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THE POTENTIAL-RAMSEY NUMBER OF K_n AND K_t^{-k}

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Abstract. A nonincreasing sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is a graphic sequence if it is realizable by a simple graph G on n vertices. In this case, G is referred to as a realization of π. Given two graphs G_1 and G_2, A. Busch et al. (2014) introduced the potential-Ramsey number of G_1 and G_2, denoted by $r_{\rm pot}(G_1, G_2)$, as the smallest nonnegative integer m such that for every m-term graphic sequence π, there is a realization G of π with $G_1 \subseteq G$ or with $G_2 \subseteq \overline{G}$, where $\overline{G}$ is the complement of G. For $t \geqslant 2$ and $0 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$, let K_t^{-k} be the graph obtained from K_t by deleting k independent edges. We determine $r_{\rm pot}(K_n, K_t^{-k})$ for $t \geqslant 3$, $1 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$ and $n \geqslant \lceil \sqrt{2k} \rceil + 2$, which gives the complete solution to a result in J. Z. Du, J. H. Yin (2021).

Keywords: graphic sequence; potentially H-graphic sequence; potential-Ramsey number

MSC 2020: 05C35, 05C07

1. Introduction

Graphs in this paper are finite, undirected and simple. Terms and notation not defined here are from [1]. A nonincreasing sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is a *graphic sequence* if it is realizable by a (simple) graph G on n vertices. In this case, G is referred to as a *realization* of π. Two well-known characterizations of graphic sequences were given by Havel and Hakimi, see [12], [13], and Erdős and Gallai, see [7]. Given a graph H, a graphic sequence π is *potentially H-graphic* if there is a realization of π containing H as a subgraph. The complementary sequence of π is denoted by $\overline{\pi} = (\overline{d_1}, \ldots, \overline{d_n}) = (n-1-d_n, \ldots, n-1-d_1)$. Additionally, we let $\sigma(\pi)$ denote the sum of the terms of π.

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Degree sequence problems can be broadly classified into two types, first described as "forcible" problems and "potential" problems by Rao in [14]. In a forcible degree sequence problem, a specified graph property must exist in every realization of the degree sequence π, while in a potential degree sequence problem, the desired property must be found in at least one realization of π. Results on forcible degree sequences are often stated as traditional problems in extremal graph theory.

There are a number of degree sequence analogues to well-known problems in extremal graph theory, including potentially graphic sequence analogues of the Turán problem, see [8], [9], [10], the Erdős-Sós conjecture, see [16], Hadwiger's conjecture, see [6], [15] and the Sauer-Spencer theorem, see [3]. Motivated in part by this previous work, Busch et al. in [2] proposed a degree sequence analogue to the classical graph Ramsey number. Given two graphs G_1 and G_2, Busch et al. in [2] defined the *potential-Ramsey number* of G_1 and G_2, denoted by $r_{\rm pot}(G_1, G_2)$, to be the smallest nonnegative integer m such that for every m-term graphic sequence π, there is a realization G of π with $G_1 \subseteq G$ or with $G_2 \subseteq \overline{G}$ (that is, either π is potentially G_1-graphic or $\overline{\pi}$ is potentially G_2-graphic). When the phrase "a realization" in the prior sentence is replaced with "all realizations", the smallest such integer m is the classical *Ramsey number* $r(G_1, G_2)$. Busch et al. in [2] gave a lower bound of $r_{\rm pot}(G, K_t)$ and determined $r_{\rm pot}(K_n, K_t)$, $r_{\rm pot}(C_n, K_t)$ and $r_{\rm pot}(P_n, K_t)$, where K_n, C_n and P_n are the complete graph, the cycle and the path, respectively, on n vertices. Du and Yin in [4] determined $r_{\rm pot}(C_n, K_r \vee \overline{K_s})$ and $r_{\rm pot}(P_n, K_r \vee \overline{K_s})$, where $\vee$ is the join operation. For $0 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$, denote by K_t^{-k} the graph obtained from K_t by deleting k independent edges. Recently, Du and Yin in [5] gave a lower bound of $r_{\rm pot}(G, K_t^{-k})$ for $0 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$, and then determined $r_{\rm pot}(K_n, K_t^{-k})$ for $1 \leqslant k \leqslant 2$. The *i-dependence number* of a graph G, denoted $\alpha^{(i)}(G)$, is the maximum order of an induced subgraph H of G with $\Delta(H) \leqslant i$, where $\Delta(H)$ is the maximum degree of H.

Theorem 1.1 ([2]). *Let G be a graph of order t with no isolated vertices such that $\alpha^{(1)}(G) \leqslant t-1$, and let $n \geqslant 2$. Then $r_{\rm pot}(K_n, G) \geqslant \max\{2n+t-\alpha^{(1)}(G)-2,\ n+t-2\}$.*

Theorem 1.2 ([2]). *For $n \geqslant t \geqslant 3$, $r_{\rm pot}(K_n, K_t) = 2n + t - 4$ except when $n = t = 3$, in which case $r_{\rm pot}(K_3, K_3) = 6$.*

Theorem 1.3 ([5]). *Let $t \geqslant 2$ and $0 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$, and let G be a graph of order n with no isolated vertices such that $\alpha^{(i)}(G) \leqslant n - 1$ for $1 \leqslant i \leqslant \triangle(G) - 1$ and $i + \lfloor \frac{2k}{i} \rfloor \leqslant 2t - 3$. Then*

$$r_{\rm pot}(G, K_t^{-k}) \geqslant \max\Big\{\max\Big\{f(i)\colon\ 1 \leqslant i \leqslant \Delta(G)-1 \text{ and } i+\Big\lfloor \frac{2k}{i} \Big\rfloor \leqslant 2t-3\Big\},\ n+t-c\Big\},$$

where

$$f(i)=\begin{cases}2t+n-\alpha^{(i)}(G)-\left\lfloor\dfrac{2k}{i}\right\rfloor-1 & \text{if } i \text{ is even or if } \lfloor\frac{2k}{i}\rfloor \text{ is odd or}\\ & \text{if } i\mid 2k+1 \text{ does not hold,}\\ 2t+n-\alpha^{(i)}(G)-\left\lfloor\dfrac{2k}{i}\right\rfloor-2 & \text{otherwise,}\end{cases}$$

and

$$c=\begin{cases}2 & \text{if } k=0,\\ 3 & \text{if } k\geqslant 1.\end{cases}$$

Theorem 1.4 ([5]). *If $t\geqslant 3$, $1\leqslant k\leqslant 2$ and $n\geqslant 4$, then*

$$r_{\rm pot}(K_n,K_t^{-k})=\begin{cases}2n+t-4 & \text{if } t<n+k,\\ 2t+n-k-4 & \text{if } t\geqslant n+k.\end{cases}$$

For $1\leqslant k\leqslant\lfloor\frac{t}{2}\rfloor$, we now determine $r_{\rm pot}(K_n,K_t^{-k})$ completely.

Theorem 1.5. *If $t\geqslant 3$, $1\leqslant k\leqslant\lfloor\frac{t}{2}\rfloor$ and $n\geqslant\lceil\sqrt{2k}\rceil+2$, then*

$$r_{\rm pot}(K_n,K_t^{-k})$$
$$=\begin{cases}2n+t-4 & \text{if } t<n+\lfloor\sqrt{2k}\rfloor+\left\lfloor\dfrac{2k}{\lfloor\sqrt{2k}\rfloor}\right\rfloor-2,\\ 2t+n-\lfloor\sqrt{2k}\rfloor-\left\lfloor\dfrac{2k}{\lfloor\sqrt{2k}\rfloor}\right\rfloor-2 & \text{if } t\geqslant n+\lfloor\sqrt{2k}\rfloor+\left\lfloor\dfrac{2k}{\lfloor\sqrt{2k}\rfloor}\right\rfloor-2.\end{cases}$$

Clearly, Theorem 1.5 is an extension of Theorem 1.4.

2. Proof of Theorem 1.5

In order to prove Theorem 1.5, we need some useful lemmas and known results. Let $\pi=(d_1,\ldots,d_n)$ be a nonincreasing sequence of nonnegative integers and let $d_1'\geqslant\ldots\geqslant d_{n-1}'$ be a rearrangement in nonincreasing order of $d_2-1,\ldots,d_{d_1+1}-1$, $d_{d_1+2},\ldots,d_n$. We say that $\pi'=(d_1',\ldots,d_{n-1}')$ is the *residual sequence* of π.

Theorem 2.1 ([12], [13]). *Let $\pi=(d_1,\ldots,d_n)$ be a nonincreasing sequence of nonnegative integers. Then π is graphic if and only if π' is graphic.*

Theorem 2.2 ([11]). *If $\pi=(d_1,\ldots,d_n)$ is a potentially H-graphic sequence, then π has a realization containing H on those vertices with degrees $d_1,\ldots,d_{|V(H)|}$.*

Theorem 2.3 ([17]). *Let $p \geqslant n$ and $\pi = (d_1, \ldots, d_p)$ be a graphic sequence.*

(1) *If $d_n \geqslant n-1$ and $d_{2n} \geqslant n-2$, then π is potentially K_n-graphic.*

(2) *If $d_n \geqslant n-1$ and $d_i \geqslant 2(n-1)-i$ for $i = 1, \ldots, n-2$, then π is potentially K_n-graphic.*

Theorem 2.4 ([18]). *Let $n \geqslant t$ and $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_{t-2} \geqslant t-1$ and $d_t \geqslant t-2$. If $d_i \geqslant 2t-3-i$ for $i = 1, \ldots, t-1$, then π is potentially K_t^{-1}-graphic.*

Let $1 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$, and let $\pi = (\varrho_1, \ldots, \varrho_t, d_{t+1}, \ldots, d_n)$ be a sequence of nonnegative integers so that $\sigma(\pi)$ is even, $n-t \geqslant \varrho_1 \geqslant \ldots \geqslant \varrho_{t-2k} \geqslant 0$, $n-t+1 \geqslant \varrho_{t-2k+1} \geqslant \ldots \geqslant \varrho_t \geqslant 1$, $\varrho_{t-2k} \geqslant \varrho_{t-2k+1}-1$ and $\varrho_t + t - 2 = d_{t+1} = \ldots = d_{t+\varrho_1+1} \geqslant d_{t+\varrho_1+2} \geqslant \ldots \geqslant d_n$. We define sequences $\pi_0, \ldots, \pi_t$ as follows. Let $\pi_0 = \pi$. Let

$$\pi_1 = (\varrho_2, \ldots, \varrho_t, d_{t+1}^{(1)}, \ldots, d_n^{(1)}),$$

where $d_{t+1}^{(1)} \geqslant \ldots \geqslant d_n^{(1)}$ is a rearrangement in the nonincreasing order of $d_{t+1}-1, \ldots, d_{t+\varrho_1}-1, d_{t+\varrho_1+1}, \ldots, d_n$. For $2 \leqslant i \leqslant t$, given

$$\pi_{i-1} = (\varrho_i, \ldots, \varrho_t, d_{t+1}^{(i-1)}, \ldots, d_n^{(i-1)}),$$

let

$$\pi_i = (\varrho_{i+1}, \ldots, \varrho_t, d_{t+1}^{(i)}, \ldots, d_n^{(i)}),$$

where $d_{t+1}^{(i)} \geqslant \ldots \geqslant d_n^{(i)}$ is a rearrangement in the nonincreasing order of $d_{t+1}^{(i-1)}-1, \ldots, d_{t+\varrho_i}^{(i-1)}-1, d_{t+\varrho_i+1}^{(i-1)}, \ldots, d_n^{(i-1)}$.

Lemma 2.1 ([17]). *For each i, the definition of $\pi_i = (\varrho_{i+1}, \ldots, \varrho_t, d_{t+1}^{(i)}, \ldots, d_n^{(i)})$ is as above. Let $t_i = \max\{j\colon d_{t+1}^{(i)} - d_{t+j}^{(i)} \leqslant 1\}$. Then $t_t \geqslant t_{t-1} \geqslant \ldots \geqslant t_0 \geqslant \varrho_1 + 1$.*

Lemma 2.2 ([5]). *The definition of $\pi_t = (d_{t+1}^{(t)}, \ldots, d_n^{(t)})$ is as above. Let $d_{t+1} = l$ and $d_{t+1}^{(t)} = m$. If $t_t \leqslant m$, then $\varrho_1 + \ldots + \varrho_t \leqslant (l-m+1)m-1$.*

Lemma 2.3 ([5]). *The definition of $\pi_t = (d_{t+1}^{(t)}, \ldots, d_n^{(t)})$ is as above. If π_t is graphic, then π has a realization G so that the t vertices with degrees $\varrho_1, \ldots, \varrho_t$ form an independent set of G.*

Lemma 2.4 ([17]). *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, where $d_1 = t$ and $\sigma(\pi)$ is even. If $d_{t+1} \geqslant t-1$, then π is graphic.*

Lemma 2.5. *Let* $\lfloor\sqrt{2k}\rfloor = b$ *and* $c = \lfloor\sqrt{2k}\rfloor + \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor$. *If* $k \geqslant 1$ *is an integer and* $a = c+1$, *then* $a^2 - 2a - 3 \leqslant 8k$ *with equality if and only if* $2k = b^2 + 2b$ *and* $a = 2b+3$.

P r o o f. If $k = 1$, then $b = 1$, $c = 3$ and $a = 4$, implying that $a^2 - 2a - 3 = 5 < 8$. Assume $k \geqslant 2$. We prove Lemma 2.5 by the parity of b.

Case 1: b is odd. It follows from $b \leqslant \sqrt{2k} < b+1$ that $b^2 + 1 \leqslant 2k \leqslant (b+1)^2 - 2$. If $b^2 + 1 \leqslant 2k \leqslant b^2 + b - 2$, then $k \geqslant 5$ and $\lfloor\frac{2k}{b}\rfloor = b$. Hence, $a = c + 1 = 2b + 1$ and $a^2 - 2a - 3 = 4b^2 - 4$. By $2k \geqslant b^2 + 1$, we have $8k \geqslant 4(b^2 + 1) = 4b^2 + 4 > a^2 - 2a - 3$. If $b^2 + b \leqslant 2k \leqslant b^2 + 2b - 1$, then $\lfloor\frac{2k}{b}\rfloor = b + 1$. Hence, $a = 2b + 2$ and $a^2 - 2a - 3 = 4b^2 + 4b - 3$. By $2k \geqslant b^2 + b$, we have $8k \geqslant 4b^2 + 4b > a^2 - 2a - 3$.

Case 2: b is even. It follows from $b \leqslant \sqrt{2k} < b+1$ that $b^2 \leqslant 2k \leqslant b^2 + 2b$. If $b^2 \leqslant 2k \leqslant b^2 + b - 2$, then $k \geqslant 2$ and $\lfloor\frac{2k}{b}\rfloor = b$. Hence, $a = 2b+1$ and $a^2 - 2a - 3 = 4b^2 - 4$. By $2k \geqslant b^2$, we have $8k \geqslant 4b^2 > a^2 - 2a - 3$. If $b^2 + b \leqslant 2k \leqslant b^2 + 2b - 2$, then $k \geqslant 2$ and $\lfloor\frac{2k}{b}\rfloor = b + 1$. Hence, $a = 2b + 2$ and $a^2 - 2a - 3 = 4b^2 + 4b - 3$. By $2k \geqslant b^2 + b$, we have $8k \geqslant 4b^2 + 4b > a^2 - 2a - 3$. If $2k = b^2 + 2b$, then $k \geqslant 4$ and $\lfloor\frac{2k}{b}\rfloor = b + 2$. Hence, $a = c + 1 = 2b + 3$ and $a^2 - 2a - 3 = 4b^2 + 8b = 8k$. □

Lemma 2.6. *Let* $c = \lfloor\sqrt{2k}\rfloor + \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor$ *and* $c' = \lfloor\sqrt{2(k-1)}\rfloor + \lfloor\frac{2(k-1)}{\lfloor\sqrt{2(k-1)}\rfloor}\rfloor$. *Then* $c' + 1 \geqslant c$ *for* $k \geqslant 2$.

P r o o f. It is obvious for $k = 2$. We assume $k \geqslant 3$. Let $b = \lfloor\sqrt{2k}\rfloor$ and $b' = \lfloor\sqrt{2(k-1)}\rfloor$. If $b = b'$, then $c \leqslant b' + \lfloor\frac{2(k-1)+b'}{b'}\rfloor = c' + 1$ as $2k \leqslant 2(k-1) + b'$. If $b = b' + 1$, then $\lfloor\frac{2k}{b}\rfloor \leqslant \lfloor\frac{2(k-1)}{b'}\rfloor$, implying that $c' + 1 \geqslant c$. Clearly, $b > b' + 1$ does not hold. □

Lemma 2.7. *Let* $n \geqslant t \geqslant 3$, $1 \leqslant k \leqslant \lfloor\frac{t}{2}\rfloor$ *and* $\pi = (d_1, \ldots, d_n)$ *be a graphic sequence with* $n - 2 \geqslant d_1 \geqslant \ldots \geqslant d_t = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$. *If* $d_t \geqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 1$, *then* π *is potentially* K_t^{-k}*-graphic.*

P r o o f. Put $d_t = l$. We first claim that $l \geqslant t - 1$. By $t \geqslant 2k$, it is easy to check that $l \geqslant t - 1$ for $k = 1, 2, 3, 4$. Assume $k \geqslant 5$. By $t \geqslant 2k \geqslant 10$ and $\lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor \leqslant 2k/\lfloor\sqrt{2k}\rfloor \leqslant 2k/(\sqrt{2k} - 1) \leqslant \sqrt{2k} + 2$, we have that $l \geqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 1 \geqslant 2t - \sqrt{t} - (\sqrt{t} + 2) - 1 = 2t - 2\sqrt{t} - 3 = (t-1) + (\sqrt{t} - 1)^2 - 3 \geqslant t - 1$. Therefore, $l \geqslant t - 1$. If $l \geqslant 2t - 3$, by Theorem 2.3 (2), then π is potentially K_t-graphic, implying that π is potentially K_t^{-k}-graphic. Assume $l \leqslant 2t - 4$. Let $\pi_0 = (\varrho_1, \ldots, \varrho_t, d_{t+1}, \ldots, d_n)$, where

$$\varrho_i = \begin{cases} d_i - (t-1) & \text{if } 1 \leqslant i \leqslant t - 2k, \\ d_i - (t-2) & \text{if } t - 2k + 1 \leqslant i \leqslant t. \end{cases}$$

By Lemma 2.3, we only need to prove that π_t is graphic. Put $d_{t+1}^{(t)} = m$. If $t_t \geqslant m+1$, then π_t is graphic by Lemma 2.4. Assume $t_t \leqslant m$. Clearly,

$$\varrho_1 + \ldots + \varrho_t = d_1 + \ldots + d_t - t(t-1) + 2k \geqslant tl - t(t-1) + 2k.$$

By Lemma 2.2, we get $(l-m+1)m - 1 \geqslant tl - t(t-1) + 2k$. It is easy to see that $(l-m+1)m-1$, considered as a function of m, attains its maximum value when $m = \frac{1}{2}(l+1)$. Hence, $(\frac{1}{2}(l+1))^2 - 1 \geqslant tl - t(t-1) + 2k$, i.e., $l^2 - (4t-2)l \geqslant -4t(t-1) + 8k + 3$. Let $a = \lfloor\sqrt{2k}\rfloor + \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor + 1$. Clearly, $a \geqslant 4$. By $l \geqslant 2t-a$, we can see that $l^2 - (4t-2)l$ attains its maximum value when $l = 2t-a$. Hence, $(2t-a)^2 - (4t-2)(2t-a) \geqslant -4t(t-1) + 8k + 3$, i.e., $a^2 - 2a - 3 \geqslant 8k$. By Lemma 2.5, we have $a^2 - 2a - 3 = 8k$, $2k = \lfloor\sqrt{2k}\rfloor^2 + 2\lfloor\sqrt{2k}\rfloor$ and $a = 2\lfloor\sqrt{2k}\rfloor + 3$. Then we must have $l = 2t-a$, $d_1 = \ldots = d_{d_1+2} = l$, $m = t_t = \frac{1}{2}(2t-a+1) = t - \lfloor\sqrt{2k}\rfloor - 1$, $d_{t+1}^{(t)} = m$, and $d_{t+2}^{(t)} = \ldots = d_{t+t_t}^{(t)} = m-1$. Thus, $\pi_t = (t - \lfloor\sqrt{2k}\rfloor - 1, (t - \lfloor\sqrt{2k}\rfloor - 2)^{t-\lfloor\sqrt{2k}\rfloor-2}, d_{2t-a+3}, \ldots, d_n)$, where $d_{2t-a+3} \leqslant t - \lfloor\sqrt{2k}\rfloor - 3$. Let π_t' be the residual sequence of π_t. Then $\pi_t' = ((t-\lfloor\sqrt{2k}\rfloor - 3)^{t-\lfloor\sqrt{2k}\rfloor-2}, d_{2t-a+3}', \ldots, d_n')$. It follows from $\sigma(\pi_t')$ being even and Lemma 2.4 that π_t' is graphic, and hence so is π_t by Theorem 2.1. □

Lemma 2.8. *Let $n \geqslant t \geqslant 3$, $1 \leqslant k \leqslant \lfloor\frac{t}{2}\rfloor$ and $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $n-2 \geqslant d_1 \geqslant \ldots \geqslant d_t = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$. If $d_{t-3} \geqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 1$ and $d_t \geqslant t-1$, then π is potentially K_t^{-k}-graphic.*

Proof. Let $d_{t-3} = l$ and $d_t = x$. If $k = 1$, by $l \geqslant 2t-4$ and Theorem 2.4, then π is potentially K_t^{-1}-graphic. Assume $k \geqslant 2$. Then $t \geqslant 2k \geqslant 4$. Let $\pi_0 = (\varrho_1, \ldots, \varrho_t, d_{t+1}, \ldots, d_n)$, where

$$\varrho_i = \begin{cases} d_i - (t-1) & \text{if } 1 \leqslant i \leqslant t-2k, \\ d_i - (t-2) & \text{if } t-2k+1 \leqslant i \leqslant t. \end{cases}$$

By Lemma 2.3, we only need to prove that π_t is graphic. Put $d_{t+1}^{(t)} = m$. If $t_t \geqslant m+1$, then π_t is graphic by Lemma 2.4. Assume $t_t \leqslant m$. If $x \geqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 1$, then π is potentially K_t^{-k}-graphic by Lemma 2.7. Assume $x \leqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 2$. Then $t \geqslant 5$ by $x \geqslant t-1$. Clearly, $\varrho_1 + \ldots + \varrho_t \geqslant (t-3)l + 3x - t(t-1) + 2k$. By Lemma 2.2, we get

$$(x-m+1)m - 1 \geqslant (t-3)l + 3x - t(t-1) + 2k,$$

that is

$$(1) \qquad (x-m+1)m - 1 + 3(l-x) \geqslant tl - t(t-1) + 2k.$$

If $l \geqslant 2t-3$, by Theorem 2.4, then π is potentially K_t^{-k}-graphic. Assume $l \leqslant 2t-4$. If $t=5$, then $k=2$, $l=5$ or 6, and $x=4$, it follows from (1) that $(5-m)m \geqslant 7$ or 9, which is impossible. Assume $t \geqslant 6$. We now show that $(\frac{1}{2}(l+1))^2-1 \leqslant tl-t(t-1)+2k$, that is, $l^2-(4t-2)l \leqslant -4t(t-1)+8k+3$. Let $a=\lfloor\sqrt{2k}\rfloor+\lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor+1$. Clearly, $a \geqslant 5$. By Lemma 2.5, we have $a^2-2a-3 \leqslant 8k$, that is, $(2t-a)^2-(4t-2)(2t-a) \leqslant -4t(t-1)+8k+3$. Since $2t-a \leqslant l \leqslant 2t-4$, we can see that $l^2-(4t-2)l$ attains its maximum value when $l=2t-a$. Hence, $l^2-(4t-2)l \leqslant (2t-a)^2-(4t-2)(2t-a) \leqslant -4t(t-1)+8k+3$. Thus, $(\frac{1}{2}(l+1))^2-1 \leqslant tl-t(t-1)+2k$. By (1), we have $(x-m+1)m-1+3(l-x) \geqslant (\frac{1}{2}(l+1))^2-1$, i.e., $(x-m+1)m-3x \geqslant (\frac{1}{2}(l+1))^2-3l$. Since $(x-m+1)m-3x$ attains its maximum value when $m=\frac{1}{2}(x+1)$, we have $(\frac{1}{2}(x+1))^2-3x \geqslant (\frac{1}{2}(l+1))^2-3l$, i.e., $x^2-10x \geqslant l^2-10l$, which is impossible as $l > x \geqslant t-1 \geqslant 5$. □

Lemma 2.9. *Let $n \geqslant t \geqslant 3$, $1 \leqslant k \leqslant \lfloor\frac{t}{2}\rfloor$ and $\pi=(d_1,\ldots,d_n)$ be a graphic sequence. If $d_{t-3} \geqslant 2t-\lfloor\sqrt{2k}\rfloor-\lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor-1$ and $d_t \geqslant t-1$, then π is potentially K_t^{-k}-graphic.*

Proof. We use induction on k. If $k=1$, by $d_{t-3} \geqslant 2t-4$ and Theorem 2.4, then π is potentially K_t^{-1}-graphic. Assume that $k \geqslant 2$ and Lemma 2.9 holds for $k-1$. If $d_1=n-1$ or if there exists an integer j, $t \leqslant j \leqslant d_1+1$, such that $d_j > d_{j+1}$, by Lemma 2.6, then the residual sequence $\pi'=(d_1',\ldots,d_{n-1}')$ satisfies $d_i'=d_{i+1}-1$ for $1 \leqslant i \leqslant t-1$, $d_{t-4}'=d_{t-3}-1 \geqslant 2t-\lfloor\sqrt{2k}\rfloor-\lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor-2 \geqslant 2(t-1)-\lfloor\sqrt{2(k-1)}\rfloor-\lfloor\frac{2(k-1)}{\lfloor\sqrt{2(k-1)}\rfloor}\rfloor-1$ and $d_{t-1}'=d_t-1 \geqslant (t-1)-1$. By $t-1 \geqslant 2(k-1)$ and the induction hypothesis, π' is potentially $K_{t-1}^{-(k-1)}$-graphic. Thus, by Theorems 2.1 and 2.2, π is potentially K_t^{-k}-graphic. So we may assume that $n-2 \geqslant d_1 \geqslant \ldots \geqslant d_t = \ldots = d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_n$. By Lemma 2.8, π is also potentially K_t^{-k}-graphic. □

Lemma 2.10. *If $t \geqslant 3$, $1 \leqslant k \leqslant \lfloor\frac{t}{2}\rfloor$ and $n \geqslant \lceil\sqrt{2k}\rceil+2$, then*

$$r_{\rm pot}(K_n, K_t^{-k}) \geqslant \begin{cases} 2n+t-4 & \text{if } t < n+\lfloor\sqrt{2k}\rfloor+\left\lfloor\dfrac{2k}{\lfloor\sqrt{2k}\rfloor}\right\rfloor-2, \\ 2t+n-\lfloor\sqrt{2k}\rfloor-\left\lfloor\dfrac{2k}{\lfloor\sqrt{2k}\rfloor}\right\rfloor-2 & \text{if } t \geqslant n+\lfloor\sqrt{2k}\rfloor+\left\lfloor\dfrac{2k}{\lfloor\sqrt{2k}\rfloor}\right\rfloor-2. \end{cases}$$

Proof. By $\alpha^{(1)}(K_t^{-k})=2$ and Theorem 1.1, $r_{\rm pot}(K_n,K_t^{-k}) \geqslant 2n+t-4$. Let $\lfloor\sqrt{2k}\rfloor=b$. Then $1 \leqslant b \leqslant n-2$ (by $n \geqslant \lceil\sqrt{2k}\rceil+2$). Clearly, $b^2 \leqslant 2k \leqslant (b+1)^2-1=b^2+2b$. We consider two cases in terms of the parity of b.

Case 1: b is even. If $b^2 \leqslant 2k \leqslant b^2+b-2$, then $b+\lfloor \frac{2k}{b} \rfloor = 2b \leqslant b^2 \leqslant 2k \leqslant t \leqslant 2t-3$. If $b^2+b \leqslant 2k \leqslant b^2+2b$, then $b+\lfloor \frac{2k}{b} \rfloor \leqslant 2b+2 \leqslant b^2+b \leqslant 2k \leqslant t \leqslant 2t-3$. It follows from $\alpha^{(b)}(K_n) = b+1 \leqslant n-1$ and Theorem 1.3 that $r_{\rm pot}(K_n, K_t^{-k}) \geqslant 2t+n-(b+1)-\lfloor \frac{2k}{b} \rfloor -1 = 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2$.

Case 2: b is odd. Then $b^2+1 \leqslant 2k \leqslant b^2+2b-1$. Clearly, $\lceil \sqrt{2k} \rceil = \lfloor \sqrt{2k} \rfloor + 1$ by $b^2 < 2k < (b+1)^2$ and $n \geqslant \lceil \sqrt{2k} \rceil + 2 = b+3$. If $b^2+1 \leqslant 2k \leqslant b^2+b-2$, i.e., $(b+1)(b-1)+2 \leqslant 2k \leqslant (b+1)(b-1)+(b-1)$, then $b \geqslant 3$, $\lfloor \frac{2k}{b+1} \rfloor = b-1$ and $\lfloor \frac{2k}{b} \rfloor = b$. Hence, $(b+1)+\lfloor \frac{2k}{b+1} \rfloor = b+\lfloor \frac{2k}{b} \rfloor = 2b \leqslant b^2+1 \leqslant 2k \leqslant t \leqslant 2t-3$. By $\alpha^{(b+1)}(K_n) = b+2 \leqslant n-1$ and Theorem 1.3, $r_{\rm pot}(K_n, K_t^{-k}) \geqslant 2t+n-(b+2)-\lfloor \frac{2k}{b+1} \rfloor - 1 = 2t+n-(b+1)-\lfloor \frac{2k}{b+1} \rfloor - 2 = 2t+n-b-\lfloor \frac{2k}{b} \rfloor - 2 = 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2$. If $b^2+b \leqslant 2k \leqslant b^2+2b-1$, i.e., $(b+1)b \leqslant 2k \leqslant (b+1)b+(b-1)$, then $b \geqslant 1$, $\lfloor \frac{2k}{b+1} \rfloor = b$ and $\lfloor \frac{2k}{b} \rfloor = b+1$. Hence, $(b+1)+\lfloor \frac{2k}{b+1} \rfloor = b+\lfloor \frac{2k}{b} \rfloor = 2b+1 \leqslant \max\{3, b^2+b\} \leqslant \max\{3, 2k\} \leqslant t \leqslant 2t-3$. By $\alpha^{(b+1)}(K_n) = b+2 \leqslant n-1$ and Theorem 1.3, $r_{\rm pot}(K_n, K_t^{-k}) \geqslant 2t+n-(\lfloor \sqrt{2k} \rfloor + 2) - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor + 1} \rfloor - 1 = 2t+n-(\lfloor \sqrt{2k} \rfloor + 2) - \big(\lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - \big) - 1 = 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2$. □

We now prove Theorem 1.5.

P r o o f of Theorem 1.5. Let $t \geqslant 3$, $1 \leqslant k \leqslant \lfloor \frac{t}{2} \rfloor$ and $n \geqslant \lceil \sqrt{2k} \rceil + 2$. Clearly,

$$\max\Big\{2n+t-4, 2t+n-\lfloor \sqrt{2k} \rfloor - \Big\lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \Big\rfloor - 2\Big\}$$
$$= \begin{cases} 2n+t-4 & \text{if } t < n+\lfloor \sqrt{2k} \rfloor + \Big\lfloor \dfrac{2k}{\lfloor \sqrt{2k} \rfloor} \Big\rfloor - 2, \\ 2t+n-\lfloor \sqrt{2k} \rfloor - \Big\lfloor \dfrac{2k}{\lfloor \sqrt{2k} \rfloor} \Big\rfloor - 2 & \text{if } t \geqslant n+\lfloor \sqrt{2k} \rfloor + \Big\lfloor \dfrac{2k}{\lfloor \sqrt{2k} \rfloor} \Big\rfloor - 2. \end{cases}$$

Firstly, by Lemma 2.10, $r_{\rm pot}(K_n, K_t^{-k}) \geqslant \max\{2n+t-4, 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2\}$.

Now, we prove that $r_{\rm pot}(K_n, K_t^{-k}) \leqslant \max\{2n+t-4, 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2\}$. We use induction on k. Theorem 1.4 is the case $1 \leqslant k \leqslant 2$ of Theorem 1.5. Assume $k \geqslant 3$. We consider two cases.

Case 1: $2n+t-4 \leqslant 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2$. Let $\pi = (d_1, \ldots, d_m)$ be a graphic sequence with $m = 2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2$. If $\overline{d_t} \leqslant t-2$, then $d_{m+1-t} = m-1-\overline{d_t} \geqslant \big(2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2\big) - 1 - (t-2) \geqslant (2n+t-4)-1-(t-2) = 2n-3$. By $m+1-t = (2t+n-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 2) + 1 - t \geqslant (2n+t-4)+1-t \geqslant n$, we have $d_n \geqslant d_{m+1-t} \geqslant 2n-3 > n-1$. Hence, π is potentially K_n-graphic by Theorem 2.3 (2). If $d_n \leqslant n-2$, then $\overline{d_{m+1-n}} = \overline{d_{2t-\lfloor \sqrt{2k} \rfloor - \lfloor 2k/\lfloor \sqrt{2k} \rfloor \rfloor - 1}} = m-1-d_n \geqslant 2t-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 1$. By $m+1-n \geqslant (2n+t-4)+1-n = n+t-3 \geqslant t$, we have $\overline{d_t} \geqslant \overline{d_{2t-\lfloor \sqrt{2k} \rfloor - \lfloor 2k/\lfloor \sqrt{2k} \rfloor \rfloor - 1}} \geqslant 2t-\lfloor \sqrt{2k} \rfloor - \lfloor \frac{2k}{\lfloor \sqrt{2k} \rfloor} \rfloor - 1 \geqslant t-1$. By

Lemma 2.9, $\overline{\pi}$ is potentially K_t^{-k}-graphic. Assume $d_n \geqslant n-1$ and $\overline{d_t} \geqslant t-1$. Clearly, $t \geqslant 2k \geqslant 6$ and $m \geqslant 2n+t-4 > 2n$. If $d_{2n} \geqslant n-2$, then π is potentially K_n-graphic by Theorem 2.3 (1). If $d_{2n} \leqslant n-3$, then $\overline{d_{m+1-2n}} = m-1-d_{2n} \geqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor$. By $m+1-2n \geqslant (2n+t-4)+1-2n = t-3$, we have $\overline{d_{t-3}} \geqslant \overline{d_{m+1-2n}} \geqslant 2t - \lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor$. Thus, by Lemma 2.9, $\overline{\pi}$ is potentially K_t^{-k}-graphic. Therefore, $r_{pot}(K_n, K_t^{-k}) \leqslant 2t+n-\lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 2$.

Case 2: $2n+t-4 \geqslant 2t+n-\lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 1$. It follows from Lemma 2.6 that $2n+t-4 \geqslant 2t+n-\lfloor\sqrt{2k}\rfloor - \lfloor\frac{2k}{\lfloor\sqrt{2k}\rfloor}\rfloor - 1 \geqslant 2t+n-\lfloor\sqrt{2(k-1)}\rfloor - \lfloor\frac{2(k-1)}{\lfloor\sqrt{2(k-1)}\rfloor}\rfloor - 2$. Thus, by the induction hypothesis,

$$\begin{aligned} r_{\rm pot}(K_n, K_t^{-k}) &\leqslant r_{\rm pot}(K_n, K_t^{-(k-1)}) \\ &\leqslant \max\Big\{2n+t-4, 2t+n-\lfloor\sqrt{2(k-1)}\rfloor - \Big\lfloor\frac{2(k-1)}{\lfloor\sqrt{2(k-1)}\rfloor}\Big\rfloor - 2\Big\} \\ &= 2n+t-4. \end{aligned}$$

□

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