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ON (n,m)-A-NORMAL AND (n,m)-A-QUASINORMAL SEMI-HILBERTIAN SPACE OPERATORS

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Abstract. The purpose of the paper is to introduce and study a new class of operators on semi-Hilbertian spaces, i.e. spaces generated by positive semi-definite sesquilinear forms. Let $\mathcal{H}$ be a Hilbert space and let A be a positive bounded operator on $\mathcal{H}$. The semi-inner product $\langle h \mid k\rangle_A := \langle Ah \mid k\rangle$, $h,k \in \mathcal{H}$, induces a semi-norm $\|\cdot\|_A$. This makes $\mathcal{H}$ into a semi-Hilbertian space. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (n,m)-A-normal if $[T^n, (T^{\sharp_A})^m] := T^n(T^{\sharp_A})^m - (T^{\sharp_A})^m T^n = 0$ for some positive integers n and m.

Keywords: semi-Hilbertian space; A-normal operator; (n,m)-normal operator; (n,m)-quasinormal operator

MSC 2020: 54E40, 47B99

1. Introduction and preliminaries

Throughout this paper, let $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ be a complex Hilbert space equipped with the norm $\|\cdot\|$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and let $\mathcal{B}(\mathcal{H})^+$ be the *cone of positive operators* of $\mathcal{B}(\mathcal{H})$ defined as

$$\mathcal{B}(\mathcal{H})^+ := \{A \in \mathcal{B}(\mathcal{H})\colon \langle Ah \mid h\rangle \geqslant 0 \ \forall\, h \in \mathcal{H}\}.$$

For every $T \in \mathcal{B}(\mathcal{H})$ its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$, and its adjoint by T^*. If $\mathcal{M}$ is a linear subspace of $\mathcal{H}$, then $\overline{\mathcal{M}}$ stands for its closure in the norm topology of $\mathcal{H}$. We denote the orthogonal projection onto a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ by $P_{\mathcal{M}}$. The positive operator $A \in \mathcal{B}(\mathcal{H})$ defines a positive semi-definite sesquilinear form $\langle\cdot|\cdot\rangle_A\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ given by $\langle h \mid k\rangle_A = \langle Ah \mid k\rangle$. Note that $\langle\cdot|\cdot\rangle_A$ defines a semi-inner product on $\mathcal{H}$, and the semi-norm induced by it is

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given by $\|h\|_A = \sqrt{\langle h \mid h\rangle_A}$ for every $h \in \mathcal{H}$. Observe that $\|h\|_A = 0$ if and only if $h \in \mathcal{N}(A)$. Then $\|\cdot\|_A$ is a norm if and only if A is injective, and the semi-normed space $(\mathcal{H}, \|\cdot\|_A)$ is a complete space if and only if $\mathcal{R}(A)$ is closed.

The above semi-norm induces a semi-norm on the subspace $\mathcal{B}^A(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ consisting of all $T \in \mathcal{B}(\mathcal{H})$ so that for some $c > 0$ and for all $h \in \mathcal{H}$, $\|Th\|_A \leqslant c\|h\|_A$. Indeed, if $T \in \mathcal{B}^A(\mathcal{H})$, then

$$\|T\|_A := \sup\Big\{\frac{\|Th\|_A}{\|h\|_A}, h \notin \mathcal{N}(A)\Big\}.$$

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-*adjoint operator* of T if for every $h, k \in \mathcal{H}$ we have $\langle Th \mid k\rangle_A = \langle h \mid Sk\rangle_A$, that is, $AS = T^*A$. If T is an A-adjoint of itself, then T is called an A-*selfadjoint operator*.

Generally, the existence of an A-adjoint operator is not guaranteed. The set of all operators that admit A-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. An application of the Douglas theorem (see [13]) shows that

$$\begin{aligned}\mathcal{B}_A(\mathcal{H}) &= \{T \in \mathcal{B}(\mathcal{H})\colon\ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}\\ &= \{T \in \mathcal{B}(\mathcal{H})\colon\ \exists\, c > 0\colon\ \|ATx\| \leqslant c\|Ax\|\ \forall\, x \in \mathcal{H}\}.\end{aligned}$$

Note that $\mathcal{B}_A(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Moreover, the inclusions $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}^A(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ hold with equality if A is one-to-one and has a closed range. If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished A-adjoint operator of T, which is denoted by $T^{\sharp_A}$. Note that $T^{\sharp_A} = A^\dagger T^*A$ in which $A^\dagger$ is the Moore-Penrose inverse of A. It was observed that the A-adjoint operator $T^{\sharp_A}$ satisfies

$$AT^{\sharp_A} = T^*A, \quad \mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$$

and

$$\mathcal{N}(T^{\sharp_A}) = \mathcal{N}(T^*A).$$

For $T, S \in \mathcal{B}_A(\mathcal{H})$, it is easy to see that $\|TS\|_A \leqslant \|T\|_A\|S\|_A$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$.

Notice that if $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}TP_{\overline{\mathcal{R}(A)}}$ and $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$. (For more detail on the concepts cited above see [5], [4], [6].)

In [17] it was observed that if $T \in \mathcal{B}_A(\mathcal{H})$ is such that $TA = AT$, then $T^{\sharp_A} = PT^*$. For an arbitrary operator $T \in \mathcal{B}_A(\mathcal{H})$, we can write

$$\mathrm{Re}_A(T) := \frac{1}{2}(T + T^{\sharp_A}) \quad \text{and} \quad \mathrm{Im}_A(T) := \frac{1}{2\mathrm{i}}(T - T^{\sharp_A}).$$

The concept of n-normal operators as a generalization of normal operators on Hilbert spaces has been introduced and studied by Jibril (see [15]) and Alzuraiqi et al. (see [3]). The class of n-power normal operators is denoted by $[n\mathbf{N}]$. An operator T is called *n-power normal* if $[T^n, T^*] = 0$ (equivalently $T^nT^* = T^*T^n$). Very recently, several papers have appeared on n-normal operators. We refer the interested reader to [12], [11], [16] for the complete details.

In [1] and [2], the authors introduced and studied the classes of (n,m)-normal powers and (n,m)-power quasinormal operators as follows: An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *(n,m)-power normal* if $T^n(T^m)^* = (T^m)^*T^n$ and it is said to be *(n,m)-power quasinormal* if $T^n(T^*)^mT = (T^*)^mTT^n$ where n, m are two nonnegative integers. We refer the interested reader to [11] for the complete details on (n,m)-power normal operators.

The classes of normal, (α,β)-normal, and n-power quasinormal operators, isometries, partial isometries, unitary operators etc. on Hilbert spaces have been generalized to semi-Hilbertian spaces by many authors in many papers. (See, for more details, [5]–[7], [9], [10], [14], [17], [18], [21].)

An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be

(1) A-normal if $T^{\sharp_A}T = TT^{\sharp_A}$ (see [17]),
(2) (α,β)-A-normal if $\beta^2T^{\sharp_A}T \geqslant_A TT^{\sharp_A} \geqslant_A \alpha^2T^{\sharp_A}T$ for $0 \leqslant \alpha \leqslant 1 \leqslant \beta$ (see [9]),
(3) (A,n)-power-quasinormal if $T^n(T^{\sharp_A}T) = (TT^{\sharp_A})T^n$ (see [14]),
(4) an A-isometry if $T^{\sharp_A}T = P_{\overline{\mathcal{R}(A)}}$ (see [5]),
(5) A-unitary if $T^{\sharp_A}T = (T^{\sharp_A})^{\sharp_A}T^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}$, i.e. T and $T^{\sharp_A}$ are A-isometries (see [5]).

From now on, A denotes a positive operator on $\mathcal{H}$, i.e. $A \in \mathcal{B}(\mathcal{H})^+$.

This paper is devoted to the study of some new classes of operators on semi-Hilbertian spaces called (n,m)-A-normal operators and (n,m)-A-quasinormal operators. Some properties of these classes are investigated.

2. (n,m)-A-normal operators

In this section, the class of (n,m)-A-normal operators as a generalization of the classes of A-normal operators is introduced. In addition, we study several properties of members of this class of operators.

Definition 2.1. Let $T \in \mathcal{B}_A(\mathcal{H})$. We say that T is *(n,m)-A-normal* if

$$[T^n, (T^{\sharp_A})^m] := T^n(T^{\sharp_A})^m - (T^{\sharp_A})^mT^n = 0 \tag{2.1}$$

for some positive integers n and m. The set of all operators which are (n,m)-A-normal is denoted by $[(n,m)\mathbf{N}]_A$.

R e m a r k 2.1. We make the following observations:

(1) Every A-normal operator is an (n,m)-A-normal for all $n,m \in \mathbb{N}$.
(2) If $n=m=1$, every $(1,1)$-A-normal operator is an A-normal operator.
(3) If $T \in [(1,m)\mathbf{N}]_A$ then $T \in [(n,m)\mathbf{N}]_A$ and if $T \in [(n,1)\mathbf{N}]_A$ then $T \in [(n,m)\mathbf{N}]_A$.
(4) If $T \in [(n,m)\mathbf{N}]_A$ then $T \in [(2n,m)\mathbf{N}]_A \cap [(n,2m)\mathbf{N}]_A \cap [(2n,2m)\mathbf{N}]_A$.

R e m a r k 2.2. In the following example we present an operator that is (n,m)-A-normal for some positive integers n and m but is not an A-normal operator.

E x a m p l e 2.1. Let $T = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be operators acting on two-dimensional Hilbert space $\mathbb{C}^2$. A simple calculation shows that $T^{\sharp_A} = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix}$. Moreover, $T^{\sharp_A}T \neq TT^{\sharp_A}$ and $T^{\sharp_A}T^2 = T^2T^{\sharp_A}$. Therefore T is a $(2,1)$-A-normal but not an A-normal operator.

In [17], Theorem 2.1 it was observed that if $T \in \mathcal{B}_A(\mathcal{H})$ then T is A-normal if and only if

$$\|Th\|_A = \|T^{\sharp_A}h\|_A \ \forall\, h \in \mathcal{H} \quad \text{and} \quad \mathcal{R}(TT^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}.$$

In the following theorem, we generalize this characterization to (n,m)-A-normal operators.

Theorem 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then T is an (n,m)-A-normal operator for some positive integers n and m if and only if T satisfies the conditions:*

$$\langle (T^{\sharp_A})^m h \mid (T^{\sharp_A})^n h\rangle_A = \langle (T^n h \mid T^m h\rangle_A \quad \forall\, h \in \mathcal{H}, \tag{1}$$

$$\mathcal{R}(T^n(T^{\sharp_A})^m) \subseteq \overline{\mathcal{R}(A)}. \tag{2}$$

P r o o f. Assume that T is an (n,m)-A-normal operator and we need to proof that T satisfies the conditions (1) and (2). In fact, we have

$$\begin{aligned} \langle [T^n,(T^{\sharp_A})^m]h \mid h\rangle_A = 0 &\Rightarrow \langle T^n(T^{\sharp_A})^m h \mid h\rangle_A - \langle (T^{\sharp_A})^m T^n h \mid h\rangle_A = 0 \\ &\Rightarrow \langle (T^{\sharp_A})^m h \mid T^{*n}Ah\rangle - \langle A(T^{\sharp_A})^m T^n h \mid h\rangle = 0 \\ &\Rightarrow \langle (T^{\sharp_A})^m h \mid (T^{\sharp_A})^n h\rangle_A - \langle T^n h \mid T^m h\rangle_A = 0 \\ &\Rightarrow \langle (T^{\sharp_A})^m h \mid (T^{\sharp_A})^n h\rangle_A = \langle T^n h \mid T^m h\rangle_A. \end{aligned}$$

Moreover, the condition $[T^n,(T^{\sharp_A})^m] = 0$ implies that $T^n(T^{\sharp_A})^m = (T^{\sharp_A})^m T^n$. Therefore

$$\mathcal{R}(T^n(T^{\sharp_A})^m) = \mathcal{R}((T^{\sharp_A})^m T^n) \subseteq \mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}.$$

Conversely, assume that T satisfies the conditions (1) and (2) and we prove that T is an (n,m)-A-normal operator. From the condition (1), a simple computation shows that

$$\begin{aligned}\langle (T^{\sharp_A})^m h \mid (T^{\sharp_A})^n h\rangle_A - \langle T^n h \mid T^m h\rangle_A = 0 \\ \Rightarrow \langle T^n (T^{\sharp_A})^m h \mid h\rangle_A - \langle (T^{\sharp_A})^m T^n h \mid h\rangle_A = 0 \\ \Rightarrow \langle [T^n, (T^{\sharp_A})^m] h \mid h\rangle_A = 0,\end{aligned}$$

which implies that $\mathcal{R}([T^n, (T^{\sharp_A})^m]) \subseteq \mathcal{N}(A)$.

On the other hand, since the condition (2) holds, it follows that

$$\mathcal{R}([T^n, (T^{\sharp_A})^m]) \subseteq \overline{\mathcal{R}(A)} = \mathcal{N}(A)^{\perp}.$$

We deduce that $[T^n, (T^{\sharp_A})^m] = 0$ which means that the operator T is (n,m)-A-normal. □

R e m a r k 2.3. If $n = m = 1$, then Theorem 2.1 coincides with Theorem 2.1 of [17].

The following proposition discusses the relation between (n,m)-A-normal operators and (m,n)-A-normal operators.

Proposition 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $\mathcal{N}(A)^{\perp}$ is an invariant subspace of T. Then the following statements are equivalent.*
(1) *T is an (n,m)-A-normal operator.*
(2) *T is an (m,n)-A-normal operator.*

P r o o f. (1) $\Rightarrow$ (2) Assume that T is an (n,m)-A-normal operator. It follows that

$$T^n (T^{\sharp_A})^m - (T^{\sharp_A})^m T^n = 0.$$

Then

$$\begin{aligned}T^n (T^{\sharp_A})^m - (T^{\sharp_A})^m T^n = 0 \\ \Rightarrow [(T^{\sharp_A})^{\sharp_A}]^m (T^n)^{\sharp_A} - (T^n)^{\sharp_A} [(T^{\sharp_A})^{\sharp_A}]^m = 0 \\ \Rightarrow (P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}})^m (T^n)^{\sharp_A} - (T^n)^{\sharp_A} (P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}})^m = 0 \\ \Rightarrow P_{\overline{\mathcal{R}(A)}} (T^m (T^n)^{\sharp_A} - (T^n)^{\sharp_A} T^m) = 0.\end{aligned}$$

This means that $(T^m (T^n)^{\sharp_A} - (T^n)^{\sharp_A} T^m) h \in \mathcal{N}(A)$ for all $h \in \mathcal{H}$.

On the other hand, this fact and $\mathcal{R}(T^{\sharp_A n}) \subset \mathcal{R}(T^{\sharp_A}) \subset \overline{\mathcal{R}(A)}$ and the assumption that $\mathcal{N}(A)^{\perp}$ is an invariant subspace for T imply that $(T^m (T^n)^{\sharp_A} - (T^n)^{\sharp_A} T^m) h \in \overline{\mathcal{R}(A)}$ for all $h \in \mathcal{H}$. Consequently, $(T^m (T^n)^{\sharp_A} - (T^n)^{\sharp_A} T^m) h = 0$ for all $h \in \mathcal{H}$. Therefore $[T^m, (T^{\sharp_A})^n] = 0$. Hence $T^{\sharp_A}$ is an (m,n)-A-normal operator.

(2) $\Rightarrow$ (1) By the same way hence we omit it. □

It is well known that if $T \in \mathcal{B}_A(\mathcal{H})$ is A-normal, then T^n is A-normal. In the following theorem, we extend this result to an (n,m)-A-normal operator as follows.

Theorem 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$. If T is an (n,m)-A-normal operator then the following statements hold:*

(i) *T^j is A-normal where j is the least common multiple of n and m, i.e. $j = LCM(n,m)$,*

(ii) *T^{nm} is an A-normal operator.*

Proof. (i) Assume that T is (n,m)-A-normal that is $T^n(T^{\sharp_A})^m = (T^{\sharp_A})^m T^n$. Let $j = pn$ and $j = qm$. By computation we get

$$\begin{aligned} T^j(T^j)^{\sharp_A} &= T^{pn}((T^{\sharp_A})^{qm} = (T^n)^p((T^{\sharp_A})^m)^q \\ &= \underbrace{T^n \ldots T^n}_{p\text{-times}} \underbrace{(T^{\sharp_A})^m \ldots (T^{\sharp_A})^m}_{q\text{-times}} \\ &= \underbrace{(T^{\sharp_A})^m \ldots (T^{\sharp_A})^m}_{q\text{-times}} \underbrace{T^n \ldots T^n}_{p\text{-times}} \\ &= (T^{\sharp_A})^{qm} T^{np} = (T^{qm})^{\sharp_A} T^{np} = (T^j)^{\sharp_A} T^j, \end{aligned}$$

which means that T^j is A-normal.

(ii) This statement is proved in the same way as the statement (i). □

Proposition 2.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$, $X = T^n + (T^{\sharp_A})^m$, $Y = T^n - (T^{\sharp_A})^m$ and $Z = T^n(T^{\sharp_A})^m$. The following statements hold:*

(1) *T is (n,m)-A-normal if and only if $[X,Y] = 0$.*

(2) *If $T \in [(n,m)\mathbf{N}]_A$, then $[Z,X] = [Z,Y] = 0$.*

(3) *$T \in [(n,m)\mathbf{N}]_A$ if and only if $[T^n, X] = 0$.*

(4) *$T \in [(n,m)\mathbf{N}]_A$ if and only if $[T^n, Y] = 0$.*

Proof. (1)

$$\begin{aligned} [X,Y] = XY - YX = 0 &\Leftrightarrow ((T^n + (T^{\sharp_A})^m)(T^n - (T^{\sharp_A})^m)) \\ &\qquad - ((T^n - (T^{\sharp_A})^m)(T^n + (T^{\sharp_A})^m)) = 0 \\ &\Leftrightarrow T^{2n} - T^n(T^{\sharp_A})^m + (T^{\sharp_A})^m T^n - (T^{\sharp_A})^{2m} \\ &\qquad - T^{2n} - T^n(T^{\sharp_A})^m + (T^{\sharp_A})^m T^n - (T^{\sharp})^{2m} = 0 \\ &\Leftrightarrow T^n(T^{\sharp_A})^m - (T^{\sharp})^m T^n = 0 \\ &\Leftrightarrow [T^n, (T^{\sharp_A})^m] = 0. \end{aligned}$$

Hence $[X,Y] = 0$ if and only if T is (n,m)-A-normal.

Proofs of the statements (2), (3) and (4) are straightforward. □

Proposition 2.3. *Let $T, V \in \mathcal{B}_A(\mathcal{H})$ be such that $\mathcal{N}(A)^\perp$ is an invariant subspace for both T and V. If T is an (n,m)-A-normal operator and V is an A-isometry, then $VTV^{\sharp_A}$ is an (n,m)-A-normal operator.*

Proof. Since V is an A-isometry then $V^{\sharp_A}V = P_{\overline{\mathcal{R}(A)}}$. Moreover from the fact that $\mathcal{N}(A)^\perp$ is an invariant subspace for T we have $P_{\overline{\mathcal{R}(A)}}T = TP_{\overline{\mathcal{R}(A)}}$ which implies that $T^{\sharp_A}P_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}T^{\sharp_A}$ since $P^{\sharp_A}_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}$. In a similar way we have

$$VP_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}V \quad \text{and} \quad V^{\sharp_A}P_{\overline{\mathcal{R}(A)}} = P_{\overline{\mathcal{R}(A)}}V^{\sharp_A}.$$

It is easy to check that

$$\begin{aligned}
(VTV^{\sharp_A})^j &= \underbrace{(VTV^{\sharp_A})(VTV^{\sharp_A})\ldots(VTV^{\sharp_A})}_{j\text{-times}} \\
&= (VTP_{\overline{\mathcal{R}(A)}}TV^{\sharp_A})\ldots(VTV^{\sharp_A}) \\
&= P_{\overline{\mathcal{R}(A)}}VT^2V^{\sharp_A}\ldots(VTV^{\sharp_A}) \\
&\ \ \vdots \\
&= P_{\overline{\mathcal{R}(A)}}VT^jV^{\sharp_A}.
\end{aligned}$$

The same arguments yield

$$\begin{aligned}
(VTV^{\sharp_A})^{\sharp_A j} &= \underbrace{(VTV^{\sharp_A})^{\sharp_A}(VTV^{\sharp_A})^{\sharp_A}\ldots(VTV^{\sharp_A})^{\sharp_A}}_{j\text{-times}} \\
&= (P_{\overline{\mathcal{R}(A)}}VP_{\overline{\mathcal{R}(A)}}T^{\sharp_A}V^{\sharp_A})\ldots(P_{\overline{\mathcal{R}(A)}}VP_{\overline{\mathcal{R}(A)}}T^{\sharp_A}V^{\sharp_A}) \\
&\ \ \vdots \\
&= P_{\overline{\mathcal{R}(A)}}V(T^{\sharp_A})^jV^{\sharp_A}.
\end{aligned}$$

From the above calculation, we deduce that

$$\begin{aligned}
&\langle\{(VTV^{\sharp_A})^{\sharp_A}\}^m h \mid \{(VTV^{\sharp_A})^{\sharp_A}\}^n h\rangle_A \\
&\quad = \langle P_{\overline{\mathcal{R}(A)}}V(T^{\sharp_A})^mV^{\sharp_A}h \mid P_{\overline{\mathcal{R}(A)}}V(T^{\sharp_A})^nV^{\sharp_A}h\rangle_A \\
&\quad = \langle (T^{\sharp_A})^mV^{\sharp_A}h \mid (T^{\sharp_A})^nV^{\sharp_A}h\rangle_A.
\end{aligned} \tag{2.2}$$

It is also easy to show that

$$\begin{aligned}
\langle (VTV^{\sharp_A})^n h \mid (VTV^{\sharp_A})^m h\rangle_A &= \langle P_{\overline{\mathcal{R}(A)}}VT^nV^{\sharp_A}h \mid P_{\overline{\mathcal{R}(A)}}VT^mV^{\sharp_A}h\rangle_A \\
&= \langle T^nV^{\sharp_A}h \mid T^mV^{\sharp_A}h\rangle_A.
\end{aligned} \tag{2.3}$$

Since T is (n, m)-A-normal, by combining (2.2) and (2.3) we have

$$\langle \{(VTV^{\sharp_A})^{\sharp_A}\}^m h \mid \{(VTV^{\sharp_A})^{\sharp_A}\}^n h\rangle_A = \langle (VTV^{\sharp_A})^n h \mid (VTV^{\sharp_A})^m h\rangle_A \quad \forall\, h \in \mathcal{H}.$$

On the other hand, we have

$$\begin{aligned}\mathcal{R}((VTV^{\sharp_A})^n\{(VTV^{\sharp_A})^{\sharp_A}\}^m) &= \mathcal{R}(P_{\overline{\mathcal{R}(A)}}VT^nV^{\sharp_A}P_{\overline{\mathcal{R}(A)}}V(T^{\sharp_A})^mV^{\sharp_A})\\ &= \mathcal{R}(P_{\overline{\mathcal{R}(A)}}VT^n(T^{\sharp_A})^mV^{\sharp_A})\\ &\subseteq \mathcal{R}(P_{\overline{\mathcal{R}(A)}}) \subseteq \overline{\mathcal{R}(A)}.\end{aligned}$$

In view of Theorem 2.1, it follows that $VTV^{\sharp_A}$ is (n, m)-A-normal operator. □

Proposition 2.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_A(\mathcal{H})$ be such that $TS = ST$ and $ST^{\sharp_A} = T^{\sharp_A}S$. If T is (n, n)-A-normal, the following statements hold:*

(1) *If S is an A-self adjoint, then TS is an (n, n)-A-normal operator.*

(2) *If S is an A-normal operator, then TS is an (n, n)-A-normal operator.*

Proof. (1) Let $h \in \mathcal{H}$, under the assumption that S is A-self-adjoint ($AS = S^*A$) and the statement (1) of Theorem 2.1 we have

$$\begin{aligned}\langle (TS)^{\sharp_A n}h \mid (TS)^{\sharp_A n}h\rangle_A &= \langle (S)^{\sharp_A n}(T)^{\sharp_A n}h \mid (S)^{\sharp_A n}(T)^{\sharp_A n}h\rangle_A\\ &= \langle A(S)^{\sharp_A n}(T)^{\sharp_A n}h \mid (S)^{\sharp_A n}(T)^{\sharp_A n}h\rangle\\ &= \langle (S^*)^n A(T)^{\sharp_A n}h \mid (S)^{\sharp_A n}(T)^{\sharp_A n}h\rangle\\ &= \langle A(S)^n(T)^{\sharp_A n}h \mid (S)^{\sharp_A n}(T)^{\sharp_A n}h\rangle\\ &= \langle A(T)^{\sharp_A n}S^n h \mid (S)^{\sharp_A n}(T)^{\sharp_A n}h\rangle\\ &= \langle (T)^{\sharp_A n}S^n h \mid A(S)^{\sharp_A n}(T)^{\sharp_A n}h\rangle\\ &= \langle (T)^{\sharp_A n}S^n h \mid (T)^{\sharp_A n}S^n h\rangle_A\\ &= \langle T^nS^nh \mid T^nS^nh\rangle_A\\ &= \langle (TS)^n h \mid (TS)^n h\rangle_A.\end{aligned}$$

On the other hand, we have

$$\mathcal{R}((TS)^n(TS)^{\sharp_A n}) = \mathcal{R}(T^nT^{\sharp_A n}S^nS^{\sharp_A n}) \subseteq \overline{\mathcal{R}(A)}.$$

This means that TS is an (n, n)-A-normal operator by Theorem 2.1.

(2) Let S be an A-normal operator then $SS^{\sharp_A} = S^{\sharp_A}S$ and because T is an (n,n)-A-normal operator we get the relations

$$\begin{aligned}
\langle (ST)^{\sharp_A n}h \mid (ST)^{\sharp_A n}h\rangle_A &= \langle S^{\sharp_A n}T^{\sharp_A n}h \mid S^{\sharp_A n}T^{\sharp_A n}h\rangle_A \\
&= \langle AS^{\sharp_A n}T^{\sharp_A n}h \mid S^{\sharp_A n}T^{\sharp_A n}h\rangle \\
&= \langle S^{*n}AT^{\sharp_A n}h \mid S^{\sharp_A n}T^{\sharp_A n}h\rangle \\
&= \langle T^{\sharp_A n}h \mid S^{n}S^{\sharp_A n}T^{\sharp_A n}h\rangle_A \\
&= \langle T^{\sharp_A n}h \mid (S^{\sharp_A})^{n}S^{n}T^{\sharp_A n}h\rangle_A \\
&= \langle S^{n}T^{\sharp_A n}h \mid S^{n}T^{\sharp_A n}h\rangle_A \\
&= \langle T^{\sharp_A n}S^{n}h \mid T^{\sharp_A n}S^{n}h\rangle_A \\
&= \langle T^{n}S^{n}h \mid T^{n}S^{n}h\rangle_A \quad \text{(since } T \text{ is } (n,n)\text{-}A\text{-normal)} \\
&= \langle (TS)^{n}h \mid (TS)^{n}h\rangle_A.
\end{aligned}$$

On the other hand, based on the (n,n)-A-normality of T we get the inclusion

$$\mathcal{R}((TS)^n(TS)^{\sharp_A n}) = \mathcal{R}(T^nS^nT^{\sharp_A n}S^{\sharp_A n}) \subseteq \mathcal{R}(T^nT^{\sharp_A n}) \subseteq \overline{\mathcal{R}(A)}.$$

From the items (1) and (2) of Theorem 2.1, the operator TS is an (n,n)-A-normal operator. □

In the following proposition, we study the relation between the classes $[(2,m)\mathbf{N}]_A$ and $[(3,m)\mathbf{N}]_A$.

Proposition 2.5. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $T \in [(2,m)\mathbf{N}]_A \cap [(3,m)\mathbf{N}]_A$ for some positive integer m, then $T \in [(n,m)\mathbf{N}]_A$ for all positive integers $n \geqslant 4$.*

P r o o f. It is obvious from Definition 2.1 that if $T \in [(2,m)\mathbf{N}]_A$ then $T \in [(4,m)\mathbf{N}]_A$. However, $T \in [(2,m)\mathbf{N}]_A \cap [(3,m)\mathbf{N}]_A$ implies that $T \in [(5,m)\mathbf{N}]_A$.

Assume that $T \in [(n,m)\mathbf{N}]_A$ for $n \geqslant 5$, that is,

$$T^n(T^{\sharp_A})^m = (T^{\sharp_A})^mT^n.$$

Then we have

$$\begin{aligned}
[T^{n+1}, (T^{\sharp_A})^m] &= T^{n+1}(T^{\sharp_A})^m - (T^{\sharp_A})^mT^{n+1} \\
&= T(T^{\sharp_A})^mT^n - (T^{\sharp_A})^mT^{n+1} \\
&= T(T^{\sharp_A})^mT^2T^{n-2} - (T^{\sharp_A})^mT^{n+1} \\
&= T^3(T^{\sharp_A})^mT^{n-2} - (T^{\sharp_A})^mT^{n+1} \\
&= (T^{\sharp_A})^mT^{n+1} - (T^{\sharp_A})^mT^{n+1} = 0.
\end{aligned}$$

This means that $T \in [(n+1,m)\mathbf{N}]_A$. The proof is complete. □

Proposition 2.6. *Let $T \in \mathcal{B}_A(\mathcal{H})$. If $T \in [(n,m)\mathbf{N}]_A \cap [(n+1,m)\mathbf{N}]_A$, then $T \in [(n+2,m)\mathbf{N}]_A$ for some positive integers n and m. In particular $T \in [(j,m)\mathbf{N}]_A$ for all $j \geqslant n$.*

Proof. Let $T \in [(n,m)\mathbf{N}]_A \cap [(n+1,m)\mathbf{N}]_A$, then it follows that

$$T^n(T^{\sharp_A})^m - (T^{\sharp_A})^m T^n = 0 \quad \text{and} \quad T^{n+1}(T^{\sharp_A})^m - (T^{\sharp_A})^m T^{n+1} = 0.$$

Note that

$$\begin{aligned}[T^{n+2},(T^{\sharp_A})^m] &= T^{n+2}(T^{\sharp_A})^m - (T^{\sharp_A})^m T^{n+2} \\ &= TT^{n+1}(T^{\sharp_A})^m - (T^{\sharp_A})^m T^{n+2} \\ &= T(T^{\sharp_A})^m T^{n+1} - (T^{\sharp_A})^m T^{n+2} \\ &= TT^n(T^{\sharp_A})^m T - (T^{\sharp_A})^m T^{n+2} \\ &= (T^{\sharp_A})^m T^{n+2} - (T^{\sharp_A})^m T^{n+2} = 0.\end{aligned}$$

Hence $T \in [(n+2,m)\mathbf{N}]_A$. By repeating this process we can prove that $T \in [(j,m)\mathbf{N}]_A$ for all $j \geqslant n$. □

Proposition 2.7. *Let $T \in \mathcal{B}_A(\mathcal{H})$. If $T \in [(n,m)\mathbf{N}]_A \cap (n+1,m)\mathbf{N}]_A$ is one-to-one, then $T \in [(1,m)\mathbf{N}]_A$.*

Proof. Let $T \in [(n,m)\mathbf{N}]_A \cap [(n+1,m)\mathbf{N}]_A$, then it follows that,

$$T^n(T(T^{\sharp_A})^m - (T^{\sharp A})^m T) = 0.$$

Since T is one-to-one, then so is T^n and it follows that $T(T^{\sharp_A})^m - (T^{\sharp_A})^m T = 0$. Therefore $T \in [(1,m)\mathbf{N}]_A$. □

Proposition 2.8. *Let $T \in \mathcal{B}_A(\mathcal{H})$. The following statements are equivalent.*

(1) *If $T \in [(n,2)\mathbf{N}]_A \cap [(n,3)\mathbf{N}]_A$ for some positive integer n, then $T \in [(n,m)\mathbf{N}]_A$ for all positive integers $m \geqslant 4$.*

(2) *If $T \in [(n,m)\mathbf{N}]_A \cap [(n,m+1)\mathbf{N}]_A$, then $T \in [(n,m+2)\mathbf{N}]_A$ for some positive integers n, m. In particular $T \in [(n,j)\mathbf{N}]_A$ for all $j \geqslant m$.*

Proof. The proof follows by applying Proposition 2.1 and Proposition 2.5. □

Proposition 2.9. *Let $T \in \mathcal{B}_A(\mathcal{H})$. If $T \in [(n,m)\mathbf{N}]_A \cap [(n,m+1)\mathbf{N}]_A$ is such that $T^{\sharp_A}$ is one-to one, then $T \in [(n,1)\mathbf{N}]_A = [n\mathbf{N}]_A$.*

P r o o f. Since $T \in [(n,m)\mathbf{N}]_A \cap [(n,m+1)\mathbf{N}]_A$, it follows that

$$(T^{\sharp_A})^m(T^nT^{\sharp_A} - T^{\sharp_A}T^n) = 0.$$

If $T^{\sharp_A}$ is one-to-one, then so is $(T^{\sharp_A})^m$ and we obtain $T^nT^{\sharp_A} - T^{\sharp_A}T^n = 0$. Consequently $T \in [(n,1)\mathbf{N}]_A$. □

In [19], Theorem 2.4 it was proved that if T is (n,m)-power normal such that T^m is a partial isometry, then T is $(n+m,m)$-power normal. In the following theorem we extend this result to (n,m)-A-normal operators.

Theorem 2.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be (n,m)-A-normal for some positive integers n and m. The following statements hold:*
(1) If $n \geqslant m$ and $T^m(T^{\sharp_A})^mT^m = T^m$, then $T \in [(n+m,m)\mathbf{N}]_A$.
(2) If $m \geqslant n$ and $(T^{\sharp_A})^nT^n(T^{\sharp_A})^n = (T^{\sharp_A})^n$, then $T \in [(n,m+n)\mathbf{N}]_A$.

P r o o f. (1) Under the assumption that $T^m(T^{\sharp_A})^mT^m = T^m$, it follows that

$$T^m(T^{\sharp_A})^mT^n = T^n \quad \text{and} \quad T^n(T^{\sharp_A})^mT^m = T^n \quad \text{for } n \geqslant m,$$

which means that $T^n(T^{\sharp_A})^mT^m = T^m(T^{\sharp_A})^mT^n$. Since T is (n,m)-A normal, we get

$$(T^{\sharp})^mT^{n+m} = T^{n+m}(T^{\sharp_A})^m.$$

So, $T \in [(m+n,m)\mathbf{N}]_A$.

(2) In same way, under the assumption $(T^{\sharp_A})^nT^n(T^{\sharp_A})^n = (T^{\sharp_A})^n$, it follows that

$$(T^{\sharp_A})^nT^n(T^{\sharp_A})^m = (T^{\sharp_A})^m \quad \text{and} \quad (T^{\sharp_A})^mT^n(T^{\sharp_A})^n = (T^{\sharp_A})^m \quad \text{for } m \geqslant n,$$

which means that $(T^{\sharp_A})^nT^n(T^{\sharp_A})^m = (T^{\sharp_A})^mT^n(T^{\sharp_A})^n$. Since T is (n,m)-A normal, we get

$$(T^{\sharp})^{m+n}T^n = T^n(T^{\sharp_A})^{n+m}.$$

So, $T \in [(n,m+n)\mathbf{N}]_A$ and the proof is complete. □

Proposition 2.10. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be an (n,m)-A-normal operator for some positive integers n and m. Then T satisfies the relation $T^{2n}(T^{\sharp_A})^{2m} = (T^n(T^{\sharp_A})^m)^2$.*

P r o o f. Since T is an (n,m)-A-normal operator, it follows that

$$T^{2n}(T^{\sharp_A})^{2m} = T^nT^n(T^{\sharp_A})^m(T^{\sharp_A})^m = \underbrace{T^n(T^{\sharp_A})^m}\,\underbrace{T^n(T^{\sharp_A})^m} = (T^n(T^{\sharp_A})^m)^2.$$

□

The idea of the following proposition is inspired by [20].

Proposition 2.11. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $AT = TA$. If T is an n-normal operator, then T is an (n,m)-A-normal operator for $m \in \mathbb{N}$.*

Proof. Indeed, since T^n is normal and $T^mT^n = T^nT^m$, it follows from the Fuglede theorem (see [14]) that $T^{*m}T^n = T^nT^{*m}$. Taking in consideration that under the assumptions we have $P_{\overline{\mathcal{R}(A)}}T = TP_{\overline{\mathcal{R}(A)}}$ and $T^{\sharp_A} = P_{\overline{\mathcal{R}(A)}}T^*$. Then

$$\begin{aligned}[T^n, (T^{\sharp_A})^m] &= T^n(T^{\sharp_A})^m - (T^{\sharp_A})^mT^n \\ &= T^n(P_{\overline{\mathcal{R}(A)}}T^*)^m - (P_{\overline{\mathcal{R}(A)}}T^*)^mT^n \\ &= P_{\overline{\mathcal{R}(A)}}[T^n, T^{*m}] = 0.\end{aligned}$$

Therefore T is (n,m)-A-normal. □

Corollary 2.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $AT = TA$. If T is an (n,m)-normal operator, then T is a (j,r)-A-normal operator where $r \in \mathbb{N}$ and j is the least common multiple of n and m.*

Proof. Since T is (n,m)-normal, it was observed in [11], Lemma 4.4 that T^j is a normal operator where $j = LCM(n,m)$. By applying Proposition 2.11 we get that T is a (j,r)-A-normal operator. □

3. (n,m)-A-quasinormal operators

In [8] the author has introduced the class of (n,m)-A-quasinormal operators as follows. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be (n,m)-A-quasinormal if T satisfies

$$[T^n, (T^{\sharp_A})^mT] := T^n(T^{\sharp_A})^mT - (T^{\sharp_A})^mTT^n = 0$$

for some positive integers n and m. This class of operators is denoted by $[(n,m)\mathbf{QN}]_A$.

Remark 3.1. Clearly, the class of (n,m)-A-quasinormal operators includes the class of (n,m)-A-normal one, i.e. the following inclusion holds

$$[(n,m)\mathbf{N}]_A \subset [(n,m)\mathbf{QN}]_A.$$

We give an example to show that there exists an (n,m)-A-quasinormal operator which is not (n,m)-A-normal for some positive integers n and m.

E x a m p l e 3.1. Let T be a unilateral shift, that is, if $\mathcal{H} = l^2$, the matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad \text{and} \quad A = I_{l^2} \text{ (the identity operator).}$$

It is easily verified that $[T^2, T^{\sharp_A}] \neq 0$ and $[T^2, T^{\sharp_A}T] = 0$. So that T is not a $(2,1)$-A-normal operator but it is a $(2,1)$-A-quasinormal operator.

The following theorem gives a characterization of (n,m)-A-quasinormal operators.

Theorem 3.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then T is an (n,m)-A-quasinormal operator for some positive integers n and m if and only if T satisfies the following conditions:*

$$\langle (T^{\sharp_A})^m Th \mid (T^{\sharp_A})^n h\rangle_A = \langle T^n Th \mid T^m h\rangle_A \quad \forall\, h \in \mathcal{H}, \tag{1}$$

$$\mathcal{R}(T^n (T^{\sharp_A})^m T) \subseteq \overline{\mathcal{R}(A)}. \tag{2}$$

P r o o f. We omit the proof, since the techniques are similar to those of Theorem 2.1. □

R e m a r k 3.2. Theorem 3.1 is an improved version of [8], Lemma 4.4.

Proposition 3.1. *Let $T \in \mathcal{B}_A(\mathcal{H})$ and $S \in \mathcal{B}_A(\mathcal{H})$ be (n,m)-A-normal operators. Then their product ST is an (n,m)-A-normal operator if the conditions $ST = TS$, $ST^{\sharp_A} = T^{\sharp_A}S$ and $TS^{\sharp_A} = S^{\sharp_A}T$ are satisfied.*

P r o o f. It is

$$\begin{aligned}(TS)^n((TS)^{\sharp_A})^m(TS) &= T^n S^n (T^{\sharp_A})^m (S^{\sharp_A})^m TS = T^n (T^{\sharp_A})^m T S^n (S^{\sharp_A})^m S \\ &= (T^{\sharp_A})^m T T^n (S^{\sharp_A})^m S S^n = ((TS)^{\sharp_A})^m (TS)(TS)^n.\end{aligned}$$

Therefore TS is an (n,m)-A-quasinormal operator. □

R e m a r k 3.3. Proposition 3.1 is an improved version of [8], Proposition 4.5.

Proposition 3.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$. If $T \in [(n,m)\mathbf{QN}]_A \cap [(n+1,m)\mathbf{QN}]_A$, then $T \in [(n+2,m)\mathbf{QN}]_A$.*

P r o o f. Assume that $T \in [(n,m)\mathbf{QN}]_A \cap [(n+1,m)\mathbf{QN}]_A$, it follows that

$$T^{n+1}(T^{\sharp_A})^m T - (T^{\sharp_A})^m T T^{n+1} = 0 \quad \text{and} \quad T^n(T^{\sharp_A})^m T - (T^{\sharp_A})^m T T^n = 0.$$

On the other hand, we have

$$\begin{aligned} T^{n+2}(T^{\sharp_A})^m T - (T^{\sharp_A})^m T T^{n+2} &= T(T^{\sharp_A})^m T T^{n+1} - (T^{\sharp_A})^m T T^{n+2} \\ &= T^{n+1}(T^{\sharp_A})^m T T - (T^{\sharp_A})^m T T^{n+2} \\ &= (T^{\sharp_A})^m T T^{n+2} - (T^{\sharp_A})^m T T^{n+2} = 0. \end{aligned}$$

□

In [19] it was proved that if $T \in [(n,m)\mathbf{QN}]$ such that T^m is a partial isometry, then $T \in [(n+m,m)\mathbf{QN}]$ for $n \geqslant m$. We extend this result to the class of $[(n,m)\mathbf{QN}]_A$ as follows.

Theorem 3.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $T \in [(n,m)\mathbf{QN}]$ for some positive integers n and m. If $T^m(T^{\sharp_A})^m T^m = T^m$ for $n \geqslant m$, then $T \in [(n+m,m)\mathbf{QN}]_A$.*

P r o o f. (1) Assume that T^m satisfies $T^m(T^{\sharp_A})^m T^m = T^m$ for $m \geqslant n$, then we have

$$T^m(T^{\sharp_A})^m T T^{m-1} = T^m. \tag{3.1}$$

Multiplying (3.1) from the left by T^{n-m} and from the right by T we get

$$T^n((T^{\sharp_A})^m T) T^m = T^{n+1}. \tag{3.2}$$

Multiplying (3.1) from the right by T^{n-m+1} we get

$$T^m((T^{\sharp_A})^m T) T^n = T^{n+1}. \tag{3.3}$$

Combining (3.2), (3.3) and using the fact that $T \in [(n,m)\mathbf{QN}]$ we obtain

$$T^{n+m}((T^{\sharp_A})^m T) = ((T^{\sharp_A})^m T) T^{n+m}.$$

Therefore $T \in [(n+m,m)\mathbf{QN}]_A$ as required. □

Proposition 3.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$, n and m positive integers. The following statements hold:*

(1) *If $T \in [(n,m)\mathbf{QN}]_A \cap [(n+1,m)\mathbf{QN}]_A$ such that T is one-to-one, then $T \in [(1,m)\mathbf{QN}]_A$.*
(2) *If $T \in [(n,m)\mathbf{QN}]_A \cap [(n,m+1)\mathbf{QN}]_A$ such that T^* is one-to-one and $\overline{\mathcal{R}(T^{\sharp_A})^m T)} = \overline{\mathcal{R}(A)}$, then $T \in [(n,1)\mathbf{N}]_A$.*

P r o o f. (1) Under the assumption $T \in [(n,m)\mathbf{QN}]_A \cap [(n+1,m)\mathbf{QN}]_A$, it follows that

$$T^n(T(T^{\sharp_A})^m T - (T^{\sharp_A})^m TT) = 0.$$

If T is injective, then so is T^n and we have $T(T^{\sharp_A})^m T - (T^{\sharp_A})^m TT = 0$. Hence, $T \in [(1,m)\mathbf{QN}]_A$.

(2) Since $T \in [(n,m)\mathbf{QN}]_A \cap [(n,m+1)\mathbf{QN}]_A$, we have

$$\begin{aligned} T^n(T^{\sharp_A})^{m+1}T - (T^{\sharp_A})^{m+1}TT^n &= 0 \\ &\Rightarrow T^nT^{\sharp_A}(T^{\sharp_A})^m T - T^{\sharp_A}(T^{\sharp_A})^m TT^n = 0 \\ &\Rightarrow (T^nT^{\sharp_A} - T^{\sharp_A}T^n)(T^{\sharp_A})^m T = 0 \\ &\Rightarrow (T^nT^{\sharp_A} - T^{\sharp_A}T^n) \equiv 0 \quad \text{on } \overline{\mathcal{R}((T^{\sharp_A})^m T)} = \overline{\mathcal{R}(A)}. \end{aligned}$$

On the other hand, since $T \in \mathcal{B}_A(\mathcal{H})$, we have $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. Moreover, by the assumption that T^* is injective we obtain $\mathcal{N}(T^{\sharp_A}) = \mathcal{N}(A)$. If $h \in \mathcal{N}(A)$ it follows from the above observation that

$$(T^nT^{\sharp_A} - T^{\sharp_A}T^n)h = T^nT^{\sharp_A}h - T^{\sharp_A}T^nh = 0.$$

Consequently, $(T^nT^{\sharp_A} - T^{\sharp_A}T^n) = 0$ on $\mathcal{H}$. Therefore $T \in [(n,1)\mathbf{N}]_A$. □

Proposition 3.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $T \in [(2,m)\mathbf{QN}]_A \cap [(3,m)\mathbf{QN}]_A$ for some positive integer m, then $T \in [(n,m)\mathbf{QN}]_A$ for all positive integers $n \geqslant 4$.*

P r o o f. We prove the assertion by using the mathematical induction. Since $T \in [(2,m)\mathbf{QN}]_A \cap [(3,m)\mathbf{QN}]_A$, it follows immediately that

$$T^4(T^{\sharp_A})^m T - (T^{\sharp_A})^m TT^4 = 0 \quad \text{and} \quad T^5(T^{\sharp_A})^m T - (T^{\sharp_A})^m TT^5 = 0.$$

Now assume that the result is true for $n \geqslant 5$, that is,

$$T^n(T^{\sharp_A})^m T - (T^{\sharp_A})^m TT^n = 0,$$

then

$$\begin{aligned} T^{n+1}(T^{\sharp_A})^m T - (T^{\sharp_A})^m TT^{n+1} &= T(T^{\sharp_A})^m TT^n - (T^{\sharp_A})^m TT^{n+1} \\ &= T^3(T^{\sharp_A})^m TT^{n-2} - (T^{\sharp_A})^m TT^{n+1} \\ &= (T^{\sharp_A})^m TT^{n+1} - (T^{\sharp_A})^m TT^{n+1} = 0. \end{aligned}$$

Therefore $T \in [(n+1,m)\mathbf{QN}]_A$. The proof is complete. □

Now we discuss the (n,m)-A-quasinormality of an operator under some commutation conditions on its real and imaginary part.

Theorem 3.3. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $\mathcal{R}(T^{m-1})$ is dense. If $TA = AT$. Then the following statements are equivalent.*
(1) $T \in [(n,m)\mathbf{QN}]_A$.
(2) $C_{m,A}$ *commutes with* $\mathrm{Re}_A(T^n)$ *and* $\mathrm{Im}_A(T^n)$, *where* $C_{m,A} = \sqrt{(T^{\sharp_A})^m T^m}$.

Proof. Since T is (n,m)-A-quasinormal, it follows that

$$T^n (T^{\sharp_A})^m T = (T^{\sharp_A})^m T T^n.$$

Hence,

$$T^n (T^{\sharp_A})^m T^m = (T^{\sharp_A})^m T^m T^n.$$

From the conditions that $TA = AT$ and $\mathcal{N}(A)^\perp$ is an invariant subspace for T, we observe that

$$TP_{\overline{\mathcal{R}(A)}} = TP_{\overline{\mathcal{R}(A)}}, \quad T^{\sharp_A} P_{\overline{\mathcal{R}(A)}} = T^{\sharp_A} P_{\overline{\mathcal{R}(A)}} \quad \text{and} \quad T^{\sharp_A} = P_{\overline{\mathcal{R}(A)}} T^*.$$

Therefore, $C_{m,A}$ is a nonnegative definite operator and by elementary calculation we get

$$C_{m,A}^2 \mathrm{Re}_A(T^n) = \mathrm{Re}_A(T^n) C_{m,A}^2.$$

Consequently,

$$C_{m,A} \mathrm{Re}_A(T^n) = \mathrm{Re}_A(T^n) C_{m,A}.$$

In a similar way we can prove that $C_{m,A}\mathrm{Im}_A(T^n) = \mathrm{Im}_A(T^n)C_{m,A}$. Conversely, assume that $C_{m,A}\mathrm{Re}_A(T^n) = \mathrm{Re}_A(T^n)C_{m,A}$ and $C_{m,A}\mathrm{Im}_A(T^n) = \mathrm{Im}_A(T^n)C_{m,A}$. Then

$$C_{m,A}^2 \mathrm{Re}_A(T^n) = \mathrm{Re}_A(T^n) C_{m,A}^2 \quad \text{and} \quad C_{m,A}^2 \mathrm{Im}_A(T^n) = \mathrm{Im}_A(T^n) C_{m,A}^2.$$

Hence

$$C_{m,A}^2 (\mathrm{Re}_A(T^n) + \mathrm{i}\mathrm{Im}_A(T^n)) = (\mathrm{Re}_A(T^n) + \mathrm{i}\mathrm{Im}_A(T^n)) C_{m,A}^2,$$

and therefore

$$C_{m,A}^2 T^n = T^n C_{m,A}^2.$$

On the other hand, we have

$$\begin{aligned} C_{m,A}^2 T^n = T^n C_{m,A}^2 &\Leftrightarrow (T^{\sharp_A})^m T^m T^n - T^n (T^{\sharp_A})^m T^m = 0 \\ &\Leftrightarrow ((T^{\sharp_A})^m T T^n - T^n (T^{\sharp_A})^m T) T^{m-1} = 0 \\ &\Leftrightarrow (T^{\sharp_A})^m T T^n - T^n (T^{\sharp_A})^m T = 0 \quad (\overline{\mathcal{R}(T^{m-1})} = \mathcal{H}). \end{aligned}$$

Therefore $T \in [(n,m)\mathbf{QN}]_A$. □

Theorem 3.4. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be such that $\mathcal{R}(T^{m-1})$ is dense and $TA = AT$. If T satisfies the conditions*

(i) *$B_{m,A}$ commutes with $\mathrm{Re}_A(T^m)$ and $\mathrm{Im}_A(T^m)$,*

(ii) *$C^2_{m,A}T^n = T^n B^2_{m,A}$, where $B_{m,A} = \sqrt{T^m(T^{\sharp_A})^m}$.*

Then T is an (m,m)-A-quasinormal operator.

P r o o f. Since

$$B_{m,A}\mathrm{Re}_A(T^m) = \mathrm{Re}_A(T^m)B_{m,A} \quad \text{and} \quad B_{m,A}\mathrm{Im}_A(T^m) = \mathrm{Im}_A(T^m)B_{m,A},$$

it follows that

$$\begin{cases} B^2_{m,A}T^m + B^2(T^m)^{\sharp_A} = T^m B^2_{m,A} + (T^m)^{\sharp_A}B^2_{m,A}, \\ B^2_{m,A}T^m - B^2_{m,A}(T^m)^{\sharp_A} = T^m B^2_{m,A} - (T^m)^{\sharp_A}B^2_{m,A}. \end{cases}$$

This gives

$$B^2_{m,A}T^m = T^m B^2_{m,A} = C^2_{m,A}T^m.$$

On the other hand, we have

$$\begin{aligned} B^2_{m,A}T^m = C^2_{m,A}T^m &\Rightarrow T^m(T^{\sharp_A})^m T^m - (T^{\sharp_A})^m T^m T^m = 0 \\ &\Rightarrow (T^m(T^{\sharp_A})^m T - (T^{\sharp_A})^m T T^m)T^{m-1} = 0 \\ &\Rightarrow T^m(T^{\sharp_A})^m T - (T^{\sharp_A})^m T T^m = 0 \quad \text{on } \overline{\mathcal{R}(T^{m-1})} = \mathcal{H}. \end{aligned}$$

Therefore $T^m(T^{\sharp_A})^m T - (T^{\sharp_A})^m T T^m = 0$ and T is an (m,m)-A-quasinormal operator. □

Proposition 3.5. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be (n,m)-A-quasinormal, then*

$$(T^{\sharp_A})^{2m}T^{2n} = ((T^{\sharp_A})^m T^n)^2.$$

P r o o f. Since T is (n,m)-A-quasinormal, it follows that

$$T^n(T^{\sharp_A})^m T = (T^{\sharp_A})^m T T^n.$$

On the other hand, we have

$$\begin{aligned} (T^{\sharp_A})^{2m}T^{2n} &= (T^{\sharp_A})^m(T^{\sharp_A})^m T^n T^n = (T^{\sharp_A})^m(T^{\sharp_A})^m T T^n T^{n-1} \\ &= (T^{\sharp_A})^m T^n (T^{\sharp_A})^m T^n = ((T^{\sharp_A})^m T^n)^2. \end{aligned}$$

□

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