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Lulu Yin; Hongwei Liu; Jun Yang

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MODIFIED GOLDEN RATIO ALGORITHMS FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS AND VARIATIONAL INEQUALITIES

Lulu Yin, Hongwei Liu, Xi'an, Jun Yang, Xianyang

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Abstract. We propose a modification of the golden ratio algorithm for solving pseudomonotone equilibrium problems with a Lipschitz-type condition in Hilbert spaces. A new non-monotone stepsize rule is used in the method. Without such an additional condition, the theorem of weak convergence is proved. Furthermore, with strongly pseudomonotone condition, the R-linear convergence rate of the method is established. The results obtained are applied to a variational inequality problem, and the convergence rate of the problem under the condition of error bound is considered. Finally, numerical experiments on several specific problems and comparison with other algorithms show the superiority of the algorithm.

Keywords: equilibrium problem; strongly pseudomonotone bifunctions; Lipschitz-type condition; variational inequality; error bound

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $f\colon \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be a bifunction with $f(x,x)=0$ for all $x\in C$, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and the norm in $\mathcal{H}$, respectively. The equilibrium problem (EP) (see [24], [3]) for the bifunction f on C is to find $x^*\in C$ such that

$$f(x^*,y)\geqslant 0\quad \forall\, y\in C \tag{1.1}$$

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which is also known as the Fan inequality [6]. We denote the solution set of equilibrium problem (1.1) by EP(f). Problem (EP) serves as a general mathematical model which unifies numerous known models in a simple form, such as optimization problems, variational inequalities, fixed-point problems, Nash equilibrium problems, and many other models [2], [4], [5], [20]. Thanks to the importance of the equilibrium problem and its applications both in theoretical and applied sciences, many authors have extensively investigated it in recent years (see, for example, [4], [23], [27], [19], [7], [21], [28], [17], [12], [8], [9], [11], [10], [29]). One of the most popular methods is the proximal point method [23], [27], [19], but it is not feasible to solve the pseudomonotone equilibrium problem. Another method is the proximal-like method (the extragradient method) [7]. With the aid of the idea of Korpelevich extragradient method [21], this method was extended by Tran et al. [28]:

$$
\begin{cases} x_0 \in C,\ y_n = \arg\min\Big\{\lambda f(x_n, y) + \dfrac{1}{2}\|x_n - y\|^2,\ y \in C\Big\}, \\ x_{n+1} = \arg\min\Big\{\lambda f(y_n, y) + \dfrac{1}{2}\|x_n - y\|^2,\ y \in C\Big\}, \end{cases} \tag{1.2}
$$

where λ is a suitable parameter. It is shown that the sequence $\{x_n\}$ generated by (1.2) converges to the solution of the equilibrium problem under appropriate assumptions. In each iteration, it is necessary to calculate two strongly convex programming problems of the algorithm. However, the evaluation of the subprograms involved in the algorithm may be expensive in the case where the bifunction and/or the feasible set have complicated structures. There is a vast literature concerning the study and improvement of this algorithm (to name but a few, we address the reader to [17], [12], [8], [9], [11], [10], [29]).

Based on the seminal work of Malitsky in the variational inequality [22], Vinh has proposed a new iterative algorithm to solve pseudomonotone equilibrium problems in real Hilbert spaces; please refer to [29], Algorithm 3.1 for more details. The weak convergence of the algorithm is established under suitable conditions. The algorithm only calculates one strongly convex programming problem for each iteration. It needs to know the Lipschitz-type constants of bifunctions in the equilibrium problem, although these constants are usually unknown or difficult to estimate. In order to overcome this drawback, Vinh [29], Algorithm 4.1 designed another algorithm, which uses a non-summable variable step sequence. Under these new rules, Vinh established the strong convergence theorem of his algorithm.

Very recently, Yang and Liu [30] introduced a new gradient method under the assumptions that equilibrium bifunctions are pseudomonotone and satisfy a certain

Lipschitz-type condition. The form is as follows:

$$
\begin{cases} x_n = (1-\delta)y_n + \delta x_{n-1}, \\ y_{n+1} = \operatorname{prox}_{\lambda_n f(y_n,\cdot)}(x_n), \\ \lambda_{n+1} = \min\left\{\lambda_n, \dfrac{\alpha\mu\theta(\|y_{n-1}-y_n\|^2 + \|y_n - y_{n+1}\|^2)}{4\delta[f(y_{n-1},y_{n+1}) - f(y_{n-1},y_n) - f(y_n,y_{n+1})]_+}\right\}. \end{cases} \tag{1.3}
$$

It is easy to see that this method uses a new stepsize and does not require the knowledge of the Lipschitz-type constants of the bifunction. Under suitable conditions, Yang and Liu established the weak and strong convergence of the iterative sequence generated by the algorithm (1.3) without its rate of convergence. Inspired by Yang and Liu [30], on the basis of reducing parameters and dependence on the initial stepsize, this paper proposes a modified golden ratio method to solve the equilibrium problem (1.1) in which a non-monotone stepsize strategy is specifically selected. In addition, the convergence theorem and convergence rate of this method are established.

The remainder of this paper is organized as follows. In Section 2, we present some definitions and preliminaries which will be needed throughout the paper. In Section 3, we put forward our algorithm and establish the convergence theorem and R-linear convergence rate of the algorithm. In Section 4, the proposed algorithm is applied to variational inequality problems. In Section 5, several numerical experiments are reported to show the behavior of the new algorithm.

2. Preliminaries

We now provide some basic concepts, definitions and lemmas that will be used in later proofs.

Definition 2.1. A bifunction $f\colon C\times C\to\mathbb{R}$ is labeled as:

(i) strongly monotone on C if there exists a constant $\gamma>0$ such that $f(x,y) + f(y,x) \leqslant -\gamma\|x-y\|^2$ for all $x,y\in C$;
(ii) monotone on C, if $f(x,y)+f(y,x)\leqslant 0$ for all $x,y\in C$;
(iii) pseudomonotone on C, if $f(x,y)\geqslant 0 \Rightarrow f(y,x)\leqslant 0$ for all $x,y\in C$;
(iv) strongly pseudomonotone on C, if $f(x,y)\geqslant 0\Rightarrow f(y,x)\leqslant -\gamma\|x-y\|^2$ for all $x,y\in C$, where $\gamma>0$.

From the above definitions, it is easy to see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). We say that a bifunction $f\colon C\times C\to\mathbb{R}$ satisfies Lipschitz-type condition on C if there exist constants $c_1>0$ and $c_1>0$, such that $f(x,y)+f(y,z)\geqslant f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2$ for all $x,y,z\in C$.

Definition 2.2. A mapping $g\colon C \to \mathbb{R}$ is called subdifferentiable at $x \in C$ if there exists a vector $w \in \mathcal{H}$ such that $g(y) - g(x) \geqslant \langle w, y - x\rangle$ for all $y \in C$.

For a proper, convex and lower semicontinuous function $g\colon C \to \mathbb{R}$ and $\lambda > 0$, the proximal mapping of g associated with λ is defined by

$$\operatorname{prox}_{\lambda g}(x) = \arg\min\Big\{\lambda g(y) + \frac{1}{2}\|x - y\|^2\colon\ y \in C\Big\}, \quad x \in \mathcal{H}. \tag{2.1}$$

The following lemma gives an important property of the proximal mapping.

Lemma 2.1 ([1]). *For all $x \in \mathcal{H}$, $y \in C$ and $\lambda > 0$, the following inequality holds:*

$$\lambda\{g(y) - g(\operatorname{prox}_{\lambda g}(x))\} \geqslant \langle x - \operatorname{prox}_{\lambda g}(x), y - \operatorname{prox}_{\lambda g}(x)\rangle. \tag{2.2}$$

Remark 2.1. From Lemma 2.1, we note that if $x = \operatorname{prox}_{\lambda g}(x)$, then

$$x = \arg\min\{g(y)\colon\ y \in C\} := \Big\{x \in C\colon\ g(x) = \min_{y \in C} g(y)\Big\}.$$

Lemma 2.2. *Let $\delta \in \mathbb{R}$ and $u, v \in \mathcal{H}$. Then*

$$\|(1-\delta)u + \delta v\| = (1-\delta)\|u\|^2 + \delta\|v\|^2 - \delta(1-\delta)\|u - v\|. \tag{2.3}$$

Lemma 2.3 (Peter-Paul inequality). *If $a, b \in \mathbb{R}$ and $\varepsilon > 0$, then*

$$2ab \leqslant \frac{a^2}{\varepsilon} + \varepsilon b^2. \tag{2.4}$$

Lemma 2.4 (Opial). *Let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that $x_n \rightharpoonup x$. Then,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \forall\, y \neq x. \tag{2.5}$$

Lemma 2.5. *Let $\{a_n\}$, $\{b_n\}$ be two non-negative real sequences such that there exists $N > 0$ for all $n \geqslant N$, $a_{n+1} \leqslant a_n - b_n$. Then $\{a_n\}$ is bounded and $\lim_{n\to\infty} b_n = 0$.*

The following identity (cosine rule) appears several times and we will use it to simplify our convergence analysis. For all $x, y, z \in \mathcal{H}$,

$$2\langle x - y, x - z\rangle = \|x - y\|^2 + \|x - z\|^2 - \|y - z\|^2. \tag{2.6}$$

3. Main part

In this section, with the help of a non-monotone stepsize strategy, which is mainly due to Yang and Liu [30], we propose an iterative algorithm for the equilibrium problem (1.1) in real Hilbert spaces. In what follows we always assume that $\mathrm{EP}(f) \neq \emptyset$, and we add the following condition:

(A1) f is pseudomonotone on C;

(A1′) f is strongly pseudomonotone on C;

(A2) f satisfies Lipschitz-type condition on C;

(A3) $f(x, \cdot)$ is convex and subdifferentiable on C for each fixed $x \in \mathcal{H}$;

(A4) $\limsup\limits_{n\to\infty} f(x_n, y) \leqslant f(x, y)$ for all $y \in C$ for every sequence $\{x_n\}$ which converges weakly to x;

Remark 3.1. It is noteworthy that the strong pseudomonotonicity assumption of the bifunction f implies that problem (EP) has a unique solution (see, e.g., [25], Proposition 1).

For the equilibrium problem (1.1), this paper designs the following algorithm:

Algorithm 3.1.

Step 0. Choose $\lambda_0 = \lambda_1 > 0$, $x_0, y_0, y_1 \in C$, $\mu \in (0, 1)$, $\theta \in (1/(2-\mu), 1)$. Choose a non-negative real sequence $\{p_n\}$ such that $\sum\limits_{n=0}^{\infty} p_n < \infty$.

Step 1. Given the current iterates x_{n-1}, y_{n-1}, y_n, compute

$$\delta_n = \min\Big\{\frac{1}{2}\sqrt{1 + 4\theta\frac{\lambda_n}{\lambda_{n-1}}} - 1, 1\Big\}, \quad x_n = (1-\delta_n)y_n + \delta_n x_{n-1},$$

$$y_{n+1} = \arg\min\Big\{\lambda_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2, y \in C\Big\} = \mathrm{prox}_{\lambda_n f(y_n, \cdot)}(x_n).$$

If $y_{n+1} = x_n = y_n$, then stop: x_n is a solution. Otherwise, go to Step 2.

Step 2. Compute

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))}, \ \lambda_n + p_n\Big\} \\ \qquad\qquad \text{if } f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}) > 0, \\ \lambda_n + p_n \qquad \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return to Step 1.

Lemma 3.1. *Let $\{\lambda_n\}$ be the sequence of steps generated by Algorithm 3.1. Then $\lim\limits_{n\to\infty} \lambda_n = \lambda$ and $\min\{\frac{1}{4}\mu/\max\{c_1, c_2\}, \lambda_0\} \leqslant \lambda \leqslant \lambda_0 + P$, where $P = \sum\limits_{n=0}^{\infty} p_n$.*

P r o o f. First, we prove that the sequence $\{\lambda_n\}$ generated by Algorithm 3.1 is bounded. Since f satisfies the Lipschitz-type condition with constants c_1 and c_2, in the case of $f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}) > 0$, we have

$$\begin{aligned} &\frac{\mu(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))} \\ &\qquad \geqslant \frac{\mu(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta_n(c_1\|y_n - y_{n-1}\|^2 + c_2\|y_{n+1} - y_n\|^2)} \\ &\qquad \geqslant \frac{\mu(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta_n \max\{c_1, c_2\}(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)} \\ &\qquad \geqslant \frac{\mu}{4\max\{c_1, c_2\}}. \end{aligned}$$

Using the definition of λ_{n+1} and the derivation of mathematical induction, the sequence $\{\lambda_n\}$ has a lower bound $\min\{\frac{1}{4}\mu/\max\{c_1, c_2\}, \lambda_0\}$ and an upper bound λ_0+P.

Next, we verify that sequence $\{\lambda_n\}$ is convergent. From the definition of $\{\lambda_n\}$, we come to

$$\sum_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+ \leqslant \sum_{n=0}^{\infty} p_n < \infty, \tag{3.1}$$

where $(\lambda_{n+1} - \lambda_n)^+ = \max\{0, \lambda_{n+1} - \lambda_n\}$, $(\lambda_{n+1} - \lambda_n)^- = \max\{0, -(\lambda_{n+1} - \lambda_n)\}$. According to the inequality (3.1), the convergence of the positive term series $\sum\limits_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+$ can be obtained. On the other hand, we verify the convergence of the positive series $\sum\limits_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^-$. Assume that $\sum\limits_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^- = -\infty$. Since

$$\lambda_{n+1} - \lambda_n = (\lambda_{n+1} - \lambda_n)^+ - (\lambda_{n+1} - \lambda_n)^-, \tag{3.2}$$

we get

$$\lambda_{k+1} - \lambda_0 = \sum_{n=0}^{k}(\lambda_{n+1} - \lambda_n)^+ - \sum_{n=0}^{k}(\lambda_{n+1} - \lambda_n)^-. \tag{3.3}$$

In (3.3), let $k \to \infty$, we have $\lambda_k \to -\infty$, which is impossible. Combining the convergence of the series $\sum\limits_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^+$ and $\sum\limits_{n=0}^{\infty}(\lambda_{n+1} - \lambda_n)^-$, let $k \to \infty$ in (3.3), we can deduce $\lim\limits_{n\to\infty}\lambda_n = \lambda$.

After what we have discussed above, we can easily come to the conclusion that $\min\{\frac{1}{4}\mu/\max\{c_1, c_2\}, \lambda_0\} \leqslant \lambda \leqslant \lambda_0 + P$. $\square$

R e m a r k 3.2. It is apparent that $\lambda > 0$. The sequence $\{\lambda_n\}$ generated by Algorithm 3.1 is not monotonically decreasing, which reduces the dependence on the initial step size λ_0. When $p_n \equiv 0$, the sequence of steps $\{\lambda_n\}$ is a monotonically decreasing sequence.

R e m a r k 3.3. Through the representation of δ_n and $\lim_{n\to\infty} \lambda_n = \lambda$, the limit of δ_n exists, denoted as δ. That is, $\lim_{n\to\infty} \delta_n = \delta$. Because of $\theta \in (1/(2-\mu), 1)$, it is evident that there exists $N_0 \geqslant 0$ such that for all $n \geqslant N_0$, $0 < \delta_n < 1$.

3.1. Weak converge. Now we prove Algorithm 3.1 converges weakly to the solution of (1.1) when f is a pseudomonotone bifunction.

Theorem 3.1. *Assume that* (A1)–(A4) *hold. Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 3.1 *converge weakly to the solution of the equilibrium problem.*

P r o o f. Followed by $y_{n+1} = \operatorname{prox}_{\lambda_n f(y_n,\cdot)}(x_n)$ and Lemma 2.1, we obtain

$$\lambda_n(f(y_n, y) - f(y_n, y_{n+1})) \geqslant \langle x_n - y_{n+1}, y - y_{n+1}\rangle \quad \forall\, y \in C. \tag{3.4}$$

Analogously, for the previous iterate we have

$$\lambda_{n-1}(f(y_{n-1}, y) - f(y_{n-1}, y_n)) \geqslant \langle x_{n-1} - y_n, y - y_n\rangle \quad \forall\, y \in C.$$

Particularly, substituting $y = y_{n+1}$ into the last inequality means

$$\lambda_{n-1}(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)) \geqslant \langle x_{n-1} - y_n, y_{n+1} - y_n\rangle. \tag{3.5}$$

Since $x_n = (1-\delta_n)y_n + \delta_n x_{n-1}$, we arrive at

$$y_n - x_n = \frac{\delta_n}{1-\delta_n}(x_n - x_{n-1}) = \delta_n(y_n - x_{n-1}). \tag{3.6}$$

Using (3.5), (3.6) and $\lambda_n > 0$, we come to the following relationship:

$$\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n)) \geqslant \frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\langle x_n - y_n, y_{n+1} - y_n\rangle. \tag{3.7}$$

Summation of (3.4) and (3.7) gives us

$$\begin{aligned} 2\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1})) + 2\lambda_n f(y_n, y) \\ \geqslant 2\langle x_n - y_{n+1}, y - y_{n+1}\rangle + 2\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\langle x_n - y_n, y_{n+1} - y_n\rangle \\ \geqslant \|x_n - y_{n+1}\|^2 + \|y_{n+1} - y\|^2 - \|x_n - y\|^2 \\ + \frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}(\|x_n - y_n\|^2 + \|y_{n+1} - y_n\|^2 - \|x_n - y_{n+1}\|^2). \end{aligned} \tag{3.8}$$

Owing to the given form of λ_{n+1}, we derive

$$\begin{aligned}&2\lambda_n(f(y_{n-1}, y_{n+1}) - f(y_{n-1}, y_n) - f(y_n, y_{n+1}))\\ &\qquad\qquad \leqslant \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}(\|y_{n-1} - y_n\|^2 + \|y_n - y_{n+1}\|^2).\end{aligned}$$

It follows from the last inequality and (3.8) that

$$\begin{aligned}\|y_{n+1} - y\|^2 &+ \Big(\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_{n+1} - y_n\|^2\\ &\leqslant \|x_n - y\|^2 + \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n-1} - y_n\|^2 + 2\lambda_n f(y_n, y)\\ &\quad + \Big(\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - 1\Big)\|x_n - y_{n+1}\|^2 - \frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\|x_n - y_n\|^2.\end{aligned}\tag{3.9}$$

From Remark 3.3 and the relation $x_n = (1 - \delta_n)y_n + \delta_n x_{n-1}$, invoking Lemma 2.2, we have for all $n \geqslant N_0$,

$$\begin{aligned}\|y_{n+1} - y\|^2 &= \frac{1}{1 - \delta_{n+1}}\|x_{n+1} - y\|^2 - \frac{\delta_{n+1}}{1 - \delta_{n+1}}\|x_n - y\|^2\\ &\quad + \frac{\delta_{n+1}}{(1 - \delta_{n+1})^2}\|x_{n+1} - x_n\|^2\\ &= \frac{1}{1 - \delta_{n+1}}\|x_{n+1} - y\|^2 - \frac{\delta_{n+1}}{1 - \delta_{n+1}}\|x_n - y\|^2 + \delta_{n+1}\|y_{n+1} - x_n\|^2.\end{aligned}$$

Injecting this last equality into (3.9) implies that for all $n \geqslant N_0$,

$$\begin{aligned}\frac{1}{1 - \delta_{n+1}}\|x_{n+1} - y\|^2 &+ \Big(\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_{n+1} - y_n\|^2\\ &\leqslant \frac{1}{1 - \delta_{n+1}}\|x_n - y\|^2 + \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n-1} - y_n\|^2 + 2\lambda_n f(y_n, y)\\ &\quad + \Big(\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - 1 - \delta_{n+1}\Big)\|x_n - y_{n+1}\|^2 - \frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\|x_n - y_n\|^2.\end{aligned}\tag{3.10}$$

By the definition of δ_n, we have that there exists $N_1 \geqslant 0$ such that

$$1 + \delta_n - \frac{1}{\delta_n}\theta\frac{\lambda_n}{\lambda_{n-1}} = 0 \quad \forall\, n \geqslant N_1.$$

Noting that $\theta \in (1/(2 - \mu), 1)$, we obtain

$$\lim_{n\to\infty}\Big(\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n}\Big) = \frac{1}{\delta} - \frac{1}{\delta}\theta = \frac{1}{\delta}(1 - \theta) > 0.$$

Then there exists N_2 ($\geqslant N_1$) such that for all $n \geqslant N_2$,

$$\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - 1 - \delta_{n+1} = \frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} > 0. \tag{3.11}$$

Choose $\eta = 1-\mu > 0$. Combining the relationships (3.10), (3.11) and $\|x_n - y_{n+1}\|^2 \leqslant (1+1/\eta)\|x_n - y_n\|^2 + (1+\eta)\|y_n - y_{n+1}\|^2$ yields

$$\begin{aligned}
&\|x_{n+1} - y\|^2 + (1-\delta_{n+1}) \\
&\qquad \times \Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_{n+1} - y_n\|^2 \\
&\quad \leqslant \|x_n - y\|^2 + \frac{1}{2\delta_n}(1-\delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n-1} - y_n\|^2 + 2(1-\delta_{n+1})\lambda_n f(y_n, y) \\
&\qquad - (1-\delta_{n+1})\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n - y_n\|^2.
\end{aligned} \tag{3.12}$$

From the facts that $\lim\limits_{n\to\infty}\lambda_n = \lambda > 0$, $\mu \in (0,1)$, $\theta \in (1/(2-\mu), 1)$ and $\eta = 1-\mu$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}(1-\delta_{n+1})\Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big) \\
= (1-\delta)\frac{1}{\delta}\Big((1+\eta)\theta - \eta - \frac{\mu}{2}\Big) > 0,
\end{aligned}$$

$$\lim_{n\to\infty}\frac{1}{2\delta_n}(1-\delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}} = \frac{1}{2\delta}(1-\delta)\mu > 0,$$

$$\begin{aligned}
\lim_{n\to\infty}(1-\delta_{n+1})\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big) \\
= (1-\delta)\frac{1}{\delta}\Big(\Big(1+\frac{1}{\eta}\Big)\theta - \frac{1}{\eta}\Big) > 0,
\end{aligned}$$

and

$$(1-\delta)\frac{1}{\delta}\Big((1+\eta)\theta - \eta - \frac{\mu}{2}\Big) - \frac{1}{2\delta}(1-\delta)\mu > 0.$$

Due to the denseness of rational numbers, there exists $\varrho > 0$ such that

$$(1-\delta)\frac{1}{\delta}\Big((1+\eta)\theta - \eta - \frac{\mu}{2}\Big) > \varrho > \frac{1}{2\delta}(1-\delta)\mu.$$

Hence, there exists N_3 ($\geqslant N_2$) such that for all $n \geqslant N_3$,

$$\begin{aligned}
(1-\delta_{n+1})\Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big) \\
> \varrho > \frac{1}{2\delta_n}(1-\delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}} > 0,
\end{aligned} \tag{3.13}$$

$$(1-\delta_{n+1})\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big) > 0. \tag{3.14}$$

On account of the relations (3.12)–(3.14), we obtain for all $n \geqslant N_3$ that

$$\begin{aligned}(3.15)\qquad &\|x_{n+1} - y\|^2 + \varrho\|y_{n+1} - y_n\|^2 \\ &\leqslant \|x_n - y\|^2 + \varrho\|y_{n-1} - y_n\|^2 \\ &\quad - (1 - \delta_{n+1})\Big(\Big(1 + \frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n - y_n\|^2 \\ &\quad + 2(1 - \delta_{n+1})\lambda_n f(y_n, y) \quad \forall\, y \in C.\end{aligned}$$

Note that for each $u \in \mathrm{EP}(f)$, we have that $f(u, y_n) \geqslant 0$, because $y_n \in C$. Furthermore, since f is pseudomonotone, we derive $f(y_n, u) \leqslant 0$. Hence, using relation (3.15) for $y = u \in C$ and $n \geqslant N_3$, setting

$$\begin{aligned}a_n &= \|x_n - u\|^2 + \varrho\|y_{n-1} - y_n\|^2, \\ b_n &= (1 - \delta_{n+1})\Big(\Big(1 + \frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n - y_n\|^2,\end{aligned}$$

we deduce that $a_{n+1} \leqslant a_n - b_n$ for all $n \geqslant N_3$. From Lemma 2.5, we can conclude that $\{a_n\}$ is bounded, $\lim\limits_{n\to\infty} b_n = 0$ and the limit of $\{a_n\}$ exists. Moreover, by the definition of b_n and $\lim\limits_{n\to\infty} \lambda_n = \lambda > 0$, we obtain $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$. In virtue of the relations (3.6), $\|x_n - y_{n+1}\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n+1}\|$ and $\|y_{n+1} - y_n\| \leqslant \|y_{n+1} - x_n\| + \|x_n - y_n\|$, we see that

$$(3.16)\qquad \lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - y_{n+1}\| = \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

Furthermore, by (3.16) and the existence of $\lim\limits_{n\to\infty} a_n$, we can deduce that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \|x_n - u\|^2$. This implies that the sequence $\{x_n\}$ is bounded and so $\{y_n\}$ is bounded. Thus there exists a subsequence $\{x_{n_k}\}$ that converges weakly to some $x^* \in \mathcal{H}$. Then $y_{n_k} \rightharpoonup x^*$, $y_{n_k+1} \rightharpoonup x^*$ and $x^* \in C$. Now we prove that $x^* \in \mathrm{EP}(f)$. Indeed, it follows from relation (3.4) that

$$(3.17)\qquad \lambda_{n_k}(f(y_{n_k}, y) - f(y_{n_k}, y_{n_k+1})) \geqslant \langle x_{n_k} - y_{n_k+1}, y - y_{n_k+1}\rangle \quad \forall\, y \in C.$$

Using the Lipschitz-type condition of f, we have

$$\begin{aligned}\lambda_{n_k} f(y_{n_k}, y_{n_k+1}) &\geqslant \lambda_{n_k}(f(y_{n_k-1}, y_{n_k+1}) - f(y_{n_k-1}, y_{n_k})) \\ &\quad - \lambda_{n_k} c_1\|y_{n_k} - y_{n_k-1}\|^2 - \lambda_{n_k} c_2\|y_{n_k} - y_{n_k+1}\|^2.\end{aligned}$$

Combining the relations (3.7) and the last inequality, we arrive at

$$\begin{aligned}(3.18)\qquad \lambda_{n_k} f(y_{n_k}, y_{n_k+1}) &\geqslant \frac{\lambda_{n_k}}{\lambda_{n_k-1}}\frac{1}{\delta_{n_k}}\langle x_{n_k} - y_{n_k}, y_{n_k+1} - y_{n_k}\rangle \\ &\quad - \lambda_{n_k} c_1\|y_{n_k} - y_{n_k-1}\|^2 - \lambda_{n_k} c_2\|y_{n_k} - y_{n_k+1}\|^2.\end{aligned}$$

From the relations (3.17) and (3.18), it follows that

$$
\begin{aligned}
f(y_{n_k}, y) \geqslant \frac{1}{\lambda_{n_k-1}} \frac{1}{\delta_{n_k}} \langle x_{n_k} - y_{n_k}, y_{n_k+1} - y_{n_k} \rangle + \frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k+1}, y - y_{n_k+1} \rangle \\
- c_1 \|y_{n_k} - y_{n_k-1}\|^2 - c_2 \|y_{n_k} - y_{n_k+1}\|^2,
\end{aligned}
$$

Let $k \to \infty$, using the facts (3.16), $\{x_n\}$ is bounded, $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and the assumption (A4), we obtain $f(x^*, y) \geqslant 0$ for all $y \in C$. That is $x^* \in \mathrm{EP}(f)$.

Next, we prove that the whole sequence $\{x_n\}$ converges weakly to x^*. Assume that $\{x_n\}$ has at least two weak cluster points $x^* \in \mathrm{EP}(f)$ and $\bar{x} \in \mathrm{EP}(f)$ such that $x^* \neq \bar{x}$. Let $\{x_{n_i}\}$ be a sequence such that x_{n_i} converges weakly to $\bar{x}$ as $i \to \infty$, noting the fact that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in \mathrm{EP}(f)$. From Lemma 2.4, we have

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| = \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| < \liminf_{i\to\infty} \|x_{n_i} - x^*\| \\
&= \lim_{n\to\infty} \|x_n - x^*\| = \lim_{k\to\infty} \|x_{n_k} - x^*\| = \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|,
\end{aligned}
$$

which is impossible. Thus, we obtain that $x_n \rightharpoonup x^*$. Since $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have $y_n \rightharpoonup x^*$. This completes the proof. $\square$

Since the proof of the following results is very similar to the proof of Yang and Liu [30], Theorem 3.2, we have omitted it here.

Theorem 3.2. *Assume that* (A1′)–(A3) *hold and* $\mathrm{EP}(f) \neq \emptyset$. *Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 3.1 *converge strongly to the unique solution* u *of the equilibrium problem.*

3.2. *R*-linear convergence rate. Now, we establish the R-convergence rate of the algorithm by using the strong pseudomonotonicity and the Lipschitz-type condition of the bifunction. Let us briefly recall two fundamental concepts of convergence rate in [26], Chapter 9. A sequence $\{x_n\}$ in $\mathcal{H}$ converges to x^* in norm. We say that

(a) $\{x_n\}$ converges to x^* with R-linear convergence rate if

$$
\limsup_{n\to\infty} \|x_n - x^*\|^{1/n} < 1,
$$

(b) $\{x_n\}$ converges to x^* with Q-linear convergence rate if there exists $\mu \in (0, 1)$ such that

$$
\|x_{n+1} - x^*\| \leqslant \mu \|x_n - x^*\|
$$

for all sufficiently large n.

Note that Q-linear convergence rate implies R-linear convergence rate, see [26], Chapter 9. The inverse in general is not true.

Theorem 3.3. *Assume that* (A1$'$), (A2), *and* (A3) *hold. Then the sequence* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 3.1 *converge at least R-linearly to the unique solution of problem* (EP).

P r o o f. The strong pseudomonotonicity assumption of the bifunction f implies that problem (EP) has a unique solution denoted by u. Since $u \in \mathrm{EP}(f)$ and $y_n \in C$, we obtain that $f(u, y_n) \geqslant 0$. Thus, from the strong pseudomonotonicity of f, we derive $f(y_n, u) \leqslant -\gamma\|y_n - u\|^2$. Then, in formula (3.15), letting $y = u \in C$, we obtain

$$
\begin{aligned}
(3.19)\quad & \|x_{n+1} - u\|^2 + (1 - \delta_{n+1}) \\
& \times \Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_{n+1} - y_n\|^2 \\
& \leqslant \|x_n - u\|^2 + \frac{1}{2\delta_n}(1 - \delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n-1} - y_n\|^2 \\
& \quad - 2(1 - \delta_{n+1})\lambda_n\gamma\|y_n - u\| \\
& \quad - (1 - \delta_{n+1})\Big(\Big(1 + \frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n - y_n\|^2,
\end{aligned}
$$

where γ is the modulus of strong pseudomonotonicity of f. Moreover, from the definition of x_n and Lemma 2.2, we obtain

$$
\begin{aligned}
(3.20)\quad \|y_n - u\|^2 &= \frac{1}{1 - \delta_n}\|x_n - u\|^2 - \frac{\delta_n}{1 - \delta_n}\|x_{n-1} - u\|^2 + \frac{\delta_n}{(1 - \delta_n)^2}\|x_n - x_{n-1}\|^2 \\
&\geqslant \frac{1}{1 - \delta_n}\|x_n - u\|^2 - \frac{\delta_n}{1 - \delta_n}\|x_{n-1} - u\|^2.
\end{aligned}
$$

Thus, from the relations (3.19) and (3.20), we see that

$$
\begin{aligned}
(3.21)\quad & \|x_{n+1} - u\|^2 + (1 - \delta_{n+1}) \\
& \times \Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_{n+1} - y_n\|^2 \\
& \leqslant \Big(1 - 2(1 - \delta_{n+1})\frac{1}{1 - \delta_n}\lambda_n\gamma\Big)\|x_n - u\|^2 \\
& \quad + \frac{1}{2\delta_n}(1 - \delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n-1} - y_n\|^2 \\
& \quad - (1 - \delta_{n+1})\Big(\Big(1 + \frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n - y_n\|^2 \\
& \quad + 2(1 - \delta_{n+1})\frac{\delta_n}{1 - \delta_n}\lambda_n\gamma\|x_{n-1} - u\|^2.
\end{aligned}
$$

Let ζ be a real number such that $0 < \zeta < (\delta^{-1} - 1)/(\delta^{-1} + 1)$. Consider the limits

$$\lim_{n\to\infty} 2(1-\delta_{n+1})\frac{1}{1-\delta_n}\lambda_n\gamma = 2\lambda\gamma > 2\lambda\gamma(1-\zeta),$$
$$\lim_{n\to\infty} 2\frac{\delta_n}{1-\delta_n}(1-\delta_{n+1})\lambda_n\gamma = 2\delta\lambda\gamma < 2\delta\lambda\gamma(1+\zeta).$$

By analogy with the proof of relation (3.13) and using the density of rational numbers, there exists $\varrho_1, \varrho_2 > 0$, such that

$$(1-\delta)\frac{1}{\delta}\Big(1 - \frac{\mu}{2}(1+\eta)(1-\theta)\Big) > \varrho_2 > \varrho_1 > \frac{1}{2\delta}(1-\delta)\mu.$$

These facts imply that there exists N_4 ($\geqslant N_3$), such that for all $n \geqslant N_4$,

$$(1-\delta_{n+1})\Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}} - \frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big) > \varrho_2 > \varrho_1 > \frac{1}{2\delta_n}(1-\delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}}, \tag{3.22}$$

$$2(1-\delta_{n+1})\frac{1}{1-\delta_n}\lambda_n\gamma > 2\lambda\gamma(1-\zeta), \tag{3.23}$$

$$2\frac{\delta_n}{1-\delta_n}(1-\delta_{n+1})\lambda_n\gamma < 2\delta\lambda\gamma(1+\zeta). \tag{3.24}$$

In virtue of the relations (3.21)–(3.24), we have that for all $n \geqslant N_4$,

$$\|x_{n+1} - u\|^2 + \varrho_2\|y_{n+1} - y_n\|^2 \leqslant (1 - 2(1-\zeta)\lambda\gamma)\|x_n - u\|^2 + 2\delta(1+\zeta)\lambda\gamma\|x_{n-1} - u\|^2 + \varrho_1\|y_{n-1} - y_n\|^2. \tag{3.25}$$

Setting $T_n = \|x_n - u\|^2$, $E_n = \varrho_2\|y_n - y_{n-1}\|^2$ and $\alpha = 2\lambda\gamma(1-\zeta) > 0$, the inequality (3.25) can be rewritten as

$$T_{n+1} + E_{n+1} \leqslant (1-\alpha)T_n + \frac{1+\zeta}{1-\zeta}\delta\alpha T_{n-1} + \frac{\varrho_1}{\varrho_2}E_n. \tag{3.26}$$

Let $\gamma_1 > 0$ and $\gamma_2 > 0$. Now, we can rewrite relation (3.26) in the following form:

$$T_{n+1} + \gamma_1 T_n + E_{n+1} \leqslant \gamma_2(T_n + \gamma_1 T_{n-1}) + \frac{\varrho_1}{\varrho_2}E_n + (1 - \alpha - \gamma_2 + \gamma_1)T_n + \Big(\frac{1+\zeta}{1-\zeta}\delta\alpha - \gamma_1\gamma_2\Big)T_{n-1}. \tag{3.27}$$

Choose $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\begin{cases} 1 - \alpha - \gamma_2 + \gamma_1 = 0, \\ \dfrac{1+\zeta}{1-\zeta}\delta\alpha - \gamma_1\gamma_2 = 0. \end{cases} \tag{3.28}$$

By a straightforward computation, we obtain

$$\gamma_1 = \frac{1}{2}\Big(\alpha - 1 + \sqrt{(\alpha-1)^2} + 4\frac{1+\zeta}{1-\zeta}\delta\alpha\Big),$$
$$\gamma_2 = \frac{1}{2}\left(1 - \alpha + \sqrt{(\alpha-1)^2 + 4\frac{1+\zeta}{1-\zeta}\delta\alpha}\right).$$

Then, we studied the following function:

$$h(t) = \frac{1}{2}\left(1 - t + \sqrt{(t-1)^2 + 4\frac{1+\zeta}{1-\zeta}\delta t}\right), \quad t \in [0,\infty),$$

whose derivative is given by the following formula:

$$\begin{aligned} h'(t) = &-\frac{1}{2} + \Big(t - 1 + 2\frac{1+\zeta}{1-\zeta}\delta\Big)\left(2\sqrt{(t-1)^2 + 4\frac{1+\zeta}{1-\zeta}\delta t}\right)^{-1} \\ = &\,4\frac{1+\zeta}{1-\zeta}\delta\Big(\frac{1+\zeta}{1-\zeta}\delta - 1\Big)\left(2\sqrt{(t-1)^2 + 4\frac{1+\zeta}{1-\zeta}\delta t}\right)^{-1} \\ &\times\left(t - 1 + 2\frac{1+\zeta}{1-\zeta}\delta + \sqrt{(t-1)^2 + 4\frac{1+\zeta}{1-\zeta}\delta t}\right)^{-1} \\ < &\,0. \end{aligned}$$

Since $0 < \zeta < (\delta^{-1} - 1)/(\delta^{-1} + 1)$, $h(t)$ is non-increasing on $[0,\infty)$. Therefore, $0 < \gamma_2 = h(\alpha) < h(0) = 1$ is established. Now, set $\varepsilon = \max\{\varrho_1/\varrho_2, \gamma_2\}$. It is known from the definitions of ϱ_1 and ϱ_2 that $0 < \varrho_1/\varrho_2 < 1$, so $\varepsilon \in (0,1)$. Combining (3.27) and (3.28), we obtain

$$\begin{aligned} T_{n+1} + \gamma_1 T_n + E_{n+1} &\leqslant \gamma_2(T_n + \gamma_1 T_{n-1}) + \frac{\varrho_1}{\varrho_2}E_n \\ &\leqslant \varepsilon(T_n + \gamma_1 T_{n-1} + E_n) \quad \forall\, n \geqslant N_4. \end{aligned} \tag{3.29}$$

Therefore, we deduce by induction that

$$T_{n+1} + \gamma_1 T_n + E_{n+1} \leqslant \varepsilon^{n-N_4+1}(T_{N_4} + \gamma_1 T_{N_4-1} + E_{N_4}). \tag{3.30}$$

From the relations $T_{n+1} \leqslant T_{n+1} + \gamma_1 T_n + E_{n+1}$ and (3.30), there exists a positive constant M such that for all $n \geqslant N_4$,

$$\begin{aligned} \|x_n - u\|^2 &\leqslant \varepsilon^{n-N_4+1}(T_{N_4} + \gamma_1 T_{N_4-1} + E_{N_4}) \\ &= \varepsilon^n \frac{T_{N_4} + \gamma_1 T_{N_4-1} + E_{N_4}}{\varepsilon^{N_4-1}} = \varepsilon^n M. \end{aligned} \tag{3.31}$$

This is $\limsup\limits_{n\to 0} \|x_n - u\|^{1/n} = \varepsilon^{1/2} < 1$. Hence, the sequence $\{x_n\}$ R-linearly converges to the unique solution of the problem (EP).

Similarly, the following formula is established:

$$E_{n+1} = \varrho_2 \|y_{n+1} - y_n\|^2 \leqslant T_{n+1} + \gamma_1 T_n + E_{n+1} \leqslant \varepsilon^n M.$$

Therefore, we deduce that $\|y_{n+1} - y_n\| \leqslant \sqrt{M/\varrho_2}\varepsilon^{n/2}$. That is,

$$\limsup_{n\to 0} \|y_{n+1} - y_n\|^{1/n} = \varepsilon^{1/2} < 1.$$

Thus, the sequence $\{y_{n+1} - y_n\}$ is R-linearly convergent. Since

$$\begin{aligned}
\|y_n - y_{n+p}\| &\leqslant \|y_n - y_{n+1} + y_{n+1} - \ldots - y_{n+p-1} + y_{n+p}\| \\
&\leqslant \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + \ldots + \|y_{n+p-1} + y_{n+p}\| \\
&\leqslant \sqrt{\frac{M}{\varrho_2}}(\varepsilon^{n/2} + \varepsilon^{(n+1)/2} + \ldots + \varepsilon^{(n+p-1)/2}) \\
&= \sqrt{\frac{M}{\varrho_2}}\frac{\varepsilon^{n/2}}{1-\sqrt{\varepsilon}}(1-\varepsilon^{p/2}) < \sqrt{\frac{M}{\varrho_2}}\frac{\varepsilon^{n/2}}{1-\sqrt{\varepsilon}},
\end{aligned}$$

it is obvious that $\{y_n\}$ is a Cauchy sequence. Therefore, $\{y_n\}$ strongly converges to u. In the above formula, seting $p \to \infty$, we deduce

$$\|y_n - u\| \leqslant \sqrt{\frac{M}{\varrho_2}}\frac{\varepsilon^{n/2}}{1-\sqrt{\varepsilon}}.$$

Hence, $\limsup_{n\to\infty} \|y_{n+1} - u\|^{1/n} = \varepsilon^{1/2} < 1$. That is, the sequence $\{y_n\}$ R-linearly converges to the unique solution of the problem $\mathrm{EP}(f)$. This completes the proof. □

4. Application and analysis of variational inequality

In this section, inspired by the work of [22], [14], [13], and [16], we consider the application of the above results to variational inequality problems. The classical variational inequality problem (VI) is to find $x^* \in C$ such that

$$\langle F(x^*), y - x^*\rangle \geqslant 0 \quad \forall\, y \in C, \tag{4.1}$$

where C is a nonempty closed convex set in a real Hilbert space $\mathcal{H}$, $F\colon \mathcal{H} \to \mathcal{H}$ is some given mapping. We denote the set of solutions of the problem (4.1) by $\mathrm{VI}(F, C)$.

It has been universally acknowledged that $x^* \in \mathrm{VI}(F, C)$ if and only if it satisfies the following projection equation

$$x^* = P_C(x^* - \lambda F(x^*)),$$

where λ is any positive real number. For solving the variational inequality, we suppose that $\mathrm{VI}(F, C) \neq \emptyset$ and we investigate problem (VI) under the following widely used conditions:

(B1) F is pseudo-monotone on C, i.e., $\langle F(y), x - y\rangle \geqslant 0 \to \langle F(x), x - y\rangle \geqslant 0$ for all $x, y \in \mathcal{H}$;

(B1′) F is strongly pseudomonotone on C, i.e., $\langle F(y), x - y\rangle \geqslant 0 \to \langle F(x), x - y\rangle \geqslant \gamma\|x - y\|^2$ for all $x, y \in \mathcal{H}$;

(B2) F is Lipschitz continuous on C with constant L, i.e., there exists $L > 0$ such that $\|F(x) - F(y)\| \leqslant L\|x - y\|$ for all $x, y \in \mathcal{H}$;

(B3) F is sequentially weakly continuous on C, i.e., for each sequence $\{x_n\}$: $\{x_n\}$ converges weakly to x implies $\{F(x_n)\}$ converges weakly to $\{F(x)\}$.

We define $f(x, y) := \langle F(x), y - x\rangle$ for all $x, y \in C$; then simple algebra shows that (ignoring constant terms)

$$\mathrm{prox}_{\lambda_n f(y_n,\cdot)}(x_n) = P_C(x_n - \lambda_n F(y_n)).$$

Therefore, the equilibrium problem (1.1) becomes the variational inequality problem. In this framework we consider the algorithm as follows:

Algorithm 4.1.

Step 0. Choose $\lambda_0 = \lambda_1 > 0$, $x_0, y_0, y_1 \in \mathcal{H}$, $\mu \in (0, 1)$, $\theta \in (1/(2 - \mu), 1)$. Choose a non-negative real sequence $\{p_n\}$ such that $\sum\limits_{n=0}^{\infty} p_n < \infty$.

Step 1. Given the current iterates x_{n-1}, y_{n-1}, y_n, compute

$$\delta_n = \min\Big\{\frac{1}{2}\sqrt{1 + 4\theta\frac{\lambda_n}{\lambda_{n-1}}} - 1, 1\Big\},$$
$$x_n = (1 - \delta_n)y_n + \delta_n x_{n-1},$$
$$y_{n+1} = P_C(x_n - \lambda_n F(y_n)).$$

If $y_{n+1} = x_n = y_n$, then stop: x_n is a solution. Otherwise, go to Step 2.

Step 2. Compute

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu(\|y_n - y_{n-1}\|^2 + \|y_{n+1} - y_n\|^2)}{4\delta_n\langle F(y_{n-1}) - F(y_n), y_{n+1} - y_n\rangle}, \lambda_n + p_n\Big\} & \text{if } \langle F(y_{n-1}) - F(y_n), y_{n+1} - y_n\rangle > 0, \\ \lambda_n + p_n & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return to Step 1.

Theorem 4.1. *Assume that* (B1)–(B3) *hold. Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 4.1 *converge weakly to the solution of the variational inequality problem.*

Proof. See Yang and Liu [30], Theorem 4.2 for detailed proof of this theorem. □

Similarly, from Theorem 3.2, the following theorem can be directly obtained.

Theorem 4.2. *Suppose* (B1′), (B2) *are satisfied and* $\mathrm{VI}(F,C) \neq \emptyset$. *Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 4.1 *R-linearly converge to the unique solution of the problem* (VI).

In particular, for many (VI) methods it is worth noticing that the linear convergence rate could be derived under some additional assumptions. The most commonly used tool is to use the error bound. For the investigation of the error bound, we recommend the reader to refer to [18].

Let us fix some $\lambda > 0$ and define the natural residual $r(x,\lambda) = x - P_C(x - \lambda F(x))$. Through the characteristics of the solution to the variational inequality problem, it is plain that $x \in \mathrm{VI}(F,C) \Leftrightarrow r(x,\lambda) = 0$.

We say that problem (4.1) satisfies an error bound condition if the variational inequality problem has a solution and there exist positive constants σ_1 and σ_2 such that

$$\mathrm{dist}(x, \mathrm{VI}(F,C)) \leqslant \sigma_1 \|r(x,\lambda)\| \quad \forall\, x\colon\ \|r(x,\lambda)\| \leqslant \sigma_2. \tag{4.2}$$

Remark 4.1. No doubt, it is not an easy task to decide whether (4.2) holds for a particular problem. When F is strongly pseudomonotone and satisfies the Lipschitz condition, the error bound condition (4.2) holds, see [18] for details.

In the analysis below, our study focus on showing the convergence rate of the sequences $\{x_n\}$ and $\{y_n\}$. For the same reason we assume that the function F is pseudomonotone and satisfies the Lipschitz condition. Choose any $\lambda > 0$ such that $\lambda_n > \lambda$ for all n. Since $\lambda \to \|r(x_n,\lambda)\|$ is non-decreasing, we can get

$$\|r(x_n,\lambda)\| \leqslant \|r(x_n,\lambda_n)\|.$$

Using $y_{n+1} = P_C(x_n - \lambda_n F(y_n))$, the non-expansibility of P_C, the Lipschitz continuity of mapping F and the property of the triangle inequality, we obtain

$$\begin{aligned} \|r(x_n,\lambda)\| &\leqslant \|r(x_n,\lambda_n)\| = \|x_n - P_C(x_n - \lambda_n F(x_n))\| \\ &= \|x_n - y_{n+1} + y_{n+1} - P_C(x_n - \lambda_n F(x_n))\| \\ &\leqslant \|x_n - y_{n+1}\| + \|P_C(x_n - \lambda_n F(y_n)) - P_C(x_n - \lambda_n F(x_n))\| \\ &\leqslant \|x_n - y_n + y_n - y_{n+1}\| + \lambda_n L\|x_n - y_n\| \\ &\leqslant \|y_{n+1} - y_n\| + (1 + \lambda_n L)\|x_n - y_n\|. \end{aligned} \tag{4.3}$$

Using the relationship (4.3) and the inequality $(a+b)^2 \leqslant a^2+b^2$, we can get

$$\|r(x_n,\lambda)\|^2 \leqslant 2\|y_{n+1}-y_n\|^2 + 2(1+\lambda_n L)^2\|x_n-y_n\|. \tag{4.4}$$

Obviously, for all $n \geqslant N_3$ and $\tau \in (0,1)$, using inequalities (3.13) and (3.14), there exist $G_1, G_2 > 0$, such that

$$(1-\delta_{n+1})\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big) > 2G_1,$$
$$(1-\delta_{n+1})\frac{\tau}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}} > 2G_2.$$

The above relations ensure that there is $m \in (0,1)$ with $m\sigma_1^2 \leqslant G_1$, $m\sigma_1^2 \times (1+\lambda_n L)^2 \leqslant G_2$. Hence, we have

$$\begin{aligned}(1-\delta_{n+1})&\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n-y_n\|^2\\ &+ (1-\delta_{n+1})\frac{\tau}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n+1}-y_n\|^2\\ &\geqslant 2G_1\|x_n-y_n\|^2 + 2G_2\|y_{n+1}-y_n\|^2\\ &\geqslant m\sigma_1^2(2\|x_n-y_n\|^2 + 2(1+\lambda_n L)^2\|y_{n+1}-y_n\|^2)\\ &\geqslant m\sigma_1^2\|r(x_n,\lambda)\|^2.\end{aligned} \tag{4.5}$$

The existence of so many constants in (4.5) is to prepare for the subsequent proof. In order to continue to study the convergence rate of the algorithm, we have to modify Algorithm 3.1, so we take the stepsize by

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu((1-m)\|y_n-y_{n-1}\|^2 + (1-\tau)\|y_{n+1}-y_n\|^2)}{4\delta_n\langle F(y_{n-1})-F(y_n), y_{n+1}-y_n\rangle}, \lambda_n+p_n\Big\} \\ \qquad\qquad\qquad\qquad \text{if } \langle F(y_{n-1})-F(y_n), y_{n+1}-y_n\rangle > 0,\\ \lambda_n+p_n \qquad\qquad\quad \text{otherwise.}\end{cases} \tag{4.6}$$

This modification basically means that we have limited the step size slightly. From the relations (3.8) and (4.6) it follows that

$$\begin{aligned}\frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}&((1-m)\|y_{n-1}-y_n\|^2 + (1-\tau)\|y_n-y_{n+1}\|^2) + 2\lambda_n\langle F(y_n), y-y_n\rangle\\ &\geqslant 2\lambda_n\langle F(y_{n-1})-F(y_n), y_{n+1}-y_n\rangle + 2\lambda_n\langle F(y_n), y-y_n\rangle\\ &\geqslant \|x_n-y_{n+1}\|^2 + \|y_{n+1}-y\|^2 - \|x_n-y\|^2\\ &\quad + \frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}(\|x_n-y_n\|^2 + \|y_{n+1}-y_n\|^2 - \|x_n-y_{n+1}\|^2).\end{aligned} \tag{4.7}$$

By injecting the inequality (4.7) into (3.12), we have

$$\begin{aligned}
(4.8)\quad & \|x_{n+1}-y\|^2+(1-\delta_{n+1}) \\
&\quad \times\Big((1+\eta)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n}-\eta\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}-\frac{1}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_{n+1}-y_n\|^2 \\
&\leqslant \|x_n-y\|^2+\frac{1-m}{2\delta_n}(1-\delta_{n+1})\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n-1}-y_n\|^2 \\
&\quad -(1-\delta_{n+1})\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n}-\frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n-y_n\|^2 \\
&\quad -(1-\delta_{n+1})\frac{\tau}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n+1}-y_n\|^2 \\
&\quad +2(1-\delta_{n+1})\lambda_n\langle F(y_n),y-y_n\rangle.
\end{aligned}$$

From $u\in\mathrm{VI}(F,C)$ and $y_n\in C$, we get $F(u,y_n-u)\geqslant 0$. Furthermore, using the pseudomonotonicity of F, we have $F(y_n,u-y_n)\leqslant 0$. In the formula (3.23), let $y=u\in C$. With the same treatment of inequality (3.13) and (3.14), for any $n\geqslant N_3$, the previous inequality can be rewritten as

$$\begin{aligned}
(4.9)\quad & \|x_{n+1}-u\|^2+\varrho\|y_{n+1}-y_n\|^2 \\
&\leqslant \|x_n-u\|^2+(1-m)\varrho\|y_{n-1}-y_n\|^2 \\
&\quad -(1-\delta_{n+1})\Big(\Big(1+\frac{1}{\eta}\Big)\frac{1}{\delta_{n+1}}\theta\frac{\lambda_{n+1}}{\lambda_n}-\frac{1}{\eta}\frac{1}{\delta_n}\frac{\lambda_n}{\lambda_{n-1}}\Big)\|x_n-y_n\|^2 \\
&\quad -(1-\delta_{n+1})\frac{\tau}{2\delta_n}\mu\frac{\lambda_n}{\lambda_{n+1}}\|y_{n+1}-y_n\|^2.
\end{aligned}$$

Using (4.5), we get

$$\begin{aligned}
(4.10)\quad & \|x_{n+1}-u\|^2+\varrho\|y_{n+1}-y_n\|^2 \\
&\leqslant \|x_n-u\|^2+(1-m)\varrho\|y_{n-1}-y_n\|^2-m\sigma_1^2\|r(x_n,\lambda)\|^2.
\end{aligned}$$

Now, taking $u=P_{\mathrm{VI}(F,C)}(x_n)\in\mathrm{VI}(F,C)$, we get

$$(4.11)\qquad d(x_n,\mathrm{VI}(F,C))=\|x_n-u\|,\quad d(x_{n+1},\mathrm{VI}(F,C))\leqslant\|x_{n+1}-u\|.$$

According to the relationship (4.9) and $m\in(0,1)$, we know that $\|x_n-y_n\|\to 0$, $\|y_{n+1}-y_n\|\to 0$ when $n\to\infty$. Therefore, from the relationship (4.3), we see that the sequence $\{\|r(x_n,\lambda)\|\}$ converges to 0. Therefore, for any $n\geqslant N_5$, $\|r(x_n,\lambda)\|\leqslant\sigma_2$ can be derived. Combining the relational expressions (4.10), (4.11) and error bounds (4.2), it is obvious that

$$
\begin{aligned}
(4.12)\qquad & d(x_{n+1}, \mathrm{VI}(F,C))^2 + \varrho\|y_{n+1} - y_n\|^2 \\
&\leqslant \|x_{n+1} - u\|^2 + \varrho\|y_{n+1} - y_n\|^2 \\
&\leqslant \|x_n - u\|^2 + (1-m)\varrho\|y_{n-1} - y_n\|^2 - m\sigma_1^2\|r(x_n,\lambda)\|^2 \\
&\leqslant d(x_n, \mathrm{VI}(F,C))^2 + (1-m)\varrho\|y_{n-1} - y_n\|^2 - md(x_n, \mathrm{VI}(F,C))^2 \\
&= (1-m)(d(x_n, \mathrm{VI}(F,C))^2 + \varrho\|y_{n-1} - y_n\|^2).
\end{aligned}
$$

Set $a_n = d(x_n, \mathrm{VI}(F,C))^2 + \varrho\|y_{n-1} - y_n\|$. Therefore, we obtain $a_{n+1} \leqslant (1-m)a_n$. The remaining proof is similar to the proof of Theorem 3.3. Therefore, the following conclusions can be drawn through mathematical induction.

Theorem 4.3. *Under the assumptions* (B1′), (B2) *and error bound condition* (4.2), *the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm* 4.1 *replacing the original step rule with the step rule of* (4.6) *R-linearly converge to a certain solution of problem* (4.1).

5. Numerical experiments

In this section, we provide numerical experiments and compare the proposed algorithm with other existing algorithms in [22], [30], [15] to illustrate the advantages of our algorithm. First, we compare Algorithm 3.1 with the Algorithm 3.1 in [30] and Algorithm 3.1 in [15]. Then we compare Algorithm 4.1 with the Algorithm 4.1 in [30] and Algorithm 1 in [22]. In the numerical results reported below, Iter. and Time. denote the number of iterations and the CPU time in seconds, respectively.

Since $x \in \mathrm{EP}(f)$ if and only if the natural residual $r(x,\lambda) = 0$, the error bound can be used to construct practical stopping rules for these methods so that the final iterate will satisfy any prescribed level of accuracy. It is natural to use the following stopping criteria:

- ▷ Alg. 3.1, Alg. 3.1 in [30], Alg. 3.1 in [15], $\|y_n - \mathrm{prox}_{f(y_n,\cdot)}(y_n)\| \leqslant \varepsilon$.
- ▷ Alg. 4.1, Alg. 4.1 in [30], Alg. 1 in [22], $\|y_n - P_C(y_n - F(y_n))\| \leqslant \varepsilon$.

For Algorithm 3.1 and Algorithm 4.1, we take $\mu = 0.8$, $\lambda_0 = 0.9$ and $p_n = 1/(1+n)^2$. For Algorithm 1 in [22], we take $\lambda_1 = 0.2$ and $\phi = 0.1 + 0.9\varphi$. For Algorithm 4.1 in [30], we choose $\alpha = \mu = 0.98$, $\theta = 0.75$ and $\delta = 0.53$. For Algorithm 3.1 in [15], we use $\mu = 0.45\varphi$ and $\lambda_1 = 3$.

Problem 5.1. Consider the equilibrium problem given in [30], [15], where the bifunction $f\colon \mathbb{R}^m \times \mathbb{R}^m \to R$ is defined for every $x, y \in \mathbb{R}^m$ by

$$f(x, y) = \langle Px + Qy + q, y - x\rangle,$$

where the vector $q \in \mathbb{R}^m$ is chosen randomly with its elements in $[-m, m]$, and the matrices P and Q are two square matrices of order m such that Q is symmetric positive semidefinite and $Q - P$ is negative semidefinite. In this case, the bifunction satisfies the conditions (A1)–(A4) . For Algorithm 3.1 in [30], we take $\lambda_0 = 1/\|P - Q\|$. For Problem 5.1, we take $\varepsilon = 10^{-3}$.

To illustrate our algorithms, we suppose that the feasible set $C \subset \mathbb{R}^m$ has the form of

$$C = \{x \in \mathbb{R}^m\colon\ -2 \leqslant x_i \leqslant 5,\ i = 1, \dots, m\},$$

where $m = 5, 50, 200$. We take the same starting point $y_1 = x_0 = y_0 = (1, \dots, 1)$ for all algorithms. For every m, as shown in Table 1, we have generated two random samples with different choice of P, Q and q. Comparing Algorithm 3.1 with the Algorithm 3.1 in [30] and Algorithm 3.1 in [15] in Table 1, we can see that our algorithm performs better.

m	Algorithm 3.1		Algorithm 3.1 in [30]		Algorithm 3.1 in [15]	
	Iter.	Time	Iter.	Time	Iter.	Time
5	56	1.21	67	1.48	83	1.93
	33	0.73	46	1.17	56	1.36
50	137	3.99	204	5.63	263	7.68
	178	5.54	365	10.87	446	13.43
200	238	55.04	360	80.93	428	99.26
	217	50.64	338	81.54	446	106.13

Table 1. Problem 5.1.

Problem 5.2. The second problem is the Kojima-Shindo Nonlinear Complementarity Problem (NCP) where $n = 4$ and the mapping F is defined by

$$F(x_1, x_2, x_3, x_4) = \begin{bmatrix} 3x_1 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}.$$

The feasible set is $C = \{\mathbb{R}_4^+ \mid x_1 + x_2 + x_3 + x_4 = 4\}$. We choose the same starting points $y_1 = x_0 = y_0 = (1, 1, 1, 1)$, $y_1 = x_0 = y_0 = (1, 1, 0, 1)$ and $y_1 = x_0 =$

$y_0 = (2, 0, 0, 2)$. We take $\lambda_1 = 0.8$ for Algorithm 4.1 in [30]. In this case, we take $\varepsilon = 10^{-6}$. The numerical results are shown in Table 2. From these results we can see that Algorithm 4.1 in [30] and Algorithm 1 in [22] are much more expensive than our method.

x_0	Algorithm 4.1		Algorithm 4.1 in [30]		Algorithm 1 in [22]	
	Iter.	Time	Iter.	Time	Iter.	Time
(1,1,1,1)	51	0.45	65	0.67	127	1.14
(1,1,0,1)	5	0.03	13	0.09	60	0.39
(2,0,0,2)	13	0.09	54	0.48	120	1.48

Table 2. Problem 5.2.

Problem 5.3. The third example is classical. The feasible set is $C = \mathbb{R}^m$ and $F(x) = Ax$, where A is a square matrix $m \times m$ given by condition

$$a_{i,j} = \begin{cases} -1 & \text{if } j = m + 1 - i \text{ and } j > i, \\ 1 & \text{if } j = m + 1 - i \text{ and } j < i, \\ 0 & \text{otherwise.} \end{cases}$$

This is a classical example of a problem, where the usual gradient method does not converge. For even m, the zero vector is the solution of Problem 5.3. For the Algorithm 4.1 in [30], we take $\lambda_0 = 0.4$. For all tests, we choose $x_0 = (1, 1, \dots, 1)$. We take $\varepsilon = 10^{-6}$. The numerical results are shown in Table 3. One can see that Algorithm 4.1 substantially outperforms Algorithm 4.1 in [30] and Algorithm 1 in [22].

m	Algorithm 4.1		Algorithm 4.1 in [30]		Algorithm 1 in [22]	
	Iter.	Time	Iter.	Time	Iter.	Time
100	57	0	682	0.34	148	0.03
1000	65	0.22	730	1.21	157	0.359
2000	66	0.99	745	7.92	160	1.59

Table 3. Problem 5.3.

6. Conclusions

In this paper, we deal with the convergence results for equilibrium problems involving the pseudomonotone and Lipschitz-type bifunction in a real Hilbert space. This method uses a new non-monotonic stepsize. The convergence and the R-linear convergence rate of the algorithms have been obtained. Moreover, the method is applied to a variational inequality. The numerical experiments are reported to illustrate the computational effectiveness of the proposed algorithm.

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Authors' addresses: *Lulu Yin*, *Hongwei Liu* (corresponding author), School of Mathematics and Statistics, Xidian University, Xi'an, Shaanxi, 710126, P. R. China, e-mail: `yinluluxidian@163.com`, `hwliu@mail.xidian.edu.cn`; *Jun Yang*, School of Mathematics and Information Science, Xianyang Normal University, Xianyang 712000, P. R. China, e-mail: `xysyyangjun@163.com`.