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INEQUALITIES FOR TAYLOR SERIES INVOLVING THE DIVISOR FUNCTION

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Abstract. Let

$$T(q)=\sum_{k=1}^{\infty}d(k)q^k,\quad |q|<1,$$

where $d(k)$ denotes the number of positive divisors of the natural number k. We present monotonicity properties of functions defined in terms of T. More specifically, we prove that

$$H(q)=T(q)-\frac{\log(1-q)}{\log(q)}$$

is strictly increasing on $(0,1)$, while

$$F(q)=\frac{1-q}{q}H(q)$$

is strictly decreasing on $(0,1)$. These results are then applied to obtain various inequalities, one of which states that the double inequality

$$\alpha\frac{q}{1-q}+\frac{\log(1-q)}{\log(q)}<T(q)<\beta\frac{q}{1-q}+\frac{\log(1-q)}{\log(q)},\quad 0<q<1,$$

holds with the best possible constant factors $\alpha=\gamma$ and $\beta=1$. Here, γ denotes Euler's constant. This refines a result of Salem, who proved the inequalities with $\alpha=\frac{1}{2}$ and $\beta=1$.

Keywords: divisor function; infinite series; inequality; monotonicity; q-digamma function; Euler's constant

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1. Introduction

In this paper, we study the Taylor series

$$T(q)=\sum_{k=1}^{\infty} d(k)q^k, \quad |q|<1,$$

where $d(k)$ denotes the number of positive divisors of the natural number k. It is well-known that the function T has a close connection to Lambert series. We have

$$T(q)=\sum_{k=1}^{\infty} \frac{q^k}{1-q^k}, \quad |q|<1. \tag{1.1}$$

A proof of (1.1) and further information on Lambert series can be found in [6], Section 58 C. Another series representation for T was given by Clausen in 1828, see [5],

$$T(q)=\sum_{k=1}^{\infty} \frac{1+q^k}{1-q^k} q^{k^2}, \quad |q|<1.$$

In 1899, Landau in [8] proved that T can be used to determine the value of a series involving the classical Fibonacci numbers, defined by $F_0=F_1=1$, $F_n=F_{n-1}+F_{n-2}$ $(n\geqslant 2)$,

$$\sum_{k=1}^{\infty} \frac{1}{F_{2k}}=\sqrt{5}(T(c)-T(c^2))=1.53537\ldots, \quad c=\Big(\frac{\sqrt{5}-1}{2}\Big)^2.$$

Stimulated by his work on the analysis of data structure, Uchimura in [17] presented in 1981 the following result:

$$T(q)=(q;q)_\infty \sum_{k=1}^{\infty} \frac{kq^k}{(q;q)_k}, \quad |q|<1,$$

where $(a;q)_k$ is the q-shifted factorial,

$$(a;q)_k=\prod_{j=0}^{k-1}(1-aq^j), \quad (a;q)_\infty=\prod_{j=0}^{\infty}(1-aq^j).$$

A related result was given by Merca, see [9]. In 2015, he proved

$$T(q)=\frac{1}{(q;q)_\infty}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{kq^{\binom{k+1}{2}}}{(q;q)_k}, \quad |q|<1,$$

and one year later, he showed that there is a relationship between partitions and T,

$$T(q) = \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty} (s_{\rm o}(k) - s_{\rm e}(k)) q^k, \quad |q| < 1.$$

Here, $s_{\rm o}(k)$ and $s_{\rm e}(k)$ denote the number of parts in all partitions of k into odd and even number of distinct parts, respectively; see [10].

The q-digamma function is the logarithmic derivative of the q-gamma function, $\psi_q = \Gamma_q'/\Gamma_q$. The properties of Γ_q and ψ_q were investigated by numerous authors. For detailed information on these functions we refer to Askey (see [2]), Salem (see [13], [14]), Salem and Alzahrani (see [16]) and the references cited therein. In view of the series representation

$$\psi_q(x) = -\log(1-q) + \log(q) \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k}, \quad 0 < q < 1, \ x > 0,$$

we conclude from (1.1) that T can be expressed in terms of $\psi_q(1)$,

$$T(q) = \frac{\psi_q(1) + \log(1-q)}{\log(q)}. \tag{1.2}$$

The work on this paper has been inspired by an interesting double inequality discovered by Salem, see [15]. He proved

$$0 < 1 - \frac{1-q}{q\log(q)} \psi_q(1) < \frac{1}{2}, \quad 0 < q < 1. \tag{1.3}$$

Using (1.2) and (1.3) we obtain elegant upper and lower bounds for $T(q)$. We have

$$\alpha \frac{q}{1-q} + \frac{\log(1-q)}{\log(q)} < T(q) < \beta \frac{q}{1-q} + \frac{\log(1-q)}{\log(q)}, \quad 0 < q < 1 \tag{1.4}$$

with $\alpha = \frac{1}{2}$ and $\beta = 1$. It is natural to ask whether these inequalities can be refined. More precisely, we look for the largest number α and the smallest number β such that (1.4) is valid. Here, we solve this problem. It turns out that $\beta = 1$ is the best possible constant on the right-hand side of (1.4), but the factor $\frac{1}{2}$ on the left-hand side can be replaced by a larger number, namely by Euler's constant $\gamma = 0.57721\ldots$ This reveals a connection between the divisor function and "the third number of holy trinity (π, e, γ) of mathematical constants", see [3], page 302.

In the next section, we collect several lemmas. Monotonicity properties of the functions

$$H(q) = T(q) - \frac{\log(1-q)}{\log(q)} \quad \text{and} \quad F(q) = \frac{1-q}{q}\Big(T(q) - \frac{\log(1-q)}{\log(q)}\Big) \tag{1.5}$$

are given in Section 3. Finally, in Section 4, we apply the monotonicity of F to prove (1.4) with $\alpha=\gamma$, $\beta=1$ and we present sharp upper and lower bounds for the three Taylor series

$$\sum_{k=1}^{\infty}(d(k+1)-d(k))q^k, \quad \sum_{k=1}^{\infty}\sum_{j=1}^{k}d(j)q^k, \quad \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{d(j)}{k-j}q^k. \tag{1.6}$$

The algebraic and numerical computations have been carried out using the computer program MAPLE 13.

2. Lemmas

Throughout the paper, we maintain the notation introduced in this section. The following nine lemmas play an important role in the proof of Theorem 3.2 given in Section 3. We define for real numbers $q\in(0,1)$, $x>0$ and integers $n\geqslant 1$,

$$C_q(n)=\sum_{j=1}^{n}\sigma_q(j) \quad \text{and} \quad D_q(n)=\sum_{j=1}^{n}\varrho_q(j)$$

with

$$\sigma_q(j)=\int_j^{j+1}\varphi_q(x)\,\mathrm{d}x-\varphi_q(j+1), \quad \varrho_q(j)=\int_j^{j+1}\varphi_q(x)\,\mathrm{d}x-\frac{1}{2}(\varphi_q(j)+\varphi_q(j+1))$$

and

$$\varphi_q(x)=\frac{q^x}{(1-q^x)^2}(q^x-qx+x-1).$$

We note that elementary properties of the expression $q^x-qx+x-1$ lead to proofs of the classical arithmetic mean — geometric mean inequality and the inequalities of Hölder and Minkowski; see [4], Chapter 1, Section 14. In Section 3, we show that the derivative of F (defined in (1.5)) can be expressed in terms of φ_q. Geometrically, $C_q(n)$ is the error of approximating the integral $\int_1^{n+1}\varphi_q(x)\,\mathrm{d}x$ using a special implementation of the rectangular rule. Likewise, $D_q(n)$ is the error of approximating the same integral using the trapezoidal rule.

Lemma 2.1. *Let $x\geqslant 1$. Then, $q\mapsto\varphi_q(x)$ is increasing on $(0,1)$.*

Proof. We have for $q\in(0,1)$,

$$\begin{aligned}\frac{(1-q^x)^3}{x^2(x^2-1)q^{x-1}}\frac{\partial}{\partial q}\varphi_q(x)&=\frac{q^x}{x(x-1)}-\frac{q^{x+1}}{x(x+1)}+\frac{1-q}{x^2-1}-\frac{1+q}{x(x^2-1)}\\&=\int_q^1\int_t^1(1-s)s^{x-2}\,\mathrm{d}s\,\mathrm{d}t\geqslant 0.\end{aligned}$$

□

Lemma 2.2. *Let $q \in (0,1)$.*

(i) *There exists a number $N_q > 1$ such that φ_q is strictly concave on $[1, N_q]$ and strictly convex on $[N_q, \infty)$.*

(ii) *There exists a number $M_q \in (1, N_q)$ such that φ_q is strictly increasing on $[1, M_q]$ and strictly decreasing on $[M_q, \infty)$.*

P r o o f. (i) Let $x \geqslant 1$. Differentiation gives

$$\varphi_q''(x) = \frac{-q^x \log(q)}{(1-q^x)^4} a_q(x) \tag{2.1}$$

with

$$a_q(x) = (1-q)\Big(-x(1+4q^x+q^{2x})\log(q) - 2(1-q^{2x}) + \frac{(1-q^{2x})\log(q)}{1-q}\Big).$$

We have

$$a_q''(x) = 4q^x \log^2(q)(1-q)b_q(x)$$

with

$$b_q(x) = \frac{-\log(q)}{1-q}(x(1-q)(1+q^x)+q^x) + q^x - 2.$$

Using

$$\frac{-\log(q)}{1-q} \geqslant \frac{2}{1+q}$$

and

$$x(1-q)(1+q^x) - (1+q)(1-q^x) = x(x^2-1)\int_q^1 \int_t^1 (1-s)s^{x-2}\,\mathrm{d}s\,\mathrm{d}t \geqslant 0$$

gives

$$b_q(x) \geqslant \frac{2}{1+q}((1+q)(1-q^x)+q^x) + q^x - 2 = \frac{(1-q)q^x}{1+q} > 0.$$

It follows that a_q is strictly convex on $[1,\infty)$. Since

$$a_q(1) = -q(1-q)(3+q)\int_q^1 \frac{(1-t)(t^2+t+6)}{t^2(t+3)^2}\,\mathrm{d}t < 0 \quad \text{and} \quad \lim_{x\to\infty} a_q(x) = \infty,$$

we obtain that there exists a number $N_q > 1$ such that a_q is negative on $(1, N_q)$ and positive on (N_q, ∞). From (2.1) we conclude that φ_q is strictly concave on $[1, N_q]$ and strictly convex on $[N_q, \infty)$.

(ii) We have

$$\varphi_q'(x) = \frac{-q^x \log(q)}{(1-q^x)^2}\Big(\frac{-x(1-q)(1+q^x)}{1-q^x} + 1 - \frac{1-q}{\log(q)}\Big). \tag{2.2}$$

It follows that

$$\varphi_q'(1) = \frac{q}{(1-q)^2}(q\log(q) + 1 - q) > 0 \quad \text{and} \quad \lim_{x\to\infty} \varphi_q'(x) = 0.$$

Since φ_q' is strictly decreasing on $(1, N_q]$ and strictly increasing on $[N_q, \infty)$, there exists a number $M_q \in (1, N_q)$ such that $\varphi_q' > 0$ on $[1, M_q)$ and $\varphi_q' < 0$ on (M_q, ∞). This implies that φ_q is strictly increasing on $[1, M_q]$ and strictly decreasing on $[M_q, \infty)$. □

Lemma 2.3. *Let $q \in (0,1)$. If there exists an integer $m \geqslant 1$ such that $C_q(m) > 0$, then $C_q(n) \geqslant C_q(m)$ for $n \geqslant m$.*

P r o o f. We claim that $M_q \leqslant m+1$. Suppose that this were false; then $m + 1 < M_q$. An application of Lemma 2.2 (ii) gives for $j \in \{1,\dots,m\}$: $\sigma_q(j) < 0$. Thus, $C_q(m) < 0$, contradicting our assumption.

Let $r \geqslant 1$ be an integer. Then, $m+r \geqslant M_q$, so Lemma 2.2 (ii) yields $\sigma_q(m+r) > 0$. Since

$$C_q(m+r) - C_q(m+r-1) = \sigma_q(m+r),$$

we obtain $C_q(m) < C_q(m+1) < C_q(m+2) < \dots$ □

Computer plots of the graphs of $q \mapsto C_q(1)$ and $q \mapsto C_q(39)$ lead to the numbers 0.117 and 0.91 given in the next lemma.

Lemma 2.4.
(i) *If $q \in (0, 0.117]$, then $C_q(1) > 0$.*
(ii) *If $q \in (0.117, 0.91]$, then $C_q(39) > 0$.*

P r o o f. (i) Let $q \in (0, 0.117]$. We have

$$C_q(1) = \int_1^2 \varphi_q(x)\,\mathrm{d}x - \varphi_q(2) = \frac{U(q)}{(q+1)^2\log^2(q)} \tag{2.3}$$

with

$$U(q) = -(1-q^2 + q\log(q))q\log(q) - (q+1)^2(q-1-\log(q))\log(1+q).$$

Applying $\log(1+q) < q$ and $q - 1 - \log(q) > 0$ gives

$$U(q) > -(1-q^2+q\log(q))q\log(q) - q(q+1)^2(q-1-\log(q)) = qV(-\log(q)), \tag{2.4}$$

where

$$V(y) = 1 + (1-2y-y^2)\mathrm{e}^{-y} - (1+2y)\mathrm{e}^{-2y} - \mathrm{e}^{-3y}.$$

Since $0 < q \leqslant 0.117$, we get $y = -\log(q) \geqslant -\log(0.117) = 2.145\ldots$ Using

$$V'(y) = (y^2 - 3)\mathrm{e}^{-y} + 4y\mathrm{e}^{-2y} + 3\mathrm{e}^{-3y} > 0$$

yields

$$V(y) \geqslant V(-\log(0.117)) = 0.0022\ldots \tag{2.5}$$

From (2.3), (2.4) and (2.5) we obtain $C_q(1) > 0$.

(ii) Let $q \in (0.117, 0.91]$. Applying Lemma 2.1 gives that

$$C_q(39) = \int_1^{40} \varphi_q(x)\,\mathrm{d}x - \sum_{k=1}^{40} \varphi_q(k) = W_1(q) - W_2(q)$$

is the difference of two increasing functions. Let $0.117 \leqslant r \leqslant q \leqslant s \leqslant 0.91$. Then,

$$C_q(39) \geqslant W_1(r) - W_2(s) = W(r, s), \quad \text{say.}$$

We set

$$\begin{aligned} r_k &= 0.117 + \frac{k}{10^3}, & r'_k &= 0.835 + \frac{k}{2 \cdot 10^4}, & r''_k &= 0.9 + \frac{k}{5 \cdot 10^5}, \\ s_k &= 0.117 + \frac{k+1}{10^3}, & s'_k &= 0.835 + \frac{k+1}{2 \cdot 10^4}, & s''_k &= 0.9 + \frac{k+1}{5 \cdot 10^5}. \end{aligned}$$

Then,

$$[0.117, 0.91] = \bigcup_{k=0}^{717} [r_k, s_k] \cup \bigcup_{k=0}^{1299} [r'_k, s'_k] \cup \bigcup_{k=0}^{4999} [r''_k, s''_k].$$

It follows that there exists an integer m such that

$$\begin{aligned} &q \in [r_m, s_m] \text{ with } m \in \{0, 1, \ldots, 717\}, \quad \text{or} \\ &q \in [r'_m, s'_m] \text{ with } m \in \{0, 1, \ldots, 1299\}, \quad \text{or} \\ &q \in [r''_m, s''_m] \text{ with } m \in \{0, 1, \ldots, 4999\}. \end{aligned}$$

Then, we have

$$C_q(39) \geqslant W(r_m, s_m) \quad \text{or} \quad C_q(39) \geqslant W(r'_m, s'_m) \quad \text{or} \quad C_q(39) \geqslant W(r''_m, s''_m).$$

By direct computation, we find

$$\begin{aligned} W(r_k, s_k) &> 0, \quad k = 0, 1, \ldots, 717, \\ W(r'_k, s'_k) &> 0, \quad k = 0, 1, \ldots, 1299, \\ W(r''_k, s''_k) &> 0, \quad k = 0, 1, \ldots, 4999. \end{aligned}$$

This yields $C_q(39) > 0$. □

Lemma 2.5. *Let* $q \in [0.91, 1)$. *Then,* $N_q \geqslant 14$.

P r o o f. In view of Lemma 2.2 (i), it suffices to show that $\varphi_q''(14) < 0$. Using (2.1) we conclude that we have to prove that $a_q(14) < 0$, or, equivalently, $G(q) < 0$, where

$$G(q) = -\log(q)\Delta(q) + 2(1-q)(q^{28} - 1)$$

with

$$\Delta(q) = 13 - 14q + 56q^{14} - 56q^{15} + 15q^{28} - 14q^{29}.$$

Next, we apply Sturm's theorem to determine the number of distinct roots of a polynomial in an interval; see [18]. We obtain that Δ has precisely one zero on $[0.91, 1]$. Since $\Delta(0.91) = 1.76\ldots$ and $\Delta(1) = 0$, we conclude that Δ is positive on $[0.91, 1)$. Using this result and

$$-\log(q) \leqslant 1 - q + \frac{11}{20}(1-q)^2, \quad 0.91 \leqslant q < 1,$$

yields

$$\frac{G(q)}{1-q} \leqslant \Big(1 + \frac{11}{20}(1-q)\Big)\Delta(q) + 2(q^{28} - 1) = G_0(q), \quad \text{say}.$$

An application of Sturm's theorem gives that G_0 has precisely one zero on $[0.91, 1]$. We have $G_0(0.91) = -0.0028\ldots$ and $G_0(1) = 0$. It follows that G_0 and G are negative on $[0.91, 1)$. □

Lemma 2.6. *Let* $q \in (0, 1)$ *and let* $j \geqslant 1$ *be an integer.*

(i) *If* $N_q \in [j, j+1]$ *and* $\varphi_q'(x) \geqslant -\omega$ *for* $x \in [j, j+1]$ *with* $\omega \geqslant 0$, *then* $\varrho_q(j) \geqslant -\frac{1}{2}\omega$.

(ii) *If* φ_q *is convex on* $[j, j+1]$, *then*

$$\varrho_q(j) \geqslant -\frac{1}{8}(\varphi_q'(j+1) - \varphi_q'(j)). \tag{2.6}$$

P r o o f. (i) We consider two cases.

Case 1: $M_q \leqslant j$. We have

$$\frac{1}{2}\omega \geqslant \frac{1}{2}\int_j^{j+1} (-\varphi_q'(x))\,\mathrm{d}x = \frac{\varphi_q(j) - \varphi_q(j+1)}{2}. \tag{2.7}$$

Applying (2.7) and Lemma 2.2 (ii) gives

$$\frac{1}{2}\omega + \varrho_q(j) \geqslant \int_j^{j+1} \varphi_q(x)\,\mathrm{d}x - \varphi_q(j+1) \geqslant 0.$$

Case 2: $j < M_q$. We have $j < M_q < N_q \leqslant j+1$ and $\varphi_q(j) \leqslant \varphi_q(M_q)$, $\varphi_q(j+1) \leqslant \varphi_q(M_q)$. Let $\varepsilon = M_q - j > 0$ and $\delta = j+1-M_q > 0$. Then, $\varepsilon + \delta = 1$ and

$$\begin{aligned}\frac{\varphi_q(j)+\varphi_q(j+1)}{2} &= \varepsilon\frac{\varphi_q(j)+\varphi_q(j+1)}{2} + \delta\frac{\varphi_q(j)+\varphi_q(j+1)}{2}\\ &\leqslant \varepsilon\frac{\varphi_q(j)+\varphi_q(M)}{2} + \delta\frac{\varphi_q(M)+\varphi_q(j+1)}{2}.\end{aligned} \tag{2.8}$$

Since φ_q is concave on $[j, M_q]$, we conclude from the Hermite-Hadamard inequality that

$$\varepsilon\frac{\varphi_q(j)+\varphi_q(M_q)}{2} \leqslant \int_j^{M_q} \varphi_q(x)\,\mathrm{d}x. \tag{2.9}$$

Moreover, since φ_q is decreasing on $[M_q, j+1]$ and $\varphi_q' + \omega \geqslant 0$ on $[M_q, j+1]$, we obtain

$$\begin{aligned}-\frac{1}{2}\omega + \delta\frac{\varphi_q(M_q)+\varphi_q(j+1)}{2} &\leqslant -\frac{1}{2}\omega\delta^2 + \delta\frac{\varphi_q(M_q)+\varphi_q(j+1)}{2}\\ &= \frac{1}{2}\delta\Big(2\varphi_q(j+1) - \int_{M_q}^{j+1} (\varphi_q'(x)+\omega)\,\mathrm{d}x\Big)\\ &\leqslant \delta\varphi_q(j+1) \leqslant \int_{M_q}^{j+1} \varphi_q(x)\,\mathrm{d}x.\end{aligned} \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) gives $\varrho_q(j) \geqslant -\frac{1}{2}\omega$.

(ii) We have

$$\begin{aligned}&\frac{(b-a)^2}{8}(f'(b)-f'(a)) - (b-a)\frac{f(a)+f(b)}{2} + \int_a^b f(x)\,\mathrm{d}x\\ &\quad= \int_a^{(a+b)/2}\Big(\frac{a+b}{2} - x\Big)(f'(x)-f'(a))\,\mathrm{d}x\\ &\qquad+ \int_{(a+b)/2}^{b}\Big(x - \frac{a+b}{2}\Big)(f'(b)-f'(x))\,\mathrm{d}x.\end{aligned} \tag{2.11}$$

Applying (2.11) with $f = \varphi_q$, $a = j$, $b = j+1$ gives (2.6). □

Lemma 2.7. *Let $q \in [0.91, 1)$. The function*

$$\Theta_q(x) = \frac{-q^x\log(q)}{(1-q^x)^2}\Big(\frac{-x(1-q)(1+q^x)}{1-q^x} + q - \frac{1-q}{\log(q)}\Big) \tag{2.12}$$

is increasing on $(0, \infty)$.

P r o o f. Let $x>0$. We set $s=1-q^x$. Then, $s\in(0,1)$ and

$$\Theta_q(x)=\eta_q(s)$$

with

$$\eta_q(s)=\frac{1-s}{s^3}((1-q)(2-s)\log(1-s)-s(q\log(q)+q-1)).$$

Using $q\log(q)/(1-q)>-1$ gives

$$\begin{aligned}\frac{s^4}{s(2-s)(1-q)}\eta_q'(s)&=\frac{(s^2-6s+6)\log(1-s)}{s(s-2)}+\frac{q\log(q)}{1-q}-2\\&>\frac{(s^2-6s+6)\log(1-s)}{s(s-2)}-3\\&=\frac{s^2-6s+6}{s(2-s)}\int_0^s\frac{t^4}{(1-t)(t^2-6t+6)^2}\,\mathrm{d}t>0.\end{aligned}$$

Since

$$\Theta_q'(x)=-(1-s)\log(q)\eta_q'(s),$$

we conclude that $\Theta_q'(x)>0$. □

Lemma 2.8. *Let $q\in[0.91,1)$ and $x\geqslant 1$. Then, $\varphi_q'(x)\geqslant -0.035$.*

P r o o f. Applying Lemmas 2.2 (i), 2.5, 2.7 and (2.2), (2.12) leads to

$$\varphi_q'(x)\geqslant\varphi_q'(N_q)=\Theta_q(N_q)-\frac{(1-q)\log(q)q^{N_q}}{(1-q^{N_q})^2}>\Theta_q(N_q)\geqslant\Theta_q(14). \tag{2.13}$$

Using $(1-q)/\log(q)\leqslant -q$ yields

$$-\Theta_q(14)\leqslant\frac{-q^{14}\log(q)}{(1-q^{14})^2}\Big(\frac{14(1-q)(1+q^{14})}{1-q^{14}}-2q\Big)=h_1(q)(h_2(q)+h_3(q)) \tag{2.14}$$

with

$$h_1(q)=\frac{-q^{14}\log(q)}{1-q^{14}},\quad h_2(q)=\frac{1-q}{1-q^{14}},\quad h_3(q)=\frac{1}{1-q^{14}}\Big(\frac{14(1-q)(1+q^{14})}{1-q^{14}}-q-1\Big).$$

From the integral representations

$$\begin{aligned}h_1'(q)&=\frac{q^{13}}{(1-q^{14})^2}\int_{q^{14}}^1\frac{1-t}{t}\,\mathrm{d}t,\\h_2'(q)&=\frac{-182}{(1-q^{14})^2}\int_q^1(1-t)t^{12}\,\mathrm{d}t,\\(1-q^{14})^3h_3'(q)&=-15+574q^{13}-600q^{14}+210q^{27}-169q^{28}\\&=-38220\int_q^1 y^{12}\int_y^1\int_t^1(27-26s)s^{12}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}y\end{aligned}$$

we conclude that h_1 is increasing and that h_2 and h_3 are decreasing on $(0,1)$. The functions h_1 and h_2 are positive on $(0,1)$, and since $\lim_{q\to 1} h_3(q) = 0$, also h_3 is positive on $(0,1)$. Using $\lim_{q\to 1} h_1(q) = \frac{1}{14}$, we obtain for $q \in [0.91, 1)$,

$$h_1(q)(h_2(q) + h_3(q)) \leqslant \frac{1}{14}(h_2(0.91) + h_3(0.91)) = 0.034\ldots \tag{2.15}$$

From (2.13), (2.14) and (2.15) we find $\varphi_q'(x) \geqslant -0.035$ for $x \geqslant 1$. □

Lemma 2.9. *Let $q \in [0.91, 1)$. Then, $D_q(10) > 0.036$.*

Proof. We have

$$D_q(10) - 0.036 = \left(\int_1^{11} \varphi_q(x)\,\mathrm{d}x - 0.036\right) - \left(\sum_{k=1}^{10} \varphi_q(k) + \frac{1}{2}\varphi_q(11)\right) = J_1(q) - J_2(q).$$

Applying Lemma 2.1 gives that J_1 and J_2 are increasing on $[0.91, 1)$. If $0.91 \leqslant r \leqslant q \leqslant s \leqslant 1$, then

$$D_q(10) - 0.036 \geqslant J_1(r) - J_2(s) = J(r,s), \quad \text{say.}$$

We set

$$r_k = 0.91 + \frac{k}{10^4}, \quad s_k = 0.91 + \frac{k+1}{10^4}.$$

Then,

$$[0.91, 1] = \bigcup_{k=0}^{898} [r_k, s_k] \cup [0.9999, 1].$$

Let $q \in [0.91, 0.9999]$. Then there exists an integer $m \in \{0, 1, \ldots, 898\}$ such that $q \in [r_m, s_m]$. Since $J(r_k, s_k) > 0$ for $k = 0, 1, \ldots, 898$, we obtain $D_q(10) - 0.036 \geqslant J(r_m, s_m) > 0$.

Let $q \in [0.9999, 1)$. Using

$$J_2(1) = \lim_{q\to 1-} J_2(q) = \frac{208609}{55440}$$

leads to $D_q(10) - 0.036 \geqslant J(0.9999, 1) = 0.0013\ldots$ □

3. Monotonicity theorems

We prove monotonicity properties of the two functions defined in (1.5).

Theorem 3.1. *The function*

$$H(q) = T(q) - \frac{\log(1-q)}{\log(q)}$$

is positive and strictly increasing on $(0,1)$.

P r o o f. Let $0 < q < 1$. Using (1.1) gives

$$qH'(q) = \sum_{k=1}^{\infty} k \frac{q^k}{(1-q^k)^2} + \frac{q\log(q) + (1-q)\log(1-q)}{(1-q)\log^2(q)}. \tag{3.1}$$

Let

$$K_q(x) = \frac{xq^x}{(1-q^x)^2}, \quad x > 0. \tag{3.2}$$

Since

$$K_q'(x) = -\frac{q^x(1+q^x)}{(1-q^x)^3} \int_{q^x}^{1} \frac{y^2+1}{y(y+1)^2}\,\mathrm{d}y < 0,$$

we conclude that K_q is strictly decreasing on $(0,\infty)$, so we get

$$\begin{aligned} \sum_{k=1}^{\infty} K_q(k) > \int_1^{\infty} K_q(x)\,\mathrm{d}x &= \frac{\log(1-q^x)}{\log^2(q)} + \frac{xq^x}{(1-q^x)\log(q)}\Bigg|_{x=1}^{x=\infty} \\ &= -\frac{q\log(q) + (1-q)\log(1-q)}{(1-q)\log^2(q)}. \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3) we obtain $H'(q) > 0$. Thus, H is strictly increasing on $(0,1)$ with $H(q) > \lim_{p\to 0} H(p) = 0$ for $q \in (0,1)$. □

With the help of the results given in the previous section, we are able to prove the following result.

Theorem 3.2. *The function*

$$F(q) = \frac{1-q}{q} H(q) = \frac{1-q}{q}\Big(T(q) - \frac{\log(1-q)}{\log(q)}\Big)$$

is strictly decreasing on $(0,1)$.

Proof. Let $q \in (0,1)$. Then,

$$q^2 F'(q) = -T(q) + q(1-q)T'(q) - q^2\Big(\frac{(1-q)\log(1-q)}{q\log(q)}\Big)' = \sum_{k=1}^{\infty} \varphi_q(k) - A_q$$

with

$$A_q = \frac{-1}{\log^2(q)}((1-q+\log(q))\log(1-q) + q\log(q)).$$

Let

$$\Phi_q(x) = \frac{(q^{x+1} - (1+\log(q))q^x + 1 - q + \log(q))\log(1-q^x) + xq^x(1-q)\log(q)}{(1-q^x)\log^2(q)}.$$

Since

$$\Phi_q'(x) = \varphi_q(x), \quad \Phi_q(1) = -A_q \quad \text{and} \quad \lim_{x\to\infty} \Phi_q(x) = 0,$$

we obtain

$$A_q = \int_1^{\infty} \varphi_q(x)\,\mathrm{d}x.$$

It follows that $F'(q) < 0$ is equivalent to

$$\sum_{k=1}^{\infty} \varphi_q(k) < \int_1^{\infty} \varphi_q(x)\,\mathrm{d}x. \tag{3.4}$$

To prove (3.4) we consider two cases.

Case 1: $0 < q \leqslant 0.91$. From Lemmas 2.3 and 2.4 we obtain

$$C_q(n) \geqslant C_q(1) > 0 \quad \text{for } q \in (0, 0.117],\ n \geqslant 1$$

and

$$C_q(n) \geqslant C_q(39) > 0 \quad \text{for } q \in (0.117, 0.91],\ n \geqslant 39.$$

Thus, for $q \in (0, 0.91]$,

$$0 < \lim_{n\to\infty} C_q(n) = \int_1^{\infty} \varphi_q(x)\,\mathrm{d}x - \sum_{k=1}^{\infty} \varphi_q(k).$$

Case 2: $0.91 < q < 1$. Let $\widetilde{N}_q$ be an integer such that $\widetilde{N}_q < N_q \leqslant \widetilde{N}_q + 1$. From Lemma 2.5 we obtain $\widetilde{N}_q \geqslant 13$. An application of Lemma 2.2 (i) and the Hermite-Hadamard inequality gives $\varrho_q(j) \geqslant 0$ for $j = 11, \ldots, \widetilde{N}_q - 1$. This result and Lemma 2.9 yield

$$D_q(\widetilde{N}_q - 1) = D_q(10) + \sum_{j=11}^{\widetilde{N}_q - 1} \varrho_q(j) \geqslant D_q(10) > 0.036. \tag{3.5}$$

From Lemmas 2.6 (i) and 2.8 we obtain

$$\varrho_q(\widetilde{N}_q) \geqslant -\frac{1}{2} \cdot 0.035. \tag{3.6}$$

Next, we apply Lemma 2.6 (ii). Since φ_q is convex on $[N_q, \infty)$, we obtain for $j \geqslant \widetilde{N}_q + 1$,

$$\varrho_q(j) \geqslant -\frac{1}{8}\big(\varphi_q'(j+1) - \varphi_q'(j)\big).$$

Using this inequality and Lemma 2.8 leads to

$$\sum_{j=\widetilde{N}_q+1}^{\infty} \varrho_q(j) \geqslant -\frac{1}{8} \sum_{j=\widetilde{N}_q+1}^{\infty} (\varphi_q'(j+1) - \varphi_q'(j)) = \frac{1}{8}\varphi_q'(\widetilde{N}_q + 1) \geqslant -\frac{1}{8} \cdot 0.035. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) gives

$$\begin{aligned}\sum_{j=1}^{\infty} \varrho_q(j) &= D_q(\widetilde{N}_q - 1) + \varrho_q(\widetilde{N}_q) + \sum_{j=\widetilde{N}_q+1}^{\infty} \varrho_q(j) \\ &> 0.036 - \frac{1}{2} \cdot 0.035 - \frac{1}{8} \cdot 0.035 = 0.014\ldots\end{aligned} \tag{3.8}$$

We have

$$\sum_{k=1}^{m} \varrho_q(k) = \int_1^{m+1} \varphi_q(x)\,\mathrm{d}x - \sum_{k=1}^{m+1} \varphi_q(k) + \frac{1}{2}\varphi_q(m+1). \tag{3.9}$$

Since $\lim\limits_{x\to\infty} \varphi_q(x) = 0$, we conclude from (3.8) and (3.9) that (3.4) holds. □

An application of Theorems 3.1 and 3.2 leads to upper and lower bounds for the ratio $H(r)/H(s)$.

Corollary 3.3. *For all real numbers r and s with $0 < r < s < 1$ we have*

$$\frac{r(1-s)}{s(1-r)} < \frac{H(r)}{H(s)} < 1.$$

4. Inequalities

We show that the monotonicity property of the function F (defined in (1.5)) can be used to obtain sharp upper and lower bounds for $T(q)$ and the Taylor series given in (1.6). First, we present the best possible constant factors in the double inequality (1.4).

Theorem 4.1. *For all real numbers $q \in (0,1)$ we have*

$$\alpha \frac{q}{1-q} + \frac{\log(1-q)}{\log(q)} < T(q) < \beta \frac{q}{1-q} + \frac{\log(1-q)}{\log(q)} \tag{4.1}$$

with the best possible constant factors $\alpha = \gamma$ and $\beta = 1$.

P r o o f. The inequalities (4.1) are equivalent to

$$\alpha < F(q) < \beta, \quad 0 < q < 1. \tag{4.2}$$

Since

$$\lim_{q \to 0} \frac{(1-q)\log(1-q)}{q\log(q)} = 0,$$

we find

$$\lim_{q \to 0} F(q) = \lim_{q \to 0} \frac{1-q}{q} T(q) = d(1) = 1. \tag{4.3}$$

From (1.2) we obtain

$$F(q) = \frac{1-q}{q\log(q)} \psi_q(1).$$

We have

$$\lim_{q \to 1} \frac{1-q}{q\log(q)} = -1 \quad \text{and} \quad \lim_{q \to 1} \psi_q(1) = \psi(1) = -\gamma;$$

see [7]. It follows that

$$\lim_{q \to 1} F(q) = \gamma. \tag{4.4}$$

Using the limit relations (4.3), (4.4) and Theorem 3.2, we conclude that (4.2) holds with the best possible bounds $\alpha = \gamma$ and $\beta = 1$. □

Next, we offer inequalities for the Taylor series whose coefficients are $d(k+1) - d(k)$ ($k = 1, 2, \ldots$). We mention an interesting property of this difference which was discovered by Turán, see [11], page 39. For each $c > 0$ there exists a natural number k such that $d(k+1) - d(k) > c$.

Theorem 4.2. *For all real numbers $q \in (0,1)$ we have*

$$\alpha_0 + \frac{(1-q)\log(1-q)}{q\log(q)} < \sum_{k=1}^{\infty} (d(k+1) - d(k)) q^k < \beta_0 + \frac{(1-q)\log(1-q)}{q\log(q)} \tag{4.5}$$

with the best possible constants $\alpha_0 = \gamma - 1$ and $\beta_0 = 0$.

P r o o f. Let $q \in (0,1)$. We define

$$F_0(q) = \sum_{k=1}^{\infty}(d(k+1) - d(k))q^k - \frac{(1-q)\log(1-q)}{q\log(q)}.$$

Since

$$1 + \sum_{k=1}^{\infty}(d(k+1) - d(k))q^k = \sum_{k=0}^{\infty} d(k+1)q^k - \sum_{k=1}^{\infty} d(k)q^k = \Big(\frac{1}{q} - 1\Big)T(q),$$

we get

$$F_0(q) = F(q) - 1.$$

Applying Theorem 3.2 and the limit relations

$$\lim_{q\to 0} F_0(q) = 0 \quad \text{and} \quad \lim_{q\to 1} F_0(q) = \gamma - 1,$$

we obtain (4.5) with the best possible constants $\alpha_0 = \gamma - 1$ and $\beta_0 = 0$. □

The coefficients of the series given in the following theorem are the partial sums of the divisor function which are related to the floor function. We have

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} [n/k],$$

where $[x]$ denotes the greatest integer less than or equal to x. These sums have a nice geometric interpretation. They give the exact number of lattice points in the area $x > 0$, $y > 0$, $xy \leqslant n$; see [12], page 131. The study of the average order of $d(k)$ dates back to Dirichlet and was continued by Hardy, Landau and others; see [1], Section 3.5.

Theorem 4.3. *For all real numbers $q \in (0,1)$ we have*

$$\lambda\frac{q}{(1-q)^2} + \frac{\log(1-q)}{(1-q)\log(q)} < \sum_{k=1}^{\infty}\sum_{j=1}^{k} d(j)q^k < \mu\frac{q}{(1-q)^2} + \frac{\log(1-q)}{(1-q)\log(q)} \tag{4.6}$$

with the best possible constant factors $\lambda = \gamma$ and $\mu = 1$.

P r o o f. Let $q \in (0,1)$ and

$$L(q) = \frac{(1-q)^2}{q}\Big(\sum_{k=1}^{\infty}\sum_{j=1}^{k} d(j)q^k - \frac{\log(1-q)}{(1-q)\log(q)}\Big).$$

Since

$$\frac{1}{1-q}T(q) = \sum_{k=1}^{\infty}\sum_{j=1}^{k} d(j)q^k,$$

we find $L(q) = F(q)$. Applying Theorem 3.2, (4.3) and (4.4) leads to (4.6) with the best possible constant factors $\lambda = \gamma$ and $\mu = 1$. $\square$

We conclude the paper with a companion to (4.6).

Theorem 4.4. *For all real numbers $q \in (0,1)$ we have*

$$\lambda_0 \frac{q\log(1-q)}{1-q} - \frac{\log^2(1-q)}{\log(q)} < \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} \frac{d(j)}{k-j}q^k < \mu_0 \frac{q\log(1-q)}{1-q} - \frac{\log^2(1-q)}{\log(q)} \tag{4.7}$$

with the best possible constant factors $\lambda_0 = -\gamma$ and $\mu_0 = -1$.

P r o o f. Let $q \in (0,1)$. Using

$$-\log(1-q)T(q) = \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} \frac{d(j)}{k-j}q^k$$

gives for

$$L_0(q) = \frac{1-q}{q\log(1-q)}\left(\sum_{k=2}^{\infty}\sum_{j=1}^{k-1} \frac{d(j)}{k-j}q^k + \frac{\log^2(1-q)}{\log(q)}\right)$$

the representation $L_0(q) = -F(q)$. From Theorem 3.2 and (4.3), (4.4) we conclude that (4.7) is valid with the smallest constant $\lambda_0 = -\gamma$ and the largest constant $\mu_0 = -1$. $\square$

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