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GENERALIZATIONS ON THE RESULTS OF CAO AND ZHANG

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Abstract. We establish some uniqueness results for meromorphic functions when two nonlinear differential polynomials $P(f) \prod_{i=1}^k (f^{(i)})^{n_i}$ and $P(g) \prod_{i=1}^k (g^{(i)})^{n_i}$ share a nonzero polynomial with certain degree and our results improve and generalize some recent results in Y.-H. Cao, X.-B. Zhang (2012). Also we exhibit two examples to show that the conditions used in the results are sharp.

Keywords: meromorphic function; uniqueness; weighted sharing; differential polynomial

MSC 2020: 30D35

1. INTRODUCTION AND PRELIMINARY RESULTS

In this entire paper we mean by meromorphic functions those complex valued functions which have poles as the only singularities in \mathbb{C} . In this paper we use the standard notations of the value distribution theory (see [8]). We define the function $T(r)$ by $T(r) = \max\{T(r, f), T(r, g)\}$. The function $S(r)$ is defined by $S(r) = o(T(r))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite linear measure. If $T(r, a) = S(r, f)$, then we say that $a(z)$ is a small function with respect to $f(z)$. If $f(z_0) = z_0$, then z_0 is called a fixed point of $f(z)$.

Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a -points of $f(z)$, where an a -point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If we have for two meromorphic functions $f(z)$ and $g(z)$ that $E_k(a; f) = E_k(a; g)$, then we say that $f(z)$ and $g(z)$ share a with weight k . The IM and CM sharing correspond to the weight 0 and ∞ , respectively. If $a(z)$ is a small function we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l depending on whether $f(z) - a(z)$ and $g(z) - a(z)$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$, respectively.

The following well known theorem in value distribution theory was posed by Hayman (see [8]) and settled by several authors almost at the same time, see [3]–[5].

Theorem A. *Let $f(z)$ be a transcendental meromorphic function and $n \in \mathbb{N}$. Then $f^n(z)f'(z) = 1$ has infinitely many solutions.*

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua in [6], and Yang and Hua in [16] obtained the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant entire (or meromorphic) functions and $n \in \mathbb{N}$ such that $n \geq 6$ (or $n \geq 11$, respectively). If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$, $c, c_1, c_2 \in \mathbb{C}$ such that $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ such that $t^{n+1} = 1$.*

In 2002 Fang and Qiu (see [7]) considered the uniqueness problems of entire or meromorphic functions having fixed points and they obtained the following result.

Theorem C. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic (or entire) functions and $n \in \mathbb{N}$ such that $n \geq 11$ (or $n \geq 6$, respectively). If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$ and $g(z) = c_2e^{-cz^2}$, $c, c_1, c_2 \in \mathbb{C}$ such that $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ such that $t^{n+1} = 1$.*

We now recall the following results due to Xu et al. (see [13]) or Zhang and Li (see [20]), respectively.

Theorem D. *Let $f(z)$ be a transcendental meromorphic function and $k \in \mathbb{N}$, $n \in \mathbb{N} \setminus \{1\}$. Then $f^n(z)f^{(k)}(z)$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Also the following recent results are due to Cao and Zhang, see [4].

Theorem E. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions whose zeros are of multiplicities at least k , where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n > \max\{2k - 1, k + 4/k + 4\}$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds:*

- (i) $f^n(z)f^{(k)}(z) \equiv g^n(z)g^{(k)}(z)$;
- (ii) $f(z) = c_1e^{cz^2}$ and $g(z) = c_2e^{-cz^2}$, where $c, c_1, c_2 \in \mathbb{C}$ such that $4(c_1c_2)^{n+1}c^2 = -1$.

Theorem F. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros are of multiplicities at least k , where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n > \max\{2k - 1, k + 4/k + 4\}$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share 1 CM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds:*

- (i) $f^n(z)f^{(k)}(z) \equiv g^n(z)g^{(k)}(z)$;
- (ii) $f(z) = c_3e^{dz}$, $g(z) = c_4e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k(c_3c_4)^{n+1}d^{2k} = 1$.

Theorem G. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros are of multiplicities at least $k + 1$, where $k \in \mathbb{N}$ with $1 \leq k \leq 5$. Let $n \in \mathbb{N}$ such that $n \geq 10$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share 1 CM, $f^{(k)}(z)$ and $g^{(k)}(z)$ share 0 CM, $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds:

- (i) $f(z) \equiv tg(z)$, $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+1} = 1$;
- (ii) $f(z) = c_3e^{dz}$, $g(z) = c_4e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k(c_3c_4)^{n+1}d^{2k} = 1$.

Now the following questions are inquisitive to any researcher:

Question 1. Is it possible to reduce the lower bound of n in Theorems E–G?

Question 2. Is it possible to weaken more the condition “Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions whose zeros are of multiplicities at least $k + 1$, where $k \in \mathbb{N}$ ” in Theorem G?

Question 3. Does Theorem G hold for $k \geq 6$?

Question 4. Can one further deduce generalized forms of Theorems E–G?

2. MAIN RESULTS AND SOME DEFINITIONS

Throughout this paper, for the sake of simplicity we use the following notations

$$n_i^* = \begin{cases} 0 & \text{if } n_i = 0, \\ 1 & \text{if } n_i \neq 0, \end{cases} \quad \text{and} \quad n_i^{**} = \begin{cases} 0 & \text{if } n_i = 0, \\ n_i & \text{if } n_i \neq 0, \end{cases}$$

where $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \dots, k - 1$ and $k, n_k \in \mathbb{N}$. Also we use $t = \sum_{i=1}^k n_i^*$, $m = \sum_{i=1}^k in_i^*$, $s = \sum_{i=1}^k n_i^{**}$, $m_1 = \sum_{i=1}^k in_i^{**}$ and $n^* = \min\{i : i \in \{1, \dots, k\} \text{ with } n_i \neq 0\}$.

In this paper we use $P(z)$ to denote an arbitrary non-constant polynomial of degree n ,

$$(2.1) \quad P(z) = a_n(z - c_1)^{d_1}(z - c_2)^{d_2} \dots (z - c_{s_1})^{d_{s_1}},$$

where $a_n \in \mathbb{C} \setminus \{0\}$ and $c_j \in \mathbb{C}$ ($j = 1, 2, \dots, s_1$) are distinct; d_1, d_2, \dots, d_{s_1} , $n \in \mathbb{N}$ with $\sum_{i=1}^{s_1} d_i = n$. Let $d = \max\{d_1, d_2, \dots, d_{s_1}\}$ and c be the corresponding zero of $P(z)$ with multiplicity d . We define

$$P_1(z) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s_1} (z - c_i)^{d_i} = b_{m_2}z^{m_2} + b_{m_2-1}z^{m_2-1} + \dots + b_0,$$

where $a_n = b_{m_2}$ and $m_2 = n - d$. Obviously $P(z) = (z - c)^d P_1(z)$. We also use $P_2(z_1)$ as an arbitrary nonzero polynomial defined by

$$P_2(z_1) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s_1} (z_1 + c - c_i)^{d_i} = e_{m_2} z_1^{m_2} + e_{m_2-1} z_1^{m_2-1} + \dots + e_0,$$

where $z_1 = z - c$ and $\deg(P_2) = m_2 \geq 0$. Obviously $P(z) = z_1^d P_2(z_1)$. Suppose $\Gamma_1 = m_3 + m_4$ and $\Gamma_2 = m_3 + 2m_4$, where m_3 is the number of simple zeros of $P_1(z)$ and m_4 is the number of multiple zeros of $P_1(z)$. We define $k^* \in \mathbb{N}$ as

$$(2.2) \quad k^* = \begin{cases} k & \text{if } P_2(z_1) \equiv e_i z_1^i \neq 0, \\ k + 1 & \text{if } P_2(z_1) \not\equiv e_i z_1^i \neq 0 \end{cases}$$

for $i \in \{0, 1, 2, \dots, m_2\}$. Again we use $p(z)$ to denote a nonzero polynomial defined by

$$(2.3) \quad p(z) = a(z - z_1)^{l_1} (z - z_2)^{l_2} \dots (z - z_{t_1})^{l_{t_1}},$$

where $a \in \mathbb{C} \cup \{0\}$, $z_i \in \mathbb{C}$, $i = 1, 2, \dots, t_1$, are distinct and $l_1, l_2, \dots, l_{t_1} \in \mathbb{N}$ such that either $\sum_{i=1}^{t_1} l_i \leq n + s - 1$ or $l_i \leq n - 1$ for all $i = 1, 2, \dots, t_1$.

Throughout the paper we consider $\mathcal{F}(z) = \prod_{i=1}^k (f^{(i)}(z))^{n_i}$ and $\mathcal{F}_1(z) = \prod_{i=1}^k (f_1^{(i)}(z))^{n_i}$, where $f_1(z) = f(z) - c$; $\mathcal{G}(z)$ and $\mathcal{G}_1(z)$ are defined similarly.

Henceforth, we obtain the following results, keeping all the possible answers of the above questions, into background, which significantly improves and generalizes Theorems E, F and G.

Theorem 2.1. *Let $f(z)$ be a transcendental meromorphic function such that zeros of $f(z) - c$ are of multiplicities at least k^* , where k^* is defined in (2.2), and let $a(z) (\neq 0, \infty)$ be a small function of $f(z)$. Also let $n, s, n_k \in \mathbb{N}$ and $n_i, \Gamma_1 \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k - 1$. If $n > s + \Gamma_1 + 1/k^*$, then $P(f(z))\mathcal{F}(z) - a(z)$ has infinitely many zeros, where $P(z)$ is defined as in (2.1).*

Theorem 2.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k , where $k \in \mathbb{N}$. Let $P(z)$ and $p(z)$ be defined as in (2.1) and (2.3), respectively, and let $n, m, m_1, k_1, s, t, n_k \in \mathbb{N}$, $n_i, \Gamma_2 \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k - 1$, be such that*

$$n \geq 4\Gamma_2 + 2m + 2s + 1 + \frac{m_1}{2} + \frac{2}{k^*} \quad \text{and} \quad k_1 = \left[\frac{3 + m_1 - s}{n + s + m_1 - 2m - 1} \right] + 3.$$

If $P(f(z))\mathcal{F}(z) - p(z)$, $P(g(z))\mathcal{G}(z) - p(z)$ share $(0, k_1)$ and $f(z)$, $g(z)$ share $(\infty, 0)$, then one of the following conclusions holds:

- (1) $f(z) - c \equiv t(g(z) - c)$ with $t^{d_0} = 1$, where $d_0 = \gcd(d + p)$: $p \in \{0, 1, \dots, m_2\}$ with $e_p \neq 0$,
- (2) $P(f(z))\mathcal{F}(z) \equiv P(g(z))\mathcal{G}(z)$.

Theorem 2.3. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k^* , where k^* is defined in (2.2). Let $P(z)$ and $p(z)$ be defined as in (2.1) and (2.3), respectively, and let $n, m, m_1, s, t, n_k \in \mathbb{N}$, $n_i, m_2, \Gamma_2 \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k - 1$, be such that

$$n \geq 4\Gamma_2 + 2m + 2s + 1 + \frac{m_1}{2} + \frac{2}{k^*} \quad \text{and} \quad k_1 = \left[\frac{3 + m_1 - s}{n + s + m_1 - 2m - 1} \right] + 3.$$

Suppose $(k - 1)s - m_1 < 0$ when at least one of n_1, n_2, \dots, n_{k-1} is nonzero. If $P(f(z))\mathcal{F}(z) - p(z)$, $P(g(z))\mathcal{G}(z) - p(z)$ share $(0, k_1)$ and $f(z)$, $g(z)$ share $(\infty, 0)$, then one of the following cases holds:

- (1) If $P_2(z_1) \equiv e_i z_1^i \neq 0$ for some $i \in \{0, 1, 2, \dots, m_1\}$ and $f^{(n^*)}(z)$, $g^{(n^*)}(z)$ share $(0, \infty)$, then $f(z) - c \equiv t(g(z) - c)$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{d+s+i} = 1$ for some $i \in \{0, 1, 2, \dots, m_1\}$.
- (2) If $P_2(z_1) \not\equiv e_i z_1^i$ for $i \in \{0, 1, 2, \dots, m_1\}$, $(f^{(i)}(z))^{n_i^*}$, $(g^{(i)}(z))^{n_i^*}$ share $(0, \infty)$, where $i = 1, 2, \dots, k$, and $f(z)$, $g(z)$ share $(c, 0)$, then $f(z) - c \equiv t(g(z) - c)$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{d+s} = 1$.

Remark 2.1. Our results generalise Theorems E, F and G in different directions. For examples we consider $P(f(z))$ instead of $f^n(z)$ and $\mathcal{F}(z)$ instead of $f^{(k)}(z)$.

Remark 2.2. Let us take $d = n$, $c = 0$, $P_2(z_1) = 1$ and $n^* = k$. Then from Theorem 2.2 we can easily get a theorem which is the improvement of Theorem E and Theorem F.

Remark 2.3. Let us take $d = n$, $c = 0$, $P_2(z_1) = 1$ and $n^* = k$. Clearly $k^* = k$. Then from Theorem 2.3 we can easily get a theorem which is the improvement of Theorem G. Consequently Theorem G holds when zeros of $f(z)$ and $g(z)$ are of multiplicities at least k , where $k \in \mathbb{N}$.

Remark 2.4. It is easy to see that the condition ‘‘Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$ ’’ in Theorem 2.3 is sharp by the following example.

Example 2.1. Let $f(z) = c_1 e^{az}$ and $g(z) = c_2 e^{-az}$, where $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^{n+2} = -c_2^{n+2}$ and $n \geq 14$. Note that

$$\mathcal{F}(z) = f'(z)f''(z) = c_1^2 a^3 e^{2az} \quad \text{and} \quad \mathcal{G}(z) = g'(z)g''(z) = -c_2^2 a^3 e^{-2az}.$$

Since $f(z)$ and $g(z)$ have no zeros, it follows that the condition “Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions having zeros of multiplicities at least $k \in \mathbb{N}$ ” does not hold. Here we see that $f(z), g(z)$ share ∞ CM and $f'(z), g'(z)$ share 0 CM. On the other hand we see that

$$f^n(z)f'(z)f''(z) - p(z) = c_1^{n+2} a^3 (e^{a(n+2)z} - 1)$$

and

$$g^n(z)g'(z)g''(z) - p(z) = -c_2^{n+2} a^3 (e^{-a(n+2)z} - 1),$$

where $p(z) = c_1^{n+2} a^3$. Clearly $f^n(z)f'(z)f''(z) - p(z)$ and $g^n(z)g'(z)g''(z) - p(z)$ share $(0, \infty)$, but $f(z) \not\equiv tg(z)$, where $t \in \mathbb{C} \setminus \{0\}$ with $t^{n+2} = 1$.

Remark 2.5. It is easy to see that the conditions “ $(f^{(i)}(z))^{n_i^*}, (g^{(i)}(z))^{n_i^*}$ share $(0, \infty)$, where $i = 1, 2, \dots, k$ ” and “ $f(z), g(z)$ share $(c, 0)$ ” in Theorem 2.3 are sharp by the following example.

Example 2.2. Let

$$P(z) = z^n((n+2)z - (n+1)), \quad f(z) = \frac{1 - h^{n+1}(z)}{1 - h^{n+2}(z)} \quad \text{and} \quad g(z) = h(z) \frac{1 - h^{n+1}(z)}{1 - h^{n+2}(z)},$$

where $h(z) = e^z - 1$ and $n \in \mathbb{N}$ with $n \geq 10$. Observe that $f(z)$ and $g(z)$ share (∞, ∞) but $f(z)$ and $g(z)$ do not share the value 0. Note that

$$f'(z) = \frac{h^n(z)h'(z)((n+2)h(z) - h^{n+2}(z) - (n+1))}{(1 - h^{n+2}(z))^2}$$

and

$$g'(z) = \frac{h'(z)(1 + (n+1)h^{n+2}(z) - (n+2)h^{n+1}(z))}{(1 - h^{n+2}(z))^2}.$$

This shows that $f'(z)$ and $g'(z)$ do not share the value 0. Also we observe that $f^{n+1}(z)(f(z) - 1) \equiv g^{n+1}(z)(g(z) - 1)$, i.e., $f^n(z)((n+2)f(z) - (n+1))f'(z) \equiv g^n(z)((n+2)g(z) - (n+1))g'(z)$. Therefore $f^n(z)((n+2)f(z) - (n+1))f'(z)$ and $g^n(z)((n+2)g(z) - (n+1))g'(z)$ share $(1, \infty)$, but $f(z) \not\equiv tg(z)$, where $t \in \mathbb{C} \setminus \{0\}$ with $t^{n+2} = 1$.

Remark 2.6. The above example shows that the conclusion (2) in Theorem 2.2 cannot be removed.

We now explain some definitions and notations which are used in the paper.

Definition 2.1 ([12]). Let $p \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. $N(r, a; f | \geq p)$ ($\overline{N}(r, a; f | \geq p)$) denotes the *counting function* (*reduced counting function*) of those a -points of $f(z)$ whose multiplicities are not less than p . $N(r, a; f | \leq p)$ ($\overline{N}(r, a; f | \leq p)$) denotes the counting function (*reduced counting function*) of those a -points of $f(z)$ whose multiplicities are not greater than p .

Definition 2.2. We denote by $\overline{N}(r, a; f | = k)$ the *reduced counting function* of those a -points of $f(z)$ whose multiplicities are exactly k , where $k \in \mathbb{N} \setminus \{1\}$.

Definition 2.3 ([18]). For $a \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$, we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f | \geq 2) + \dots + \overline{N}(r, a; f | \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 2.4 ([1]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that $f(z)$ and $g(z)$ share the value 1 IM. Let z_0 be a 1-point of $f(z)$ with multiplicity p , a 1-point of $g(z)$ with multiplicity q . We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of $f(z)$ and $g(z)$, where $p > q$, and by $\overline{N}_E^{(2)}(r, 1; f)$ the counting function of those 1-points of $f(z)$ and $g(z)$, where $p = q \geq 2$, and each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$ and $\overline{N}_E^{(2)}(r, 1; g)$.

Definition 2.5 ([10]). Let $f(z)$ and $g(z)$ share the value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a -points of $f(z)$ whose multiplicities differ from the multiplicities of the corresponding a -points of $g(z)$. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

3. LEMMAS

By the non-constant meromorphic functions $F(z)$ and $G(z)$, we construct the functions

$$(3.1) \quad H(z) = \left(\frac{F''(z)}{F'(z)} - \frac{2F'(z)}{F(z) - 1} \right) - \left(\frac{G''(z)}{G'(z)} - \frac{2G'(z)}{G(z) - 1} \right)$$

and

$$(3.2) \quad \begin{aligned} V(z) &= \left(\frac{F'(z)}{F(z) - 1} - \frac{F'(z)}{F(z)} \right) - \left(\frac{G'(z)}{G(z) - 1} - \frac{G'(z)}{G(z)} \right) \\ &= \frac{F'(z)}{F(z)(F(z) - 1)} - \frac{G'(z)}{G(z)(G(z) - 1)}. \end{aligned}$$

Lemma 3.1 ([15]). Let $f(z)$ be a non-constant meromorphic function and let $a_n(z) (\neq 0)$, $a_{n-1}(z), \dots, a_0(z)$ be the small functions of $f(z)$. Then $T\left(r, \sum_{i=0}^n a_i f^i\right) = nT(r, f) + S(r, f)$.

Lemma 3.2 ([19]). Let $f(z)$ be a non-constant meromorphic function and $k, p \in \mathbb{N}$, then $N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)$.

Lemma 3.3 ([11]). If $N(r, 0; f^{(k)} | f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}(z)$ which are not the zeros of $f(z)$, where a zero of $f^{(k)}(z)$ is counted according to its multiplicity, then

$$N(r, 0; f^{(k)} | f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f | < k) + k\overline{N}(r, 0; f | \geq k) + S(r, f).$$

Lemma 3.4 ([17], Theorem 1.24). Let $f(z)$ be a non-constant meromorphic function and let $k \in \mathbb{N}$. If $f^{(k)}(z) \not\equiv 0$, then

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 3.5 ([17]). Let $f_j(z)$, $j = 1, 2, 3$, be meromorphic and $f_1(z)$ be non-constant. Suppose that $\sum_{j=1}^3 f_j(z) \equiv 1$ and $\sum_{j=1}^3 N(r, 0; f_j) + 2 \sum_{j=1}^3 \overline{N}(r, \infty; f_j) < (\lambda + o(1))T_1(r)$ as $r \rightarrow \infty$, $r \in I$, where I is a set of infinite linear measure, $\lambda < 1$ and $T_1(r) = \max_{1 \leq j \leq 3} T(r, f_j)$. Then $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

Lemma 3.6 ([8]). Let $f(z)$ be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two small functions of $f(z)$. Then

$$T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 3.7 ([8]). Suppose that $f(z)$ is a non-constant meromorphic function and $k \in \mathbb{N} \setminus \{1\}$. If $N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, f'/f)$, then $f(z) = e^{az+b}$, where $a (\neq 0)$, $b \in \mathbb{C}$.

Lemma 3.8. Let $f(z)$ be a transcendental meromorphic function and $n, n_k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \dots, k-1$. Then $\varphi(z) = P(f(z))\mathcal{F}(z)$ is non-constant, where $P(z)$ is defined by (2.1).

Proof. If possible, let $\varphi(z)$ be constant. Then $\overline{N}(r, 0; P(f)) = S(r, f)$ and $\overline{N}(r, \infty; f) = S(r, f)$. If $s_1 \geq 2$, by the second fundamental theorem we arrive at a contradiction.

Next we suppose $s_1 = 1$, i.e., $P(z) = a_n(z-c)^n$. Therefore $\varphi(z) = a_n f_1^n(z)\mathcal{F}_1(z)$. Clearly

$$\frac{1}{f_1^{n+s}(z)} \equiv a_n \frac{\mathcal{F}_1(z)}{f_1^s(z)} \frac{1}{\varphi(z)}.$$

Using Lemma 3.1, we now see that

$$\begin{aligned}
(n+s)T(r, f_1) &\leq T\left(r, \frac{\mathcal{F}_1}{f_1^s}\right) + T\left(r, \frac{1}{\varphi}\right) + O(1) \leq \sum_{i=1}^k n_i^{**} T\left(r, \frac{f_1^{(i)}}{f_1}\right) + O(1) \\
&\leq \sum_{i=1}^k n_i^{**} N\left(r, \infty; \frac{f_1^{(i)}}{f_1}\right) + S(r, f_1) \\
&\leq \sum_{i=1}^k n_i^{**} (N_i(r, 0; f_1) + i\overline{N}(r, \infty; f_1)) + S(r, f_1) = S(r, f_1),
\end{aligned}$$

which is not possible. Consequently $\varphi(z)$ is non-constant. Thus the proof is complete. \square

Lemma 3.9. *Let $f(z)$ be a non-constant meromorphic function and $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \dots, k-1$ be such, that $n > s$. If $\varphi(z) = P(f(z))\mathcal{F}(z)$, then*

$$(n-s)T(r, f) \leq T(r, \varphi) - sN(r, \infty; f) - N(r, 0; \mathcal{F}) + S(r, f).$$

Proof. Note that

$$N(r, \infty; \varphi) = N(r, \infty; P(f)) + sN(r, \infty; f) + m_1\overline{N}(r, \infty; f),$$

i.e.,

$$N(r, \infty; P(f)) = N(r, \infty; \varphi) - sN(r, \infty; f) - m_1\overline{N}(r, \infty; f) + S(r, f).$$

Also

$$\begin{aligned}
m(r, P(f)) &= m\left(r, \frac{\varphi}{\mathcal{F}}\right) \leq m(r, \varphi) + m\left(r, \frac{1}{\mathcal{F}}\right) + S(r, f) \\
&= m(r, \varphi) + T(r, \mathcal{F}) - N(r, 0; \mathcal{F}) + S(r, f) \\
&= m(r, \varphi) + N(r, \infty; \mathcal{F}) + m(r, \mathcal{F}) - N(r, 0; \mathcal{F}) + S(r, f) \\
&\leq m(r, \varphi) + sN(r, \infty; f) + m_1\overline{N}(r, \infty; f) + m\left(r, \frac{\mathcal{F}}{f^s}\right) \\
&\quad + m(r, f^s) - N(r, 0; \mathcal{F}) + S(r, f) \\
&= m(r, \varphi) + sT(r, f) + m_1\overline{N}(r, \infty; f) - N(r, 0; \mathcal{F}) + S(r, f).
\end{aligned}$$

Now

$$\begin{aligned}
nT(r, f) &= N(r, \infty; P(f)) + m(r, P(f)) \\
&\leq T(r, \varphi) + sT(r, f) - sN(r, \infty; f) - N(r, 0; \mathcal{F}) + S(r, f),
\end{aligned}$$

i.e.,

$$(n-s)T(r, f) \leq T(r, \varphi) - sN(r, \infty; f) - N(r, 0; \mathcal{F}) + S(r, f).$$

Thus the proof is complete. \square

Lemma 3.10. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k^* , where k^* is defined by (2.2). Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k-1$, be such that $n > 2\Gamma_1 + 2/k^* + s + t + m$. Let $F(z) = P(f(z))\mathcal{F}(z)/p(z)$ and $G(z) = P(g(z))\mathcal{G}(z)/p(z)$, where $p(z)$ is a nonzero polynomial and $P(z)$ is defined by (2.1). If $f(z), g(z)$ share $(\infty, 0)$ and $H(z) \equiv 0$, then one of the following three cases holds:

- (1) $P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \equiv p^2(z)$, where $P(f(z))\mathcal{F}(z) - p(z)$ and $P(g(z))\mathcal{G}(z) - p(z)$ share $(0, \infty)$,
- (2) $(f(z) - c) \equiv t(g(z) - c)$, $t^{d_0} = 1$, where $d_0 = \gcd(d + p: p \in \{0, 1, \dots, m_2\})$ with $e_p \neq 0$,
- (3) $P(f(z))\mathcal{F}(z) \equiv P(g(z))\mathcal{G}(z)$.

Proof. Since $H \equiv 0$, by integration we get

$$\frac{F'(z)}{(F(z) - 1)^2} \equiv l \frac{G'(z)}{(G(z) - 1)^2},$$

i.e.,

$$\begin{aligned} & \left(\frac{P(f(z))\mathcal{F}(z) - p(z)}{p(z)} \right)' \left(\frac{P(f(z))\mathcal{F}(z) - p(z)}{p(z)} \right)^{-2} \\ & \equiv l \left(\frac{P(g(z))\mathcal{G}(z) - p(z)}{p(z)} \right)' \left(\frac{P(g(z))\mathcal{G}(z) - p(z)}{p(z)} \right)^{-2}, \quad l \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

This shows that

$$\frac{P(f(z))\mathcal{F}(z) - p(z)}{p(z)} \quad \text{and} \quad \frac{P(g(z))\mathcal{G}(z) - p(z)}{p(z)}$$

share $(0, \infty)$. Therefore $P(f(z))\mathcal{F}(z) - p(z)$ and $P(g(z))\mathcal{G}(z) - p(z)$ share $(0, \infty)$. Again by integration we obtain

$$(3.3) \quad \frac{1}{F(z) - 1} \equiv \frac{bG(z) + a - b}{G(z) - 1},$$

where $a, b \in \mathbb{C} \setminus \{0\}$ and $a \neq 0$. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, then from (3.3) we have $F(z) \equiv -a/(G(z) - a - 1)$. Therefore $\overline{N}(r, a + 1; G) = \overline{N}(r, \infty; F) \leq \overline{N}(r, \infty; f) + S(r, f)$. So in view of Lemma 3.9 and the second fundamental theorem we get

$$\begin{aligned} (n - s)T(r, g) & \leq T(r, P(g)\mathcal{G}) - sN(r, \infty; g) - N(r, 0; \mathcal{G}) + S(r, g) \\ & \leq T(r, G) - sN(r, \infty; g) - N(r, 0; \mathcal{G}) + S(r, g) \\ & \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a + 1; G) \\ & \quad - sN(r, \infty; g) - N(r, 0; \mathcal{G}) + S(r, g) \end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r, 0; P_1(g)) + \overline{N}(r, 0; g - c) + \overline{N}(r, 0; \mathcal{G}) \\
&\quad + \overline{N}(r, \infty; f) - N(r, 0; \mathcal{G}) + S(r, g) \\
&\leq \overline{N}(r, \infty; g) + \Gamma_1 T(r, g) + \frac{1}{k^*} T(r, g) + S(r, g) \\
&\leq N(r, \infty; g) + \left(\Gamma_1 + \frac{1}{k^*} \right) T(r, g) + S(r, g) \\
&\leq \left(\Gamma_1 + \frac{1}{k^*} + 1 \right) T(r, g) + S(r, g)
\end{aligned}$$

and it is a contradiction as $n > \Gamma_1 + 1/k^* + s + 1$.

If $b \neq -1$, from (3.3) we obtain that $F(z) - (1+1/b) \equiv -a/(b^2(G(z) + (a-b)/b))$. So $\overline{N}(r, (b-a)/b; G) = \overline{N}(r, \infty; F) \leq \overline{N}(r, \infty; f) + S(r, f)$. Using Lemma 3.9 and the same argument as used in the case when $b = -1$ we get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b = -1$, then from (3.3) we get $F(z)G(z) \equiv 1$, i.e., $P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \equiv p^2(z)$.

If $b \neq -1$, from (3.3) we have $1/F(z) \equiv bG(z)/((1+b)G(z) - 1)$. Therefore $\overline{N}(r, 1/(1+b); G) = \overline{N}(r, 0; F)$. So in view of Lemmas 3.2, 3.9 and the second fundamental theorem, we get

$$\begin{aligned}
(n-s)T(r, g) &\leq T(r, G) - sN(r, \infty; g) - N(r, 0; \mathcal{G}) + S(r, g) \\
&\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+b}; G\right) \\
&\quad - sN(r, \infty; g) - N(r, 0; \mathcal{G}) + S(r, g) \\
&\leq \overline{N}(r, 0; P(g)) + \overline{N}(r, 0; \mathcal{G}) + \overline{N}(r, 0; F) - N(r, 0; \mathcal{G}) + S(r, g) \\
&\leq \overline{N}(r, 0; g - c) + \overline{N}(r, 0; P_1(g)) + \overline{N}(r, 0; f - c) \\
&\quad + \overline{N}(r, 0; P_1(f)) + \overline{N}(r, 0; F_1) + S(r, g) \\
&\leq \left(\Gamma_1 + \frac{1}{k^*} \right) (T(r, f) + T(r, g)) + \sum_{i=1}^k n_i^* \overline{N}(r, 0; f^{(i)}) + S(r, g) \\
&\leq \sum_{i=1}^k n_i^* (N_{i+1}(r, 0; f) + i\overline{N}(r, \infty; f)) \\
&\quad + \left(\Gamma_1 + \frac{1}{k^*} \right) (T(r, f) + T(r, g)) + S(r, f) + S(r, g) \\
&\leq \left(\Gamma_1 + \frac{1}{k^*} \right) (T(r, f) + T(r, g)) + tT(r, f) + mT(r, f) + S(r, g).
\end{aligned}$$

We suppose $T(r, f) \leq T(r, g)$ for $r \in I$. So for $r \in I$, we have

$$(n-s)T(r, g) \leq \left(2\Gamma_1 + \frac{2}{k^*} + t + m \right) T(r, g) + S(r, g),$$

which is a contradiction since $n > 2\Gamma_1 + 2/k^* + s + t + m$.

Case 3. Let $b = 0$. From (3.3) we obtain $F(z) \equiv (G(z) + a - 1)/a$. If $a \neq 1$, then we obtain $\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F)$. We can deduce a contradiction similarly as in Case 2. Therefore $a = 1$ and so we have $F(z) \equiv G(z)$. This gives

$$(3.4) \quad f_1^d(z) \left(\sum_{i=0}^{m_2} e_i f_1^i(z) \right) \mathcal{F}_1(z) \equiv g_1^d(z) \left(\sum_{i=0}^{m_2} e_i g_1^i(z) \right) \mathcal{G}_1(z).$$

Let $h(z) = f_1(z)/g_1(z)$. If $h(z)$ is a constant, by putting $f_1(z) = hg_1(z)$ in (3.4) we get

$$e_{m_2} g_1^{d+m_2}(z) (h^{d+m_2} - 1) + e_{m_2-1} g_1^{d+m_2-1}(z) (h^{d+m_2-1} - 1) + \dots + e_0 g_1^d(z) (h^d - 1) \equiv 0,$$

which gives $h^{d_0} = 1$, where $d_0 = \gcd(d + p: p \in \{0, 1, \dots, m_2\} \text{ with } e_p \neq 0)$. Thus $f_1(z) \equiv t g_1(z)$, i.e., $f(z) - c \equiv t(g(z) - c)$, $t^{d_0} = 1$, where $d_0 = \gcd(d + p: p \in \{0, 1, \dots, m_2\} \text{ with } e_p \neq 0)$.

If $h(z)$ is not constant, then we must have $P(f(z))\mathcal{F}(z) \equiv P(g(z))\mathcal{G}(z)$. Thus the proof is complete. \square

Lemma 3.11 ([8], Lemma 3.5). *Suppose that $F(z)$ is meromorphic in a domain D and set $f(z) = F'(z)/F(z)$. Then for $n \in \mathbb{N}$ we have*

$$\begin{aligned} \frac{F^{(n)}(z)}{F(z)} &= f^n(z) + \frac{n(n-1)}{2} f^{n-2}(z) f'(z) \\ &\quad + a_n f^{n-3}(z) f''(z) + b_n f^{n-4}(z) (f'(z))^2 + P_{n-3}(f(z)), \end{aligned}$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f(z))$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 3.12. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k , where $k \in \mathbb{N}$. Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \dots, k-1$. Suppose that $P(f(z))\mathcal{F}(z) - p(z)$ and $P(g(z))\mathcal{G}(z) - p(z)$ share $(0, \infty)$, and $f(z), g(z)$ share $(\infty, 0)$, where $P(z)$ and $p(z)$ are defined in (2.1) and (2.3), respectively. Then $P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \not\equiv p^2(z)$.*

Proof. Suppose

$$(3.5) \quad P(f(z))\mathcal{F}(z)P(g(z))\mathcal{G}(z) \equiv p^2(z).$$

Since $f(z)$ and $g(z)$ share $(\infty, 0)$, from (3.5) we claim that $f(z)$ and $g(z)$ are transcendental entire functions.

Suppose that $P(z)$ is a non-constant polynomial. For the sake of simplicity we may assume that $P_1(z) = a_n(z - c_{m_2})^{m_2}$, where $d + m_2 = n$. Obviously $c \neq c_{m_2}$. By (3.5), we have $N(r, c; f) = O(\log r)$ and $N(r, c_{m_2}; f) = O(\log r)$. So by the second fundamental theorem we obtain $T(r, f) \leq \overline{N}(r, c; f) + \overline{N}(r, c_{m_2}; f) + \overline{N}(r, \infty; f) + S(r, f) = S(r, f)$, which is not possible. Therefore $P(z)$ must be of the form $a_n(z - c)^n$ and so (3.5) reduces to the form

$$(3.6) \quad a_n^2(f(z) - c)^n \mathcal{F}(z)(g(z) - c)^n \mathcal{G}(z) \equiv p^2(z), \text{ i.e., } f_1^n(z) \mathcal{F}_1(z) g_1^n(z) \mathcal{G}_1(z) \equiv p_1^2(z),$$

where $p_1(z) = p(z)/a_n$. We now consider the following two cases.

Case 1. Let $\deg(p_1) \in \mathbb{N}$. Then from (3.6) we see that $N(r, 0; f_1^n) = O(\log r)$ and $N(r, 0; g_1^n) = O(\log r)$. Let

$$(3.7) \quad F_1(z) = \frac{f_1^n(z) \mathcal{F}_1(z)}{p_1(z)} \quad \text{and} \quad G_1(z) = \frac{g_1^n(z) \mathcal{G}_1(z)}{p_1(z)}.$$

Then (3.6) reduces to

$$(3.8) \quad F_1(z) G_1(z) \equiv 1.$$

If $F_1(z) \equiv e G_1(z)$, where $e \in \mathbb{C} \setminus \{0\}$, then $F_1(z)$ must be a constant, which is not possible by Lemma 3.8. So $F_1(z) \not\equiv e G_1(z)$. Let

$$(3.9) \quad \Phi(z) = \frac{f_1^n(z) \mathcal{F}_1(z) - p_1(z)}{g_1^n(z) \mathcal{G}_1(z) - p_1(z)}.$$

Since $f_1(z)$ and $g_1(z)$ are transcendental entire functions, it follows that $f_1^n(z) \mathcal{F}_1(z) - p_1(z) \neq \infty$ and $g_1^n(z) \mathcal{G}_1(z) - p_1(z) \neq \infty$. Also since $f_1^n(z) \mathcal{F}_1(z) - p_1(z)$ and $g_1^n(z) \mathcal{G}_1(z) - p_1(z)$ share $(0, \infty)$, we deduce from (3.9) that

$$(3.10) \quad \Phi(z) \equiv e^{\beta^*(z)},$$

where β^* is an entire function. Let $f_{11}(z) = F_1(z)$, $f_{21}(z) = -e^{\beta^*(z)} G_1(z)$ and $f_{31}(z) = e^{\beta^*(z)}$, where $f_{11}(z)$ is transcendental. Now from (3.10), we have $f_{11}(z) + f_{21}(z) + f_{31}(z) \equiv 1$. Also, by Lemma 3.4 we get

$$\begin{aligned} & \sum_{j=1}^3 N(r, 0; f_{j1}) + 2 \sum_{j=1}^3 \overline{N}(r, \infty; f_{j1}) \\ & \leq N(r, 0; F_1) + N(r, 0; e^{\beta^*} G_1) + O(\log r) \leq (\lambda + o(1)) T_1(r) \end{aligned}$$

as $r \rightarrow \infty$, $r \in I$, $\lambda < 1$ and $T_1(r) = \max_{1 \leq j \leq 3} T(r, f_{j1})$. So by Lemma 3.5, we obtain either $e^{\beta^*(z)} G_1(z) \equiv -1$ or $e^{\beta^*(z)} \equiv 1$. But the only possibility is that

$e^{\beta^*(z)}G_1(z) \equiv -1$ otherwise $F_1(z) \equiv G_1(z)$, which is possible. Then $g_1^n(z)\mathcal{G}_1(z) \equiv -e^{-\beta^*(z)}p_1(z)$. Also from (3.6) we obtain $f_1^n(z)\mathcal{F}_1(z) \equiv -e^{\beta^*(z)}p_1(z)$. Therefore $f_1^n(z)\mathcal{F}_1(z)$ and $g_1^n(z)\mathcal{G}_1(z)$ share $(0, \infty)$. As $f_1(z)$ and $g_1(z)$ have finitely many zeros, we can assume that

$$(3.11) \quad f_1(z) = h_1(z)e^{\alpha(z)} \quad \text{and} \quad g_1(z) = h_2(z)e^{\beta(z)},$$

where $h_1(z), h_2(z)$ are non-constant polynomials and $\alpha(z), \beta(z)$ are two non-constant entire functions. Let

$$\alpha_1(z) = \frac{f_1'(z)}{f_1(z)} = \alpha'(z) + \frac{h_1'(z)}{h_1(z)} \quad \text{and} \quad \beta_1(z) = \frac{g_1'(z)}{g_1(z)} = \beta'(z) + \frac{h_2'(z)}{h_2(z)}.$$

Now from (3.11) and Lemma 3.11 we have

$$(3.12) \quad f_1^n(z)\mathcal{F}_1(z) \equiv h_1^n(z) \prod_{i=1}^k (h_1(z)(\alpha'(z))^i + P_{i-1}(\alpha'(z), h_1'(z)))^{n_i} e^{(n+s)\alpha(z)}$$

and

$$(3.13) \quad g_1^n(z)\mathcal{G}_1(z) \equiv h_2^n(z) \prod_{i=1}^k (h_2(z)(\beta'(z))^i + Q_{i-1}(\beta'(z), h_2'(z)))^{n_i} e^{(n+s)\beta(z)},$$

respectively, where $P_{i-1}(\alpha'(z), h_1'(z))$ and $Q_{i-1}(\beta'(z), h_2'(z))$ are differential polynomials in $\alpha'(z), h_1'(z)$ and $\beta'(z), h_2'(z)$, respectively. We now consider the following two subcases.

Subcase 1.1. Let $k \geq 2$. First we suppose that both $\alpha(z)$ and $\beta(z)$ are transcendental entire functions. Clearly both $\alpha_1(z)$ and $\beta_1(z)$ are transcendental meromorphic functions. Note that $S(r, \alpha_1) = S(r, f_1'/f_1)$ and $S(r, \beta_1) = S(r, g_1'/g_1)$. Moreover, from (3.6) we have $N(r, 0; f_1^{(k)}) = O(\log r)$ and $N(r, 0; g_1^{(k)}) = O(\log r)$. From this and using (3.11), we have

$$(3.14) \quad N(r, \infty; f_1) + N(r, 0; f_1) + N(r, 0; f_1^{(k)}) = S(r, \alpha_1) = S\left(r, \frac{f_1'}{f_1}\right)$$

and

$$(3.15) \quad N(r, \infty; g_1) + N(r, 0; g_1) + N(r, 0; g_1^{(k)}) = S(r, \beta_1) = S\left(r, \frac{g_1'}{g_1}\right).$$

Using (3.14), (3.15) and Lemma 3.7, we get $f_1(z) = e^{a^*z+b^*}$ and $g_1(z) = e^{c^*z+d^*}$, where $a^* (\neq 0), b^*, c^* (\neq 0), d^* \in \mathbb{C}$, which is possible as zeros of $f_1(z)$ and $g_1(z)$ are of multiplicities at least k .

Next we suppose that both $\alpha(z)$ and $\beta(z)$ are non-constant polynomials. Since $f_1^n(z)\mathcal{F}_1(z) \equiv -e^{\beta^*(z)}p_1(z)$ and $g_1^n(z)\mathcal{G}_1(z) \equiv -e^{-\beta^*(z)}p_1(z)$, from (3.12) and (3.13) we have

$$(3.16) \quad \begin{aligned} f_1^n(z)\mathcal{F}_1(z) &\equiv h_1^n(z) \prod_{i=1}^k (h_1(z)(\alpha'(z))^i + P_{i-1}(\alpha'(z), h_1'(z)))^{n_i} e^{(n+s)\alpha(z)} \\ &\equiv Ap_1(z)e^{(n+s)\alpha(z)} \end{aligned}$$

and

$$(3.17) \quad \begin{aligned} g_1^n(z)\mathcal{G}_1(z) &\equiv h_2^n(z) \prod_{i=1}^k (h_2(z)(\beta'(z))^i + Q_{i-1}(\beta'(z), h_2'(z)))^{n_i} e^{(n+s)\beta(z)} \\ &\equiv Bp_1(z)e^{(n+s)\beta(z)}, \end{aligned}$$

respectively, where $A, B \in \mathbb{C} \setminus \{0\}$. Now from (3.6), (3.16) and (3.17) we deduce that $\alpha(z) + \beta(z) \in \mathbb{C}$, i.e., $\alpha'(z) \equiv -\beta'(z)$ and so $\deg(\alpha) = \deg(\beta)$. Note that $\deg(\alpha), \deg(\beta) \in \mathbb{N}$. Since either $\deg(p_1) \leq n + s - 1$ or zeros of $p_1(z)$ are of multiplicities at most $n - 1$ from (3.16) or (3.17) we arrive at a contradiction.

Finally we suppose that one of $\alpha(z)$ and $\beta(z)$ is transcendental and the other one is polynomial. For the sake of simplicity we assume that $\beta(z)$ is a polynomial. In this case we get a contradiction from (3.17).

Subcase 1.2. Let $k = 1$. From (3.11) we deduce that

$$(3.18) \quad f_1^n(z)(f_1'(z))^{n_1} \equiv h_1^n(z)(h_1(z)\alpha'(z) + h_1'(z))^{n_1} e^{(n+n_1)\alpha(z)}$$

and

$$(3.19) \quad g_1^n(z)(g_1'(z))^{n_1} \equiv h_2^n(z)(h_2(z)\beta'(z) + h_2'(z))^{n_1} e^{(n+n_1)\beta(z)}.$$

First we suppose that both of $\alpha(z)$ and $\beta(z)$ are transcendental. Then from (3.6), (3.18) and (3.19) we get

$$(3.20) \quad \begin{aligned} &(h_1(z)h_2(z))^n (h_1(z)\alpha'(z) + h_1'(z))^{n_1} \\ &\quad \times (h_2(z)\beta'(z) + h_2'(z))^{n_1} e^{(n+n_1)(\alpha(z)+\beta(z))} \equiv p_1^2(z). \end{aligned}$$

Let $\alpha(z) + \beta(z) = \gamma(z)$ and $s_2 = n + n_1$. We claim that $\gamma(z) \notin \mathbb{C}$. If not, suppose $\gamma \in \mathbb{C}$. Then $\alpha'(z) \equiv -\beta'(z)$ and so from (3.20) we have

$$(3.21) \quad H_{2n_1}(z)(\alpha'(z))^{2n_1} + H_{2n_1-1}(z)(\alpha'(z))^{2n_1-1} + \dots + H_0(z) \equiv 0,$$

where $H_0(z), H_1(z), \dots, H_{2n_1}(z) (\not\equiv 0)$ are polynomials. Since a transcendental entire function is non-algebraic, from (3.21) we arrive at a contradiction. Hence $\gamma \notin \mathbb{C}$.

Now (3.20) reduces to

$$(3.22) \quad (h_1(z)h_2(z))^n(h_1(z)\alpha'(z) + h_1'(z))^{n_1} \\ \times (h_2(z)(\gamma'(z) - \alpha'(z)) + h_2'(z))^{n_1} e^{s_2\gamma(z)} \equiv p_1^2(z).$$

We have $T(r, \gamma') = m(r, s_2\gamma') + O(1) = m(r, (e^{s_2\gamma})'/e^{s_2\gamma}) = S(r, e^{s_2\gamma})$. Thus from (3.22) we get

$$T(r, e^{s_2\gamma}) \leq T\left(r, \frac{p_1^2}{(h_1h_2)^n(h_1\alpha' + h_1')^{n_1}(h_2(\gamma' - \alpha') + h_2')^{n_1}}\right) + O(1) \\ \leq n_1T(r, \alpha') + n_1T(r, \gamma' - \alpha') + O(\log r) + O(1) \\ \leq 2n_1T(r, \alpha') + S(r, \alpha') + S(r, e^{s_2\gamma}),$$

implying that $T(r, e^{s_2\gamma}) = O(T(r, \alpha'))$ and so $S(r, e^{s_2\gamma})$ can be replaced by $S(r, \alpha')$. Thus $T(r, \gamma') = S(r, \alpha')$ and so $\gamma'(z)$ is a small function with respect to $\alpha'(z)$. In view of (3.22) and by Lemma 3.6, we get

$$T(r, \alpha') \leq \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; h_1\alpha' + h_1') + \overline{N}(r, 0; h_2(\gamma' - \alpha') + h_2') + S(r, \alpha') \\ \leq O(\log r) + S(r, \alpha')$$

and it shows that $\alpha'(z)$ is a polynomial and consequently $\alpha(z)$ is a polynomial. Similarly we can prove that $\beta(z)$ is also a polynomial. This contradicts that $\alpha(z)$ and $\beta(z)$ are both transcendental.

Next suppose that both $\alpha(z)$ and $\beta(z)$ are polynomials. Since $f_1^n(z)\mathcal{F}_1(z) \equiv -e^{\beta^*(z)}p_1(z)$ and $g_1^n(z)\mathcal{G}_1(z) \equiv -e^{-\beta^*(z)}p_1(z)$, from (3.18) and (3.19), we have

$$(3.23) \quad f_1^n(z)(f_1'(z))^{n_1} \equiv h_1^n(z)(h_1(z)\alpha'(z) + h_1'(z))^{n_1} e^{s_2\alpha(z)} \equiv A_1p_1(z)e^{s_2\alpha(z)}$$

and

$$(3.24) \quad g_1^n(z)(g_1'(z))^{n_1} \equiv h_2^n(z)(h_2(z)\beta'(z) + h_2'(z))^{n_1} e^{s_2\beta(z)} \equiv B_1p_1(z)e^{s_2\beta(z)},$$

respectively, where $A_1, B_1 \in \mathbb{C} \setminus \{0\}$. Now from (3.6), (3.23) and (3.24) we deduce that $\alpha(z) + \beta(z) \in \mathbb{C}$, i.e., $\alpha'(z) \equiv -\beta'(z)$ and so $\deg(\alpha) = \deg(\beta)$. Note that $\deg(\alpha), \deg(\beta) \in \mathbb{N}$. Since either $\deg(p_1) \leq n + s - 1$ or zeros of $p_1(z)$ are of multiplicities at most $n - 1$, from (3.23) or (3.24) we arrive at a contradiction.

Finally we suppose that one of $\alpha(z)$ and $\beta(z)$ is transcendental and the other one is polynomial. For the sake of simplicity we assume that $\beta(z)$ is a polynomial. In this case we get a contradiction from (3.24).

Case 2. Let $p_1(z) \equiv b \in \mathbb{C} \setminus \{0\}$. Then (3.6) reduces to $f_1^n(z)\mathcal{F}_1(z)g_1^n(z)\mathcal{G}_1(z) \equiv b^2$. This shows that both $f_1(z)$ and $g_1(z)$ have no zeros. But this is not possible as zeros of $f_1(z)$ and $g_1(z)$ are of multiplicities at least $k (\geq 1)$. Thus the proof is complete. \square

Lemma 3.13 ([9]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Suppose that $f(z)$ and $g(z)$ share $(0, \infty)$, (∞, ∞) ; $f^{(k)}(z)$ and $g^{(k)}(z)$ share $(0, \infty)$ for $k = 1, 2, \dots, 6$. Then $f(z)$ and $g(z)$ satisfy one of the following cases:*

- (i) $f(z) \equiv tg(z)$, where $t \in \mathbb{C} \setminus \{0\}$,
- (ii) $f(z) = e^{az+b}$ and $g(z) = e^{cz+d}$, where $a, b, c, d \in \mathbb{C} \setminus \{0\}$ such that $ac \neq 0$,
- (iii) $f(z) = a/(1 - be^{\alpha(z)})$ and $g(z) = a/(e^{-\alpha(z)} - b)$, where $a, b \in \mathbb{C} \setminus \{0\}$, α is a non-constant entire function,
- (iv) $f(z) = a(1 - be^{cz})$ and $g(z) = d(e^{-cz} - b)$, where $a, b, c, d \in \mathbb{C} \setminus \{0\}$.

Lemma 3.14. *Let*

$$Q_1(x) = n_1(x-1)(x-2)\dots(x-k+1) + 2n_2x(x-2)\dots(x-k+1) \\ + \dots + kn_kx(x-1)\dots(x-k+2)$$

and

$$Q_2(x) = x(x-1)(x-2)\dots(x-k+1),$$

where $n_k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k-1$, but at least one of n_1, n_2, \dots, n_{k-1} is nonzero. Suppose $(k-1)s - m_1 < 0$. Then all the roots of the equation $(sx - m_1) \times Q_1(x) - \lambda Q_2(x) = 0$, where $\lambda \in \mathbb{R}$, lie in the interval $(-\infty, k-1)$.

Proof. By the given conditions we have $k \geq 2$ and $1 < m_1/s < k$. Also we see that $m_1/s \neq 2, 3, \dots, k-1$. Therefore $js - m_1 < 0$ for $j = 1, 2, \dots, k-1$ and $ks - m_1 > 0$. Let $f(x) = x^{n_1}(x-1)^{2n_2}\dots(x-k+1)^{kn_k}$. Then $f'(x) = x^{n_1-1}(x-1)^{2n_2-1}\dots(x-k+1)^{kn_k-1}Q_1(x)$. By Rolle's theorem, we can say that each of the $(k-1)$ intervals $(0, 1), (1, 2), \dots, (k-2, k-1)$ contains at least one real root of the equation $f'(x) = 0$.

Let α_i , $i = 1, 2, \dots, k-1$, be the roots of the equation $Q_1(x) = 0$ such that $i-1 < \alpha_i < i$ for $i = 1, 2, \dots, k-1$. Then $Q_1(x) = m_1(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{k-1})$.

Let $F(x) = (sx - m_1)Q_1(x) - \lambda Q_2(x)$. Now we consider the following three cases.

Case 1. Let $sm_1 - \lambda < 0$. We now consider the following two subcases.

Subcase 1.1. Suppose k is an odd positive integer. Note that $F(-\infty) > 0$, $F(0) < 0$, $F(1) > 0$, $F(2) < 0$, $F(3) > 0, \dots, F(k-2) > 0$, $F(k-1) < 0$. Therefore each of the intervals $(-\infty, 0), (0, 1), (1, 2), \dots, (k-2, k-1)$ contains a real root of the equation $F(x) = 0$. Since the equation is of degree k , all its roots are real and simple. Therefore all the roots of the equation $F(x) = 0$ lie in the interval $(-\infty, k-1)$.

Subcase 1.2. Suppose that k is an even positive integer. Note that $F(-\infty) < 0$, $F(0) > 0$, $F(1) < 0$, $F(2) > 0$, $F(3) < 0, \dots, F(k-2) > 0$, $F(k-1) < 0$. Therefore each of the intervals $(-\infty, 0), (0, 1), (1, 2), \dots, (k-2, k-1)$ contains a real root of the equation $F(x) = 0$. Since the equation is of degree k , all its roots are real and simple. Therefore all the roots of the equation $F(x) = 0$ lie in the interval $(-\infty, k-1)$.

Case 2. Let $sm_1 - \lambda > 0$. We omit the proof since it can be carried out in the line of the proof of Case 1.

Case 3. Let $sm_1 - \lambda = 0$. In this case the equation $F(x) = 0$ is of degree $k - 1$. Consequently each of the intervals $(0, 1), (1, 2), \dots, (k - 2, k - 1)$ contains a real root of the equation $F(x) = 0$. Since the equation is of degree $k - 1$, all its roots are real and simple. Therefore all the roots of the equation $F(x) = 0$ lie in the interval $(0, k - 1)$. Thus the proof is complete. \square

Lemma 3.15. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k , where $k \in \mathbb{N}$. Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}, i = 1, 2, \dots, k - 1$. Suppose $(k - 1)s - m_1 < 0$ when at least one of n_1, n_2, \dots, n_{k-1} is nonzero. Also we assume that $f^{(n^*)}(z), g^{(n^*)}(z)$ share $(0, \infty)$ and $f(z), g(z)$ share $(\infty, 0)$. Now when $(f(z) - c)^n \mathcal{F}(z) \equiv (g(z) - c)^n \mathcal{G}(z)$, then $(f(z) - c) \equiv t(g(z) - c)$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{n+s} = 1$.*

Proof. Suppose that

$$(3.25) \quad f_1^n(z) \mathcal{F}_1(z) \equiv g_1^n(z) \mathcal{G}_1(z), \quad \text{i.e.,} \quad f_1^n(z)/g_1^n(z) \equiv \mathcal{G}_1(z)/\mathcal{F}_1(z).$$

Since $f_1(z)$ and $g_1(z)$ share $(\infty, 0)$, it follows from (3.25) that $f_1(z)$ and $g_1(z)$ share (∞, ∞) and so $(f_1^{(i)}(z))^{n_i^*}$ and $(g_1^{(i)}(z))^{n_i^*}$ share (∞, ∞) , where $i = 1, 2, \dots, k$. Again since $f^{(n^*)}(z)$ and $g^{(n^*)}(z)$ share $(0, \infty)$, it follows that $f_1^{(n^*)}(z)$ and $g_1^{(n^*)}(z)$ share $(0, \infty)$. Suppose $n^* = k$. Then from (3.25) we have $f_1^n(z)(f_1^{(k)}(z))^{n_k} \equiv g_1^n(z)(g_1^{(k)}(z))^{n_k}$, and so $f_1(z)$ and $g_1(z)$ share $(0, \infty)$. Next we suppose $n^* < k$. For the sake of simplicity we assume that $n^* = 1$. Let z_{11} be a zero of $f_1(z)$ of multiplicity $p_{11} (\geq k)$. Then z_{11} is a zero of $f_1'(z)$ of multiplicity $p_{11} - 1 (\geq 1)$. Since $f_1'(z)$ and $g_1'(z)$ share $(0, \infty)$, it follows that z_{11} is a zero of $g_1'(z)$ of multiplicity $p_{11} - 1 (\geq 1)$. Clearly z_{11} is a zero of both $f_1^{(i)}(z)$ and $g_1^{(i)}(z)$ of multiplicity $p_{11} - i$, where $i \in \{1, 2, \dots, k\}$. Consequently z_{11} is a zero of both $\mathcal{F}_1(z)$ and $\mathcal{G}_1(z)$ of multiplicity $p_{11}s - m_1$. Note that z_{11} is a zero of $f_1^n(z)\mathcal{F}_1(z)$ of multiplicity $p_{11}(n + s) - m_1$. Therefore, from (3.25) we see that z_{11} must be a zero of $g_1(z)$ of multiplicity p_{11} . Hence $f_1(z)$ and $g_1(z)$ share $(0, \infty)$. Let $h_1(z) = f_1(z)/g_1(z)$ and $h_2(z) = \mathcal{F}_1(z)/\mathcal{G}_1(z)$. Then $h_1(z)$ and $h_2(z) \neq 0, \infty$. Now (3.25) yields

$$(3.26) \quad h_1^n(z)h_2(z) \equiv 1.$$

First we suppose that $h_1(z)$ is a non-constant entire function. Clearly $h_2(z)$ is also a non-constant entire function. Let $F_1(z) = h_1^n(z)$ and $G_1(z) = h_2(z)$. Also from (3.26), we get

$$(3.27) \quad F_1(z)G_1(z) \equiv 1.$$

Clearly $F_1(z) \not\equiv d_0 G_1(z)$, $d_0 \in \mathbb{C} \setminus \{0\}$, otherwise we have $F_1 \in \mathbb{C} \setminus \{0\}$ from (3.27) and so $h_1 \in \mathbb{C} \setminus \{0\}$. Since $F_1(z)$ and $G_1(z) \neq 0, \infty$, there exist two non-constant entire functions $\alpha(z)$ and $\beta(z)$ such that $F_1(z) = e^{\alpha(z)}$ and $G_1(z) = e^{\beta(z)}$. Now from (3.27) we see that $\alpha + \beta \in \mathbb{C}$ and so $\alpha'(z) \equiv -\beta'(z)$. Note that $F_1'(z) = \alpha'(z)e^{\alpha(z)}$ and $G_1'(z) = \beta'(z)e^{\beta(z)}$. This shows that $F_1'(z)$ and $G_1'(z)$ share $(0, \infty)$. Note that $F_1(z), G_1(z) \neq 0, \infty$ and $F_1(z) \not\equiv d_0 G_1(z)$, $d_0 \in \mathbb{C} \setminus \{0\}$. Now in view of Lemma 3.13 we get $F_1(z) = c_1 e^{az}$ and $G_1(z) = c_2 e^{-az}$, $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ with $c_1 c_2 = 1$. Since $(f_1(z)/g_1(z))^n = c_1 e^{az}$ it follows that

$$(3.28) \quad f_1(z)/g_1(z) = t_1 e^{(a/n)z} = t_1 e^{cz},$$

where $c, t_1 \in \mathbb{C} \setminus \{0\}$ such that $t_1^n = c_1$ and $c = a/n$. Also we have

$$(3.29) \quad \mathcal{F}_1(z)/\mathcal{G}_1(z) = c_2 e^{-az}.$$

Let

$$(3.30) \quad \Phi_1(z) = \frac{\mathcal{F}_1'(z)}{\mathcal{F}_1(z)} - \frac{\mathcal{G}_1'(z)}{\mathcal{G}_1(z)}.$$

Using (3.29), we deduce that

$$(3.31) \quad \Phi_1(z) = -a.$$

Noting $g_1^{(0)}(z) = g_1(z)$, we calculate from (3.28) that

$$\begin{aligned} f_1^{(j)}(z) &= t_1 \sum_{i=0}^j j C_i (e^{cz})^{(j-i)} g_1^{(i)}(z) = t_1 e^{cz} (c^j g_1(z) + j c^{j-1} g_1'(z) + \frac{1}{2} j(j-1) c^{j-2} g_1''(z) \\ &\quad + \dots + j c g_1^{(j-1)}(z) + g_1^{(j)}(z)). \end{aligned}$$

Consequently we have

$$\begin{aligned} (f_1^{(j)}(z))^{n_j} &= t_1^{n_j} e^{c n_j z} \left((g_1^{(j)}(z))^{n_j} + j n_j c g_1^{(j-1)}(z) (g_1^{(j)}(z))^{n_j-1} \right. \\ &\quad \left. + \sum_{\lambda} P_{1\lambda} g_1^{p_0^\lambda}(z) (g_1'(z))^{p_1^\lambda} \dots (g_1^{(j)}(z))^{p_j^\lambda} \right), \end{aligned}$$

where $P_{1\lambda} \in \mathbb{C} \setminus \{0\}$ and $p_0^\lambda, p_1^\lambda, \dots, p_j^\lambda \in \mathbb{N} \cup \{0\}$ such that $p_i^\lambda \leq n_j$, where $i = 0, 1, \dots, j-1$, $p_j^\lambda < n_j$ and $p_0^\lambda + p_1^\lambda + \dots + p_j^\lambda = n_j$. Therefore

$$(3.32) \quad \begin{aligned} \mathcal{F}_1(z) &= t_1^s e^{csz} \left(\mathcal{G}_1(z) + c \sum_{j=1}^k j n_j g_1^{(j-1)}(z) (g_1^{(j)}(z))^{n_j-1} \prod_{\substack{i=1 \\ i \neq j}}^k (g_1^{(i)}(z))^{n_i} \right. \\ &\quad \left. + \sum_{\lambda} Q_{1\lambda} g_1^{q_0^\lambda}(z) (g_1'(z))^{q_1^\lambda} \dots (g_1^{(k)}(z))^{q_k^\lambda} \right), \end{aligned}$$

where $Q_{1\lambda} \in \mathbb{C} \setminus \{0\}$ and $q_0^\lambda, q_1^\lambda, \dots, q_k^\lambda \in \mathbb{N} \cup \{0\}$ such that $q_i^\lambda \leq s$, where $i = 0, 1, \dots, k-1$, $q_k^\lambda < s$ and $q_0^\lambda + q_1^\lambda + \dots + q_k^\lambda = s$. It is clear that $0 \leq q_1^\lambda + 2q_2^\lambda + \dots + kq_k^\lambda \leq m_1 - 2$. Note that

$$(3.33) \quad \mathcal{F}'_1(z) = t_1^s e^{csz} \left(\mathcal{G}'_1(z) + c(m_1 + s)\mathcal{G}_1(z) \right. \\ \left. + \sum_{\lambda} R_{1\lambda} g_1^{r_0^\lambda}(z) (g'_1(z))^{r_1^\lambda} \dots (g_1^{(k)}(z))^{r_k^\lambda} \right) \\ + cst_1^s e^{csz} \sum_{\lambda} Q_{1\lambda} g_1^{q_0^\lambda}(z) (g'_1(z))^{q_1^\lambda} \dots (g_1^{(k)}(z))^{q_k^\lambda},$$

where $R_{1\lambda} \in \mathbb{C} \setminus \{0\}$ and $r_0^\lambda, r_1^\lambda, \dots, r_k^\lambda \in \mathbb{N} \cup \{0\}$ such that $r_i^\lambda \leq s$, where $i = 0, 1, \dots, k-1$, $r_k^\lambda < s$ and $r_0^\lambda + r_1^\lambda + \dots + r_k^\lambda = s$. It is clear that $0 \leq r_1^\lambda + 2r_2^\lambda + \dots + kr_k^\lambda \leq m_1 - 1$. Now from (3.30), (3.32) and (3.33) we have

$$(3.34) \quad \Phi_1(z) = \frac{1}{F_3(z) + \mathcal{G}_1^2(z)} \left(H_1(z) + c(m_1 + s)\mathcal{G}_1^2(z) \right. \\ \left. - c \sum_{j=1}^k j n_j g_1^{(j-1)}(z) (g_1^{(j)}(z))^{n_j-1} \prod_{\substack{i=1 \\ i \neq j}}^k (g_1^{(i)}(z))^{n_i} \mathcal{G}'_1(z) \right),$$

where $H_1(z) = F_2(z) - G_2(z)$ with

$$F_2(z) = \mathcal{G}_1(z) \left(\sum_{\lambda} R_{1\lambda} g_1^{r_0^\lambda}(z) (g'_1(z))^{r_1^\lambda} \dots (g_1^{(k)}(z))^{r_k^\lambda} \right. \\ \left. + cs \sum_{\lambda} Q_{1\lambda} g_1^{q_0^\lambda}(z) (g'_1(z))^{q_1^\lambda} \dots (g_1^{(k)}(z))^{q_k^\lambda} \right), \\ G_2 = \mathcal{G}'_1(z) \sum_{\lambda} Q_{1\lambda} g_1^{q_0^\lambda}(z) (g'_1(z))^{q_1^\lambda} \dots (g_1^{(k)}(z))^{q_k^\lambda}$$

and

$$F_3(z) = \mathcal{G}_1(z) \left(c \sum_{j=1}^k j n_j g_1^{(j-1)}(z) (g_1^{(j)}(z))^{n_j-1} \prod_{\substack{i=1 \\ i \neq j}}^k (g_1^{(i)}(z))^{n_i} \right. \\ \left. + \sum_{\lambda} Q_{1\lambda} g_1^{q_0^\lambda}(z) (g'_1(z))^{q_1^\lambda} \dots (g_1^{(k)}(z))^{q_k^\lambda} \right).$$

Let z_p be a zero of $g_1(z)$ with multiplicity $p (\geq k)$. Then the Taylor expansion of $g_1(z)$ about z_p is

$$(3.35) \quad g_1(z) = a_p(z - z_p)^p + a_{p+1}(z - z_p)^{p+1} + \dots, \quad a_p \neq 0.$$

Therefore $g_1^{(i)}(z) = N_i a_p (z - z_p)^{p-i} + \dots$, where $N_i = p(p-1)\dots(p-i+1)$. Consequently $(g_1^{(i)}(z))^{n_i} = N_i^{n_i} a_p^{n_i} (z - z_p)^{pn_i - in_i} + \dots$ and so

$$\mathcal{G}_1(z) = \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right) (z - z_p)^{ps - m_1} + \dots$$

Note that

$$(3.36) \quad \mathcal{G}_1^2(z) = \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right)^2 (z - z_p)^{2ps - 2m_1} + \dots$$

and

$$(3.37) \quad \mathcal{G}'_1(z) = (ps - m_1) \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right) (z - z_p)^{ps - m_1 - 1} + \dots$$

Also we see that

$$\prod_{\substack{i=1, \\ i \neq j}}^k (g_1^{(i)}(z))^{n_i} = \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right) N_j^{-n_j} a_p^{-n_j} (z - z_p)^{ps - m_1 - pn_j + jn_j} + \dots$$

and

$$g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j-1} = N_{j-1} a_p N_j^{n_j-1} a_p^{n_j-1} (z - z_p)^{pn_j - jn_j + 1}.$$

Consequently,

$$g_1^{(j-1)}(z)(g_1^{(j)}(z))^{n_j-1} \prod_{\substack{i=1, \\ i \neq j}}^k (g_1^{(i)}(z))^{n_i} = \frac{1}{p-j+1} \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right) (z - z_p)^{ps - m_1 + 1} + \dots$$

and so

$$(3.38) \quad \sum_{j=1}^k j n_j^{**} g_1^{(j-1)}(z) (g_1^{(j)}(z))^{n_j-1} \prod_{\substack{i=1, \\ i \neq j}}^k (g_1^{(i)}(z))^{n_i} \mathcal{G}'_1(z) \\ = (ps - m_1) \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right)^2 \sum_{j=1}^k \frac{j n_j^{**}}{p-j+1} (z - z_p)^{2ps - 2m_1} + \dots$$

Also $F_2(z) = A_1(z - z_p)^{2ps - 2m_1 + 1} + \dots$, $G_2(z) = A_2(z - z_p)^{2ps - 2m_1 + 1} + \dots$ and $F_3(z) = A_3(z - z_p)^{2ps - 2m_1 + 1} + \dots$, where A_1, A_2, A_3 are suitable nonzero constants.

Now from (3.34), (3.36) and (3.38) we have

$$\begin{aligned}
(3.39) \quad \Phi_1(z_p) &= \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right)^{-2} \left(c(m_1 + s) \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right)^2 \right. \\
&\quad \left. - c(ps - m_1) \left(\sum_{j=1}^k \frac{jn_j}{p-j+1} \right) \left(\prod_{i=1}^k N_i^{n_i} a_p^{n_i} \right)^2 \right) \\
&= \frac{a}{n} \left(m_1 + s - (ps - m_1) \sum_{j=1}^k \frac{jn_j}{p-j+1} \right).
\end{aligned}$$

We now consider the following two cases.

Case 1. Suppose $n_1 = n_2 = \dots = n_{k-1} = 0$. Then from (3.39) we get $\Phi_1(z_p) = cn_k(p+1)/(p-k+1)$ and so from (3.31) we arrive at a contradiction.

Case 2. Suppose that at least one of n_1, n_2, \dots, n_{k-1} is nonzero. Then from (3.31) and (3.39) we have

$$(ps - m_1) \sum_{i=1}^k \frac{jn_j}{p-j+1} - (n + s + m_1) = 0,$$

i.e.,

$$(3.40) \quad (ps - m_1)Q_1(p) - (n + s + m_1)Q_2(p) = 0,$$

where $Q_1(p)$ and $Q_2(z)$ are as in Lemma 3.14. By Lemma 3.14 we see that the roots of the equation $(px - m_1)Q_1(x) - (n + s + m_1)Q_2(x) = 0$ lie in the interval $(-\infty, k-1)$. Therefore the roots of the equation (3.40) also lie in the interval $(-\infty, k-1)$ but this is not possible as z_p is a zero of g_1 with multiplicity $p \geq k$. Thus the only possibility is that $g_1(z)$ has no zeros. Since $f_1(z)$ and $g_1(z)$ share $(0, \infty)$, it follows that $f_1(z)$ and $g_1(z)$ have no zeros, which is possible as zeros of $f_1(z)$ and $g_1(z)$ are of multiplicities at least $k (\geq 1)$. Hence $h_1 \in \mathbb{C} \setminus \{0\}$. Then from (3.25) we get $h_1^{n+s} = 1$ and so $f_1(z) \equiv tg_1(z)$, i.e., $(f(z) - c) \equiv t(g(z) - c)$, $t \in \mathbb{C} \setminus \{0\}$ with $t^{n+s} = 1$. Thus the proof is complete. \square

Lemma 3.16. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k^* , where k^* is defined in (2.2). Let $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \dots, k-1$. Suppose $(k-1)s - m_1 < 0$ when at least one of n_1, n_2, \dots, n_{k-1} is nonzero. Also we assume that $f(z)$ and $g(z)$ share $(\infty, 0)$. If $f_1^d(z)P_2(f_1(z))\mathcal{F}_1(z) \equiv g_1^d(z)P_2(g_1(z))\mathcal{G}_1(z)$, then one of the following cases holds:*

- (1) *If $P_2(z_1) \equiv e_i z_1^i \neq 0$ for some $i \in \{0, 1, 2, \dots, m_1\}$ and $f^{(n^*)}(z)$, $g^{(n^*)}(z)$ share $(0, \infty)$, then $f(z) - c \equiv t(g(z) - c)$, where $t \in \mathbb{C} \setminus \{0\}$ such that $t^{d+s+i} = 1$ for some $i \in \{0, 1, 2, \dots, m_1\}$.*

- (2) If $P_2(z_1) \not\equiv e_i z_1^i$ for $i \in \{0, 1, 2, \dots, m_1\}$, $(f^{(i)}(z))^{n_i^*}$, $(g^{(i)}(z))^{n_i^*}$ share $(0, \infty)$, where $i = 1, 2, \dots, k$, and $f(z)$, $g(z)$ share $(c, 0)$, then $f(z) - c \equiv t(g(z) - c)$ for $t \in \mathbb{C} \setminus \{0\}$ such that $t^{d+s} = 1$.

Proof. Suppose

$$(3.41) \quad f_1^d(z)P_2(f_1(z))\mathcal{F}_1(z) \equiv g_1^d(z)P_2(g_1(z))\mathcal{G}_1(z),$$

i.e.,

$$(3.42) \quad \frac{P_2(f_1(z))}{P_2(g_1(z))} \equiv \frac{g_1^d(z)\mathcal{G}_1(z)}{f_1^d(z)\mathcal{F}_1(z)}.$$

We now consider the following two cases.

Case 1. Suppose $P_2(z_1) \equiv e_i z_1^i \not\equiv 0$ for some $i \in \{0, 1, 2, \dots, m_2\}$. Then the result follows from Lemma 3.15.

Case 2. Suppose $P_2(z_1) \not\equiv e_i z_1^i$ where $i \in \{0, 1, 2, \dots, m_2\}$. For the sake of simplicity we assume that $P_2(z_1) = e_{m_2} z_1^{m_2} + e_{m_2-1} z_1^{m_2-1} + \dots + e_1 z_1 + e_0$, $e_{m_2}, e_0 \neq 0$. Since $f_1(z)$ and $g_1(z)$ share $(\infty, 0)$, from (3.41) we see that $f_1(z)$ and $g_1(z)$ share (∞, ∞) . Now we prove that $f_1(z)$ and $g_1(z)$ share $(0, \infty)$. Note that $P_2(0) \neq 0$. Let z_{12} be a zero of $f_1(z)$ of multiplicity $r_{12} (\geq k+1)$. Since $f_1(z)$ and $g_1(z)$ share $(0, 0)$, z_{12} is a zero of $g_1(z)$ of multiplicity $q_{12} (\geq k+1)$. Clearly z_{12} is a zero of $f_1^{(k)}(z)$ of multiplicity $r_{12} - k$ and a zero of $g_1^{(k)}(z)$ of multiplicity $q_{12} - k$. Since $f_1^{(k)}(z)$ and $g_1^{(k)}(z)$ share $(0, \infty)$, we have $r_{12} = q_{12}$. Therefore $f_1(z)$ and $g_1(z)$ share $(0, \infty)$. Since $f_1(z)$ and $g_1(z)$ share $(0, \infty)$ and (∞, ∞) , it follows that $f_1(z) = e^{\gamma(z)} g_1(z)$, where $\gamma(z)$ is an entire function. Let

$$h_1^*(z) = \frac{P_2(f_1(z))}{P_2(g_1(z))} \quad \text{and} \quad h_2^*(z) = \frac{f_1^d(z)\mathcal{F}_1(z)}{g_1^d(z)\mathcal{G}_1(z)}.$$

Since $\mathcal{F}_1(z)$ and $\mathcal{G}_1(z)$ share $(0, \infty)$, we have $h_2(z) \neq 0, \infty$. Also from (3.41) we see that $h_1(z) \neq 0, \infty$ and

$$(3.43) \quad h_1^*(z)h_2^*(z) \equiv 1.$$

We now consider the following two subcases.

Subcase 2.1. Suppose $h_1^* \equiv b \in \mathbb{C} \setminus \{0\}$. Let $b = 1$. Then from (3.41) we have $f_1^d(z)\mathcal{F}_1(z) \equiv g_1^d(z)\mathcal{G}_1(z)$. Then the result follows from Lemma 3.15. Let $b \neq 1$. Then we have

$$(3.44) \quad \sum_{i=0}^{m_2} e_i f_1^i \equiv b \sum_{i=0}^{m_2} e_i g_1^i.$$

Since $f_1(z) = e^{\gamma(z)}g_1(z)$, from (3.44) we have

$$(3.45) \quad e_{m_2}g_1^{m_2}(z)(e^{m_2\gamma(z)} - b) + \dots + e_1g_1(z)(e^{\gamma(z)} - b) \equiv e_0(b - 1).$$

Note that $g_1(z) \not\equiv d \in \mathbb{C}$. Then from (3.45) we see that $g_1(z)$ has no zero. But this is impossible because zeros of $g_1(z)$ are of multiplicities at least $k + 1$.

Subcase 2.2. Suppose $h_1^* \notin \mathbb{C}$. Then $h_2^* \notin \mathbb{C}$. Note that $h_1^*(z) \not\equiv d_0^*h_2^*(z)$, $d_0^* \in \mathbb{C} \setminus \{0\}$. Since $h_1^*(z)$ and $h_2^*(z) \neq 0, \infty$, then there exist two non-constant entire functions $\alpha^*(z)$ and $\beta^*(z)$ such that $h_1^*(z) = e^{\alpha^*(z)}$ and $h_2^*(z) = e^{\beta^*(z)}$. Now from (3.43) we see that $\alpha^{*'}(z) \equiv -\beta^{*'}(z)$. Therefore $h_1^{*'}(z)$ and $h_2^{*'}(z)$ share $(0, \infty)$. Now in view of Lemma 3.13, we get $h_1^{*'}(z) = c_1^*e^{az}$ and $h_2^{*'}(z) = c_2^*e^{-az}$, where $a, c_1, c_2 \in \mathbb{C} \setminus \{0\}$ are such that $c_1c_2 = 1$. Therefore we have

$$(3.46) \quad \sum_{i=1}^{m_2} e_i g_1^i(z)(e^{i\gamma(z)} - c_1^*e^{az}) \equiv e_0(c_1^*e^{az} - 1).$$

Note that the zeros of $(c_1^*e^{az} - 1)$ are simple. Also from (3.46) we see that zeros of $g_1(z)$ are the zeros of $(c_1^*e^{az} - 1)$. Since zeros of $g_1(z)$ are of multiplicities at least $k + 1$, from (3.46) we arrive at a contradiction. Thus the proof is complete. \square

Lemma 3.17. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that the zeros of $f(z) - c$ and $g(z) - c$ are of multiplicities at least k^* , where k^* is defined by (2.2) and $F(z) = P(f(z))\mathcal{F}(z)/p(z)$ and $G(z) = P(g(z))\mathcal{G}(z)/p(z)$, where $n, n_k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$ for $i = 1, 2, \dots, k - 1$ are such that $n + s + m_1 > 2m + 1$ and $P(z)$ is defined by (2.1). Suppose $H(z) \not\equiv 0$. If $F(z)$ and $G(z)$ share $(1, k_1)$ except for the zeros of $p(z)$, and $f(z)$ and $g(z)$ share $(\infty, 0)$, then*

$$\begin{aligned} \overline{N}(r, \infty; f) &\leq \frac{k^*t + k^*\Gamma_1 + 1}{k^*(n + s + m_1 - 2m - 1)}(T(r, f) + T(r, g)) \\ &\quad + \frac{1}{n + s + m_1 - 2m - 1}\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

Proof. Since $H(z) \not\equiv 0$, it follows that $F \not\equiv G$. First we observe that if ∞ is Picard's exceptional value of $f(z)$, then the result follows immediately. Next we suppose that ∞ is not Picard's exceptional value of $f(z)$. Since $f(z)$ and $g(z)$ share $(\infty, 0)$, it follows that ∞ is not Picard's exceptional value of $g(z)$. We claim that $V(z) \not\equiv 0$. If possible, suppose $V(z) \equiv 0$. Then by integration we obtain $1 - 1/F(z) = A_0(1 - 1/G(z))$, where $A_0 \neq 0, 1$. Let z_{q_0} be a pole of $f(z)$ of multiplicity q_0 such that $p(z_{q_0}) \neq 0$. Since $f(z)$ and $g(z)$ share $(\infty, 0)$, we suppose that z_{q_0} is a pole of $g(z)$ of multiplicity r_0 . Therefore $1/F(z_{q_0}) = 0$ and $1/G(z_{q_0}) = 0$

and so $A_0 = 1$, which is not possible. Hence $V(z) \not\equiv 0$. Note that z_{q_0} is a pole of $F(z)$ with multiplicity $(n+s)q_0 + m_1$ and a pole of $G(z)$ with multiplicity $(n+s)r_0 + m_1$. Clearly

$$\frac{F'(z)}{F(z)(F(z) - 1)} = O((z - z_{q_0})^{(n+s)q_0 + m_1 - 1})$$

and

$$\frac{G'(z)}{G(z)(G(z) - 1)} = O((z - z_{q_0})^{(n+s)r_0 + m_1 - 1}).$$

Consequently we have

$$V(z) = O((z - z_{q_0})^{(n+m)t_0 + k - 1}),$$

where $t_0 = \min\{q_0, r_0\} \geq 1$. This shows that z_{q_0} is a zero of $V(z)$ of multiplicity at least $n + s + m_1 - 1$. Also $m(r, V) = S(r, f) + S(r, g)$. Thus using Lemma 3.1 and Lemma 3.3, we see that

$$\begin{aligned} & (n + s + m_1 - 1)\overline{N}(r, \infty; f) \\ & \leq N(r, 0; V) + O(\log r) \leq N(r, \infty; V) + S(r, f) + S(r, g) \\ & \leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\ & \leq \overline{N}(r, 0; P_1(f)) + \overline{N}(r, 0; f_1) + \sum_{i=1}^k n_i^* \overline{N}(r, 0; f_1^{(i)} \mid f_1 \neq 0) \\ & \quad + \overline{N}(r, 0; P_1(g)) + \overline{N}(r, 0; g_1) + \sum_{i=1}^k n_i^* \overline{N}(r, 0; g_1^{(i)} \mid g_1 \neq 0) \\ & \quad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\ & \leq \overline{N}(r, 0; P_1(f)) + \overline{N}(r, 0; f_1) + \sum_{i=1}^k n_i^* (i\overline{N}(r, \infty; f_1) + N_i(r, 0; f_1)) \\ & \quad + \overline{N}(r, 0; P_1(g)) + \overline{N}(r, 0; g_1) + \sum_{i=1}^k n_i^* (i\overline{N}(r, \infty; g_1) + N_i(r, 0; g_1)) \\ & \quad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\ & \leq \overline{N}(r, 0; P_1(f)) + \overline{N}(r, 0; f_1) + m\overline{N}(r, \infty; f_1) + tN(r, 0; f_1) \\ & \quad + \overline{N}(r, 0; P_1(g)) + \overline{N}(r, 0; g_1) + m\overline{N}(r, \infty; g_1) + tN(r, 0; g_1) \\ & \quad + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\ & \leq \frac{k^*t + k^*\Gamma_1 + 1}{k^*} (T(r, f) + T(r, g)) \\ & \quad + 2m\overline{N}(r, \infty; f) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \end{aligned}$$

Thus the proof is complete. □

Lemma 3.18 ([2]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions such that they share $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then*

$$\begin{aligned} \overline{N}(r, 1; f | = 2) + 2\overline{N}(r, 1; f | = 3) + \dots + (k_1 - 1)\overline{N}(r, 1; f | = k_1) + k_1\overline{N}_L(r, 1; f) \\ + (k_1 + 1)\overline{N}_L(r, 1; g) + k_1\overline{N}_E^{(k_1+1)}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g). \end{aligned}$$

4. PROOF OF THE MAIN THEOREMS

Proof of Theorem 2.1. Let $F(z) = P(f(z))\mathcal{F}(z)$. Now in view of Lemma 3.9 and using the second theorem for small functions (see [14]), we get

$$\begin{aligned} (n - s)T(r, f) &\leq T(r, F) - sN(r, \infty; f) - N(r, 0; \mathcal{F}_1) + S(r, f) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, a; F) - sN(r, \infty; f) \\ &\quad - N(r, 0; \mathcal{F}) + (\varepsilon + o(1))T(r, f) \\ &\leq \overline{N}(r, 0; P_1(f)) + \overline{N}(r, 0; f - c) + \overline{N}(r, a; F) + (\varepsilon + o(1))T(r, f) \\ &\leq (\Gamma_1 + 1/k^*)T(r, f) + \overline{N}(r, a; F) + (\varepsilon + o(1))T(r, f) \end{aligned}$$

for all $\varepsilon > 0$. Take $\varepsilon < n - s - \Gamma_1 - 1/k^*$. Since $n > s + \Gamma_1 + 1/k^*$, one can easily say that $F - a$ has infinitely many zeros. Thus the proof is complete. \square

Proof of Theorem 2.3. Let

$$F(z) = \frac{P(f(z))\mathcal{F}(z)}{p(z)} \quad \text{and} \quad G(z) = \frac{P(g(z))\mathcal{G}(z)}{p(z)}.$$

Then $F(z)$ and $G(z)$ share $(1, k_1)$ except for the zeros of $p(z)$ and $f(z)$, $g(z)$ share $(\infty, 0)$.

Case 1. Let $H(z) \not\equiv 0$. Now from (3.1) we observe that

$$(4.1) \quad \begin{aligned} N(r, \infty; H) &\leq \overline{N}_*(r, \infty; f, g) + \overline{N}_*(r, 1; F, G) + \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) \\ &\quad + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g), \end{aligned}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of $F'(z)$ which are not the zeros of $F(z)(F(z) - 1)$ and $\overline{N}_0(r, 0; G')$ is defined similarly. Let z_0 be a simple zero of $F(z) - 1$ but $p(z_0) \neq 0$. Then z_0 is a simple zero of $G(z) - 1$ and a zero of $H(z)$. Therefore $N(r, 1; F | = 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, f) + S(r, g)$ and so from (4.1) we get

$$(4.2) \quad \begin{aligned} \overline{N}(r, 1; F) &\leq N(r, 1; F | = 1) + \overline{N}(r, 1; F | \geq 2) \\ &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}_*(r, 1; F, G) \\ &\quad + \overline{N}(r, 1; F | \geq 2) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g). \end{aligned}$$

Now in view of Lemmas 3.3 and 3.18 we get

$$\begin{aligned}
(4.3) \quad & \overline{N}_0(r, 0; G') + \overline{N}(r, 1; F | \geq 2) + \overline{N}_*(r, 1; F, G) \\
& \leq \overline{N}_0(r, 0; G') + \overline{N}(r, 1; F | = 2) + \overline{N}(r, 1; F | = 3) + \dots + \overline{N}(r, 1; F | = k_1) \\
& \quad + \overline{N}_E^{(k_1+1)}(r, 1; F) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_*(r, 1; F, G) \\
& \leq \overline{N}_0(r, 0; G') + N(r, 1; G) - \overline{N}(r, 1; G) \\
& \quad - (k_1 - 2)\overline{N}_L(r, 1; F) - (k_1 - 1)\overline{N}_L(r, 1; G) \\
& \leq N(r, 0; G' | G \neq 0) - (k_1 - 2)\overline{N}_L(r, 1; F) - (k_1 - 1)\overline{N}_L(r, 1; G) \\
& \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; g) - (k_1 - 2)\overline{N}_*(r, 1; F, G) - \overline{N}_L(r, 1; G).
\end{aligned}$$

Hence using (4.2), (4.3) and Lemma 3.2, we get from the second fundamental theorem that

$$\begin{aligned}
(4.4) \quad & T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) - N_0(r, 0; F') + S(r, f) \\
& \leq 2\overline{N}(r, \infty, f) + N_2(r, 0; F) + \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, 1; F | \geq 2) \\
& \quad + \overline{N}_*(r, 1; F, G) + \overline{N}_0(r, 0; G') + S(r, f) + S(r, g) \\
& \leq 3\overline{N}(r, \infty; f) + N_2(r, 0; F) + N_2(r, 0; G) \\
& \quad - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq 3\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f_1) + N_2(r, 0; P_1(f)) + N_2(r, 0; \mathcal{F}_1) \\
& \quad + 2\overline{N}(r, 0; g_1) + N_2(r, 0; P_1(g)) + N_2(r, 0; \mathcal{G}_1) \\
& \quad - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq 3\overline{N}(r, \infty; f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r, f) + T(r, g)) + N_2(r, 0; \mathcal{F}_1) \\
& \quad + N_2(r, 0; \mathcal{G}_1) - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq 3\overline{N}(r, \infty; f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r, f) + T(r, g)) + N_2(r, 0; \mathcal{F}_1) \\
& \quad + \sum_{i=1}^k n_i^{**} N_2(r, 0; g^{(i)}) - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq 3\overline{N}(r, \infty; f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r, f) + T(r, g)) + N_2(r, 0; \mathcal{F}_1) \\
& \quad + \sum_{i=1}^k n_i^{**} N_{i+2}(r, 0; g) + \sum_{i=1}^k i n_i^{**} \overline{N}(r, \infty; g) \\
& \quad - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq (3 + m_1)\overline{N}(r, \infty; f) + \left(\Gamma_2 + \frac{2}{k^*}\right)(T(r, f) + T(r, g)) + N_2(r, 0; \mathcal{F}_1) \\
& \quad + sN(r, 0; g) - (k_1 - 2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).
\end{aligned}$$

Now using Lemmas 3.17 and 3.9 we get from (4.4)

$$\begin{aligned}
(n-s)T(r, f) &\leq T(r, F) - sN(r, \infty; f) - N(r, 0; \mathcal{F}_1) + S(r, f) \\
&\leq (3+m_1-s)\overline{N}(r, \infty; f) + \left(\Gamma_2 + \frac{2}{k^*}\right)T(r, f) + \left(\Gamma_2 + \frac{2}{k^*}\right)T(r, g) \\
&\quad + sN(r, 0; g) - (k_1-2)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g) \\
&\leq 2\frac{(k^*t+k^*\Gamma_1+1)(3+m_1-s)}{k^*(n+s+m_1-2m-1)}T(r) + \left(2\Gamma_2 + \frac{4}{k^*} + s\right)T(r) + S(r) \\
&\leq \left(2\frac{(k^*t+k^*\Gamma_1+1)(3+m_1-s)}{k^*(n+s+m_1-2m-1)} + 2\Gamma_2 + \frac{4}{k^*} + s\right)T(r) + S(r).
\end{aligned}$$

We obtain a similar inequality for $g(z)$. Combining these inequalities we obtain

$$(n-s)T(r) \leq \left(2\frac{(k^*t+k^*\Gamma_1+1)(3+m_1-s)}{k^*(n+s+m_1-2m-1)} + 2\Gamma_2 + \frac{4}{k^*} + s\right)T(r) + S(r),$$

i.e.,

$$(k^*n^2 - ((2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4)n + A)T(r) \leq S(r),$$

where

$$\begin{aligned}
A &= k^*(4m\Gamma_2 + 2\Gamma_2 + 4ms + 2s + 2s\Gamma_1 + 2ts - 2m_1s - 2s^2 - 2s\Gamma_2 \\
&\quad - 2m_1\Gamma_2 - 2m_1\Gamma_1 - 6\Gamma_1 - 6t - 2m_1t) + 8m - 6m_1 - 2s - 2.
\end{aligned}$$

Therefore

$$(4.5) \quad (n - K_1)(n - K_2)T(r) \leq S(r),$$

where

$$K_1 = \frac{(2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4 + \sqrt{L}}{2k^*}$$

and

$$K_2 = \frac{(2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4 - \sqrt{L}}{2k^*},$$

so that $L = ((2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4)^2 - 4k^*A$. Note that

$$\begin{aligned}
L &= ((2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4)^2 - 4k^*A \\
&= (k^*)^2(4\Gamma_2^2 + 4m^2 + 9s^2 + m_1^2 + 1 - 8m\Gamma_2 + 12s\Gamma_2 + 4m_1\Gamma_2 \\
&\quad - 4\Gamma_2 + 6sm_1 - 12sm - 8st - 8s\Gamma_1 - 4mm_1 - 2m_1 \\
&\quad + 24t + 24\Gamma_1 + 8tm_1 + 8m_1\Gamma_1 + 4m - 6s) \\
&\quad + 4k^*(4\Gamma_2 + 4s - 4m + 4 + 4m_1) + 16 \\
&\leq (k^*)^2(4\Gamma_2^2 + 4m^2 + 9s^2 + m_1^2 + 1 + 12s\Gamma_2 + 12m_1\Gamma_2 + 20\Gamma_2 - 6s \\
&\quad + 4mm_1 + 6sm_1 + 28m - 8m\Gamma_2 - 12sm - 8st - 8s\Gamma_1 - 2m_1) \\
&\quad + 16(k^*(\Gamma_2 + s - m + 1 + m_1) + 1)
\end{aligned}$$

$$\begin{aligned}
&\leq (k^*)^2(36\Gamma_2^2 + 4m^2 + 9s^2 + 4m_1^2 + 1 + 24m\Gamma_2 + 36s\Gamma_2 + 24m_1\Gamma_2 \\
&\quad + 12\Gamma_2 + 6ms + 8mm_1 + 4m + 6sm_1 + 6s + 4m_1) \\
&\quad + k^*(16\Gamma_2 + 6s - 16m + 16 + 14m_1) + 16 \\
&\quad + (k^*)^2(8\Gamma_2 + 4m - 24m\Gamma_2 - 24s\Gamma_2 - 6ms - 4m_1 \\
&\quad - 32\Gamma_2^2 - 8st - 4s\Gamma_1 - 3m_1^2 - 12m_1\Gamma_2 - 4mm_1 - 6s) \\
&\leq (k^*(6\Gamma_2 + 2m + 3s + 2m_1 + 1))^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
K_1 &< \frac{(2\Gamma_2 + s + 2m + 1 - m_1)k^* + 4 + \sqrt{(k^*(6\Gamma_2 + 2m + 3s + 2m_1 + 1))^2}}{2k^*} \\
&= 4\Gamma_2 + 2m + 2s + 1 + \frac{m_1}{2} + \frac{2}{k^*}.
\end{aligned}$$

Since $n \geq 4\Gamma_2 + 2m + 2s + 1 + m_1/2 + 2/k^*$, (4.5) leads to a contradiction.

Case 2. Let $H(z) \equiv 0$. Now the theorem follows from Lemmas 3.10, 3.12 and 3.16. □

P r o o f of Theorem 2.2. Using Lemmas 3.10 and 3.12, the theorem can be proved in the line of the proof of Theorem 2.3. So we omit the details. □

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