrak{g}$, i.e., $\tau \in F_{\mathrm{qtr}}(\mathfrak{g})$.*

P r o o f. Since $\mathrm{U}(\mathfrak{g}_\tau)$ is generated by $\wedge^2\mathfrak{g}_\tau$ and $\mathrm{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$ respectively, it suffices to check that $\tau^\sharp$ sends defining relations of $\mathrm{U}(\mathfrak{g}_\tau)$ to that of $\mathrm{U}(\mathfrak{g})$. Recall that the defining relations of $\mathrm{U}(\mathfrak{g}_\tau)$ are

$$\begin{gathered}\mathrm{XY}-\mathrm{YX}=[\mathrm{X},\mathrm{Y}]_F,\\ [\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau\wedge\mathrm{x}_4=\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}(\mathrm{x}_1\wedge\mathrm{x}_2)(\mathrm{x}_3\wedge\mathrm{x}_4),\end{gathered}\tag{6.4}$$

where $\mathrm{X},\mathrm{Y}\in\wedge^2\mathfrak{g}_\tau$, $\mathrm{x}_i\in\mathfrak{g}_\tau$, and $[\cdot,\cdot,\cdot]_\tau$ is given by (1.2), while the defining relation of $\mathrm{U}(\mathfrak{g})$ is

$$\mathrm{xy}-\mathrm{yx}=[\mathrm{x},\mathrm{y}]\quad\forall\,\mathrm{x},\mathrm{y}\in\mathfrak{g}.\tag{6.5}$$

By (6.1) in $\mathrm{U}(\mathfrak{g})$ we have

$$\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau^\sharp(\mathrm{x}_1\wedge\mathrm{x}_2)\tau(\mathrm{x}_3)=\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}(\tau(\mathrm{x}_1)\mathrm{x}_2-\tau(\mathrm{x}_2)\mathrm{x}_1)\tau(\mathrm{x}_3)=0\tag{6.6}$$

and

$$\begin{aligned}\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau^\sharp(\mathrm{x}_1\wedge\mathrm{x}_2)\mathrm{x}_3\ &=\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}(\tau(\mathrm{x}_1)\mathrm{x}_2\mathrm{x}_3-\tau(\mathrm{x}_2)\mathrm{x}_1\mathrm{x}_3)\\ &=\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau(\mathrm{x}_1)(\mathrm{x}_2\mathrm{x}_3-\mathrm{x}_3\mathrm{x}_2)\\ &\overset{(6.5)}{=}\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau(\mathrm{x}_1)[\mathrm{x}_2,\mathrm{x}_3]\\ &\overset{(1.2)}{=}[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau.\end{aligned}\tag{6.7}$$

By (6.6) and (6.7) we have in $\mathrm{U}(\mathfrak{g})$

$$\begin{aligned}&\tau^\sharp([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau\wedge\mathrm{x}_4)-\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau^\sharp(\mathrm{x}_1\wedge\mathrm{x}_2)\tau^\sharp(\mathrm{x}_3\wedge\mathrm{x}_4)\\ &\quad=\tau([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau)\mathrm{x}_4-\tau(\mathrm{x}_4)[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau\\ &\qquad-(\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau^\sharp(\mathrm{x}_1\wedge\mathrm{x}_2)\tau(\mathrm{x}_3))\mathrm{x}_4+\tau(\mathrm{x}_4)(\underset{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}{\circlearrowleft}\tau^\sharp(\mathrm{x}_1\wedge\mathrm{x}_2)\mathrm{x}_3)\\ &\quad=\tau([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau)\mathrm{x}_4-\tau(\mathrm{x}_4)[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau+\tau(\mathrm{x}_4)[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau\\ &\quad=\tau([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau)\mathrm{x}_4.\end{aligned}\tag{6.8}$$

By Corollary 2.1, Lemma 6.2 and (6.8) the result follows. □

As a direct application of Theorem 6.1 and Lemma 6.1 we have the following corollary.

Corollary 6.1. *Assume that* $0\neq\tau$ *is a quasi-trace function on* $\mathfrak{g}$. *Then the image of the homomorphism* $\tau^\sharp\colon\ \mathrm{U}(\mathfrak{g}_\tau)\to\mathrm{U}(\mathfrak{g})$ *equals to* $\mathrm{U}(\ker\tau)$, *the universal enveloping algebra of the Lie algebra* $\ker\tau$.

Using the homomorphism $\tau^\sharp\colon \mathrm{U}(\mathfrak{g}_\tau) \to \mathrm{U}(\mathfrak{g})$ in Theorem 6.1 we get the following corollary.

Corollary 6.2. *Assume that τ is a quasi-trace function on $\mathfrak{g}$. Let (V, ϱ) be a representation of $\mathfrak{g}$. Then the composition $\varrho_\tau := \varrho \circ \tau^\sharp\colon \mathrm{U}(\mathfrak{g}_\tau) \to \mathrm{End}(V)$ affords a representation of the 3-Lie algebra $\mathfrak{g}_\tau$ on V, and τ induces a functor from $\mathfrak{g} - \mathrm{Mod}$ to $\mathfrak{g}_\tau - \mathrm{Mod}$.*

Note that ϱ_τ is given by

$$\varrho_\tau(\mathrm{x}_1, \mathrm{x}_2) = \tau(\mathrm{x}_1)\varrho(\mathrm{x}_2) - \tau(\mathrm{x}_2)\varrho(\mathrm{x}_1) \in \mathrm{End}(V), \quad \mathrm{x}_1, \mathrm{x}_2 \in \mathfrak{g}_\tau = \mathfrak{g}. \tag{6.9}$$

Now we recall the Chevalley-Eilenberg cochain complexes of $\mathfrak{g}$. Let (V, ϱ) be a representation of $\mathfrak{g}$. The space $C^p(\mathfrak{g}, V)$ of p-cochains is $\mathrm{Hom}(\wedge^p \mathfrak{g}, V)$, while the coboundary operator $\delta_\varrho\colon C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$ is given by

$$\begin{aligned}(\delta_\varrho(f))(\mathrm{x}_1, \ldots, \mathrm{x}_{p+1}) &= \sum_{j=1}^{p+1} (-1)^{j+1} \varrho(\mathrm{x}_j) f(\mathrm{x}_1, \ldots, \widehat{\mathrm{x}_j}, \ldots, \mathrm{x}_{p+1}) \\ &\quad + \sum_{1 \leqslant j < k \leqslant p+1} (-1)^{j+k} f([\mathrm{x}_j, \mathrm{x}_k], \mathrm{x}_1, \ldots, \widehat{\mathrm{x}_j}, \ldots, \widehat{\mathrm{x}_k}, \ldots, \mathrm{x}_{p+1}).\end{aligned} \tag{6.10}$$

Let $Z^p_\varrho(\mathfrak{g}, V)$ (or $B^p_\varrho(\mathfrak{g}, V)$) be the space of p-cocycles (or p-coboundaries, respectively). Then, the pth cohomology group of $\mathfrak{g}$ (with coefficients in V) is

$$H^p_\varrho(\mathfrak{g}, V) = Z^p_\varrho(\mathfrak{g}, V)/B^p_\varrho(\mathfrak{g}, V).$$

For the representation given in Corollary 6.2 and the cohomology $\mathrm{H}^*(\mathfrak{g}_\tau, V)$ introduced in [16] (see (5.4)) we have the following corollary.

Corollary 6.3. *Let τ be a quasi-trace function on $\mathfrak{g}$ and (V, ϱ) a representation of $\mathfrak{g}$. If $\mathrm{U}(\mathfrak{g})$ is a projective module of $\mathrm{U}(\mathfrak{g}_\tau)$ via the homomorphism $\tau^\sharp$ then $\mathrm{H}^*(\mathfrak{g}_\tau, V) \cong H^*_\varrho(\mathfrak{g}, V)$.*

Proof. Recall that $H^*_\varrho(\mathfrak{g}, V) = \mathrm{Ext}^*_{\mathrm{U}(\mathfrak{g})}(\mathbb{C}, V)$. Since $\mathrm{U}(\mathfrak{g})$ is a projective module of $\mathrm{U}(\mathfrak{g}_\tau)$, by the theorem of change of rings any projective resolution of the trivial representation of $\mathfrak{g}$ is also a projective resolution of the trivial representation of $\mathfrak{g}_\tau$, and hence the result follows. □

7. Some comparison of cohomologies arising from quasi-trace functions

In this section $\mathfrak{g}$ is a Lie algebra and τ is a quasi-trace function on $\mathfrak{g}$. We fix a representation (V,ϱ) of $\mathfrak{g}$. Then we have a representation (V,ϱ_τ) of the 3-Lie algebra $\mathfrak{g}_\tau$ given by Corollary 6.2. We construct some cocycles of $\mathfrak{g}_\tau$ from those of $\mathfrak{g}$, and compare the cohomologies of $\mathfrak{g}$ and the Leibniz algebra $\wedge^2\mathfrak{g}_\tau$ associated to the 3-Lie algebra $\mathfrak{g}_\tau$.

In the case that τ is a trace function, 1-cocycles and 2-cocyles of $\mathfrak{g}_\tau$ are studied in [3] for the trivial representation and adjoint representation of $\mathfrak{g}_\tau$. Note that the trivial representation of $\mathfrak{g}_\tau$ is induced by the trivial representation of $\mathfrak{g}$ via Corollary 6.2, while the adjoint representation of $\mathfrak{g}_\tau$ cannot be induced by the adjoint representation of $\mathfrak{g}$ via Corollary 6.2 in general, see Corollary 6.1.

At first we consider 1-cocyles. Note that $C^1(\mathfrak{g},V)=\mathrm{Hom}(\mathfrak{g},V)=\mathcal{C}^0(\mathfrak{g}_\tau,V)$.

Proposition 7.1. *It holds that* $Z^1_\varrho(\mathfrak{g},V)\subseteq\mathcal{Z}^0_{\varrho_\tau}(\mathfrak{g}_\tau,V)$.

Proof. It suffices to show that

$$(d_{\varrho_\tau}(\lambda))(\mathrm{x},\mathrm{y},\mathrm{z})=\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\tau(\mathrm{x})(\delta_\varrho(\lambda))(\mathrm{y},\mathrm{z}),\tag{7.1}$$

where δ_ϱ (or d_{ϱ_τ}) is given by (6.10) (or (5.10), respectively), and $\lambda\in C^1_\varrho(\mathfrak{g},V)$, $\mathrm{x},\mathrm{y},\mathrm{z}\in\mathfrak{g}_\tau=\mathfrak{g}$. Indeed,

$$\begin{aligned}(d_{\varrho_\tau}(\lambda))(\mathrm{x},\mathrm{y},\mathrm{z})&=-\lambda([\mathrm{x},\mathrm{y},\mathrm{z}]_\tau)+\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\varrho_\tau(\mathrm{x},\mathrm{y})\lambda(\mathrm{z}) && \text{(by (5.10))}\\ &=-\lambda\big(\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\tau(\mathrm{x})[\mathrm{y},\mathrm{z}]\big)\\ &\quad+\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}(\tau(\mathrm{x})\varrho(\mathrm{y})-\tau(\mathrm{y})\varrho(\mathrm{x}))\lambda(\mathrm{z}) && \text{(by (1.2), (6.9))}\\ &=\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}(\tau(\mathrm{x})(-\lambda([\mathrm{y},\mathrm{z}])+\varrho(\mathrm{y})\lambda(\mathrm{z})-\varrho(\mathrm{z})\lambda(\mathrm{y})))\\ &=\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\tau(\mathrm{x})(\delta_\varrho(\lambda))(\mathrm{y},\mathrm{z}) && \text{(by (6.10))}\end{aligned}$$

as required. □

Proposition 7.1 generalizes Theorem 4.3 of [3]. More precisely, the identity (7.1) generalizes Lemma 4.2 of [3] where τ is a trace function on $\mathfrak{g}$.

For 2-cocycles we consider the linear map, denoted again by $\tau^\sharp$, from $C^2(\mathfrak{g},V)$ $=\mathrm{Hom}\,(\wedge^2\mathfrak{g},\,V)$ to $\mathcal{C}^1(\mathfrak{g}_\tau,V)=\mathrm{Hom}(\wedge^2\mathfrak{g}\otimes\mathfrak{g},V)$, given by

$$(\tau^\sharp(\omega))(\mathrm{x},\mathrm{y},\mathrm{z})=\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\tau(\mathrm{x})\omega(\mathrm{y},\mathrm{z}),\ \omega\in C^2(\mathfrak{g},V),\quad \mathrm{x},\mathrm{y},\mathrm{z}\in\mathfrak{g}.\tag{7.2}$$

With respect to the trivial representation and the adjoint representation of $\mathfrak{g}_\tau$, (7.2) is defined for a trace function τ in Theorems 4.2 and 4.4 of [3].

Proposition 7.2. *Let τ be a quasi-trace function on $\mathfrak{g}$ and (V,ϱ) be a representation of $\mathfrak{g}$. Then there is a morphism $H^2_\varrho(\mathfrak{g},V)\to\mathcal{H}^1_{\varrho_\tau}(\mathfrak{g}_\tau,V)$ given by $[\omega]\mapsto[\tau^\sharp(\omega)]$.*

Proof. At first we show that $d_{\varrho_\tau}(\tau^\sharp(\omega))=0$ for any $\omega\in Z^2_\varrho(\mathfrak{g},V)$, i.e., $\tau^\sharp(\omega)\in\mathcal{Z}^1(\mathfrak{g}_\tau,V)$. Since $\delta_\varrho(\omega)=0$, by (6.10) it follows that

$$0=(\delta_\varrho(\omega))(\mathrm{x},\mathrm{y},\mathrm{z})=\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\varrho(\mathrm{x})\omega(\mathrm{y},\mathrm{z})-\circlearrowleft_{\mathrm{x},\mathrm{y},\mathrm{z}}\omega([\mathrm{x},\mathrm{y}],\mathrm{z}),\quad \mathrm{x},\mathrm{y},\mathrm{z}\in\mathfrak{g}. \tag{7.3}$$

Fix any $\mathrm{x}_i\in\mathfrak{g}=\mathfrak{g}_\tau$, $1\leqslant i\leqslant 5$. Since $\tau\in F_{\mathrm{qtr}}(\mathfrak{g})$, $\tau([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau)=0$ by (2.5). So, by (7.2) it follows that

$$\tau^\sharp(\omega)([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau,\mathrm{x}_4,\mathrm{x}_5)=\tau(\mathrm{x}_4)\omega(\mathrm{x}_5,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau)+\tau(\mathrm{x}_5)\omega([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau,\mathrm{x}_4).$$

By this and similar identities we deduce from (5.10) that

$$\begin{aligned}(d_{\varrho_\tau}&(\tau^\sharp(\omega)))(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5)\\
&=-(\tau(\mathrm{x}_4)\omega(\mathrm{x}_5,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau)+\tau(\mathrm{x}_5)\omega([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3]_\tau,\mathrm{x}_4))\\
&\quad-(\tau(\mathrm{x}_3)\omega([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4]_\tau,\mathrm{x}_5)+\tau(\mathrm{x}_5)\omega(\mathrm{x}_3,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4]_\tau))\\
&\quad-(\tau(\mathrm{x}_3)\omega(\mathrm{x}_4,[\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5]_\tau)+\tau(\mathrm{x}_4)\omega([\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5]_\tau,\mathrm{x}_3))\\
&\quad+(\tau(\mathrm{x}_1)\omega(\mathrm{x}_2,[\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5]_\tau)+\tau(\mathrm{x}_2)\omega([\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5]_\tau,\mathrm{x}_1))\\
&\quad+(\tau(\mathrm{x}_1)\varrho(\mathrm{x}_2)-\tau(\mathrm{x}_2)\varrho(\mathrm{x}_1))(\circlearrowleft_{\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5}\tau(\mathrm{x}_3)\omega(\mathrm{x}_4,\mathrm{x}_5))\\
&\quad-(\tau(\mathrm{x}_3)\varrho(\mathrm{x}_4)-\tau(\mathrm{x}_4)\varrho(\mathrm{x}_3))(\circlearrowleft_{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_5}\tau(\mathrm{x}_1)\omega(\mathrm{x}_2,\mathrm{x}_5))\\
&\quad-(\tau(\mathrm{x}_4)\varrho(\mathrm{x}_5)-\tau(\mathrm{x}_5)\varrho(\mathrm{x}_4))(\circlearrowleft_{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3}\tau(\mathrm{x}_1)\omega(\mathrm{x}_2,\mathrm{x}_3))\\
&\quad-(\tau(\mathrm{x}_5)\varrho(\mathrm{x}_3)-\tau(\mathrm{x}_3)\varrho(\mathrm{x}_5))(\circlearrowleft_{\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4}\tau(\mathrm{x}_1)\omega(\mathrm{x}_2,\mathrm{x}_4)).\end{aligned} \tag{7.4}$$

By (1.2), (7.3) and anti-symmetry of ω, the right hand side of (7.4) can be rewritten as

$$\begin{aligned}(d_{\varrho_\tau}&(\tau^\sharp(\omega)))(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4,\mathrm{x}_5)\\
&=\tau(\mathrm{x}_1)\tau(\mathrm{x}_4)(\delta_\varrho(\omega))(\mathrm{x}_2,\mathrm{x}_5,\mathrm{x}_3)+\tau(\mathrm{x}_2)\tau(\mathrm{x}_4)(\delta_\varrho(\omega))(\mathrm{x}_3,\mathrm{x}_5,\mathrm{x}_1)\\
&\quad+\tau(\mathrm{x}_1)\tau(\mathrm{x}_5)(\delta_\varrho(\omega))(\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)+\tau(\mathrm{x}_2)\tau(\mathrm{x}_5)(\delta_\varrho(\omega))(\mathrm{x}_3,\mathrm{x}_1,\mathrm{x}_4)\\
&\quad+\tau(\mathrm{x}_1)\tau(\mathrm{x}_3)(\delta_\varrho(\omega))(\mathrm{x}_2,\mathrm{x}_4,\mathrm{x}_5)+\tau(\mathrm{x}_2)\tau(\mathrm{x}_3)(\delta_\varrho(\omega))(\mathrm{x}_4,\mathrm{x}_1,\mathrm{x}_5)\\
&=0,\end{aligned}$$

which means $d_{\varrho_\tau}(\tau^\sharp(\omega))=0$ as required.

Now we show that, if $[\omega_1] = [\omega_2]$ then $[\tau^\sharp(\omega_1)] = [\tau^\sharp(\omega_2)]$, where $\tau^\sharp(\omega_1)$, $\tau^\sharp(\omega_2)$ are given by (7.2). Assume that $\omega_2 - \omega_1 = \delta_\varrho(\lambda)$ for some $\lambda \in C^1_\varrho(\mathfrak{g}, V) = \mathcal{Z}^0(\mathfrak{g}_\tau, V)$. For any $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{g}$, by (7.2) it follows that

$$\begin{aligned}
&(\tau^\sharp(\omega_2))(\mathrm{x},\mathrm{y},\mathrm{z}) - (\tau^\sharp(\omega_1))(\mathrm{x},\mathrm{y},\mathrm{z}) \\
&\qquad = \underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})\omega_2(\mathrm{y},\mathrm{z}) - \underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})\omega_1(\mathrm{y},\mathrm{z}) = \underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})(\omega_2 - \omega_1)(\mathrm{y},\mathrm{z}) \\
&\qquad = \underset{\mathrm{x},\mathrm{y},\mathrm{z}}{\circlearrowleft} \tau(\mathrm{x})(\delta_\varrho(\lambda))(\mathrm{y},\mathrm{z}) = (d_{\varrho_\tau}(\lambda))(\mathrm{x},\mathrm{y},\mathrm{z}) \quad \text{(by (7.1))}.
\end{aligned}$$

So, $\tau^\sharp(\omega_2) - \tau^\sharp(\omega_1) = d_{\varrho_\tau}(\lambda)$, and hence $[\tau^\sharp(\omega_2)] = [\tau^\sharp(\omega_1)]$ as required. □

Let $\mathfrak{g}$ be a Lie algebra and τ a quasi-trace function. By Lemma 5.3 and Corollary 6.2, $(V, \varrho_\tau, -\varrho_\tau)$ is a representation of the Leibniz algebra $\wedge^2\mathfrak{g}_\tau$, whose bracket is given by (1.3). For any integer $p \geqslant 0$, define $\tau^{(p)}\colon C^p(\mathfrak{g}, V) = \mathrm{Hom}(\wedge^p\mathfrak{g}, V) \to CL^p(\wedge^2\mathfrak{g}_\tau, V) = \mathrm{Hom}(\otimes^p(\wedge^2\mathfrak{g}_\tau), V)$ by

$$\omega \mapsto \tau^{(p)}(\omega) \triangleq \omega \circ \tau^\sharp, \tag{7.5}$$

where $\tau^\sharp$ is given by (6.1). For the cochain complex $(\oplus_p C^p(\mathfrak{g}, V), \delta_\varrho)$ associated to (V, ϱ) (see (6.10)) and the cochain complex $(\oplus_p CL^p(\wedge^2\mathfrak{g}_\tau, V), \partial)$ associated to $(V, \varrho_\tau, -\varrho_\tau)$ (see (5.7)), we have the following result.

Proposition 7.3. *Let $\tau \in F_{\mathrm{qtr}}(\mathfrak{g})$ and (V, ϱ) be a representation of $\mathfrak{g}$. Then $\{\tau^{(p)}\}$ is a cochain map from $(\oplus_p C^p(\mathfrak{g}, V), \delta_\varrho)$ to $(\oplus_p CL^p(\wedge^2\mathfrak{g}_\tau, V), \partial)$. In particular, there is a map from $H^p_\varrho(\mathfrak{g}, V)$ to $HL^p_{\varrho_\tau, -\varrho_\tau}(\wedge^2\mathfrak{g}_\tau, V)$ given by $[\omega] \to [\tau^{(p)}(\omega)]$, $\omega \in Z^p_\varrho(\mathfrak{g}, V)$.*

Proof. It suffices to show that $\partial \circ \tau^{(p)} = \tau^{(p+1)} \circ \delta_\varrho$. Fix any $\mathrm{X}_i \in \wedge^2\mathfrak{g}_\tau = \wedge^2\mathfrak{g}$, $1 \leqslant i \leqslant p+1$, and any $\omega \in C^p(\mathfrak{g}, V)$. Note that $l(\mathrm{X}_i) = \varrho_\tau(\mathrm{X}_i)$, $r(\mathrm{X}_i) = -\varrho_\tau(\mathrm{X}_i)$. By (5.7) and Lemma 6.2 it follows that

$$\begin{aligned}
&((\partial \circ \tau^{(p)})(\omega))(\mathrm{X}_1, \ldots, \mathrm{X}_{p+1}) \\
&\quad = (\partial(\tau^{(p)}\omega))(\mathrm{X}_1, \ldots, \mathrm{X}_{p+1}) \\
&\quad = \sum_{j=1}^{p+1} (-1)^{j+1} \varrho_\tau(\mathrm{X}_j)\omega(\tau^\sharp(\mathrm{X}_1), \ldots, \widehat{\tau^\sharp(\mathrm{X}_j)}, \ldots, \tau^\sharp(\mathrm{X}_{p+1})) \\
&\qquad + \sum_{1 \leqslant j < k \leqslant p+1} (-1)^j \omega(\tau^\sharp(\mathrm{X}_1), \ldots, \widehat{\tau^\sharp(\mathrm{X}_j)}, \ldots, \tau^\sharp(\mathrm{X}_{k-1}), \\
&\qquad\qquad [\tau^\sharp(\mathrm{X}_j), \tau^\sharp(\mathrm{X}_k)], \tau^\sharp(\mathrm{X}_{k+1}), \ldots, \tau^\sharp(\mathrm{X}_{p+1})).
\end{aligned} \tag{7.6}$$

On the other hand, we have

$$
\begin{aligned}
(7.7)\ ((\tau^{(p+1)} \circ \delta_\varrho)(\omega))(\mathrm{X}_1, \ldots, \mathrm{X}_{p+1}) &= (\tau^{(p+1)}(\delta_\varrho \omega))(\mathrm{X}_1, \ldots, \mathrm{X}_{p+1}) = (\delta_\varrho(\omega))(\tau^\sharp(\mathrm{X}_1), \ldots, \tau^\sharp(\mathrm{X}_{p+1})) \quad \text{(by (7.5))} \\
&= \sum_{j=1}^{p+1} (-1)^{j+1} \varrho(\tau^\sharp(\mathrm{X}_j)) \omega(\tau^\sharp(\mathrm{X}_1), \ldots, \widehat{\tau^\sharp(\mathrm{X}_j)}, \ldots, \tau^\sharp(\mathrm{X}_{p+1})) \\
&\quad + \sum_{1 \leqslant j < k \leqslant p+1} (-1)^{j+k} \omega([\tau^\sharp(\mathrm{X}_j), \tau^\sharp(\mathrm{X}_k)], \\
&\qquad \tau^\sharp(\mathrm{X}_1), \ldots, \widehat{\tau^\sharp(\mathrm{X}_j)}, \ldots, \widehat{\tau^\sharp(\mathrm{X}_k)}, \ldots, \tau^\sharp(\mathrm{X}_{p+1})) \quad \text{(by (6.10))}.
\end{aligned}
$$

Since ω is anti-symmetric, by (7.6) and (7.7) we have $\partial \circ \tau^{(p)} = \tau^{(p+1)} \circ \delta_\varrho$. □

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