Keng Hao Ooi

On the Choquet integrals associated to Bessel capacities

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 2, 433–447

Persistent URL: http://dml.cz/dmlcz/150410

Czechoslovak Mathematical Journal, 72 (147) (2022), 433–447

ON THE CHOQUET INTEGRALS ASSOCIATED TO BESSEL CAPACITIES

Keng Hao Ooi, Baton Rouge

Received December 6, 2020. Published online August 4, 2021.

Abstract. We characterize the Choquet integrals associated to Bessel capacities in terms of the preduals of the Sobolev multiplier spaces. We make use of the boundedness of local Hardy-Littlewood maximal function on the preduals of the Sobolev multiplier spaces and the minimax theorem as the main tools for the characterizations.

Keywords: Choquet integral; Bessel capacity; Hardy-Littlewood maximal function

MSC 2020: 31C15, 42B25

1. Introduction

Let $\alpha > 0$, $s > 1$ be real numbers. We define the Sobolev space $W^{\alpha,s} = W^{\alpha,s}(\mathbb{R}^n)$, $n \geqslant 1$ to be the set of functions u of the type

$$u = G_\alpha * f$$

for some $f \in L^s$. Here G_α is the Bessel kernel of order α defined by

$$G_\alpha(x) := \mathcal{F}^{-1}[(1 + |\cdot|^2)^{-\alpha/2}](x),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform in $\mathbb{R}^n$. The norm of $u = G_\alpha * f \in W^{\alpha,s}$ is defined as $\|u\|_{W^{\alpha,s}} = \|f\|_{L^s}$. Recall also that the Bessel capacity $\mathrm{Cap}_{\alpha,s}(\cdot)$ associated to $W^{\alpha,s}$ is defined as

$$\mathrm{Cap}_{\alpha,s}(E) := \inf\{\|f\|_{L^s}^s \colon\ f \geqslant 0,\ G_\alpha * f \geqslant 1 \text{ on } E\}$$

for any set $E \subseteq \mathbb{R}^n$. We say that a *property holds quasi-everywhere* (q.e.) if it holds everywhere except for a set E with $\mathrm{Cap}_{\alpha,s}(E) = 0$. The notion of Choquet integrals

DOI: 10.21136/CMJ.2021.0525-20

associated to Bessel capacities will be important in this work. Assuming that f is a q.e. defined function, the Choquet integral of f is meant to be

$$\int_{\mathbb{R}^n} |f| \,\mathrm{d}C := \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n \colon\ |f(x)| > t\}) \,\mathrm{d}t.$$

We denote by $L^1(C)$ the set of all q.e. defined f with finite quantity $\|f\|_{L^1(C)} := \int_{\mathbb{R}^n} |f| \,\mathrm{d}C$. On the other hand, let $M_p^{\alpha,s}(\mathbb{R}^n)$, $1 < p < \infty$ be the Sobolev multiplier space which consists of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^{\alpha,s}} := \sup_K \biggl(\frac{\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x}{\mathrm{Cap}_{\alpha,s}(K)}\biggr)^{p^{-1}} < \infty,$$

where the supremum is taken over all compact sets K with nonzero capacity, see [9] and [7].

It has been argued in [10] that

$$A^{-1}\|f\|_{L^1(C)} \leqslant \inf\{\|\varphi\|_{\mathcal{Z}'} \colon\ 0 \leqslant \varphi \in \mathcal{Z}',\, G_\alpha * \varphi \geqslant |f| \text{ q.e.}\} \leqslant A\|f\|_{L^1(C)} \tag{1.1}$$

for a constant $A > 0$, where $\mathcal{Z}'$ is the predual of the Sobolev multiplier space $\mathcal{Z} := M_t^{\alpha,s}$, $s^{-1} + t^{-1} = 1$. We denote

$$\|f\|_{\mathcal{I}} := \inf\{\|\varphi\|_{\mathcal{Z}'} \colon\ 0 \leqslant \varphi \in \mathcal{Z}',\, G_\alpha * \varphi \geqslant |f| \text{ q.e.}\}.$$

The proof of (1.1) presented in [10] is twofold. Firstly, it is proved that

$$\|f\|_{L^1(C)} \lesssim \|f\|_{\mathcal{I}} \lesssim \inf\biggl\{\int_{\mathbb{R}^n} \varphi^s (G_\alpha * \varphi)^{1-s} \,\mathrm{d}x \colon\ 0 \leqslant \varphi \in \mathcal{Z}',\, G_\alpha * \varphi \geqslant |f| \text{ q.e.}\biggr\}, \tag{1.2}$$

where $\alpha \lesssim \beta$ denotes $\alpha \leqslant A\beta$ for a constant $A > 0$. Subsequently, it is proved that

$$\inf\biggl\{\int_{\mathbb{R}^n} \varphi^s (G_\alpha * \varphi)^{1-s} \,\mathrm{d}x \colon\ 0 \leqslant \varphi \in \mathcal{Z}',\, G_\alpha * \varphi \geqslant |f| \text{ q.e.}\biggr\} \lesssim \|f\|_{L^1(C)}. \tag{1.3}$$

The proof of $\|\cdot\|_{L^1(C)} \lesssim \|\cdot\|_{\mathcal{I}}$ is simple and goes through by the standard duality argument. However, the proof of the second $\lesssim$ in (1.2) is then somewhat technical, one needs to interpret $\mathcal{Z}$ as the solution space of the integral equation

$$u = G_\alpha * (u^t) + \frac{|f|}{M}$$

for a fixed f and $\mathcal{Z}'$ as the Köthe dual of $\mathcal{Z}$ to finish the job, see [5]. The proof of (1.3) is also technical, it uses the nontrivial "integration by parts" trick that

$$(G_\alpha * f)^s \lesssim G_\alpha * [f(G_\alpha * f)^{s-1}]$$

for $f=(G_\alpha*\mu)^{t-1}$, where $\mu\geqslant 0$ is a compactly supported measure, see [4] and [10], Lemma 3.1.

The main purpose of this paper is to give an entirely different proof of

$$\|\cdot\|_{\mathcal{I}}\lesssim\|\cdot\|_{L^1(C)}. \tag{1.4}$$

The proof that will be presented later uses some classical techniques in the standard text of nonlinear potential theory (see, e.g. [2]) without recourse to the properties of the complicated expression that

$$\inf\biggl\{\int_{\mathbb{R}^n}\varphi^s(G_\alpha*\varphi)^{1-s}\,\mathrm{d}x\colon\ 0\leqslant\varphi\in\mathcal{Z}',\ G_\alpha*\varphi\geqslant|f|\text{ q.e.}\biggr\}$$

as in (1.2) and (1.3).

Let us present all the statements that will be proved later. To begin with, we will include the proof of the left-sided estimate of (1.1) for readers' convenience:

Proposition 1.1. *For any q.e. defined function f, it follows that*

$$\|\cdot\|_{L^1(C)}\lesssim\|\cdot\|_{\mathcal{I}}.$$

As a corollary, we have:

Corollary 1.2. *The function $G_\alpha*f$ is q.e. defined for $f\in\mathcal{Z}'$.*

The following proposition extends Egorov's theorem:

Proposition 1.3. *Suppose that $\{f_n\}_1^\infty$ is a Cauchy sequence in $\mathcal{Z}'$ with limit f. Then there is a subsequence $\{f_{n_i}\}_{i=1}^\infty$ such that $\lim\limits_{i\to\infty}G_\alpha*f_{n_i}(x)=G_\alpha*f(x)$ q.e., uniformly outside an open set of arbitrarily small capacity.*

Recall that a q.e. defined *function* f is said to be *quasi-continuous* if for every $\varepsilon>0$ there is an open set G such that $\mathrm{Cap}_{\alpha,s}(G)<\varepsilon$ and the restriction of $f\big|_{G^c}$ to G^c is continuous in the induced topology. We have the following important proposition:

Proposition 1.4. *If $f\in\mathcal{Z}'$, then $G_\alpha*f$ is quasi-continuous.*

On the other hand, by locally Hardy-Littlewood maximal function we mean that

$$M^{\mathrm{loc}}(f)=\sup_{0<r<1}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|\,\mathrm{d}y$$

for a locally integrable function f. Then we have the following weak type $(1,1)$ boundedness estimate, whose proof uses the boundedness of M^{loc} on $\mathcal{Z}'$, see [9]:

Lemma 1.5. *Let $f \in \mathcal{Z}'$ be nonnegative. Set*

$$E_\lambda = \{x \in \mathbb{R}^n\colon\ M^{\mathrm{loc}}(G_\alpha * f)(x) > \lambda\}.$$

Then there is a constant A independent of f such that

$$\mathrm{Cap}_{\alpha,s}(E_\lambda) \leqslant \frac{A}{\lambda}\|f\|_{\mathcal{Z}'}$$

for all $\lambda > 0$.

The main idea of the proof of (1.4) relies on the following proposition, which resembles the classic Lebesgue's differentiation theorem:

Proposition 1.6. *Let $f \in \mathcal{Z}'$. Then the following convergence holds for q.e. x:*

$$\lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} G_\alpha * f(y)\,\mathrm{d}y = G_\alpha * f(x).$$

Moreover, the convergence is uniform outside an open set of arbitrarily small capacity.

For a technical reason, we need an auxiliary norm $\|\cdot\|_{\mathcal{J}}$ defined by

$$\|f\|_{\mathcal{J}} = \inf\{\|\varphi\|_{\mathcal{Z}'}\colon\ 0 \leqslant \varphi \in \mathcal{Z}',\ G_\alpha * \varphi \geqslant |f|\}, \tag{1.5}$$

where we drop the q.e. condition in the definition of $\|\cdot\|_{\mathcal{I}}$. By denoting

$$\|f\|_{\mathcal{M}} := \sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{d}\mu\colon\ \mu \geqslant 0,\ \mathrm{supp}(\mu) \subseteq \mathrm{supp}(f),\ \|G_\alpha * \mu\|_{\mathcal{Z}} \leqslant 1\right\}$$

for compactly supported function f, the following theorem extends the classical minimax theorem:

Theorem 1.7. *For any function f with compact support* $\mathrm{supp}\, f$ *if $f\big|_{\mathrm{supp}\, f}$ is continuous with $\min\limits_{\mathrm{supp}\, f} f > 0$, then*

$$\|f\|_{\mathcal{J}} = \|f\|_{\mathcal{M}}.$$

As a result for any compact set K it follows that $\|\chi_K\|_{\mathcal{I}} \approx \mathrm{Cap}_{\alpha,s}(K)$.

The above theorem shows that (1.4) holds for characteristic functions of compact sets K. The following theorem addresses (1.4) for general cases:

Theorem 1.8. *For any set E, the following estimate holds:*

$$\|\chi_E\|_{\mathcal{I}} \lesssim \mathrm{Cap}_{\alpha,s}(E).$$

As a result for any q.e. defined function f it follows that

$$\|f\|_{\mathcal{I}} \lesssim \|f\|_{L^1(C)}.$$

Note that it is the L^s space which plays the main role in the standard nonlinear potential theory. In a sense, the aforementioned propositions and theorems replace the role of L^s with $\mathcal{Z}'$. For instance, in contrast to Proposition 1.4, the function $G_\alpha * f$ is quasi-continuous for $f \in L^s$ (see [2], Proposition 6.1.2), meanwhile the Lebesgue's differentiation theorem holds for $f \in L^s$ in Proposition 1.6, see [2], Theorem 6.2.1. We refer the readers to the excellent text [2] for more details about the correspondence. As a simple application, we may extend the trace inequalities presented in [2], Theorem 7.2.1 and [6] to the following form:

Theorem 1.9. *Let μ be a nonnegative measure on $\mathbb{R}^n$. The following assertions regarding μ are equivalent:*

(a) *There is a constant A_1 such that*

$$\left(\int_{\mathbb{R}^n} |G_\alpha * f| \,\mathrm{d}\mu\right)^{s^{-1}} \leqslant A_1 \|f\|_{\mathcal{Z}'}^{s^{-1}}$$

for all $f \in \mathcal{Z}'$.

(b) *There is a constant A_2 such that*

$$\left(\int_{\mathbb{R}^n} |G_\alpha * \mu_K|^t \,\mathrm{d}x\right)^{t^{-1}} \leqslant A_2 \mu(K)^{t^{-1}}$$

for all compact sets K.

(c) *There is a constant A_3 such that*

$$\sup_{t>0} t\mu(\{x \in \mathbb{R}^n \colon |G_\alpha * f| \geqslant t\}) \leqslant A_3 \|f\|_{\mathcal{Z}'}$$

for all $f \in \mathcal{Z}'$.

(d) *There is a constant A_4 such that*

$$\mu(K)^{s^{-1}} \leqslant A_4 \mathrm{Cap}_{\alpha,s}(K)^{s^{-1}}$$

for all compact sets K.

(e) *There is a constant A_5 such that*

$$\left(\int_K |G_\alpha * \mu|^t \,\mathrm{d}x\right)^{t^{-1}} \leqslant A_5 \mathrm{Cap}_{\alpha,s}(K)^{t^{-1}}$$

for all compact sets K.

The least possible values of constants A_i, $i = 1, \ldots, 5$ are all equivalent to $\|G_\alpha * \mu\|_{\mathcal{Z}}$.

We conclude this section with another application. The readers may have noticed that the transition from Theorem 1.7 to Theorem 1.8 suggests that $\|\cdot\|_{L^1(C)}$ and $\|\cdot\|_{\mathcal{I}}$ have the regularity property similar to measures. This observation is true to some extent. First of all, let us denote by $\mathcal{QLSC}$ the class of functions that are both quasi-continuous and lower semi-continuous. Let $\mathcal{C}$ be the operator defined successively in the following way:
For any $f \in C_0$, define

$$\mathcal{C}(f) = \|f\|_{L^1(C)}.$$

For any $f \in \mathcal{QLSC}$, define

$$\mathcal{C}(f) = \sup_{\substack{0 \leqslant g \leqslant |f| \\ g \in C_0}} \mathcal{C}(g).$$

For any f, define

$$\mathcal{C}(f) = \inf_{\substack{h \geqslant |f| \\ h \in \mathcal{QLSC}}} \mathcal{C}(h).$$

Therefore, the operator $\mathcal{C}$ is defined as having the inner and outer regularity. One may expect that $\|\cdot\|_{L^1(C)}$ is exactly $\mathcal{C}$, unfortunately, it seems to us that they are only equivalent but not identical:

Theorem 1.10. *For any nonnegative f, $\mathcal{C}(f) \approx \|f\|_{L^1(C)}$.*

The next section will provide the proofs for all aforementioned statements. In what follows, the notation $\alpha \approx \beta$ will denote both $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$ for any two quantities α and β.

2. Proofs

Proof of Proposition 1.1. Let us denote by $\mathscr{L}^1(C)$ the subspace of $L^1(C)$ which consists of quasi-continuous functions. One can identify the dual of $\mathscr{L}^1(C)$ with the space $\mathfrak{M}$ which consists of measures μ such that

$$\|\mu\|_{\mathfrak{M}} := \sup_K \frac{|\mu|(K)}{\mathrm{Cap}_{\alpha,s}(K)},$$

where the supremum is taken over all compact sets $K \subseteq \mathbb{R}^n$ with nonzero capacity, see [6] and [9], Theorem 2.4. We note that $\mathscr{L}^1(C)$ is normable and thus it follows from Hahn-Banach theorem that for any $u \in \mathscr{L}^1(C)$ we have

$$\|u\|_{L^1(C)} \approx \sup\left\{\left|\int u \,\mathrm{d}\mu\right| : \|\mu\|_{\mathfrak{M}} \leqslant 1\right\}.$$

Let φ be a nonnegative compactly supported continuous function. Since $G_\alpha(x) = \mathcal{O}(\mathrm{e}^{-x/2})$, it is not hard to see that $G_\alpha * \varphi \in \mathscr{L}^1(C)$ and hence,

$$\int_{\mathbb{R}^n} G_\alpha * \varphi \,\mathrm{d}C \lesssim \sup_{\|\mu\|_{\mathfrak{M}} \leqslant 1} \int G_\alpha * \varphi \,\mathrm{d}|\mu| = \sup_{\|\mu\|_{\mathfrak{M}} \leqslant 1} \int (G_\alpha * |\mu|)\varphi \,\mathrm{d}x$$
$$\leqslant \|\varphi\|_{\mathcal{Z}'} \sup_{\|\mu\|_{\mathfrak{M}} \leqslant 1} \|G_\alpha * |\mu|\|_{\mathcal{Z}} \lesssim \|\varphi\|_{\mathcal{Z}'},$$

where the last $\lesssim$ follows from [8], Theorem 1.2.

Let $\varphi \in \mathcal{Z}'$ be a nonnegative function. By the density of C_0^∞ in $\mathcal{Z}'$ (see [9], Remark 3.3), we may choose a sequence $\{\varphi_n\}_{i=1}^\infty$ of C_0^∞ that converges to φ in $\mathcal{Z}'$. Since $\mathcal{Z}' \hookrightarrow L^1(\mathbb{R}^n)$ (see [9], Remark 2.1), we can further assume that $\varphi_i(x) \to \varphi(x)$ a.e. Note that $G_\alpha * \varphi(x) \leqslant \liminf_{i\to\infty} G_\alpha * \varphi_i(x)$ everywhere and hence,

$$\int_{\mathbb{R}^n} G_\alpha * \varphi \,\mathrm{d}C \leqslant \liminf_{i\to\infty} \int_{\mathbb{R}^n} G_\alpha * \varphi_i \,\mathrm{d}C \lesssim \liminf_{i\to\infty} \|\varphi_i\|_{\mathcal{Z}'} = \|\varphi\|_{\mathcal{Z}'}.$$

If we further let $G_\alpha * \varphi \geqslant f$ q.e. for an arbitrary function $f \geqslant 0$, then

$$\int_{\mathbb{R}^n} f \,\mathrm{d}C \lesssim \|\varphi\|_{\mathcal{Z}'}$$

and hence the estimate $\|f\|_{L^1(C)} \lesssim \|f\|_{\mathcal{I}}$ holds by definition. □

P r o o f of Corollary 1.2. Just note that by $\|\cdot\|_{L^1(C)} \leqslant \|\cdot\|_{\mathcal{I}}$, one has

$$\mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n \colon\ G_\alpha * f(x) \geqslant \lambda\}) \leqslant \frac{1}{\lambda}\|f\|_{\mathcal{Z}'}$$

for any $\lambda > 0$. □

P r o o f of Proposition 1.3. By Corollary 1.2, $G_\alpha * f_n(x)$ and $G_\alpha * f(x)$ are well-defined and finite on F^c for a set F with $\mathrm{Cap}_{\alpha,s}(F) = 0$. Choose $\{n_i\}_{i=1}^\infty$ such that

$$\|f_{n_i} - f\|_{\mathcal{Z}'} < 4^{-i}.$$

Set $E_i = \{x \colon\ G_\alpha * |f_{n_i} - f| > 2^{-i}\}$ and $G_m = \bigcup_{i=m}^\infty E_i$. We have

$$\mathrm{Cap}_{\alpha,s}(E_i) \lesssim \|\chi_{E_i}\|_{\mathcal{I}} \leqslant 2^i \|f_{n_i} - f\|_{\mathcal{Z}'} \leqslant 2^{-i}, \quad \text{and} \quad \mathrm{Cap}_{\alpha,s}(G_m) \leqslant \sum_{i=m}^\infty 2^{-i},$$

so

$$\mathrm{Cap}_{\alpha,s}\Bigl(\bigcap_{m=1}^\infty G_m\Bigr) = 0.$$

Note that if $x \notin G_m \cup F$, then $|G_\alpha * f_{n_i}(x) - G_\alpha * f(x)| \leqslant 2^{-i}$ for all $i \geqslant m$. The proof is complete by noting that F is contained in an open set of arbitrarily small capacity. □

P r o o f of Proposition 1.4. By Corollary 1.2 we know that $G_\alpha * f$ is well defined and finite q.e. By the density of C_0^∞ in $\mathcal{Z}'$ (see [9], Remark 3.3), we pick a sequence $\{f_i\}$ of C_0^∞ that converges to f in $\mathcal{Z}'$. Then $G_\alpha * f_i$ is a Schwartz function, and by Proposition 1.3 there is a subsequence $\{i_k\}_{k=1}^\infty$ such that $G_\alpha * f_{i_k}(x)$ converges to $G_\alpha * f(x)$ q.e. and uniformly outside an open set of arbitrarily small capacity, the proposition follows. □

P r o o f of Lemma 1.5. Let $\chi(x) = \chi_{B_1(0)}(x)/|B_1(0)|$ and $\chi_r(x) = r^{-n}\chi(x/r)$ for $x \in \mathbb{R}^n$ and $r > 0$. One may write

$$M^{\rm loc}(G_\alpha * f)(x) = \sup_{0<r<1} \chi_r * G_\alpha * f(x)$$

and hence,

$$M^{\rm loc}(G_\alpha * f)(x) \leqslant G_\alpha * M^{\rm loc} f(x).$$

As a consequence, we have

$$\{x \in \mathbb{R}^n \colon\ M^{\rm loc} f(x) > \lambda\} \subseteq \{x \in \mathbb{R}^n \colon\ G_\alpha * M^{\rm loc} f(x) > \lambda\}$$

and

$$\mathrm{Cap}_{\alpha,s}(E_\lambda) \lesssim \|\chi_{E_\lambda}\|_{\mathcal{I}} \leqslant \lambda^{-1}\|M^{\rm loc} f\|_{\mathcal{Z}'}$$

for all $\lambda > 0$. The lemma follows by the boundedness of $M^{\rm loc}$ on $\mathcal{Z}'$, see [9]. □

P r o o f of Proposition 1.6. Let χ_r be as in the proof of Lemma 1.5. By the density of C_0^∞ in $\mathcal{Z}'$ (see [9], Remark 3.3), we can choose for every $\varepsilon > 0$ an $f_0 \in \mathcal{Z}'$ such that $\|f - f_0\|_{\mathcal{Z}'} < \varepsilon$. Then $G_\alpha * f_0$ is a Schwartz function and thus $\lim\limits_{r\to 0} \chi_r * f_0(x) = f_0(x)$ for all $x \in \mathbb{R}^n$.

For $\delta > 0$ we define

$$\Omega_\delta(\varphi)(x) = \sup_{0<r<\delta} (\chi_r * \varphi)(x) - \inf_{0<r<\delta} (\chi_r * \varphi)(x)$$

for any suitable function φ. It follows that

$$\Omega_\delta(G_\alpha * f)(x) \leqslant \Omega_\delta(G_\alpha * f - G_\alpha * f_0)(x) + \Omega_\delta(G_\alpha * f_0)(x).$$

By uniform continuity, we can choose a $\delta \in (0, 1)$ so small that

$$\Omega_\delta(G_\alpha * f_0)(x) < \varepsilon$$

for all $x \in \mathbb{R}^n$. Moreover,

$$\sup_{0<r<1} |\chi_r * G_\alpha * (f - f_0)(x)| \leqslant M^{\rm loc}(G_\alpha * (f - f_0))(x),$$

and hence,

$$\Omega_\delta(G_\alpha * f)(x) \leqslant 2M^{\mathrm{loc}}(G_\alpha * (f - f_0))(x) + \varepsilon.$$

If $\varepsilon < \frac{1}{2}\lambda$, this implies that

$$\{x \in \mathbb{R}^n\colon\ \Omega_\delta(G_\alpha * f)(x) > \lambda\} \subseteq \Big\{x \in \mathbb{R}^n\colon\ M^{\mathrm{loc}}(G_\alpha * (f - f_0))(x) > \frac{1}{4}\lambda\Big\},$$

and thus, we have by Lemma 1.5 that

$$\mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n\colon\ \Omega_\delta(G_\alpha * f)(x) > \lambda\}) \lesssim \frac{1}{\lambda}\|f - f_0\|_{\mathcal{Z}'} \lesssim \frac{\varepsilon}{\lambda}.$$

Now choose $\lambda = 2^{-i}$ and $\varepsilon = 4^{-i}$ for $i = 1, 2, \ldots$, and denote the corresponding δ by δ_i. Set

$$E_i = \{x \in \mathbb{R}^n\colon\ \Omega_{\delta_i}(G_\alpha * f)(x) > 2^{-i}\},$$

then

$$\mathrm{Cap}_{\alpha,s}(E_i) \lesssim 2^{-i}.$$

If $F_m = \bigcup\limits_{i=m}^{\infty} E_i$, it follows that

$$\mathrm{Cap}_{\alpha,s}(E_m) \lesssim \sum_{i=m}^{\infty} 2^{-i} \to 0$$

as $m \to \infty$, whence

$$\mathrm{Cap}_{\alpha,s}\Big(\bigcap_{m=1}^{\infty} F_m\Big) = 0.$$

If $x \notin F_m$, we see that $\Omega_\delta(G_\alpha * f)(x) \leqslant 2^{-i}$ for $\delta \leqslant \delta_i$ and $i \geqslant m$. It follows that $\lim\limits_{r\to 0} \chi_r * G_\alpha * f(x) = G_\alpha * f(x)$ exists if $x \notin \bigcap\limits_{m=1}^{\infty} F_m$. On the other hand, for any $m = 1, 2, \ldots, \lim\limits_{r\to 0} \chi_r * G_\alpha * f(x) = G_\alpha * f(x)$ uniformly on F_m^c, the proof is now complete. □

P r o o f of Theorem 1.7. Let

$$\mathcal{M}_f = \Big\{\nu\colon\ \nu \geqslant 0,\ \mathrm{supp}(\nu) \subseteq \mathrm{supp}(f),\ \int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu = 1\Big\}$$

and

$$\mathcal{F} = \{\varphi \in \mathcal{Z}'\colon\ \varphi \geqslant 0,\ \|\varphi\|_{\mathcal{Z}'} \leqslant 1\}.$$

We also let

$$\|f\|_{\mathcal{J},1} = \Big(\sup_{\mathcal{F}} \inf_{\mathcal{M}_f} \int_{\mathbb{R}^n} G_\alpha * \varphi(x)\,\mathrm{d}\nu\Big)^{-1}$$

and

$$\|f\|_{\mathcal{M},1}=\Big(\inf_{\mathcal{M}_f}\sup_{\mathcal{F}}\int_{\mathbb{R}^n}G_\alpha*\varphi(x)\,\mathrm{d}\nu\Big)^{-1}.$$

We claim that

$$\|f\|_{\mathcal{J},1}=\|f\|_{\mathcal{M},1}. \tag{2.1}$$

The sets $\mathcal{M}_\varphi$ and $\mathcal{F}$ are convex. Viewing $\mathcal{M}_f$ as a subset of the space $\mathscr{M}(\operatorname{supp}(f))$ of measures on $\operatorname{supp}(f)$, the set $\mathcal{M}_f$ is vaguely compact by the observation that $\nu(\operatorname{supp}(f))\leqslant(\min\limits_{\operatorname{supp}(f)}f)^{-1}$ for $\nu\in\mathcal{M}_f$ and the Banach-Alaoglu theorem. The linearity of the maps

$$\varphi\to\int_{\mathbb{R}^n}G_\alpha*\varphi(x)\,\mathrm{d}\nu,\quad \nu\to\int_{\mathbb{R}^n}G_\alpha*\varphi(x)\,\mathrm{d}\nu,$$

and the continuity of the second map allow us to invoke Fan's minimax theorem (see [2], Theorem 2.4.1), and hence (2.1) follows by the minimax theorem. We are now to show that

$$\|f\|_{\mathcal{J}}=\|f\|_{\mathcal{J},1} \tag{2.2}$$

and

$$\|f\|_{\mathcal{M}}=\|f\|_{\mathcal{M},1}. \tag{2.3}$$

We begin by showing that

$$\|f\|_{\mathcal{J},1}\leqslant\|f\|_{\mathcal{J}}. \tag{2.4}$$

We could assume that $\|f\|_{\mathcal{J}}<\infty$. For any $\varepsilon>0$ there is $\varphi_\varepsilon\geqslant0$ such that $G_\alpha*\varphi_\varepsilon\geqslant f$ and

$$\|\varphi_\varepsilon\|_{\mathcal{Z}'}<\|f\|_{\mathcal{J}}+\varepsilon.$$

As a result,

$$\Big\|\frac{\varphi_\varepsilon}{\|f\|_{\mathcal{J}}+\varepsilon}\Big\|_{\mathcal{Z}'}\leqslant1.$$

For any $\nu\in\mathcal{M}_f$ we have

$$\int_{\mathbb{R}^n}G_\alpha*\Big(\frac{\varphi_\varepsilon}{\|f\|_{\mathcal{J}}+\varepsilon}\Big)(x)\,\mathrm{d}\nu\geqslant\frac{1}{\|f\|_{\mathcal{J}}+\varepsilon}.$$

Thus,

$$\|f\|_{\mathcal{J}}+\varepsilon\geqslant\Big(\int_{\mathbb{R}^n}G_\alpha*\Big(\frac{\varphi_\varepsilon}{\|f\|_{\mathcal{J}}+\varepsilon}\Big)(x)\,\mathrm{d}\nu\Big)^{-1},$$

which implies that $\|f\|_{\mathcal{J}} + \varepsilon \geqslant \|f\|_{\mathcal{J},1}$, and thus (2.4) follows. We now show that

$$\|f\|_{\mathcal{J}} \leqslant \|f\|_{\mathcal{J},1}. \tag{2.5}$$

We assume that $\|f\|_{\mathcal{J},1} < \infty$. For any $\varepsilon > 0$ there is $\psi_\varepsilon \in \mathcal{F}$ such that

$$\left(\inf_{\nu \in \mathcal{M}_\varphi} \int_{\mathbb{R}^n} G_\alpha * \psi_\varepsilon(x)\,\mathrm{d}\nu\right)^{-1} < \|f\|_{\mathcal{J},1} + \varepsilon.$$

Thus,

$$1 \leqslant \inf_{\nu \in \mathcal{M}_\varphi} \int_{\mathbb{R}^n} G_\alpha * (\psi_\varepsilon \cdot (\|f\|_{\mathcal{J},1} + \varepsilon))(x)\,\mathrm{d}\nu.$$

Fix an $x \in \operatorname{supp}(f)$ and let $\mathrm{d}\nu = \mathrm{d}\delta_x / f(x)$, where $\mathrm{d}\delta_x$ is the point mass measure at x. Then $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu = 1$ and hence,

$$1 \leqslant G_\alpha * (\psi_\varepsilon \cdot (\|\varphi\|_{\mathcal{J},1} + \varepsilon))(x) \cdot \frac{1}{f(x)}, \quad f(x) \leqslant G_\alpha * (\psi_\varepsilon \cdot (\|f\|_{\mathcal{J},1} + \varepsilon))(x).$$

Since $\|\psi_\varepsilon\|_{\mathcal{Z}'} \leqslant 1$, we get

$$\|f\|_{\mathcal{J}} \leqslant \|\psi_\varepsilon \cdot (\|f\|_{\mathcal{J},1} + \varepsilon)\|_{\mathcal{Z}'} \leqslant \|f\|_{\mathcal{J},1} + \varepsilon,$$

so (2.5) follows and hence (2.2). We are now to show (2.3). As before, we will divide the cases to

$$\|f\|_{\mathcal{M},1} \leqslant \|f\|_{\mathcal{M}} \tag{2.6}$$

and

$$\|f\|_{\mathcal{M}} \leqslant \|f\|_{\mathcal{M},1}. \tag{2.7}$$

We note that $\|f\|_{\mathcal{M},1} \geqslant 0$ since $0 \in \mathcal{F}$. Assume at the moment that

$$\|f\|_{\mathcal{M},1} < \infty, \tag{2.8}$$

we invoke the dual pair $(\mathcal{Z}, \mathcal{Z}')$, then for every $\varepsilon > 0$ there is a measure $\nu \in \mathcal{M}_f$ satisfying

$$\begin{aligned} \|f\|_{\mathcal{M},1} &< \left(\sup_{\varphi \in \mathcal{F}} \int_{\mathbb{R}^n} G_\alpha * \varphi(x)\,\mathrm{d}\nu\right)^{-1} + \varepsilon \\ &= \left(\sup_{\varphi \in \mathcal{F}} \int_{\mathbb{R}^n} \varphi(x)(G_\alpha * \nu)(x)\,\mathrm{d}x\right)^{-1} + \varepsilon = \|G_\alpha * \nu\|_{\mathcal{Z}}^{-1} + \varepsilon. \end{aligned}$$

Set $\mu = \|G_\alpha * \nu\|_{\mathcal{Z}}^{-1} \nu$. We get

$$\|f\|_{\mathcal{M},1} - \varepsilon < \|G_\alpha * \nu\|_{\mathcal{Z}}^{-1} = \int_{\mathbb{R}^n} \|G_\alpha * \nu\|_{\mathcal{Z}}^{-1} f(x)\,\mathrm{d}\nu = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}\mu \leqslant \|f\|_{\mathcal{M}},$$

so (2.6) follows. Now we justify (2.7). For any $\nu \geqslant 0$ and $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(f)$ with $\|G_\alpha * \nu\|_{\mathcal{Z}} \leqslant 1$ and $\varphi \in \mathcal{F}$, set $\mu = (\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu)^{-1}\nu$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} G_\alpha * \varphi(x)\,\mathrm{d}\mu &= \int_{\mathbb{R}^n} (G_\alpha * \mu)(x)\varphi(x)\,\mathrm{d}x = \Big(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu\Big)^{-1}\int_{\mathbb{R}^n}(G_\alpha * \nu)(x)\varphi(x)\,\mathrm{d}x\\ &\leqslant \Big(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu\Big)^{-1},\end{aligned}$$

by the dual pair $(\mathcal{Z}, \mathcal{Z}')$. Therefore,

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu \leqslant \|f\|_{\mathcal{M},1},$$

which implies (2.7), so (2.3) is established as well.

We now justify (2.8). Assume on the contrary that a sequence $\{\nu_j\} \subseteq \mathcal{M}_f$ is such that

$$\sup_{\varphi\in\mathcal{F}} \int_{\mathbb{R}^n} G_\alpha * \varphi(x)\,\mathrm{d}\nu_j \to 0.$$

By the dual pair $(\mathcal{Z}, \mathcal{Z}')$, we get immediately that

$$\|G_\alpha * \nu_j\|_{\mathcal{Z}} \to 0.$$

It follows from [8], Theorem 1.2, that $\nu_j(K) \to 0$, and hence $\{\nu_j(K)\}_{j=1}^\infty$ is bounded. By Banach-Alaoglu theorem, there exists a subnet $\{\nu_{j_k}\}$ converging vaguely to a measure ν; this measure satisfies $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu = 1$. On the other hand, we already had $\nu_{j_k}(K) \to 0$, so $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}\nu = 0$, we get a contradiction, so (2.8) follows.

In view of (1.5), we have apparently that $\|\cdot\|_{\mathcal{I}} \leqslant \|\cdot\|_{\mathcal{J}}$ and hence

$$\|\chi_K\|_{\mathcal{I}} \leqslant \|\chi_K\|_{\mathcal{M}}.$$

By [8], Theorem 1.2, it is easy to deduce that $\|\chi_K\|_{\mathcal{M}} \lesssim \operatorname{Cap}_{\alpha,s}(K)$. The other direction that $\operatorname{Cap}_{\alpha,s}(K) \lesssim \|\chi_K\|_{\mathcal{I}}$ follows by Proposition 1.1. □

P r o o f of Theorem 1.8. We note that Fatou's property of $\mathcal{Z}'$ entails the following countable subadditivity:

$$\|\chi_E\|_{\mathcal{I}} \leqslant \sum_j \|\chi_{E\cap R_j}\|_{\mathcal{I}},$$

where R_j is the annulus $\{j-1 \leqslant |x| < j\}$. On the other hand, the quasi-additivity of $\operatorname{Cap}_{\alpha,s}$ (see [1]) implies that

$$\sum_j \operatorname{Cap}_{\alpha,s}(E\cap R_j) \lesssim \operatorname{Cap}_{\alpha,s}(E).$$

Therefore, it suffices to prove the theorem under the assumption that E is a bounded set. Besides that, since $\mathrm{Cap}_{\alpha,s}$ is outer regular, we can further assume that E is a bounded open set. With such an assumption, we can find a sequence $\{\varphi_j\}$ of continuous functions and a sequence $\{K_j\}$ of compact sets such that

$$\chi_{K_1} \leqslant \varphi_1 \leqslant \chi_{K_2} \leqslant \varphi_2 \leqslant \dots$$

and $\chi_E(x) = \sup_j \varphi_j(x) = \sup_j \chi_{K_j}(x)$.

Fix an $N \in \mathbb{N}$ and let $j \geqslant N$, $\varepsilon > 0$. We choose a nonnegative $f_j \in \mathcal{Z}'$ such that

$$G_\alpha * f_j(x) \geqslant \varphi_j(x) \text{ q.e.}, \quad \|f_j\|_{\mathcal{Z}'} \leqslant \|\varphi_j\|_{\mathcal{I}} + \varepsilon.$$

Note that the sequence $\{\|f_j\|_{\mathcal{Z}'}\}$ is bounded by $\|\chi_E\| + \varepsilon$. Using the $\overline{C_0}^{\mathcal{Z}}$-$\mathcal{Z}'$ duality (see [9], Theorem 1.9) and the trivial fact that $\overline{C_0}^{\mathcal{Z}}$ is separable, we may assume by Banach-Alaoglu theorem that f_j converges weak* to an $f \in \mathcal{Z}'$. Since all the characteristic functions of sets of finite measure belong to $\overline{C_0}^{\mathcal{Z}}$, by the usual Lebesgue's differentiation theorem, we may assume that $f \geqslant 0$. For any $x \in \mathbb{R}^n$ and $r > 0$ we see that

$$\int_{B_r(x)} \varphi_N(y)\,\mathrm{d}y \leqslant \int_{\mathbb{R}^n} \chi_{B_r(x)}(y) G_\alpha * f_j(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} f_j(y) G_\alpha * \chi_{B_r(x)}(y)\,\mathrm{d}y.$$

Since $G_\alpha * \chi_{B_r(x)} \in \overline{C_0}^{\mathcal{Z}}$, by the weak* convergence we have by taking $j \to \infty$ that

$$\int_{B_r(x)} \varphi_N(y)\,\mathrm{d}y \leqslant \int_{B_r(x)} G_\alpha * f(y)\,\mathrm{d}y.$$

The continuity of φ_N implies for every x that

$$\varphi_N(x) = \lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \varphi_N(y)\,\mathrm{d}y,$$

then we use Proposition 1.6 to obtain

$$\varphi_N(x) \leqslant G_\alpha * f(x) \text{ q.e.}$$

Taking $N \to \infty$ yields

$$\chi_E(x) \leqslant G_\alpha * f(x) \text{ q.e.}$$

As a result, by a standard property of weak* convergence, the fact that $\varphi_j \leqslant \chi_{K_{j+1}}$, $\|\chi_{K_j}\|_{\mathcal{M}} \approx \mathrm{Cap}_{\alpha,s}(K_j)$, and Theorem 1.7, we deduce that

$$\begin{aligned} \|\chi_E\|_{\mathcal{I}} \leqslant \|f\|_{\mathcal{Z}'} &\lesssim \liminf_{j\to\infty} \|f_j\|_{\mathcal{Z}'} = \sup_j \|\chi_{K_j}\|_{\mathcal{I}} + \varepsilon \\ &\approx \sup_j \mathrm{Cap}_{\alpha,s}(K_j) + \varepsilon = \mathrm{Cap}_{\alpha,s}(E) + \varepsilon. \end{aligned}$$

The arbitrariness of $\varepsilon > 0$ finishes the proof of the first part of this theorem. For the part that $\|f\|_{\mathcal{I}} \lesssim \|f\|_{L^1(C)}$, we can argue as in the beginning of the proof that by countably subadditivity of $\|\cdot\|_{\mathcal{I}}$ that

$$\|f\|_{\mathcal{I}} \leqslant \sum_j \|f\chi_{\{2^{j-1}\leqslant|f|<2^j\}}\|_{\mathcal{I}} \leqslant \sum_j 2^j \operatorname{Cap}_{\alpha,s}(\{|f| \geqslant 2^{j-1}\}) \approx \|f\|_{L^1(C)}.$$

The proof of this theorem is now complete. □

P r o o f of Theorem 1.9. The equivalence between (b), (d), and (e) is known, see [8], Theorem 1.2. The implication that (a) $\to$ (c) is trivial. Therefore, it suffices to show the implications that (c) $\to$ (d) and (d) $\to$ (a).

(c) $\to$ (d): We choose an f such that $G_\alpha * f \geqslant 1$ on K. It follows from (c) that $\mu(K) \leqslant A_3\|f\|_{\mathcal{Z}'}$, then by the definition of $\|\cdot\|_{\mathcal{I}}$, we have $\mu(K) \leqslant A_3\|\chi_K\|_{\mathcal{I}}$. We invoke Theorem 1.8 to finish the proof of this implication.

(d) $\to$ (a): We first assume that $f \in C_0^\infty$. We have

$$\begin{aligned}\int_{\mathbb{R}^n} |G_\alpha * f| \,\mathrm{d}\mu &= \int_0^\infty \mu(\{x \in \mathbb{R}^n\colon\ |G_\alpha * f| \geqslant t\})\,\mathrm{d}t \\ &\leqslant \sup_K \frac{\mu(K)}{\operatorname{Cap}_{\alpha,s}(K)} \cdot \|G_\alpha * f\|_{L^1(C)} \lesssim \sup_K \frac{\mu(K)}{\operatorname{Cap}_{\alpha,s}(K)} \cdot \|f\|_{\mathcal{Z}'},\end{aligned}$$

the implication is proved by the density of C_0^∞ in $\mathcal{Z}'$. □

P r o o f of Theorem 1.10. We first prove that $\mathcal{C}(f) \lesssim \|f\|_{L^1(C)}$. Let

$$0 \leqslant \varphi \in \mathcal{Z}'$$

be such that $G_\alpha * \varphi \geqslant f$. Define $\varphi_n(x) = \min\{\varphi(x), n\}$ for $|x| \leqslant n$ and $\varphi_n(x) = 0$ for $|x| > n$, so $G_\alpha * \varphi_n$ is continuous and

$$G_\alpha * \varphi(x) = \sup_n (G_\alpha * \varphi_n)(x).$$

It follows that $G_\alpha * \varphi$ is lower semi-continuous. Together with Proposition 1.4, we see that $G_\alpha * \varphi \in \mathcal{QLSC}$, then

$$\mathcal{C}(f) \leqslant \mathcal{C}(G_\alpha * \varphi) = \sup_{\substack{0\leqslant g\leqslant G_\alpha*\varphi \\ g\in C_0}} \mathcal{C}(g) \leqslant \|G_\alpha * \varphi\|_{L^1(C)} \leqslant \|\varphi\|_{\mathcal{Z}'}.$$

Hence, $\mathcal{C}(f) \leqslant \|f\|_{\mathcal{I}} \lesssim \|f\|_{L^1(C)}$.

For the other direction, we let $h \in \mathcal{QLSC}$ be such that $h \geqslant f$. Since h is lower semi-continuous, the set $\{h \leqslant n\}$ is closed. We choose an increasing sequence $\{\varphi_n\}$

of continuous functions such that $\varphi_n = 1$ on the compact set $\{|x| \leqslant n\} \cap \{h \leqslant n\}$, apparently, we have $h\varphi_n \in L^1(C)$. Again, as h is lower semi-continuous, it is standard that

$$h(x) = \sup_{\substack{0 \leqslant g \leqslant h \\ g \in C_0}} g(x),$$

and that

$$\int h\varphi_n \, \mathrm{d}\mu = \sup_{\substack{0 \leqslant g \leqslant h \\ g \in C_0}} \int g\varphi_n \, \mathrm{d}\mu$$

for any nonnegative measure μ, see [3], Proposition 16.1. As a result, we have

$$\begin{aligned} \|f\|_{L^1(C)} &\leqslant \sup_{n \geqslant 1} \|h\varphi_n\|_{L^1(C)} \approx \sup_{\substack{n \geqslant 1 \\ \|\mu\|_{\mathfrak{M}} \leqslant 1}} \int h\varphi_n \, \mathrm{d}\mu = \sup_{\substack{0 \leqslant g \leqslant h \\ g \in C_0}} \sup_{\substack{n \geqslant 1 \\ \|\mu\|_{\mathfrak{M}} \leqslant 1}} \int g\varphi_n \, \mathrm{d}\mu \\ &\lesssim \sup_{\substack{0 \leqslant g \leqslant h \\ g \in C_0}} \|g\|_{L^1(C)} = \mathcal{C}(h). \end{aligned}$$

It follows from the definition of $\mathcal{C}$ that $\|f\|_{L^1(C)} \lesssim \mathcal{C}(f)$, the proof is now complete. □

References

[1] *D. R. Adams*: Quasi-additivity and sets of finite L^p-capacity. Pac. J. Math. *79* (1978), 283–291. zbl MR doi

[2] *D. R. Adams, L. I. Hedberg*: Function Spaces and Potential Theory. Grundlehren der Mathematischen Wissenschaften 314. Springer, Berlin, 1996. zbl MR doi

[3] *N. Dinculeanu*: Integration on Locally Compact Spaces. Monographs and Textbooks on Pure and Applied Mathematics. Noordhoff International Publishing, Leiden, 1974. zbl MR

[4] *A. Grigor'yan, I. Verbitsky*: Pointwise estimates of solutions to nonlinear equations for nonlocal operators. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) *20* (2020), 721–750. zbl MR doi

[5] *N. J. Kalton, I. E. Verbitsky*: Nonlinear equations and weighted norm inequalities. Trans. Am. Math. Soc. *351* (1999), 3441–3497. zbl MR doi

[6] *V. G. Maz'ya*: Sobolev Spaces: With Applications To Elliptic Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften 342. Springer, Berlin, 2011. zbl MR doi

[7] *V. G. Maz'ya, T. O. Shaposhnikova*: Theory of Sobolev Multipliers: With Applications To Differential and Integral Operators. Grundlehren der Mathematischen Wissenschaften 337. Springer, Berlin, 2009. zbl MR doi

[8] *V. G. Maz'ya, I. E. Verbitsky*: Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. Ark. Mat. *33* (1995), 81–115. zbl MR doi

[9] *K. H. Ooi, N. C. Phuc*: Characterizations of predual spaces to a class of Sobolev multiplier type spaces. Available at `https://arxiv.org/abs/2005.04349` (2020), 46 pages.

[10] *K. H. Ooi, N. C. Phuc*: On a capacitary strong type inequality and related capacitary estimates. Available at `https://arxiv.org/abs/2009.09291v1` (2020), 12 pages.

Author's address: Keng Hao Ooi, Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA, e-mail: `kooi1@lsu.edu`.