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Existence results for impulsive fractional differential equations with p-Laplacian via variational methods

EXISTENCE RESULTS FOR IMPULSIVE FRACTIONAL
DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN
VIA VARIATIONAL METHODS

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Abstract. This paper presents several sufficient conditions for the existence of at least one classical solution to impulsive fractional differential equations with a p-Laplacian and Dirichlet boundary conditions. Our technical approach is based on variational methods. Some recent results are extended and improved. Moreover, a concrete example of an application is presented.

Keywords: fractional p-Laplacian; impulsive effect; classical solution; variational method

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1. INTRODUCTION

We consider the impulsive fractional differential equation with a p-Laplacian and Dirichlet boundary conditions

$$(\mathrm{P}^f) \quad \begin{cases} D_{T^-}^\alpha \Phi_p(^cD_{0^+}^\alpha u(t)) + |u(t)|^{p-2}u(t) = f(t, u(t)), & t \neq t_j, \ t \in (0, T), \\ \Delta(D_{T^-}^{\alpha-1}\Phi_p(^cD_{0^+}^\alpha u))(t_j) = I_j(u(t_j)), \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (1/p, 1]$, $p > 1$, $\Phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, $D_{T^-}^\alpha$ is the right Riemann-Liouville fractional derivative of order α, $^cD_{0^+}^\alpha$ is the left Caputo fractional derivative of order α, $D_{T^-}^{\alpha-1} = D_{T^-}^{-(1-\alpha)}$ is the right Riemann-Liouville fractional integral

of order $1 - \alpha$,

$$\Delta(D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u))(t_j) = D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t_j^+) - D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t_j^-),$$
$$D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t_j^+) = \lim_{t \to t_j^+} D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t),$$
$$D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t_j^-) = \lim_{t \to t_j^-} D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t),$$

$f \colon [0, T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, and $I_j \colon \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, m$, are Lipschitz continuous functions having Lipschitz constants $L_j > 0$, i.e., $|I_j(x_2) - I_j(x_1)| \leqslant L_j|x_2 - x_1|^{p-1}$ for every $x_1, x_2 \in \mathbb{R}$, and with $I_j(0) = 0$.

Fractional calculus is a generalization of classical derivatives and integrals to an arbitrary (noninteger) order. It represents a powerful tool in applied mathematics to deal with a myriad of problems from different fields such as physics, mechanics, electricity, control theory, rheology, signal and image processing, aerodynamics, etc.; for details, see [15], [26], [28], [33] and the references therein. Recently, the existence of solutions to boundary value problems for fractional differential equations (FDEs) has been studied in many papers and we refer the reader to the papers [13], [17], [20], [21], [22], [29] and the references therein for some recent contributions. For example, in [13], Chen and Tang studied the existence and multiplicity of solutions to the fractional boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}{_0D_t^{-\beta}}(u'(t)) + \dfrac{1}{2}{_tD_T^{-\beta}}(u'(t))\right) + \lambda\nabla F(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

where $T > 0$, $\lambda > 0$, $0 \leqslant \beta < 1$, $_0D_t^{-\beta}$ and $_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β, respectively, $F \colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function, $\nabla F(t, x)$ is the gradient of F in x, and $F(t, \cdot)$ is superquadratic, asymptotically quadratic, or subquadratic. Using the Avery-Peterson fixed point theorem, Guo and Zhang (see [21]) provided sufficient conditions for the existence of multiple positive solutions to the boundary value problem in which the nonlinear terms contain the derivatives of order up to i

$$\begin{cases} ^cD^\alpha u(t) + f(t, u(t), u'(t), \ldots, u^{(i)}(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \ldots = u^{(i-1)}(0) = u^{(i+1)}(0) = \ldots = u^{(n-1)}(0) = 0, \\ u^{(i)}(1) = 0, \end{cases}$$

where $n - 1 < \alpha \leqslant n$ is a real number, $n \geqslant 2$ is a natural number, and $\alpha - i > 1$ for $0 \leqslant i \leqslant n - 1$. The function $f(t, x_0, \ldots, x_i)$ may be singular at $t = 0$.

Nonlinear boundary value problems involving the p-Laplacian arise from a variety of physical problems. They are used in non-Newtonian fluids, reaction-diffusion problems, flow through porous media, and petroleum extraction (see, e.g., [8], [9], [16], [30], [39]). Recently, several researchers have studied nonlinear problems of this type using different approaches.

The theory of impulsive differential equations provides a general framework for the mathematical modeling of many real-world phenomena; see, for instance, [12], [14]. Indeed, many dynamical systems have an impulsive behavior due to abrupt changes at certain instants during the evolution process. Impulsive differential equations are basic tools for studying these phenomena, see [2], [3], [4], [5], [6]. In relation to the variational approach to impulsive differential equations, we refer to the papers [36], [31]. See also the papers [18], [7], [32].

To the best of our knowledge, there are few results on the existence and multiplicity of solutions to impulsive fractional boundary value problems with a p-Laplacian. For details, see [11], [19], [25], [37], [38], [40] and the references therein. For example, Wang et al. (see [38]) used a variant fountain theorem to prove the existence of infinitely many nontrivial large or small energy solutions to problem (P^f). Zhao and Tang (see [40]) employed critical point theory and variational methods to study the existence and multiplicity of solutions to (P^f).

Motivated by the above discussion, in the present paper we study the existence of at least one nontrivial classical solution to problem (P^f) under an assumption on the asymptotic behavior of the nonlinear function F at zero (see Theorem 3.1). In Theorem 3.2, we present an application of Theorem 3.1. We also give some detailed remarks about our results. As a special case of our result, we obtain Theorem 4.3 for the case where f does not depend upon t. We end the paper with an example to illustrate our results.

2. Preliminaries

We are going to prove the existence of at least one nontrivial classical solution to problem (P^f). Essential to our approach is the following version of Ricceri's variational principle (see [35], Theorem 2.1) as given by Bonanno and Molica Bisci in [10].

Theorem 2.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive in X, and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional $I_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for*

every $r > \inf_X \Phi$, let φ be the function

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely, a local minimum) of I_λ in X.

We refer the interested reader to the papers [1], [17], [23], [24] in which Theorem 2.1 has been successfully used to prove the existence of at least one nontrivial solution to boundary value problems.

Next, we introduce several basic definitions, notations, lemmas, and propositions that will be used in the remainder of this paper. Let $AC[a, b]$ be the space of absolutely continuous functions on $[a, b]$ and $\Gamma(\alpha)$ be the usual Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} \, dz.$$

Definition 2.2 ([27]). Let f be a function defined on $[a, b]$ and $0 < \alpha \leqslant 1$. The *left* and *right Riemann-Liouville fractional integrals* of f of order α are defined by

$$D_{a^+}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s) \, ds, \quad t \in [a, b],$$

$$D_{b^-}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha-1} f(s) \, ds, \quad t \in [a, b],$$

provided the right-hand sides are defined pointwise on $[a, b]$.

Definition 2.3 ([27]). Let f be a function defined on $[a, b]$ and $0 < \alpha \leqslant 1$. The *left* and *right Riemann-Liouville fractional derivatives* of f of order α are defined by

$$D_{a^+}^{\alpha} f(t) = \frac{d}{dt} D_{a^+}^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t - s)^{-\alpha} f(s) \, ds, \quad t \in [a, b],$$

$$D_{b^-}^{\alpha} f(t) = -\frac{d}{dt} D_{b^-}^{\alpha-1} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (s - t)^{-\alpha} f(s) \, ds, \quad t \in [a, b].$$

Definition 2.4 ([27]). Let $f \in AC([a, b], \mathbb{R})$ and $0 < \alpha \leqslant 1$. The *left* and *right Caputo fractional derivatives* of f of order α are defined by

$$^c D_{a^+}^{\alpha} f(t) = D_{a^+}^{\alpha-1} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t - s)^{-\alpha} f'(s) \, ds, \quad t \in [a, b],$$

$$^c D_{b^-}^{\alpha} f'(t) = -D_{b^-}^{\alpha-1} f'(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (s - t)^{-\alpha} f'(s) \, ds, \quad t \in [a, b].$$

In particular, if $\alpha = 1$, we have $^c D_{a^+}^{1} f(t) = f'(t)$ and $^c D_{b^-}^{1} f(t) = -f'(t)$.

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Let $C_0^\infty([0,T], \mathbb{R})$ be the set of all functions $u \in C^\infty([0,T], \mathbb{R})$ with $u(a) = u(b) = 0$ and the norm $\|u\|_\infty = \max_{t \in [a,b]} |u(t)|$. Denote the norm of the space $L^p([0,T], \mathbb{R})$ for $1 \leqslant p < \infty$ by

$$\|u\|_{L^p} = \left(\int_a^b |u(s)|^p \, \mathrm{d}s \right)^{1/p}.$$

Definition 2.5. Let $0 < \alpha \leqslant 1$ and $1 < p < \infty$. The *fractional derivative space* $E_0^{\alpha, p}$ is defined as the closure of $C_0^\infty([0,T], \mathbb{R})$, that is,

$$E_0^{\alpha, p} = \overline{C_0^\infty([0,T], \mathbb{R})}$$

with respect to the norm

$$(2.1) \qquad \|u\|_{E_0^{\alpha, p}} = \left(\int_0^T |{}^c D_{0+}^\alpha u(t)|^p \, \mathrm{d}t + \int_0^T |u(t)|^p \, \mathrm{d}t \right)^{1/p} \qquad \text{for every } u \in E_0^{\alpha, p}.$$

R e m a r k 2.6. Note that the fractional derivative space $E_0^{\alpha, p}$ is the space of functions $u \in L^p([0,T], \mathbb{R})$ having an α-order Caputo fractional derivative ${}^c D_t^\alpha u \in L^p([0,T], \mathbb{R})$ and $u(0) = u(T) = 0$. From [27], Proposition 3.1, we know that for $0 < \alpha \leqslant 1$, the space $E_0^{\alpha, p}$ is a reflexive and separable Banach space.

The following lemma addresses the boundedness of the Caputo fractional integral operators from the space $L^p([a,b], \mathbb{R})$ into itself where $1 < p < \infty$.

Lemma 2.7 ([41])**.** *Let* $0 < \alpha \leqslant 1$ *and* $1 < p < \infty$. *For any* $u \in E_0^{\alpha, p}$, *we have*

$$(2.2) \qquad \|u\|_{L^p} \leqslant \frac{T^\alpha}{\Gamma(\alpha + 1)} \|{}^c D_{0+}^\alpha u\|_{L^p}.$$

In addition, for $1/p < \alpha \leqslant 1$ *and* $1/p + 1/q = 1$, *we have*

$$(2.3) \qquad \|u\|_\infty \leqslant k \|{}^c D_{0+}^\alpha u\|_{L^p} \quad \text{where } k = \frac{T^{\alpha - 1/2}}{\Gamma(\alpha)(\alpha q - q + 1)^{1/q}}.$$

R e m a r k 2.8. In view of Lemma 2.7, it is easy to see that the norm in $E_0^{\alpha, p}$ defined by (2.1) is equivalent to the norm

$$(2.4) \qquad \|u\|_{\alpha, p} = \left(\int_0^T |{}^c D_{0+}^\alpha u(t)|^p \, \mathrm{d}t \right)^{1/p}.$$

Lemma 2.9 ([41], Proposition 3.3)**.** *Let* $1/p < \alpha \leqslant 1$. *If the sequence* $\{u_k\}$ *converges weakly to* u *in* $E_0^{\alpha, p}$, *i.e.,* $u_k \rightharpoonup u$, *then* $u_k \to u$ *in* $C([0,T], \mathbb{R})$, *i.e.,* $\|u - u_k\|_\infty \to 0$ *as* $k \to \infty$.

Definition 2.10. A function

$$u \in \left\{ u \in AC[0,T] \colon \left(\int_{t_j}^{t_{j+1}} |^cD_{0+}^\alpha u(t)|^p + |u(t)|^p) \, dt \right) < \infty, \ j = 1,2,\ldots,m \right\},$$

where

$$^cD_{0+}^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^t \frac{u'(s) \, ds}{(t-s)^\alpha},$$

if $t \in (t_j, t_{j+1})$, is called a *classical solution of* BVP (P^f) if u satisfies (P^f) and the limits $D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t_j^+)$ and $D_{T-}^{\alpha-1}\Phi_p(^cD_{0+}^\alpha u)(t_j^-)$ exist.

Definition 2.11. By a *weak solution of* the BVP (P^f), we mean a function $u \in E_0^{\alpha,p}$ such that

$$\int_0^T |^cD_{0+}^\alpha u(t)|^{p-2}(^cD_{0+}^\alpha u(t))(^cD_{0+}^\alpha v(t)) \, dt + \int_0^T |u(t)|^{p-2}u(t)v(t) \, dt$$

$$+ \sum_{j=1}^m I_j(u(t_j))v(t_j) - \int_0^T f(t,u(t))v(t) \, dt = 0$$

for every $v \in E_0^{\alpha,p}$.

The following lemma establishes the relationship between a classical solution and a weak solution of problem (P^f).

Lemma 2.12 ([40], Proposition 2.6). *If $u \in E_0^{\alpha,p}$ is a weak solution of BVP (P^f), then u is a classical solution of BVP (P^f).*

3. Main results

In this section, we formulate our main results on the existence of at least one weak solution to problem (P^f) and then invoke Lemma 2.12 to conclude that we have the existence of a classical solution. Let

$$F(t,\xi) = \int_0^\xi f(t,x) \, dx \quad \forall \, (t,\xi) \in [0,T] \times \mathbb{R}.$$

We will assume throughout that
(H) $1 > Lk^p$, where $L = \sum_{j=1}^m L_j$ and k is given in Lemma 2.7.

Theorem 3.1. *Assume that*

(S)
$$\sup_{\theta>0} \frac{\theta^p}{\int_0^T \max_{|x|\leqslant\theta} F(t,x) \, dt} > \frac{pk^p}{1 - Lk^p}.$$

Then, problem (P^f) admits at least one classical solution in $E_0^{\alpha,p}$.

P r o o f. Our aim is to apply Theorem 2.1 to problem (P^f), so let $X = E_0^{\alpha,p}$. We introduce the functionals Φ and Ψ for $u \in X$ given by

$$(3.1) \qquad \Phi(u) = \frac{1}{p}\|u\|_{E_0^{\alpha,p}}^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,\mathrm{d}s$$

and

$$(3.2) \qquad \Psi(u) = \int_0^T F(t, u(t))\,\mathrm{d}t,$$

and we set

$$I(u) = \Phi(u) - \Psi(u).$$

First we wish to prove that the functionals Φ and Ψ satisfy the conditions required in Theorem 2.1. Since X is compactly embedded in $C^0([0,T]), \mathbb{R})$, it is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$ given by

$$\Psi'(u)(v) = \int_0^T f(t, u(t))v(t)\,\mathrm{d}t \quad \text{for every } v \in X,$$

and that Ψ is sequentially weakly upper semicontinuous. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_0^T |{}^c D_{0+}^\alpha u(t)|^{p-2}({}^c D_{0+}^\alpha u(t))({}^c D_{0+}^\alpha v(t))\,\mathrm{d}t$$
$$+ \int_0^T |u(t)|^{p-2}u(t)v(t)\,\mathrm{d}t + \sum_{j=1}^m I_j(u(t_j))v(t_j) \quad \text{for every } v \in X.$$

By its definition, we see that Φ is sequentially weakly lower semicontinuous and strongly continuous. Now since $-L_j|\xi|^{p-1} \leqslant I_j(\xi) \leqslant L_j|\xi|^{p-1}$ for every $\xi \in \mathbb{R}$, $j = 1, \ldots, n$,

$$-\frac{L_j}{p}|u(t_j)|^p \leqslant \int_0^{u(t_j)} I_j(s)\,\mathrm{d}s \leqslant \frac{L_j}{p}|u(t_j)|^p.$$

In view of (2.3), for every $u \in X$, we have

$$\Phi(u) = \frac{1}{p}\|u\|_{E_0^{\alpha,p}}^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\,\mathrm{d}s \leqslant \frac{1}{p}\|u\|_{E_0^{\alpha,p}}^p + \frac{1}{p}k^p L\|{}^c D_{0+}^\alpha u\|_{L^p}^p$$

$$\leqslant \frac{1}{p}\|u\|_{E_0^{\alpha,p}}^p + \frac{1}{p}k^p L\|u\|_{\alpha,p}^p \leqslant \frac{1}{p}(1 + Lk^p)\|u\|_{E_0^{\alpha,p}}^p.$$

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Similarly,

$$\Phi(u) \geqslant \frac{1}{p}(1 - Lk^p)\|u\|^p_{E_0^{\alpha,p}},$$

so

(3.3) $$\frac{1}{p}(1 - Lk^p)\|u\|^p_{E_0^{\alpha,p}} \leqslant \Phi(u) \leqslant \frac{1}{p}(1 + Lk^p)\|u\|^p_{E_0^{\alpha,p}}.$$

From condition (H) and the first inequality in (3.3), it follows that $\lim_{\|u\|\to\infty} \Phi(u) = \infty$, that is, Φ is coercive.

From condition (S), there exists $\bar{\theta} > 0$ such that

(3.4) $$\frac{\bar{\theta}^p}{\int_0^T \max_{|x|\leqslant\bar{\theta}} F(t,x)\,\mathrm{d}t} > \frac{pk^p}{1 - Lk^p}.$$

Take

$$r = \frac{1 - Lk^p}{pk^p}\bar{\theta}^p.$$

In view of (2.3), (3.1), and (3.3), for every $r > 0$, we have

$$\begin{aligned}
\Phi^{-1}(-\infty, r) &= \{u \in X\colon \Phi(u) < r\} \\
&\subseteq \left\{u \in X\colon \|u\|^p_{\alpha,p} \leqslant \frac{pr}{1 - Lk^p}\right\} \subseteq \left\{u \in X\colon \frac{1}{k^p}\|u\|^p_\infty \leqslant \frac{pr}{1 - Lk^p}\right\} \\
&= \{u \in X\colon \|u\|^p_\infty \leqslant \bar{\theta}^p\},
\end{aligned}$$

from which it follows that

$$\sup_{\Phi(u)<r} \Psi(u) \leqslant \int_0^T \max_{|x|\leqslant\bar{\theta}} F(t,x)\,\mathrm{d}t.$$

By considering the above computations, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, we see that

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u) - \Psi(u)}{r - \Phi(u)} \\
&\leqslant \frac{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)}{r} \leqslant \frac{pk^p}{1 - Lk^p} \frac{\int_0^T \max_{|x|\leqslant\bar{\theta}} F(t,x)\,\mathrm{d}t}{\bar{\theta}^p}.
\end{aligned}$$

Thus,

(3.5) $$\varphi(r) \leqslant \frac{pk^p}{1 - LTk^p} \frac{\int_0^T \max_{|x|\leqslant\bar{\theta}} F(t,x)\,\mathrm{d}t}{\bar{\theta}^p}.$$

Consequently, in view of (3.4) and (3.5), $\varphi(r) < 1$. Hence, since $1 \in (0, 1/\varphi(r))$, Theorem 2.1 ensures that the functional I admits at least one critical point (local minimum) $\tilde{u} \in \Phi^{-1}(-\infty, r)$. Then, taking into account the fact that the weak solutions of problem (P^f) are exactly the critical points of the functional I, and applying Lemma 2.12, we have the desired conclusion. $\qquad\square$

We note that Theorem 3.1 can be used to ensure the existence of at least one weak solution to the parametric version of the problem

$$(\mathrm{P}_\lambda^f) \quad \begin{cases} D_{T^-}^\alpha \Phi_p(^c D_{0^+}^\alpha u(t)) + |u(t)|^{p-2}u(t) = \lambda f(t, u(t)), & t \neq t_j, \ t \in (0, T), \\ \Delta(D_{T^-}^{\alpha-1}\Phi_p(^c D_{0^+}^\alpha u))(t_j) = I_j(u(t_j)), \\ u(0) = u(T) = 0, \end{cases}$$

where λ is a positive parameter. More precisely, we have the following existence result.

Theorem 3.2. *For every λ small enough, i.e.,*

$$\lambda \in \left(0, \frac{1 - Lk^p}{pk^p} \sup_{\theta > 0} \frac{\theta^p}{\int_0^T \max_{|x| \leqslant \theta} F(t, x)\, \mathrm{d}t}\right),$$

problem (P_λ^f) admits at least one classical solution $u_\lambda \in E_0^{\alpha, p}$.

P r o o f. Fix λ as in the conclusion of the theorem. Then condition (S) in Theorem 3.1 is satisfied with F replaced by λF. The conclusion then follows from Theorem 3.1. $\qquad\square$

4. Discussion of the main results

In this section we discuss some implications of the above theorems. In Theorem 3.2 we looked for the critical points of the functional I_λ naturally associated with problem (P_λ^f). We note that, in general, I_λ can be unbounded. For example, in the case where $f(\xi) = 1 + |\xi|^{\gamma - p}\xi^{p-1}$ for every $\xi \in \mathbb{R}$ with $\gamma > p$, for any fixed $u \in E_0^{\alpha, p} \setminus \{0\}$ and $\iota \in \mathbb{R}$, we see that

$$I_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_0^T F(t, \iota u(t))\, \mathrm{d}t \leqslant \iota^p \frac{1 + Lk^p}{p} - \lambda \iota k T \|u\|_{\alpha, p} - \lambda \frac{\iota^\gamma}{\gamma} k^\gamma T \|u\|_{\alpha, p}^\gamma \to -\infty$$

as $\iota \to \infty$. Therefore, the condition (I_2) in [34], Theorem 2.2 is not satisfied. Hence, we can not use direct minimization to find critical points of the functional I_λ.

We wish to point out that the energy functional I_λ associated with problem (P_λ^f) is not coercive. In fact, if $F(\xi) = |\xi|^s$ with $s \in (p, \infty)$ for every $\xi \in \mathbb{R}$, then for any fixed $u \in E_0^{\alpha, p} \setminus \{0\}$ and $\iota \in \mathbb{R}$, we have

$$I_\lambda(\iota u) = \Phi(\iota u) - \lambda \int_0^T F(t, \iota u(t))\, \mathrm{d}t \leqslant \iota^p \frac{1 + Lk^p}{p} - \lambda \iota^s k^s T \|u\|_{\alpha, p}^s \to -\infty$$

as $\iota \to -\infty$.

Remark 4.1. If in Theorem 3.1, the function $f(t,x) \geqslant 0$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}$, then condition (S) takes the simpler form

$$(S_\lambda) \qquad \sup_{\theta>0} \frac{\theta^p}{\int_0^T F(t,\theta)\,dt} > \frac{pk^p}{1-Lk^p}.$$

Moreover, if

$$\limsup_{\theta\to\infty} \frac{\theta^p}{\int_0^T F(t,\theta)\,dt} > \frac{pk^p}{1-Lk^p},$$

then condition (S_λ) automatically holds.

Remark 4.2. If for fixed $\bar{\theta} > 0$,

$$\frac{\bar{\theta}^p}{\int_0^T \max_{|x|\leqslant\bar{\theta}} F(t,x)\,dt} > \frac{pk^p}{1-Lk^p},$$

then the conclusion of Theorem 3.2 holds with $\|u_\lambda\|_\infty \leqslant \bar{\theta}$ being the guaranteed classical solution in $E_0^{\alpha,p}$.

If in Theorem 3.2, we have $f(t,0) \neq 0$ for a.e. $t \in [0,T]$, then the solution obtained is clearly nontrivial. On the other hand, the nontriviality of the solution can be achieved even in the case $f(t,0) = 0$ for a.e. $t \in [0,T]$ by requiring an additional condition at zero, namely, that there are a nonempty open set $D \subseteq (0,T)$ and a set $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{t\in B} F(t,\xi)}{|\xi|^p} = \infty \quad \text{and} \quad \liminf_{\xi\to 0^+} \frac{\operatorname{ess\,inf}_{t\in D} F(t,\xi)}{|\xi|^p} > -\infty.$$

To see this, let $0 < \bar{\lambda} < \lambda^*$ where

$$\lambda^* = \frac{1-Lk^p}{pk^p} \sup_{\theta>0} \frac{\theta^p}{\int_0^T \max_{|x|\leqslant\theta} F(t,x)\,dt}.$$

Then, there exists $\bar{\theta} > 0$ such that

$$\frac{pk^p}{1-Lk^p}\bar{\lambda} < \frac{\bar{\theta}^p}{\int_0^T \max_{|x|\leqslant\bar{\theta}} F(t,x)\,dt}.$$

By Theorem 2.1, for every $\lambda \in (0,\bar{\lambda})$ there exists a critical point of I_λ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where

$$r_\lambda = \frac{1-Lk^p}{pk^p}\bar{\theta}^p.$$

In particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$. To show that the function u_λ cannot be trivial, we will show that

$$(4.1) \qquad \limsup_{\|u\| \to 0^+} \frac{\Psi(u)}{\Phi(u)} = \infty.$$

Due to our assumptions at zero, we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants ζ and κ with $\zeta > 0$ such that

$$\lim_{n \to \infty} \frac{\mathrm{ess\ inf}_{t \in B} F(t, \xi_n)}{|\xi_n|^p} = \infty \quad \text{and} \quad \underset{t \in D}{\mathrm{ess\ inf}}\, F(t, \xi) \geqslant \kappa|\xi|^p$$

for $\xi \in [0, \zeta]$. Now, fix a set $C \subset B$ of positive measure and a function $v \in E_0^{\alpha, p}$ such that:

(i) $v(t) \in [0, 1]$ for every $t \in [0, T]$;
(ii) $v(t) = 1$ for every $t \in C$;
(iii) $v(t) = 0$ for every $t \in (0, T) \setminus D$.

Fix $Y > 0$ and consider a positive number η with

$$Y < \frac{\eta \, \mathrm{meas}(C) + \kappa \int_{D \setminus C} |v(t)|^p \, dt}{(1 + Lk^p)p^{-1}\|v\|_{\alpha, p}^p}.$$

Then, there exists $n_0 \in \mathbb{N}$ such that $\xi_n < \zeta$ and

$$\underset{t \in B}{\mathrm{ess\ inf}}\, F(t, \xi_n) \geqslant \eta|\xi_n|^p \quad \text{for } n > n_0.$$

Now, using the fact that $0 \leqslant \xi_n v(t) < \zeta$ for n large enough, by (3.3), we have

$$\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \frac{\int_C F(t, \xi_n) \, dt + \int_{D \setminus C} F(t, \xi_n v(t)) \, dt}{\Phi(\xi_n v)} > \frac{\eta \, \mathrm{meas}(C) + \kappa \int_{D \setminus C} |v(t)|^p \, dt}{(1 + Lk^p)p^{-1}\|v\|_{\alpha, p}^p} > Y.$$

Since Y can be taken arbitrarily large,

$$\lim_{k \to \infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = \infty,$$

from which (4.1) clearly follows. Hence, there exists a sequence $\{w_n\} \subset E_0^{\alpha, p}$ strongly converging to zero with $w_n \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we conclude that

$$(4.2) \qquad\qquad I_\lambda(u_\lambda) < 0,$$

and so u_λ is nontrivial. From (4.2) we can easily observe that the map

$$(4.3) \qquad\qquad (0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is negative. We claim that

$$(4.4) \qquad\qquad \lim_{\lambda \to 0^+} \|u_\lambda\|_{\alpha, p} = 0.$$

To see this, consider the fact that Φ is coercive and for $\lambda \in (0, \lambda^*)$, the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, and there exists a positive constant $\mathcal{L}$ such that $\|u_\lambda\|_\infty \leqslant \mathcal{L}$ for every $\lambda \in (0, \lambda^*)$. It is then easy to see that there exists a positive constant $\mathcal{M}$ such that

$$(4.5) \qquad\qquad \left| \int_0^T f(t, u_\lambda(t)) u_\lambda(t) \, \mathrm{d}t \right| \leqslant \mathcal{M} \|u_\lambda\|_{\alpha, p} \leqslant \mathcal{M}\mathcal{L}$$

for every $\lambda \in (0, \lambda^*)$. Since u_λ is a critical point of I_λ, we have $I_\lambda'(u_\lambda)(v) = 0$ for any $v \in X$ and every $\lambda \in (0, \lambda^*)$. In particular $I_\lambda'(u_\lambda)(u_\lambda) = 0$; that is,

$$(4.6) \qquad\qquad \Phi'(u_\lambda)(u_\lambda) = \lambda \int_0^T f(t, u_\lambda(t)) u_\lambda(t) \, \mathrm{d}t$$

for every $\lambda \in (0, \lambda^*)$. Then, since

$$0 \leqslant (1 - Lk^p) \|u_\lambda\|_{\alpha, p}^p \leqslant \Phi'(u_\lambda)(u_\lambda),$$

from (4.6) we see that

$$0 \leqslant (1 - Lk^p) \|u_\lambda\|_{\alpha, p}^p \leqslant \Phi'(u_\lambda)(u_\lambda) \leqslant \lambda \int_0^T f(t, u_\lambda(t)) u_\lambda(t) \, \mathrm{d}t$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^+$, (4.5) implies that (4.4) holds. Hence,

$$(4.7) \qquad\qquad \lim_{\lambda \to 0^+} \|u_\lambda\|_\infty = 0.$$

Finally, we wish to show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. To do this, first note that for any $u \in E_0^{\alpha, p}$,

$$(4.8) \qquad\qquad I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right).$$

Now, fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_1} and u_{λ_2} be the global minima of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Let

$$m_{\lambda_i} = \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v) \right) \quad \text{for } i = 1, 2.$$

Clearly, (4.3) together with (4.8) and the fact that $\lambda > 0$ implies

(4.9) $$m_{\lambda_i} < 0 \quad \text{for } i = 1, 2.$$

Moreover,

(4.10) $$m_{\lambda_2} \leqslant m_{\lambda_1},$$

since $0 < \lambda_1 < \lambda_2$. Then, by considering (4.8)–(4.10), we see that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leqslant \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. Since $\lambda < \lambda^*$ is arbitrary, $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. Theorem 3.2 above is a bifurcation result in the sense that the pair $(0, 0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in E_0^{\alpha,p} \times (0, \infty) \colon u_\lambda \text{ is a nontrivial classical solution of } (\mathrm{P}_\lambda^f)\}$$

in $E_0^{\alpha,p} \times \mathbb{R}$. In order to see this, recall from above that

$$\|u_\lambda\|_{\alpha,p} \to 0 \quad \text{as } \lambda \to 0.$$

Hence, there exist two sequences $\{u_j^i\}$, $i = 1, 2$, in $E_0^{\alpha,p}$ and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ such that

$$\lambda_i \to 0^+ \quad \text{and} \quad \|u_j^i\|_{\alpha,p} \to 0$$

as $j \to \infty$ for $i = 1, 2$. Moreover, due to the fact that, as shown above, the map $(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$ with $\lambda_1 \neq \lambda_2$, the solutions u_{λ_1} and u_{λ_2} are different.

If f is non-negative, then the solution obtained in Theorem 3.2 is non-negative. To see this, let u_0 be a nontrivial classical solution of problem (P_λ^f). Assume, for the sake of a contradiction, that the set $\mathcal{A} = \{t \in (0, T) \colon u_0(t) < 0\}$ is nonempty and of positive measure. Set $\overline{v}(t) = \min\{0, u_0(t)\}$ for all $t \in [0, T]$. Clearly, $\overline{v} \in E_0^{\alpha,p}$ and

$$\int_0^T |{}^c D_{0+}^\alpha u_0(t)|^{p-2} ({}^c D_{0+}^\alpha u_0(t)) ({}^c D_{0+}^\alpha \overline{v}(t)) \, \mathrm{d}t + \int_0^T |u_0(t)|^{p-2} u_0(t) \overline{v}(t) \, \mathrm{d}t$$

$$+ \sum_{j=1}^m I_j(u_0(t_j)) \overline{v}(t_j) - \lambda \int_0^T f(t, u_0(t)) \overline{v}(t) \, \mathrm{d}t = 0.$$

Thus,

$$0 \leqslant (1 - Lk^p)\|u_0\|_{\mathcal{A}}^p \leqslant \int_0^T |{}^cD_{0+}^\alpha u_0(t)|^p \, \mathrm{d}t + \int_0^T |u_0(t)|^p \, \mathrm{d}t + \sum_{j=1}^m I_j(u_0(t_j))(u_0(t_j)$$

$$= \lambda \int_0^T f(t, u_0(t))u_0(t) \, \mathrm{d}t \leqslant 0.$$

Hence, since $Lk^p < 1$, $u_0 = 0$ in $\mathcal{A}$, and this is a contradiction.

The final theorem in this paper is concerned with a particular case of our results, namely, where $f(t, u)$ is independent of t.

Theorem 4.3. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a non-negative continuous function,* $F(\xi) = \int_0^\xi f(s) \, \mathrm{d}s$ *for all* $\xi \in \mathbb{R}$, *and assume that*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} = \infty.$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{1 - Lk^p}{pk^p} \sup_{\theta > 0} \frac{\theta^p}{F(\theta)}\right),$$

the problem

$$\begin{cases} D_{T-}^\alpha \Phi_p({}^cD_{0+}^\alpha u(t)) + |u(t)|^{p-2}u(t) = \lambda f(u(t)), & t \neq t_j, \ t \in (0, T), \\ \Delta(D_{T-}^{\alpha-1}\Phi_p({}^cD_{0+}^\alpha u))(t_j) = I_j(u(t_j)), \\ u(0) = u(T) = 0 \end{cases}$$

admits at least one nontrivial classical solution $u_\lambda \in E_0^{\alpha, p}$ *such that*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\alpha, p} = 0.$$

In addition, the function

$$\lambda \to \frac{1}{p}\|u\|_{\alpha, p}^p - \sum_{j=1}^m \int_0^{u(t_j)} I_j(s) \, \mathrm{d}s - \lambda \int_0^T F(u(t)) \, \mathrm{d}t$$

is negative and strictly decreasing in Λ.

As a special case of Theorem 4.3 with $p = 2$, we have the following result.

Corollary 4.4. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a non-negative continuous function, set* $F(\xi) = \int_0^\xi f(s) \, \mathrm{d}s$ *for all* $\xi \in \mathbb{R}$, *and assume that*

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = \infty.$$

Then, for each

$$\lambda \in \Lambda = \left(0, \frac{1 - L\overline{k}^2}{p\overline{k}^2} \sup_{\theta > 0} \frac{\theta^2}{F(\theta)}\right), \quad \text{where } \overline{k} = \frac{T^{\alpha - 1/2}}{\Gamma(\alpha)(2\alpha - 1)^{1/2}},$$

the problem

$$\begin{cases} D_{T-}^{\alpha}(^cD_{0+}^{\alpha}u(t)) + u(t) = \lambda f(u(t)), & t \neq t_j, \ t \in (0, T), \\ \Delta(D_{T-}^{\alpha-1}(^cD_{0+}^{\alpha}u))(t_j) = I_j(u(t_j)), \\ u(0) = u(T) = 0 \end{cases}$$

admits at least one nontrivial classical solution $u_\lambda \in E_0^{\alpha,2}$ such that

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{\alpha,2} = 0$$

and the real function

$$\lambda \to \frac{1}{2}\|u\|_{\alpha,2}^2 - \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)\,\mathrm{d}s - \lambda \int_0^T F(u(t))\,\mathrm{d}t$$

is negative and strictly decreasing in Λ.

We conclude this paper with an example to illustrate our results.

E x a m p l e 4.5.   Consider the problem

(4.11)
$$\begin{cases} D_{T-}^{2/3}\Phi_{5/3}(^cD_{0+}^{2/3}u(t)) + |u(t)|u(t) = \lambda f(u(t)), & t \neq \frac{1}{2}, \ t \in (0, 1), \\ \Delta(D_{T-}^{-1/3}\Phi_{5/3}(^cD_{0+}^{2/3}u))(t_1) = \frac{\Gamma^{5/3}(\frac{2}{3})}{2\sqrt[3]{36}}|u(\frac{1}{2})|^{2/3}, \\ u(0) = u(1) = 0, \end{cases}$$

where $f(\xi) = 3\xi^2$ for all $\xi \in \mathbb{R}$. Here, $\alpha = \frac{2}{3}$, $p = \frac{5}{3}$, $T = 1$, $F(\xi) = \xi^3$, and a direct computation shows

$$k = \frac{\sqrt[5]{36}}{\Gamma(\frac{2}{3})}.$$

All conditions of Theorem 4.3 are satisfied and $\Lambda = (0, \infty)$, so for each $\lambda \in (0, \infty)$, problem (4.11) admits at least one nontrivial classical solution $u_\lambda \in E_0^{2/3,5/3}$ such that $\lim_{\lambda \to 0^+} \|u_\lambda\|_{2/3,5/3} = 0$, and the function

$$\lambda \to \frac{3}{5}\|u\|_{2/3,5/3}^{5/3} - \frac{\Gamma^{5/3}(\frac{2}{3})}{2\sqrt[3]{36}} \int_0^{u(1/2)} |s|^{2/3}\,\mathrm{d}s - \lambda \int_0^1 F(u(t))\,\mathrm{d}t$$

is negative and strictly decreasing in $(0, \infty)$.

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