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THE STRONG PERSISTENCE PROPERTY AND SYMBOLIC STRONG PERSISTENCE PROPERTY

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Abstract. Let I be an ideal in a commutative Noetherian ring R. Then the ideal I has the strong persistence property if and only if $(I^{k+1}\colon {}_R I) = I^k$ for all k, and I has the symbolic strong persistence property if and only if $(I^{(k+1)}\colon {}_R I^{(1)}) = I^{(k)}$ for all k, where $I^{(k)}$ denotes the kth symbolic power of I. We study the strong persistence property for some classes of monomial ideals. In particular, we present a family of primary monomial ideals failing the strong persistence property. Finally, we show that every square-free monomial ideal has the symbolic strong persistence property.

Keywords: strong persistence property; associated prime; cover ideal; symbolic strong persistence property

MSC 2020: 13C13, 13B25, 13A30, 13P25, 05C25, 05E40

1. Introduction and preliminaries

Let R be a commutative Noetherian ring and I an ideal of R. A prime ideal $\mathfrak{p} \subset R$ is an *associated prime* of I if there exists an element v in R such that $\mathfrak{p} = (I\colon {}_R v)$, where $(I\colon {}_R v) = \{r \in R\colon rv \in I\}$. The set of associated primes of I, denoted by $\mathrm{Ass}_R(R/I)$, is the set of all prime ideals associated to I. Brodmann in [1] proved that the sequence $\{\mathrm{Ass}_R(R/I^k)\}_{k\geqslant 1}$ of associated prime ideals is stationary for large k, that is, there exists a positive integer k_0 such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all integers $k \geqslant k_0$. The minimum such k_0 is called the *index of stability* of I and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set* of associated prime ideals of I, which is denoted by $\mathrm{Ass}^\infty(I)$. There are a few exact calculations of the stable set and the index of stability for ideals, see [10] and [17] for more details. There have been several

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questions arising from Brodmann's result. An ideal I of R satisfies the *persistence property* if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I^{k+1})$ for all positive integers k. In addition, an ideal I of R has the *strong persistence property* if $(I^{k+1} :_R I) = I^k$ for all positive integers k; refer to [16] for more information. It is well-known that the strong persistence property implies the persistence property, see [15], Proposition 2.9.

Assume that I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and $x_1, \dots, x_n$ are indeterminates. Generally, finding classes of monomial ideals which either have or fail the persistence property is complicated. It has been shown in [9] that there exists a square-free monomial ideal which does not satisfy the persistence property. However, Ratliff in [20] proved that $(I^{k+1} :_R I) = I^k$ for all large k. Also, it is known by [12] that all edge ideals of finite simple graphs have the strong persistence property; this result is valid for every finite graph with loops, see [21]. Furthermore, it has been established in [7] that every polymatroidal ideal has the strong persistence property. Moreover, according to [3], the cover ideals of perfect graphs satisfy the persistence property. More recently, it has been proved in [18] that the cover ideals of some imperfect graphs have the strong persistence property, that is, cycle graphs of odd orders, wheel graphs of even orders, and helm graphs of odd orders greater than or equal to 5.

We know from [19] that if I and J are two ideals in a commutative Noetherian ring R, then we say that J is a *superficial ideal* for I if the following conditions are satisfied:

(i) $\mathcal{G}(J) \subseteq \mathcal{G}(I)$, where $\mathcal{G}(L)$ denotes the unique minimal set of monomial generators of a monomial ideal L,

(ii) $(I^{k+1} :_R J) = I^k$ for all positive integers k.

It is easy to see that an ideal I has the strong persistence property if and only if I has a superficial ideal. Besides, note that an ideal I in a commutative Noetherian ring R is called *normally torsion-free* if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I)$ for all $k \in \mathbb{N}$. It has been shown in [19], Theorem 6.10, that every normally torsion-free square-free monomial ideal has the strong persistence property.

In this direction, the notion of symbolic strong persistence property was introduced in [21]. An ideal I in a commutative Noetherian ring R has the *symbolic strong persistence property* if $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all k, where $I^{(k)}$ denotes the kth symbolic power of I. In this paper, we continue studying the symbolic strong persistence property. Symbolic powers have many nice properties, especially if I is a square-free monomial ideal. The symbolic strong persistence property was suggested to us by analogy with the strong persistence property, with the hope that symbolic powers would behave better than regular powers. This hope is borne out in one of our main theorems, Theorem 5.1, where we prove that every square-free monomial ideal has the symbolic strong persistence property, even though normally torsion-free monomial ideals need not have the symbolic strong persistence property.

This paper is organized as follows. In Section 2, in Lemmas 2.1 and 2.2, we investigate the strong persistence property of the intersection, product, and sum of two monomial ideals which are generated by two disjoint sets of variables. Next, in Corollary 2.1, we prove that every irreducible monomial ideal has the strong persistence property. In Theorem 2.1, we show that if one takes any graph G and form a new graph H by adding new vertices joining each to every vertex of G, then $J(G)$ has the strong persistence property if and only if $J(H)$ has the strong persistence property. Finally, Corollary 2.2 tells us that if the cover ideal of a finite simple graph has the strong persistence property, then the cover ideal of its whisker graph has the strong persistence property.

In Section 3, we focus on the strong persistence property of primary monomial ideals as a case study. In fact, in Proposition 3.1, we give a class of primary monomial ideals which do not satisfy the strong persistence property.

Section 4 is devoted to the strong persistence property of the cover ideal of the union of finite simple graphs. To do this, we first, in Lemma 4.2, explore the relation between associated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the associated primes of powers of the cover ideals of each of them. We finally give the main result of this section in Theorem 4.1.

Section 5 is concerned with the symbolic strong persistence property. In Proposition 5.2 we prove that if an ideal has the symbolic strong persistence property, then any power of it has the symbolic strong persistence property as well. Theorem 5.1 as the main result of the section, says that every square-free monomial ideal has the symbolic strong persistence property. Throughout this paper, we denote the unique minimal set of monomial generators of a monomial ideal I by $\mathcal{G}(I)$. Also, $R = K[x_1, \ldots, x_n]$ is a polynomial ring over a field K and $x_1, \ldots, x_n$ are indeterminates. The symbols $\mathbb{N}$ and $\mathbb{Q}$, respectively, will always denote the set of positive integers and rational numbers, respectively. A simple graph G means that G has no loop and no multiple edge. All graphs in this paper are undirected. Moreover, if G is a finite simple graph, then $J(G)$ stands for the cover ideal of G.

2. Some results on the strong persistence property

In this section, we study the strong persistence property of monomial ideals with a suitable assumption on its support. We begin with the following lemma which allows us to discuss the strong persistence property of the intersection and product of two monomial ideals which are generated by two disjoint sets of variables. To see this, we need the following definition.

Definition 2.1 ([22], Definition 6.1.5). Let $u = x_1^{a_1} \dots x_n^{a_n}$ be a monomial in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K. The *support* of u is given by $\operatorname{supp}(u) := \{x_i \colon a_i > 0\}$. In addition, for a monomial ideal I of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, we define $\operatorname{supp}(I) := \bigcup\limits_{i=1}^{m} \operatorname{supp}(u_i)$.

Lemma 2.1. *Suppose that I_1 and I_2 are two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that* $\operatorname{supp}(I_1) \cap \operatorname{supp}(I_2) = \emptyset$. *If I_1 and I_2 have the strong persistence property, then $I_1 \cap I_2$ has the strong persistence property.*

Proof. Assume that I_1 and I_2 have the strong persistence property. Since I_1 and I_2 are generated by disjoint sets of variables, Lemma 1.1 of [8] yields that $I_1^a \cap I_2^b = I_1^a I_2^b$ for any positive integers a, b. Let $k \geqslant 1$. By observing Lemma 2.1 of [11] one can deduce that

$$(I_1^{k+1} I_2^{k+1} \colon {}_R I_1) = I_2^{k+1}(I_1^{k+1} \colon {}_R I_1) \quad \text{and} \quad (I_1^k I_2^{k+1} \colon {}_R I_2) = I_1^k (I_2^{k+1} \colon {}_R I_2).$$

To end the proof, it is enough to consider the following equalities:

$$\begin{aligned}((I_1 \cap I_2)^{k+1} \colon {}_R I_1 \cap I_2) &= ((I_1 I_2)^{k+1} \colon {}_R I_1 I_2) = ((I_1^{k+1} I_2^{k+1} \colon {}_R I_1) \colon {}_R I_2) \\ &= (I_2^{k+1}(I_1^{k+1} \colon {}_R I_1) \colon {}_R I_2) = (I_1^k I_2^{k+1} \colon {}_R I_2) \\ &= I_1^k (I_2^{k+1} \colon {}_R I_2) = I_1^k I_2^k = (I_1 \cap I_2)^k.\end{aligned}$$

□

Lemma 2.2. *Suppose that I_1 and I_2 are two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that* $\operatorname{supp}(I_1) \cap \operatorname{supp}(I_2) = \emptyset$. *Then $I_1 + I_2$ has the strong persistence property if and only if I_1 or I_2 has the strong persistence property.*

Proof. The backward implication can be immediately deduced from [19], Theorem 3.2. To establish the forward implication, suppose, on the contrary, that I_1 and I_2 do not satisfy the strong persistence property. This implies that there exist a positive integer k_1 (or k_2) and a monomial m_1 (or m_2) such that $m_1 \in \mathcal{G}(I_1^{k_1+1} \colon {}_R I_1) \setminus \mathcal{G}(I_1^{k_1})$ (or $m_2 \in \mathcal{G}(I_2^{k_2+1} \colon {}_R I_2) \setminus \mathcal{G}(I_2^{k_2})$). Take the nonnegative integer a_1 (or a_2) such that $m_1 \in I_1^{a_1} \setminus I_1^{a_1+1}$ (or $m_2 \in I_2^{a_2} \setminus I_2^{a_2+1}$). This gives that $a_1 \leqslant k_1 - 1$ (or $a_2 \leqslant k_2 - 1$). Put $I := I_1 + I_2$, $m := m_1 m_2$ and $b := a_1 + a_2$. Thus, one has $m \in I^b$. Note that $m \notin I_1^i \cap I_2^j$ for either $i > a_1$ or $j > a_2$, so by Lemma 1.1 of [8], $m \notin I^{b+1}$. To conclude the proof, it is enough to show that $m \in (I^{b+2} \colon {}_R I)$, this contradicts the assumption that I has the strong persistence property. So, take a monomial $u \in I = I_1 + I_2$. Without loss of generality, we can assume that $u \in I_1$. Thus, $um = (um_1)m_2 \in I_1^{k_1+1} I_2^{a_2} \subseteq (I_1 + I_2)^{k_1+1+a_2}$ and since $k_1 > a_1$, we are done. □

An application of Lemma 2.2 is the corollary below.

Corollary 2.1. *Every irreducible primary monomial ideal has the strong persistence property.*

Proof. Let $k \geqslant 1$ be an integer, and recall that by Proposition 6.1.7 of [22], a monomial ideal is primary irreducible ideal if and only if it has the form $Q = (x_{i_1}^{\alpha_1}, \ldots, x_{i_t}^{\alpha_t})$ in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K with $\alpha_1, \ldots, \alpha_t$ being positive integers and $\{x_{i_1}, \ldots, x_{i_t}\} \subseteq \{x_1, \ldots, x_n\}$. Set $I_1 := (x_{i_1}^{\alpha_1})$ and $I_2 = (x_{i_2}^{\alpha_2}, \ldots, x_{i_t}^{\alpha_t})$. Since I_1 has the strong persistence property, Lemma 2.2 implies that Q has the strong persistence property, as desired. $\square$

By a repeated application of Lemma 2.1 of [18], we have the following result.

Lemma 2.3. *Let I and J be two monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$, $\mathcal{G}(J) = \{v_1, \ldots, v_s\}$, and h a monomial in R such that for each $i = 1, \ldots, m$ and $j = 1, \ldots, s$, $\gcd(h, v_j) = 1$, $\gcd(v_j, u_i) = 1$, and $h \in I$. If I has the strong persistence property, then $L := JI + hR$ has the strong persistence property.*

Remark 2.1. It should be noted that Lemma 2.3 may be false if we consider the ideal L as $L = JI + H$ with H not a principal monomial ideal. To see this, assume that L is the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane, that is, $L \subset R = K[x_1, x_2, x_3, x_4, x_5, x_6]$, one has

$$\begin{aligned} L = (&x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, \\ &x_3x_4x_5, x_3x_4x_6). \end{aligned}$$

In the sequel, put $J := (x_3)$, $I := (x_1x_2, x_1x_5, x_2x_6, x_4x_5, x_4x_6)$ and $H := (x_1x_2x_4, x_1x_4x_6, x_1x_5x_6, x_2x_4x_5, x_2x_5x_6)$. It can be rapidly checked that $L = JI+H$, $H \subseteq I$, $\gcd(x_3, h) = 1$ for all $h \in \mathcal{G}(H)$, $\gcd(x_3, u) = 1$ for all $u \in \mathcal{G}(I)$. We prove that I has the strong persistence property. To do this, assume that G is the cycle graph with $V(G) = \{1, 2, 4, 5, 6\}$ and $E(G) = \{\{1, 2\}, \{2, 6\}, \{6, 4\}, \{4, 5\}, \{5, 1\}\}$. It is easy to detect that the edge ideal of G is $I(G) = (x_1x_2, x_1x_5, x_2x_6, x_4x_5, x_4x_6)$. This implies that $I(G) = I$. In addition, Theorem 7.7.14 of [22] yields that $I(G)$ has the strong persistence property, and thus the monomial ideal I has the strong persistence property. On the other hand, by using Macaulay2 (see [5]), one can detect that $(L^3 :_R L) \neq L^2$, that is, L does not satisfy the strong persistence property.

As an application of Lemma 2.1 of [18], we present Theorem 2.1. To understand the importance of this theorem, we first review some background. Recall the following definitions and theorem.

Let $G = (V(G), E(G))$ be a finite simple graph on the vertex set $V(G) := \{1, \ldots, n\}$. Then the *edge ideal* associated to G is the monomial ideal

$$I(G) = (x_i x_j \colon \ \{i, j\} \in E(G)) \subset R = K[x_1, \ldots, x_n],$$

and the *cover ideal associated to* G is the monomial ideal

$$J(G) = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j) \subset R = K[x_1, \ldots, x_n].$$

Definition 2.2 ([22], Definition 10.5.4)**.** The *cone* $C(G)$ *over the graph* G is obtained by adding a new vertex t to G and joining every vertex of G to t.

More generally, we can take any graph G and form a new graph H by adding new vertices, joining each to every vertex of G. Then $J(G)$ has the strong persistence property if and only if $J(H)$ has the strong persistence property. Explicitly:

Theorem 2.1. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w_1, \ldots, w_r\}$ with $w_i \notin V(G)$ for all $i = 1, \ldots, r$, and*

$$E(H) = E(G) \cup \{\{v, w_i\} \colon \text{ for all } v \in V(G) \text{ and for all } i = 1, \ldots, r\}.$$

Then $J(G)$ has the strong persistence property if and only if $J(H)$ has the strong persistence property.

P r o o f. We give a sketch of the proof. Assume that $V(G) = [n]$ and $V(H) = V(G) \cup \{n+1, \ldots, n+r\}$. In addition, let $R = K[x_1, \ldots, x_{n+r}]$ be the polynomial ring over a field K. Put $L := J(H)$, $I := J(G)$, $h := \prod\limits_{i=1}^{n} x_i$ and $g := \prod\limits_{i=n+1}^{n+r} x_i$. It follows from Exercise 6.1.23 of [22] that

$$\bigcap_{i=1}^{n} \bigcap_{j=n+1}^{n+r} (x_i, x_j) = \left(\prod_{i=1}^{n} x_i, \prod_{j=n+1}^{n+r} x_j \right).$$

Also, it is easy to see that $L = gIR + hR$. Suppose that I has the strong persistence property. Now, Lemma 2.1 of [18] yields that L has the strong persistence property, as claimed. Conversely, let $J(H)$ have the strong persistence property. Put $\mathfrak{p} := (x_1, \ldots, x_n)$. Since $I = L(\mathfrak{p})$, where $L(\mathfrak{p})$ denotes the monomial localization of L with respect to $\mathfrak{p}$, the claim follows at once from Theorem 4.7 of [19]. $\square$

Here, we concentrate on the notion of a whisker graph. Our aim is to explore the strong persistence property of the cover ideal of a whisker graph. To do this, we state the subsequent definition.

Definition 2.3 ([22], Definition 7.3.10). Let G_0 be a graph on the vertex set $Y = \{y_1, \ldots, y_n\}$ and take a new set of variables $X = \{x_1, \ldots, x_n\}$. The *whisker graph* or *suspension* of G_0, denoted by $G_0 \cup W(Y)$, is the graph obtained from G_0 by attaching to each vertex y_i a new vertex x_i and the edge $\{x_i, y_i\}$. The edge $\{x_i, y_i\}$ is called a *whisker*.

Theorem 2.7 of [18] says that if we take any graph G whose cover ideal has the strong persistence property, and if we then add a leaf to G, then the cover ideal of the new graph satisfies the strong persistence property as well.

Corollary 2.2. *With the notation of Definition* 2.3, *if the cover ideal of a finite simple graph* G_0 *has the strong persistence property, then the cover ideal of the whisker graph of* G_0 *has the strong persistence property.*

Proof. Proceed by using Theorem 2.7 of [18] repeatedly, n times. □

3. Case study: strong persistence property of primary monomial ideals

Several questions may be asked along our argument. In this section, we investigate the strong persistence property of primary monomial ideals as a case study. To accomplish this, we start with the following main question.

Question 3.1. *Which classes of monomial ideals have the strong persistence property?*

We present a class of primary monomial ideals which do not satisfy the strong persistence property. To do this, consider the monomial ideals

$$I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$$

in the polynomial ring $R = K[x, y]$, $n \geqslant 4$ over a field K. The idea is to describe the monomials in $(I_n^d)_{nd+i}$ for $i \geqslant 0$, where subscripts denote the degree in R. We list all the monomials in $(I_n^d)_{nd+i}$ in lexicographic order with respect to $x < y$, i.e., $x^{nd+i}, x^{nd+i-1}y, \ldots, y^{nd+i}$. In order to simplify the notation, we will often describe a monomial in $(I_n^d)_{nd+i}$ with its power of y because we can identify such a monomial with its power of y.

It is clear that $(I_n^d)_j = 0$ for $j < nd$ and that the monomials in $(I_n^d)_{nd}$ are exactly the products of d generators of I_n (with repetition). First of all, we consider the monomials $\{(x^n)^\alpha (y^n)^\beta\}_{\alpha+\beta=d,\, \alpha\geqslant 0,\, \beta\geqslant 0}$ which we will refer to as "subdivision points". Start with the subdivision point $(x^n)^\alpha(y^n)^\beta$. If $\alpha \geqslant 1$, then some of the x^n can be replaced by $x^{n-1}y$, and if $\beta \geqslant 1$, then some of the y^n can be replaced by xy^{n-1}. This leads to an "interval" of consecutive monomials $\{x^{n\alpha-j}y^{n\beta+j}\}_{-\beta\leqslant j\leqslant\alpha}$ about the subdivision point $(x^n)^\alpha(y^n)^\beta$. The first monomial in this interval is $\{x^{n\alpha+\beta}y^{n\beta-\beta}\}$ and the last is $\{x^{n\alpha-\alpha}y^{n\beta+\alpha}\}$. Describing this interval with the powers of y, we have an increasing sequence of consecutive powers of y, $\{y^j\}_{n\beta-\beta\leqslant j\leqslant n\beta+\alpha}$. The number of monomials in this interval is $(n\beta+\alpha)-(n\beta-\beta)+1=\alpha+\beta+1=d+1$. The previous subdivision point (if $\beta \geqslant 1$) is $x^{n(\alpha+1)}y^{n(\beta-1)}$ and so to avoid overlap of intervals we require that $n(\beta-1)+(\alpha+1)<n\beta-\beta$ or equivalently $d+1<n$, so that we have $1 \leqslant d \leqslant n-2$.

If $d=n-2$, then there is no overlap of intervals, but also no gap. If $1 \leqslant d \leqslant n-3$, between adjacent intervals

$$\{x^{n(\alpha+1)-j}y^{n(\beta-1)+j}\}_{-(\beta-1)\leqslant j\leqslant(\alpha+1)} \quad \text{and} \quad \{x^{n\alpha-j}y^{n\beta+j}\}_{-\beta\leqslant j\leqslant\alpha}$$

there is a gap. For the left-hand side interval, the largest exponent of y is $n(\beta-1)+(\alpha+1)$ and the smallest exponent of y on the right-hand side interval is $n\beta-\beta$, so the powers of y in the gap are $\{y^j\}_{n(\beta-1)+(\alpha+1)<j<n\beta-\beta}$, there being

$$(n\beta-\beta)-(n(\beta-1)+(\alpha+1))-1=n-d-2$$

monomials in the gap. As a simple check for consistency, we have $d+1$ intervals each containing $d+1$ monomials, and d gaps each containing $n-d+2$ monomials, and $(d+1)^2+d(n-d-2)=nd+1$, the total number of monomials in R_{nd}.

Now consider the monomials in $(I_n^d)_{nd+i}$, $i \geqslant 0$. Then the interval

$$\{x^{n(\alpha+1)-j}y^{n(\beta-1)+j}\}_{-(\beta-1)\leqslant j\leqslant(\alpha+1)}$$

(of cardinality $\alpha+\beta+1=d+1$) expands to include

$$\{x^{n(\alpha+1)-j+r}y^{n(\beta-1)+j+s}\}_{-(\beta-1)\leqslant j\leqslant(\alpha+1),\, r\geqslant 0,\, s\geqslant 0,\, r+s=i}$$

(of cardinality $\alpha+\beta+1+i=d+i+1$) and $\{x^{n\alpha-j}y^{n\beta+j}\}_{-\beta\leqslant j\leqslant\alpha}$ expands to include $\{x^{n\alpha-j+r}y^{n\beta+j+s}\}_{-\beta\leqslant j\leqslant\alpha,\, r\geqslant 0,\, s\geqslant 0,\, r+s=i}$ (again of cardinalities, respectively, $d+1$ and $d+i+1$). As above we will describe the gap between these intervals by giving the exponents of y in the monomials in this gap.

Lemma 3.1 (Gap Lemma).
(1) *If $i \geqslant n-d-2$, then $(I_n^d)_{nd+i}=R_{nd+i}$.*
(2) *If $0 \leqslant i \leqslant n-d-2$, then $(I_n^d)_{nd+i}$ has $d+1$ subdivision intervals each of whose cardinality is $d+i+1$, which is i more than that for $(I_n^d)_{nd}$.*

(3) *If $0 \leqslant i < n-d-2$, then the list of y-exponents of monomials in the gap between consecutive intervals*

$$\{x^{n(\alpha+1)-j+r}y^{n(\beta-1)+j+s}\}_{-(\beta-1)\leqslant j\leqslant(\alpha+1),\, r\geqslant 0,\, s\geqslant 0,\, r+s=i}$$

and

$$\{x^{n\alpha-j+r}y^{n\beta+j+s}\}_{-\beta\leqslant j\leqslant\alpha,\, r\geqslant 0,\, s\geqslant 0,\, r+s=i}$$

of $(I_n^d)_{nd+i}$, $i \geqslant 0$ contains $n-d-2-i$ consecutive values (that is, i less than for $(I_n^d)_{nd}$). Also the largest y-exponent in the gap is $n\beta-\beta-1$.

Proof. Part (2) was observed in the paragraph before the statement of the lemma. The largest exponent of y of a monomial in the first interval is $n(\beta-1)+\alpha+1+i$ (coming from $j=\alpha+1$ and $s=i$) and the smallest exponent of y in the right most interval is $n\beta-\beta$ (coming from $j=-\beta$ and $s=0$). If

$$(n\beta-\beta)-(n(\beta-1)+\alpha+1+i)-1 = n-d-2-i \leqslant 0,$$

there are no gaps and part (1) follows. Otherwise we have (3). □

Thus, we start with gap size $n-d-2$ previously obtained for $i=0$, and drop one exponent from the beginning of the list of exponents for each increase of 1 in the value of i, until $(I_n^d)_{nd+i}$ consists of all R_{nd+i} for $i=n-d-2$.

To make the notation more manageable we will refer to

$$\{x^{n\alpha-j+r}y^{n\beta+j+s}\}_{-\beta\leqslant j\leqslant\alpha,\, r\geqslant 0,\, s\geqslant 0,\, r+s=i}$$

as the (α,β) subdivision interval of $(I_n^d)_{nd+i}$ ($0 \leqslant \beta \leqslant d$, $\alpha+\beta=d$). With this notation, Lemma 3.1 (3) takes on the simpler appearance:

Lemma 3.2 (simplified Gap Lemma). *For $1 \leqslant \beta \leqslant d$, the list of y-exponents of monomials in the gap between consecutive subdivision intervals $(\alpha+1,\beta-1)$ and (α,β) of $(I_n^d)_{nd+i}$ with $0 \leqslant i < n-d-2$ contains $n-d-2-i$ consecutive values, with the largest equal to $n\beta-\beta-1$.*

Now, we calculate $I_n^{d+1} : I_n$ for a fixed $d \geqslant 1$. Note that $I_n^{d+1} : I_n$ is a monomial ideal and $I_n^d \subseteq (I_n^{d+1} : I_n)$, so we seek monomials in $I_n^{d+1} : I_n$ but not in I_n^d. If $d \geqslant n-2$, then both I_n^d and I_n^{d+1} are all of R_j in all degrees j for which they are nonzero (namely $j \geqslant nd$ for I_n^d and $j \geqslant n(d+1)$ for I_n^{d+1}) by Lemma 3.1 (1). From this, it follows that $(I_n^{d+1} : I_n) = I_n^d$ for $d \geqslant n-2$.

Now, consider $1 \leqslant d \leqslant n-3$. Let $i = n-d-3$. Then by Lemma 3.1 (3), $(I_n^d)_{nd+i}$ has gaps of size 1 and $(I_n^{d+1})_{n(d+1)+i}$ is equal to $R_{n(d+1)+i}$. The monomials in these gaps of size 1 are mapped into $(I_n^{d+1})_{n(d+1)+i} = R_{n(d+1)+i}$ when multiplied by any minimal generator of I_n (and indeed, by any element of R_n), hence lie in $(I_n^{d+1} : I_n) \setminus I_n^d$. These monomials are $\{x^{nd+n-d-3-n\beta+\beta+1}y^{n\beta-\beta-1}\}_{1\leqslant\beta\leqslant d} = \{x^{nd-d-1}y^{n-2}, x^{nd-n-d}y^{2n-3}, \ldots, x^{n-2}y^{nd-d-1}\}$.

If $0 \leqslant i < n-d-3$, then $(I_n^d)_{nd+i}$ has gaps of size $t > 1$, which are again mapped into $R_{n(d+1)+i}$ when multiplied by any minimal generator of I_n, in which $(I_n^{d+1})_{n(d+1)+i}$ has gaps of size $t-1 > 0$. Suppose that one of our gaps of $(I_n^d)_{nd+i}$ lies between subdivision intervals $(\alpha+1, \beta-1)$ and (α, β). Then by Lemma 3.2, the monomials in the gap (described by giving only their exponent of y) consist of t consecutive powers of y with the largest being $n\beta - \beta - 1$. There is not a one-to-one correspondence between the gaps of $(I_n^{d+1})_{n(d+1)+i}$ and $(I_n^d)_{nd+i}$, the former having $d+1$ gaps, and the latter having d gaps, so we have to make a choice. Choose the gap of $(I_n^{d+1})_{n(d+1)+i}$ between subdivision intervals $(\alpha+2, \beta-1)$ and $(\alpha+1, \beta)$. Then the monomials in the gap between these intervals consist of $t-1$ consecutive powers of y with the largest again being $n\beta-\beta-1$. The monomials in our gap for $(I_n^d)_{nd+i}$ are then represented by $\{y^{n\beta-\beta-t}, y^{n\beta-\beta-(t-1)}, \ldots, y^{n\beta-\beta-1}\}$ and the monomials in our gap for $(I_n^{d+1})_{n(d+1)+i}$ are represented by $\{y^{n\beta-\beta-(t-1)}, \ldots, y^{n\beta-\beta-1}\}$. Recall that x^n and $x^{n-1}y$ are minimal generators of I_n and multiplication by them raises the degree by n. Thus, if we multiply a monomial represented by y^a of the gap sequence for $(I_n^d)_{nd+i}$ by x^n, we get a monomial of the gap sequence for $(I_n^{d+1})_{n(d+1)+i}$ that is also represented by y^a, except for the first power $y^{n\beta-\beta-t}$, where we can multiply by $x^{n-1}y$ instead and get the monomial represented by $y^{n\beta-\beta-(t-1)}$, which is in the gap sequence for $(I_n^{d+1})_{n(d+1)+i}$. This shows that if $0 \leqslant i < n-d-3$, then the monomials in the gaps of $(I_n^d)_{nd+i}$ do not lie in $I_n^{d+1} : I_n$. Therefore, $(I_n^{d+1} : I_n) \setminus I_n^{d+1}$ consists only of the monomials $\{x^{nd-d-1}y^{n-2}, x^{nd-n-d}y^{2n-3}, \ldots, x^{n-2}y^{nd-d-1}\}$ of degree $nd+i$, where $i = n-d-3$ found in the previous paragraph.

In the expression $\{x^{nd+n-d-3-n\beta+\beta+1}y^{n\beta-\beta-1}\}_{1\leqslant\beta\leqslant d}$ there is an annoying apparent lack of symmetry between x and y and between α and β. This can be removed by setting $i = \beta - 1$, $j = \alpha$, $a = d-1$ so that this expression becomes $\{x^{n-2}y^{n-2}(x^{n-1})^j(y^{n-1})^i\}$ and our result becomes:

Proposition 3.1. *Let $I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$ in the polynomial ring $R = \mathbb{Q}[x,y]$, $n \geqslant 4$. Then $I_n^{a+2} : I_n \supsetneq I_n^{a+1}$ for $0 \leqslant a \leqslant n-4$ and $I_n^{a+2} : I_n = I_n^{a+1}$ for $a \geqslant n-3$. More precisely, if $0 \leqslant a \leqslant n-4$, then $(I_n^{a+2} : I_n)/I_n^{a+1}$ is an $(a+1)$-dimensional vector space with basis $\{x^{n-2}y^{n-2}(x^{n-1})^j(y^{n-1})^i\}_{i\geqslant 0,\, j\geqslant 0,\, i+j=a}$.*

It should be noted that, in particular, if $a = 0$, then Proposition 3.1 yields the result that $x^{n-2}y^{n-2}$ is the only monomial in $I_n^2 : I_n$ but not in I_n.

To clarify our discussion, we provide the following example.

Example 3.1. Let us now look at I_6^2 and $(I_6^3 : I_6) \setminus I_6^2$. This will illustrate the ideas in Lemma 3.1 and Proposition 3.1 and help ensure that we have gotten the notation straight. By direct computation one has

$$I_6^2 = (x^{12}, x^{11}y, x^{10}y^2, x^7y^5, x^6y^6, x^5y^7, x^2y^{10}, xy^{11}, y^{12}).$$

The monomials in $(I_6^2)_{12}$ are just the 9 minimal generators of I_6^2 and $(I_6^2)_j = 0$ for $j < 12$. Furthermore, the set of all monomials in $(I_6^2)_{13}$ is

$$\{x^{13}, x^{12}y, x^{11}y^2, x^{10}y^3, x^8y^5, x^7y^6, x^6y^7, x^5y^8, x^3y^{10}, x^2y^{11}, xy^{12}, y^{13}\},$$

and the set of all monomials in $(I_6^2)_{14}$ is

$$\{x^{14}, x^{13}y, x^{12}y^2, x^{11}y^3, x^{10}y^4, x^9y^5, x^8y^6, x^7y^7, x^6y^8, x^5y^9, x^4y^{10}, \\ x^3y^{11}, x^2y^{12}, xy^{13}, y^{14}\},$$

that is, all monomials of degree 14. Obviously, $(I_6^2)_j = R_j$ for $j > 14$. In the notation of Lemma 3.1, we have $n = 6$ and $d = 2$. Part (1) of this Lemma says that $(I_6^2)_{nd+i} = (I_6^2)_{12+i} = R_{12+i}$ for $i \geqslant 6 - 2 - 2 = 2$, or $(I_6^2)_j = R_j$ for $j \geqslant 14$, which is what we have observed. Furthermore, the $d+1 = 3$ subdivision intervals of $(I_6^2)_{12}$ are each of cardinality $d + i + 1 = 2 + 0 + 1 = 3$ and the subdivision intervals of $(I_6^2)_{13}$ are each of cardinality $d+i+1 = 2+1+1 = 4$, both in agreement with Lemma 3.1 part (2), increasing by 1 each time we increase i by 1. There are $d = 2$ gaps in $(I_6^2)_{12}$, namely $\{x^9y^3, x^8y^4\}$ and $\{x^4y^8, x^3y^9\}$, are each of cardinality $n-d-2-i = 6-2-2-0 = 2$ and the largest exponent of y in these gaps is $n\beta - \beta - 1 = 4, 9$ for $\beta = 1, 2$, all in agreement with Lemma 3.1 part (3). The gaps in $(I_6^2)_{13}$, namely $\{x^9y^4\}$ and $\{x^4y^9\}$, are each of cardinality $n - d - 2 - i = 6 - 2 - 2 - 1 = 1$ and the largest exponent of y in these gaps is again $n\beta - \beta - 1 = 4, 9$ for $\beta = 1, 2$, also in agreement with Lemma 3.1 part (3). We are interested in $I_6^3 : I_6$. This contains I_6^2 and potentially the gaps of $(I_6^2)_{12}$ and $(I_6^2)_{13}$. The minimal generators of I_6 are all of degree 6, and multiplication by one of these maps $(I_6^2)_{13}$ into R_{19}. But by Lemma 3.1 part (1), $(I_6^3)_{19} = R_{19}$, so the gaps of $(I_6^2)_{13}$ are in $(I_6^3 : I_6) \setminus (I_6)^2$. But the gaps of $(I_6^2)_{12}$ are not mapped into I_6^3 by all the minimal generators of I_6. The gaps of $(I_6^3)_{18}$ are $\{x^{14}y^4\}$, $\{x^9y^9\}$ and $\{x^4y^{14}\}$, and for example $x^6(x^8y^4) = x^{14}y^4 \notin (I_6)^3$, so $x^8y^4 \notin I_6^3 : I_6$. Similarly, $x^5y(x^9y^3) = x^{14}y^4 \notin (I_6)^3$ so $x^9y^3 \notin I_6^3 : I_6$ either. Instead of the gap $\{x^{14}y^4\}$ we can use $\{x^9y^9\}$. Thus $y^6(x^9y^3) = x^9y^9 \notin (I_6)^3$, again yielding that $x^9y^3 \notin I_6^3 : I_6$. We thus conclude that $(I_6^3 : I_6) \setminus (I_6)^2 = \{x^9y^4, x^4y^9\}$. (In the language of Proposition 3.1, $a = 1$.)

We are ready to present the second question.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let I be a monomial ideal in R that has the strong persistence property. Also, let Q be an irreducible primary monomial ideal of R. Does the monomial ideal $I + Q$ satisfy the strong persistence property?

We provide a counterexample. For this purpose, assume that $R = K[x, y]$ is the polynomial ring over a field K, and put $I := (x^4, x^3y, xy^3, y^5)$, and $Q = (y^4)$. We thus have $I + Q = (x^4, x^3y, xy^3, y^4)$. It follows from Proposition 3.1 that $I + Q$ does not satisfy the strong persistence property. To conclude our argument, it remains to verify that the ideal I has the strong persistence property. To see this, fix $k \geqslant 1$. Since $I^k \subseteq (I^{k+1} :_R I) \subseteq (I^{k+1} :_R xy^3)$, it is enough for us to show that $(I^{k+1} :_R xy^3) \subseteq I^k$. It is well-known that

$$I^{k+1} = \sum_{\lambda_1+\lambda_2+\lambda_3+\lambda_4=k+1} (x^4)^{\lambda_1}(x^3y)^{\lambda_2}(y^5)^{\lambda_3}(xy^3)^{\lambda_4}R.$$

If $\lambda_4 \geqslant 1$, then we have

$$((x^4)^{\lambda_1}(x^3y)^{\lambda_2}(y^5)^{\lambda_3}(xy^3)^{\lambda_4} :_R xy^3) = ((x^4)^{\lambda_1}(x^3y)^{\lambda_2}(y^5)^{\lambda_3}(xy^3)^{\lambda_4-1}) \subseteq I^k.$$

Thus, let $\lambda_4 = 0$. If $\lambda_2 \geqslant 3$, then one obtains that

$$\begin{aligned}((x^4)^{\lambda_1}(x^3y)^{\lambda_2}(y^5)^{\lambda_3} :_R xy^3) &= ((x^4)^{\lambda_1}(x^3y)^{\lambda_2-3}(y^5)^{\lambda_3}x^8) \\ &= ((x^4)^{\lambda_1+2}(x^3y)^{\lambda_2-3}(y^5)^{\lambda_3}) \subseteq I^k.\end{aligned}$$

Hence, let $\lambda_4 = 0$ and $0 \leqslant \lambda_2 \leqslant 2$. Accordingly, one may consider the following cases:

Case 1: $\lambda_4 = 0$ and $\lambda_2 = 0$. If $\lambda_1 = 0$, then $\lambda_3 = k + 1$. This leads to $((y^5)^{\lambda_3} :_R xy^3) = ((y^5)^{\lambda_3-1}y^2) \subseteq I^k$. Let $\lambda_1 \geqslant 1$. If $\lambda_3 = 0$, then $\lambda_1 = k + 1$, and thus $((x^4)^{\lambda_1} :_R xy^3) = ((x^4)^{\lambda_1-1}x^3) \subseteq I^k$. Hence, let $\lambda_3 \geqslant 1$. Due to $\lambda_1 \geqslant 1$ and $\lambda_3 \geqslant 1$, we have $((x^4)^{\lambda_1}(y^5)^{\lambda_3} :_R xy^3) = ((x^4)^{\lambda_1-1}(y^5)^{\lambda_3-1}(x^3y)y) \subseteq I^k$.

Case 2: $\lambda_4 = 0$ and $\lambda_2 = 1$. Let $\lambda_1 = 0$. Hence, $\lambda_3 = k \geqslant 1$, and so

$$((x^3y)(y^5)^{\lambda_3} :_R xy^3) = (x(xy^3)(y^5)^{\lambda_3-1}) \subseteq I^k.$$

Therefore, let $\lambda_1 \geqslant 1$. If $\lambda_3 = 0$, then $\lambda_1 = k$, and thus

$$((x^4)^{\lambda_1}(x^3y) :_R xy^3) = ((x^4)^{\lambda_1}x^2) \subseteq I^k.$$

Hence, let $\lambda_3 \geqslant 1$. Thanks to $\lambda_1 \geqslant 1$ and $\lambda_3 \geqslant 1$, one derives that

$$((x^4)^{\lambda_1}(x^3y)(y^5)^{\lambda_3} :_R xy^3) = ((x^4)^{\lambda_1-1}(x^3y)^2(y^5)^{\lambda_3-1}y) \subseteq I^k.$$

Case 3: $\lambda_4 = 0$ and $\lambda_2 = 2$. If $\lambda_3 \geqslant 1$, then

$$\begin{aligned}((x^4)^{\lambda_1}(x^3y)^2(y^5)^{\lambda_3} \colon {}_R xy^3) &= ((x^4)^{\lambda_1}(x^5y^2)(y^5)^{\lambda_3-1}y^2) \\ &= ((x^4)^{\lambda_1}(x^3y)(y^5)^{\lambda_3-1}(xy^3)x) \subseteq I^k.\end{aligned}$$

Therefore, let $\lambda_3 = 0$. This gives that $\lambda_1 = k - 1$. Hence, we gain

$$((x^4)^{\lambda_1}(x^3y)^2 \colon {}_R xy^3) = ((x^4)^{\lambda_1}x^5) = ((x^4)^{\lambda_1+1}x) \subseteq I^k.$$

This terminates our argument.

4. Strong persistence property of the cover ideals

In this section, we focus on the strong persistence property of the cover ideals of simple finite graphs. For this purpose, we consider the following lemma which examines the relation between associated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the associated primes of powers of the cover ideals of each of them, under the condition that they have only one common vertex.

It should be noted that throughout this section, all trees are nontrivial, that is, they have at least two vertices.

A repeated application of Theorem 2.5 of [18] yields the following lemma:

Lemma 4.1. *Let $G = (V(G), E(G))$ be a finite simple connected graph and T be a tree such that $|V(G) \cap V(T)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(T)$ and $E(L) := E(G) \cup E(T)$. Then*

$$\mathrm{Ass}_R(R/J(L)^s) = \mathrm{Ass}_{R_1}(R_1/J(G)^s) \cup \mathrm{Ass}_{R_2}(R_2/J(T)^s)$$

for all s, where $R_1 = K[x_\alpha \colon \alpha \in V(G)]$, $R_2 = K[x_\alpha \colon \alpha \in V(T)]$ and $R = K[x_\alpha \colon \alpha \in V(L)]$ over a field K.

The next lemma explores the relation between associated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the associated primes of powers of the cover ideals of each of them, under the condition that they have only a path in common. In fact, a repeated application of Lemma 4.1 gives the following lemma:

Lemma 4.2. *Let $G = (V(G), E(G))$ be a finite simple connected graph, $T_1, \ldots, T_r$ be some trees with $V(G) \cap V(T_i) = \{v_i\}$ for each $i = 1, \ldots, r$, $V(T_i) \cap V(T_j) = \emptyset$ for $i \neq j$, and $P = (V(P), E(P))$ be a path of G with*

$$V(P) = \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_m\} \subseteq V(G)$$

and
$$E(P)=\{\{v_i,v_{i+1}\}\colon \text{ for } i=1,\ldots,m-1\}\subseteq E(G).$$
Let $T=(V(T),E(T))$ *be the tree with*
$$V(T)=\left(\bigcup_{i=1}^{r}V(T_i)\right)\cup V(P)\quad\text{and}\quad E(T)=\left(\bigcup_{i=1}^{r}E(T_i)\right)\cup E(P).$$
Also, let $L=(V(L),E(L))$ *be the finite simple graph such that*
$$V(L):=V(G)\cup V(T)\quad\text{and}\quad E(L):=E(G)\cup E(T).$$
Then
$$\mathrm{Ass}_R(R/J(L)^s)=\mathrm{Ass}_{R'}(R'/J(G)^s)\cup\mathrm{Ass}_{R''}(R''/J(T)^s)$$
for all s*, where* $R'=K[x_\alpha\colon \alpha\in V(G)]$*,* $R''=K[x_\alpha\colon \alpha\in V(T)]$ *and* $R=K[x_\alpha\colon \alpha\in V(L)]$ *over a field* K*.*

To establish Theorem 4.1, one needs to know the following auxiliary propositions. Indeed, by considering the fact that localizing at a minimal prime inverts everything outside of it, one can deduce the following proposition:

Proposition 4.1. *Let* I *be an ideal in a commutative Noetherian ring* R*. Also, let* $I=Q_1\cap\ldots\cap Q_t\cap Q_{t+1}\cap\ldots\cap Q_r$ *be a minimal primary decomposition of* I *with* $\mathfrak{p}_i=\sqrt{Q_i}$ *for* $i=1,\ldots,r$ *and* $\mathrm{Min}(I)=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_t\}$*. Then* $I_{\mathfrak{p}_i}=(Q_i)_{\mathfrak{p}_i}$ *for* $i=1,\ldots,t$*.*

Proposition 4.2. *Let* I *be a monomial ideal in* $R=K[x_1,\ldots,x_n]$ *over a field* K *with* $\mathcal{G}(I)=\{u_1,\ldots,u_m\}$ *and* $\mathrm{Ass}_R(R/I)=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_s\}$*. Then the following statements hold.*

(i) *If* $x_i\mid u_t$ *for some* i *with* $1\leqslant i\leqslant n$ *and for some* t *with* $1\leqslant t\leqslant m$*, then there exists* j *with* $1\leqslant j\leqslant s$ *such that* $x_i\in\mathfrak{p}_j$*.*

(ii) *If* $x_i\in\mathfrak{p}_j$ *for some* i *with* $1\leqslant i\leqslant n$ *and for some* j *with* $1\leqslant j\leqslant s$*, then there exists* t *with* $1\leqslant t\leqslant m$ *such that* $x_i\mid u_t$*.*

Especially, $\bigcup_{j=1}^{s}\mathrm{supp}(\mathfrak{p}_j)=\bigcup_{t=1}^{m}\mathrm{supp}(u_t)$*.*

Proof. (i) Suppose that $I=Q_1\cap\ldots\cap Q_s$ is a minimal primary decomposition of I such that $\sqrt{Q_z}=\mathfrak{p}_z$ for all $z=1,\ldots,s$. Also, let $x_i\mid u_t$ for some $1\leqslant i\leqslant n$ and $1\leqslant t\leqslant m$. Since $u_t\in\mathcal{G}(I)$, one has $u_t\in Q_z$ for all $z=1,\ldots,s$, and so $u_t\in\mathfrak{p}_z$ for all $z=1,\ldots,s$. It follows also from $x_i\mid u_t$ that there exists a monomial v in R such that $u_t=x_iv$. Suppose on the contrary, that $x_i\notin\mathfrak{p}_z$ for all $z=1,\ldots,s$. Then one can conclude that $v \in Q_z$ for all $z = 1, \ldots, s$ because $x_i v \in Q_z$, $x_i \notin \mathfrak{p}_z$, and Q_z is primary. Therefore, $v \in \bigcap\limits_{z=1}^{s} Q_z$, that is, $v \in I$. This contradicts the minimality of u_t. We thus have, there exists some $1 \leqslant j \leqslant s$ such that $x_i \in \mathfrak{p}_j$.

(ii) Let $x_i \in \mathfrak{p}_j$ for some $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant s$. Since $\mathfrak{p}_j \in \operatorname{Ass}_R(R/I)$, there exists a monomial v in R such that $\mathfrak{p}_j = (I\colon\ _R v)$. In addition, the assumption $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$ yields that $I = \sum\limits_{r=1}^{m} u_r R$. By virtue of $\mathfrak{p}_j = (I\colon\ _R v) = \sum\limits_{r=1}^{m} (u_r R\colon\ _R v)$ and $x_i \in \mathfrak{p}_j$, one can conclude that there exists some $1 \leqslant t \leqslant m$ such that $x_i \in (u_t R\colon\ _R v)$. We thus have $u_t h = x_i v$ for a monomial h in R. If $x_i \mid h$, then $v \in I$, which is a contradiction. Therefore, one can derive that $x_i \mid u_t$, as claimed.

The last assertion is an immediate consequence of parts (i) and (ii). □

In the next theorem, we turn our attention to study the strong persistence property of the cover ideal of the union of two finite simple connected graphs.

Theorem 4.1. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $J(G)$ and $J(H)$ have the strong persistence property. Also, let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$, $E(L) := E(G) \cup E(H)$. Assume that*

$$\operatorname{Ass}_R(R/J(L)^s) = \operatorname{Ass}_{R_1}(R_1/J(G)^s) \cup \operatorname{Ass}_{R_2}(R_2/J(H)^s)$$

for all s, where $R_1 = K[x_\alpha\colon\ \alpha \in V(G)]$, $R_2 = K[x_\alpha\colon\ \alpha \in V(H)]$, and $R = K[x_\alpha\colon\ \alpha \in V(L)]$ over a field K. Then under each of the following cases, $J(L)$ has the strong persistence property:

(i) $V(G) \cap V(H) = \{v\}$,

(ii) $V(G) \cap V(H) = \{v, w\}$ *and* $E(G) \cap E(H) = \{\{v, w\}\}$,

(iii) $V(G) \cap V(H) = \{v, w, z\}$ *and* $E(G) \cap E(H) = \{\{v, w\}, \{w, z\}\}$.

P r o o f. To simplify our notation, set $I_1 := J(G)$, $I_2 := J(H)$, and $I := J(L)$. Note that $I = I_1 \cap I_2$. We require to show that $(I^{k+1}\colon\ _R I) = I^k$ for all $k \geqslant 1$. To achieve this, fix $k \geqslant 1$. In view of Exercise 6.4 of [13], it is sufficient to prove that $(I_{\mathfrak{p}}^{k+1}\colon\ _{R_{\mathfrak{p}}} I_{\mathfrak{p}}) = I_{\mathfrak{p}}^k$ for every $\mathfrak{p} \in \operatorname{Ass}_R(R/I^k)$. For this purpose, pick an arbitrary element $\mathfrak{p} \in \operatorname{Ass}_R(R/I^k)$. We therefore can consider the following two cases:

Case 1: $\mathfrak{p} \in \operatorname{Min}(I^k)$. Because $\operatorname{Min}(I^k) = \operatorname{Min}(I)$, one has $\mathfrak{p} \in \operatorname{Min}(I)$. Now, Proposition 4.1 yields that $I_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$ and we thus have $I_{\mathfrak{p}}^k = \mathfrak{p}_{\mathfrak{p}}^k$ and $I_{\mathfrak{p}}^{k+1} = \mathfrak{p}_{\mathfrak{p}}^{k+1}$. On the other hand, Corollary 2.1 implies that $(\mathfrak{p}^{k+1}\colon\ _R \mathfrak{p}) = \mathfrak{p}^k$. Hence, one derives that $(\mathfrak{p}_{\mathfrak{p}}^{k+1}\colon\ _{R_{\mathfrak{p}}} \mathfrak{p}_{\mathfrak{p}}) = \mathfrak{p}_{\mathfrak{p}}^k$ and so $(I_{\mathfrak{p}}^{k+1}\colon\ _{R_{\mathfrak{p}}} I_{\mathfrak{p}}) = I_{\mathfrak{p}}^k$, as required.

Case 2: $\mathfrak{p} \in \mathrm{Ass}_R(R/I^k) \backslash \mathrm{Min}(I^k)$. By considering the assumption, one can deduce that $\mathfrak{p} \in \mathrm{Ass}_{R_1}(R_1/I_1^k)$ or $\mathfrak{p} \in \mathrm{Ass}_{R_2}(R_2/I_2^k)$. In view of the fact that $\mathrm{Min}(I^k) = \mathrm{Min}(I) = \mathrm{Min}(I_1) \cup \mathrm{Min}(I_2)$, we get

$$\mathfrak{p} \in \mathrm{Ass}_R(R/I_1^k) \setminus \mathrm{Min}(I_1) \quad \text{or} \quad \mathfrak{p} \in \mathrm{Ass}_R(R/I_2^k) \setminus \mathrm{Min}(I_2).$$

We only demonstrate the case $\mathfrak{p} \in \mathrm{Ass}_R(R/I_1^k) \backslash \mathrm{Min}(I_1)$, while another case is proved similarly. To finish the argument, we show that $I_{\mathfrak{p}} = (I_1)_{\mathfrak{p}}$. Thanks to $I_{\mathfrak{p}} \subseteq (I_1)_{\mathfrak{p}}$, it remains to establish that $(I_1)_{\mathfrak{p}} \subseteq I_{\mathfrak{p}}$. To do this, take an arbitrary element r/s in $(I_1)_{\mathfrak{p}}$. This gives that $r/s = \alpha/\beta$ for some $\alpha \in I_1$ and $\beta \notin \mathfrak{p}$. We now have to consider the following cases:

Case 2.1: $V(G) \cap V(H) = \{v\}$. One can easily see that

$$\alpha \prod_{l \in V(H) \setminus \{v\}} x_l \in I_1 \cap I_2.$$

Furthermore, Proposition 4.2 implies that $x_l \notin \mathfrak{p}$ for any $l \in V(H) \setminus \{v\}$, and thus

$$\beta \prod_{l \in V(H) \setminus \{v\}} x_l \notin \mathfrak{p}.$$

Due to the equality

$$\frac{\alpha}{\beta} = \frac{\alpha \prod_{l \in V(H) \setminus \{v\}} x_l}{\beta \prod_{l \in V(H) \setminus \{v\}} x_l},$$

we obtain $r/s \in I_{\mathfrak{p}}$.

Case 2.2: $V(G) \cap V(H) = \{v, w\}$ and $E(G) \cap E(H) = \{\{v, w\}\}$. Since $\alpha \in I_1$, we get $\alpha \in (x_v, x_w)$, and so $x_v \mid \alpha$ or $x_w \mid \alpha$. Hence, one can readily deduce that

$$x_v \mid \alpha \prod_{l \in V(H) \setminus \{v,w\}} x_l \quad \text{or} \quad x_w \mid \alpha \prod_{l \in V(H) \setminus \{v,w\}} x_l.$$

This leads to $\alpha \prod\limits_{l \in V(H) \setminus \{v,w\}} x_l \in I_2$, and so $\alpha \prod\limits_{l \in V(H) \setminus \{v,w\}} x_l \in I_1 \cap I_2$. Based on Proposition 4.2, we derive that $x_l \notin \mathfrak{p}$ for any $l \in V(H) \setminus \{v, w\}$, and hence $\beta \prod\limits_{l \in V(H) \setminus \{v,w\}} x_l \notin \mathfrak{p}$. In the light of the equality

$$\frac{\alpha}{\beta} = \frac{\alpha \prod_{l \in V(H) \setminus \{v,w\}} x_l}{\beta \prod_{l \in V(H) \setminus \{v,w\}} x_l},$$

one can conclude that $r/s \in I_{\mathfrak{p}}$.

Case 2.3: $V(G)\cap V(H)=\{v,w,z\}$ and $E(G)\cap E(H)=\{\{v,w\},\{w,z\}\}$. On account of $\alpha\in I_1$, one has $\alpha\in(x_v,x_w)\cap(x_w,x_z)$. If $x_w\mid\alpha$, then we obtain immediately that $x_w\mid\alpha\prod\limits_{l\in V(H)\setminus\{v,w,z\}}x_l$, and so $\alpha\prod\limits_{l\in V(H)\setminus\{v,w,z\}}x_l\in I_2$. Let $x_w\nmid\alpha$. Because I_1 is the cover ideal of the graph G, one must have $x_v\mid\alpha$ and $x_z\mid\alpha$. This gives rise to $\alpha\prod\limits_{l\in V(H)\setminus\{v,w,z\}}x_l\in I_2$. It follows also from Proposition 4.2 that $x_l\notin\mathfrak{p}$ for any $l\in V(H)\setminus\{v,w,z\}$, and thus $\beta\prod\limits_{l\in V(H)\setminus\{v,w,z\}}x_l\notin\mathfrak{p}$. Now, by observing the equality

$$\frac{\alpha}{\beta}=\frac{\alpha\prod_{l\in V(H)\setminus\{v,w,z\}}x_l}{\beta\prod_{l\in V(H)\setminus\{v,w,z\}}x_l},$$

one derives that $r/s\in I_{\mathfrak{p}}$.

As the ideal I_1 has the strong persistence property, one gains $(I_1^{k+1}\colon_R I_1)=I_1^k$, and hence $((I_1)_{\mathfrak{p}}^{k+1}\colon_{R_{\mathfrak{p}}}(I_1)_{\mathfrak{p}})=(I_1)_{\mathfrak{p}}^k$. We thus have $(I_{\mathfrak{p}}^{k+1}\colon_{R_{\mathfrak{p}}}I_{\mathfrak{p}})=I_{\mathfrak{p}}^k$. This completes the proof. □

Corollary 4.1. *Let $G=(V(G),E(G))$ be a finite simple connected graph such that $J(G)$ has the strong persistence property and T be a tree. Also, let $L=(V(L),E(L))$ be the finite simple graph such that $V(L):=V(G)\cup V(T)$ and $E(L):=E(G)\cup E(T)$. Then, under each of the following cases, $J(L)$ has the strong persistence property:*

(i) $V(G)\cap V(T)=\{v\}$,

(ii) $V(G)\cap V(T)=\{v,w\}$ *and* $E(G)\cap E(T)=\{\{v,w\}\}$,

(iii) $V(G)\cap V(T)=\{v,w,z\}$ *and* $E(G)\cap E(T)=\{\{v,w\},\{w,z\}\}$.

Proof. Since T is a tree, Corollary 2.6 of [4] yields that $J(T)$ is a normally torsion-free square-free monomial ideal. It follows now from Theorem 6.10 of [19], that $J(T)$ has the strong persistence property. Hence, this claim is a direct consequence of Lemmas 4.1, 4.2, and Theorem 4.1. □

The following remark says that with the notation of Theorem 4.1, it is possible that $J(G)$, $J(H)$ and $J(L)$ have the strong persistence property, while for some s we have

$$\operatorname{Ass}_R(R/J(L)^s)\neq\operatorname{Ass}_{R_1}(R_1/J(G)^s)\cup\operatorname{Ass}_{R_2}(R_2/J(H)^s).$$

Remark 4.1. Let $G=(V(G),E(G))$ and $H=(V(H),E(H))$ be two finite simple connected graphs such that $J(G)$ and $J(H)$ have the strong persistence property, $|V(G)\cap V(H)|>2$ and $|E(G)\cap E(H)|>1$. Also, let $L=(V(L),E(L))$ be the finite simple graph such that $V(L):=V(G)\cup V(H)$ and $E(L):=E(G)\cup E(H)$. Then it is possible that $J(L)$ has the strong persistence property, while for some s we have

$$\operatorname{Ass}_{R_1}(R_1/J(G)^s)\cup\operatorname{Ass}_{R_2}(R_2/J(H)^s)\subsetneq\operatorname{Ass}_R(R/J(L)^s),$$

where $R_1 = K[x_\alpha \colon \alpha \in V(G)]$, $R_2 = K[x_\alpha \colon \alpha \in V(H)]$ and $R = K[x_\alpha \colon \alpha \in V(L)]$ are polynomial rings over a field K. As an example, consider the graph $G = (V(G), E(G))$ with $V(G) = \{1, 2, 3, 4, 5\}$ and

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}\},$$

and also the graph $H = (V(H), E(H))$ with $V(H) = \{1, 4, 5, 6\}$ and

$$E(H) = \{\{1, 5\}, \{4, 5\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$

Hence, as shown in the figure below, we have $V(L) = \{1, 2, 3, 4, 5, 6\}$ and

$$V(L) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$

It is easy to compute that

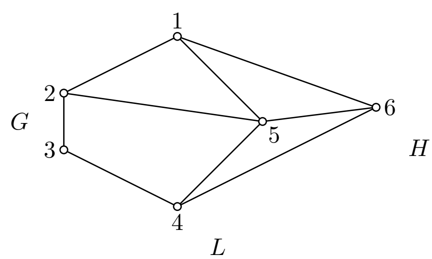
$$\begin{aligned} I_1 := J(G) &= (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \cap (x_5, x_2) \\ &= (x_2x_4x_5, x_2x_3x_5, x_1x_3x_5, x_1x_2x_4), \end{aligned}$$

and

$$\begin{aligned} I_2 := J(H) &= (x_1, x_5) \cap (x_4, x_5) \cap (x_1, x_6) \cap (x_5, x_6) \cap (x_4, x_6) \\ &= (x_5x_6, x_1x_4x_6, x_1x_4x_5), \end{aligned}$$

and

$$\begin{aligned} I := J(L) = &(x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_5, x_1) \cap (x_5, x_2) \\ &\cap (x_4, x_5) \cap (x_1, x_6) \cap (x_5, x_6) \cap (x_4, x_6) \\ = &(x_2x_4x_5x_6, x_2x_3x_5x_6, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_5, x_1x_2x_4x_5). \end{aligned}$$



In the first step, we verify that I_1 and I_2 have the strong persistence property. To accomplish this, one can immediately write $I_1 = x_5(x_2x_4, x_2x_3, x_1x_3) + x_1x_2x_4R_1$ and $I_2 = x_1x_4(x_5, x_6) + x_5x_6R_2$. Let P be the path graph with $V(P) = \{1, 2, 3, 4\}$ and $E(P) = \{\{1, 3\}, \{3, 2\}, \{2, 4\}\}$. Since the edge ideal of P is $I(P) =$

(x_2x_4, x_2x_3, x_1x_3), by virtue of Theorem 7.7.14 of [22], we gain that the monomial ideal (x_2x_4, x_2x_3, x_1x_3) has the strong persistence property. Moreover, it follows from Corollary 2.1 that the prime ideal (x_5, x_6) has the strong persistence property. Consequently, Lemma 2.3 gives that I_1 and I_2 have the strong persistence property. In the second step, we demonstrate that I has the strong persistence property. For this purpose, set $u_1 := x_1x_3x_4$, $u_2 := x_1x_3x_6$, $u_3 := x_2x_3x_6$, $u_4 := x_2x_4x_6$, $u_5 := x_1x_2x_4$ and $F := (u_1, u_2, u_3, u_4, u_5)R$. This gives rise to $I = x_5F + x_1x_2x_4x_6R$. Our strategy is to show that F has the strong persistence property. For this purpose, let $G = (V(G), E(G))$ be the graph with the vertex set $V(G) = \{1, 2, 3, 4, 6\}$ and the edge set $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 6\}, \{6, 1\}\}$. It is routine to check that G is the odd cycle graph of order 5. By using Macaulay2 (see [5]), we can deduce that F is the cover ideal of G. Also, by virtue of Corollary 2.6 of [4], F is normally torsion-free; thus, Theorem 6.10 of [19] implies that F has the strong persistence property. It follows now from Lemma 2.3 that I has the strong persistence property. Ultimately, by using Macaulay2 (see [5]), we note that

$$(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathrm{Ass}_R(R/I^3) \setminus (\mathrm{Ass}_{R_1}(R_1/I_1^3) \cup \mathrm{Ass}_{R_2}(R_2/I_2^3)).$$

This completes our discussion.

We terminate this section with the following result, which examines the relation between associated primes of powers of cover ideal of a finite simple connected graph and the associated primes of powers of the cover ideals of each connected subgraph of that graph. In fact, by using Lemma 2.11 of [3], we can conclude the following proposition:

Proposition 4.3. *Let $G = (V(G), E(G))$ be a finite simple connected graph and $H = (V(H), E(H))$ be a connected subgraph of G. Then*

$$\mathrm{Ass}_{R_1}(R_1/J(H)^s) \subseteq \mathrm{Ass}_R(R/J(G)^s),$$

where $R = K[x_\alpha \colon \alpha \in V(G)]$ and $R_1 = K[x_\alpha \colon \alpha \in V(H)]$ over a field K.

5. Some results on the symbolic strong persistence property

In this section, our aim is to prove that any square-free monomial ideal satisfies the symbolic strong persistence property. To achieve this, we start with the definition of symbolic strong persistence property of ideals.

Definition 5.1 ([21], Definition 13). Let I be an ideal in a commutative Noetherian ring R. Then we say that I has the *symbolic strong persistence* property if $(I^{(i+1)} :_R I^{(1)}) = I^{(i)}$ for each i.

The proposition below investigates the symbolic strong persistence property for powers of ideals.

Proposition 5.1. *Let R be a commutative Noetherian ring and I be an ideal of R such that I has a nonzero divisor element. Then there exists a positive integer s such that I^s has the symbolic strong persistence property.*

Proof. According to [19], Proposition 2.5, one can conclude that there exists a positive integer s such that I^s is a superficial ideal for I^s, that is, I^s has the strong persistence property. Now, the claim follows readily from Theorem 11 of [21]. $\square$

To demonstrate the subsequent results, we should state the definition of symbolic powers of an ideal.

Definition 5.2 ([22], Definition 4.3.22). Let I be an ideal of a ring R and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the minimal primes of I. Given an integer $n \geqslant 1$, the *nth symbolic power* of I is defined to be the ideal

$$I^{(n)} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r,$$

where $\mathfrak{q}_i$ is the primary component of I^n corresponding to $\mathfrak{p}_i$.

Remark 5.1. In much literature on symbolic powers a different definition is used based on all the primary decomposition, not just the minimal primes. See for example [2], page 1. If all associated primes of I are minimal (as is the case with square-free monomial ideals), then the two definitions of $I^{(n)}$ are the same. Otherwise, they are different.

The following proposition says that if an ideal has the symbolic strong persistence property, then any power of it has the symbolic strong persistence property as well.

Proposition 5.2. *Let I be an ideal in a commutative Noetherian ring R such that I has the symbolic strong persistence property. Then I^s has the symbolic strong persistence property for all positive integers s.*

Proof. Fix $s, k \geqslant 1$. It suffices to prove that $((I^s)^{(k+1)} :_R (I^s)^{(1)}) = (I^s)^{(k)}$. Let $\operatorname{Min}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Set $S := R \setminus \bigcup\limits_{i=1}^{r} \mathfrak{p}_i$. It follows from [22], Proposition 4.3.23, that $I^{(n)} = S^{-1}I^n \cap R$ for all $n \geqslant 1$. Since $\operatorname{Min}(I^n) = \operatorname{Min}(I)$ for all $n \geqslant 1$, this implies the following equalities:

$$(I^s)^{(n)} = S^{-1}(I^s)^n \cap R = S^{-1}I^{sn} \cap R = I^{(sn)}.$$

Accordingly, one obtains $(I^s)^{(k+1)} = I^{(sk+s)}$, $(I^s)^{(k)} = I^{(sk)}$ and $(I^s)^{(1)} = I^{(s)}$. Therefore, we get $((I^s)^{(k+1)} \colon {}_R (I^s)^{(1)}) = (I^{(sk+s)} \colon {}_R I^{(s)})$. Because the ideal I has the symbolic strong persistence property, it follows from [21], Proposition 12 that $(I^{(sk+s)} \colon {}_R I^{(s)}) = I^{(sk)}$, and so $((I^s)^{(k+1)} \colon {}_R (I^s)^{(1)}) = (I^s)^{(k)}$, as claimed. $\square$

Here, we turn our attention to study the symbolic strong persistence property of monomial ideals. In fact, one may ask the following question:

Does every monomial ideal satisfy the symbolic strong persistence property?

The answer is negative. To see this, we come back to Proposition 3.1. Indeed, we proved that for the monomial ideal $I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$ in the polynomial ring $R = \mathbb{Q}[x, y]$, $n \geqslant 4$, one has $(I_n^{a+2} : I_n) \supsetneq I_n^{a+1}$ for $0 \leqslant a \leqslant n-4$. On the other hand, since I_n is a (x, y)-primary monomial ideal, we have $I_n^{(k)} = I_n^k$ for all $k \geqslant 1$. Therefore, we get $(I_n^{(a+2)} : I_n^{(1)}) \supsetneq I_n^{(a+1)}$ for $0 \leqslant a \leqslant n-4$. This means that I_n does not satisfy the symbolic strong persistence property.

Based on Theorem 11 of [21], the strong persistence property implies the symbolic strong persistence property. Moreover, in view of the proof of Proposition 2.9 of [15], one can conclude that the strong persistence property implies the persistence property. Does the persistence property imply the symbolic strong persistence property? Does normally torsion-freeness imply the symbolic strong persistence property?

Our answers are negative. To accomplish this, consider the monomial ideal $I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$ in the polynomial ring $R = \mathbb{Q}[x, y]$, $n \geqslant 4$. Since $\mathrm{Ass}_R(R/I_n^k) = \{(x, y)\}$ for all $k \geqslant 1$, one can conclude that I has the persistence property, and also is normally torsion-free. While, by the argument which has been mentioned before, we get that I does not satisfy the symbolic strong persistence property.

In the sequel, our intent is to show that every square-free monomial ideal satisfies the symbolic strong persistence property. To achieve this, we require Proposition 5.3 and Lemma 5.1.

Proposition 5.3. *Every power of a primary monomial ideal is primary.*

Proof. Assume that Q is a primary monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, and fix $t \geqslant 1$. Let $x_i \mid u_1 \ldots u_t$, where $1 \leqslant i \leqslant n$ and $u_1, \ldots, u_t \in \mathcal{G}(Q)$. This yields that $x_i \mid u_s$ for some $1 \leqslant s \leqslant t$. Since Q is a primary monomial ideal, it follows from [22], Proposition 6.1.7 that there exists a positive integer k such that $x_i^k \in \mathcal{G}(Q)$, and hence $x_i^{kt} \in \mathcal{G}(Q^t)$. By setting $\alpha := kt$, one has $x_i^{\alpha} \in \mathcal{G}(Q^t)$. Once again, Proposition 6.1.7 of [22] implies that Q^t is a primary monomial ideal, as required. $\square$

Lemma 5.1. *Let $Q_1, \ldots, Q_r$ be primary monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $1 \leqslant i \neq j \leqslant r$, and $\sqrt{Q_1}, \ldots, \sqrt{Q_r}$ are incomparable with respect to inclusion. If Q_i has the strong persistence property for each $i = 1, \ldots, r$, then for all positive integers k,*

$$\left(\bigcap_{i=1}^{r} Q_i^{k+1} :_R \bigcap_{i=1}^{r} Q_i\right) = \bigcap_{i=1}^{r} Q_i^k.$$

P r o o f. We give a sketch of the proof. Assume that Q_i has the strong persistence property for each $i = 1, \ldots, r$. Fix $k \geqslant 1$ and pick u in $\left(\bigcap\limits_{i=1}^{r} Q_i^{k+1} :_R \bigcap\limits_{i=1}^{r} Q_i\right)$. Put $\mathfrak{p}_i := \sqrt{Q_i}$ for each $i = 1, \ldots, r$. Fix $1 \leqslant j \leqslant r$. One can choose an element such as $v_i \in Q_i \setminus \mathfrak{p}_j$ for $1 \leqslant i \neq j \leqslant r$. Let λ be an arbitrary element in Q_j. Therefore, $u\lambda v_1 \ldots v_{j-1} v_{j+1} \ldots v_r \in \bigcap\limits_{i=1}^{r} Q_i^{k+1} \subseteq Q_j^{k+1}$, and so $v_1 \ldots v_{j-1} v_{j+1} \ldots v_r \notin \mathfrak{p}_j$. Since Q_j is primary, Proposition 5.3 gives that Q_j^{k+1} is primary. Hence, $u\lambda \in Q_j^{k+1}$, and so $u \in (Q_j^{k+1} :_R Q_j) = Q_j^k$. Therefore, $\left(\bigcap\limits_{i=1}^{r} Q_i^{k+1} :_R \bigcap\limits_{i=1}^{r} Q_i\right) \subseteq \bigcap\limits_{i=1}^{r} Q_i^k$. To prove the reverse inclusion, one should note that $\left(\bigcap\limits_{i=1}^{r} Q_i\right)\left(\bigcap\limits_{i=1}^{r} Q_i^k\right) \subseteq \bigcap\limits_{i=1}^{r} Q_i^{k+1}$, and so $\bigcap\limits_{i=1}^{r} Q_i^k \subseteq \left(\bigcap\limits_{i=1}^{r} Q_i^{k+1} :_R \bigcap\limits_{i=1}^{r} Q_i\right)$. This finishes our argument. □

We are now in a position to express the main result of this section.

Theorem 5.1. *Every square-free monomial ideal has the symbolic strong persistence property.*

P r o o f. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K with $\mathrm{Ass}_R(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. On account of $\mathrm{Ass}_R(R/I) = \mathrm{Min}(I)$, it follows from Lemma 5.1 and Corollary 2.1 that

$$\left(\bigcap_{i=1}^{r} \mathfrak{p}_i^{k+1} :_R \bigcap_{i=1}^{r} \mathfrak{p}_i\right) = \bigcap_{i=1}^{r} \mathfrak{p}_i^k$$

for all $k \geqslant 1$. In addition, Proposition 1.4.4 of [6] implies that $I^{(k)} = \bigcap\limits_{i=1}^{r} \mathfrak{p}_i^k$ for all $k \geqslant 1$. Therefore, we have $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k \geqslant 1$, that is, I has the symbolic strong persistence property, as desired. □

Is there an ideal satisfying the symbolic strong persistence property but not being the strong persistence property?

The answer is positive. We give such an ideal. Consider the following square-free monomial ideal $\mathcal{I}$ in the polynomial $R = K[x_1, \dots, x_{12}]$ over a field K,

$$\begin{aligned}\mathcal{I} = (&x_1x_3x_6x_8x_9x_{10}x_{11}x_{12}, x_2x_4x_5x_7x_9x_{10}x_{11}x_{12}, x_1x_2x_4x_5x_7x_{10}x_{11}x_{12},\\ &x_2x_3x_5x_6x_8x_9x_{11}x_{12}, x_1x_2x_3x_6x_8x_9x_{11}x_{12}, x_2x_4x_5x_6x_7x_9x_{11}x_{12},\\ &x_1x_3x_6x_7x_8x_9x_{10}x_{12}, x_2x_3x_5x_7x_8x_9x_{10}x_{12}, x_2x_3x_4x_5x_7x_9x_{10}x_{12},\\ &x_1x_3x_4x_5x_6x_7x_{10}x_{12}, x_1x_2x_3x_4x_5x_7x_{10}x_{12}, x_1x_3x_4x_6x_8x_9x_{10}x_{11},\\ &x_1x_2x_4x_5x_7x_8x_{10}x_{11}, x_1x_3x_4x_5x_6x_8x_{10}x_{11}, x_1x_2x_4x_6x_7x_8x_9x_{11},\\ &x_1x_2x_3x_4x_6x_8x_9x_{11}).\end{aligned}$$

As we will state in Question 6.3, since $(\mathcal{I}^4\colon {}_R\mathcal{I}) \neq \mathcal{I}^3$, one gains that $\mathcal{I}$ does not satisfy the strong persistence property, whereas Theorem 5.1 shows that $\mathcal{I}$ satisfies the symbolic strong persistence property.

We terminate this paper with the following corollary. In fact, Rajaee, Al-Ayyoub and the first author have established it in [19], Theorem 6.10, and we now re-prove it by using Theorem 5.1.

Corollary 5.1. *Every normally torsion-free square-free monomial ideal has the strong persistence property.*

Proof. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\operatorname{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. In the light of Theorem 5.1, one has $(I^{(k+1)}\colon {}_RI^{(1)}) = I^{(k)}$ for all $k \geqslant 1$. On the other hand, Theorem 1.4.6 of [6] implies that $I^{(k)} = I^k$ for all $k \geqslant 1$. Cosequently, by Theorem 5.1, we get $(I^{k+1}\colon {}_RI) = I^k$ for all $k \geqslant 1$, that is, I has the strong persistence property, as required. $\square$

6. Future works

Many questions arise along these arguments for future works. We terminate this paper with several open questions which are devoted to the strong persistence property and the symbolic strong persistence property of monomial ideals. In particular, after investigating and examining the cover ideals of a plenty of graphs, we made up the questions which are related to the cover ideals of the union of two finite simple graphs. We list them as follows:

To express the following question, one has to recall the definition of simple graphs of the form $\theta_{n_1,\dots,n_k}$, which has been introduced in [14]. To do this, let $k > 1$ be an integer and $n_1, \dots, n_k$ be a sequence of positive integers. Then $\theta_{n_1,\dots,n_k}$ is the graph constructed by k paths of length $n_1, \dots, n_k$ such that only their endpoints are common. By length of a path, we mean the number of edges in the path. If $k = 2$, then $\theta_{n_1,\dots,n_k}$ will be a cycle of length $n_1 + n_2$.

Question 6.1. *With the notation above, does $J(\theta_{n_1,\dots,n_k})$ have the strong persistence property?*

Question 6.2. *Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K, and $\mathfrak{m} = (x_1, \dots, x_n)$ be the unique homogeneous maximal ideal of R. If $\mathrm{Ass}_R(R/I^s) = \mathrm{Ass}_R(R/I) \cup \{\mathfrak{m}\}$ for all $s \geqslant 2$, then does I have the strong persistence property?*

It should be noted that in general, finding a square-free monomial ideal I such that $\mathrm{Ass}_R(R/I^s) = \mathrm{Ass}_R(R/I) \cup \{\mathfrak{m}\}$ for all $s \geqslant 2$, could be really tricky. One of the well-known such classes is the cover ideals of odd cycle graphs. It has been proved in [18] that if I is the cover ideal of an odd cycle graph C_{2n+1}, then $\mathrm{Ass}_R(R/I^s) = \mathrm{Ass}_R(R/I) \cup \{\mathfrak{m}\}$ for all $s \geqslant 2$, where $R = K[x_1, \dots, x_{2n+1}]$ and $\mathfrak{m} = (x_1, \dots, x_{2n+1})$, see [18], Proposition 3.6, and also satisfies the strong persistence property, see [18], Theorem 3.3.

To formulate the following question, we need to recall the notion of a clutter. A *clutter* (or *simple hypergraph*) $\mathcal{C}$ with vertex set $X = \{x_1, \dots, x_n\}$ is a family of subsets of X, called *edges*, none of which is included in another. See [22], Definition 6.3.33 for more details. The edge ideal of a clutter is defined in [22], Definition 6.3.35, and the cover ideal of a clutter can be defined as the ideal of all monomials M such that given any edge e of $\mathcal{C}$ there is a variable x_i such that $x_i \in e$ and $x_i \mid M$. Note that the vertices of these clutters become the variables of the ring in which the edge ideal and cover ideal are allocated. Given a clutter $\mathcal{C}$ on $\{x_1, \dots, x_n\}$ with edges $e_1, \dots, e_r$, we define the *complement clutter*, denoted by $\mathcal{C}^c$, as the clutter whose edges are $\{x_1, \dots, x_n\} \setminus e_i$ for each $i = 1, \dots, r$. We denote the edge ideal of a clutter $\mathcal{C}$ by $I(\mathcal{C})$.

Question 6.3. *Does $I(\mathcal{C})$ have the strong persistence property if and only if $I(\mathcal{C}^c)$ has the strong persistence property?*

We give an example of a clutter $\mathcal{C}$, where both $I(\mathcal{C})$ and $I(\mathcal{C}^c)$ do not have the strong persistence property. Consider the graph below, from [9]. For a positive integer n, let $[n]$ denote the set $\{0, \dots, n-1\}$. We denote by P_n a path with vertex set $[n]$, with vertices in the increasing order along P_n. Let also K_3 be the complete graph whose vertex set is the group $\mathbb{Z}_3$. For $n \geqslant 4$, we define H_n as the graph obtained from the Cartesian product $P_n \,\square\, K_3$ by adding the three edges joining $(0, j)$ to $(n-1, -j)$ for $j \in \mathbb{Z}_3$. The Figure 1 below is the graph of H_4.

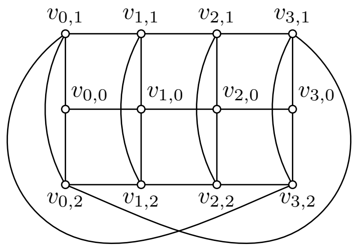


Figure 1. H_4.

Set $F := J(H_4)$. By using Macaulay2 (see [5]), F is given by

$$\begin{aligned}F = (&x_1x_3x_6x_8x_9x_{10}x_{11}x_{12}, x_2x_4x_5x_7x_9x_{10}x_{11}x_{12}, x_1x_2x_4x_5x_7x_{10}x_{11}x_{12},\\ &x_2x_3x_5x_6x_8x_9x_{11}x_{12}, x_1x_2x_3x_6x_8x_9x_{11}x_{12}, x_2x_4x_5x_6x_7x_9x_{11}x_{12},\\ &x_1x_3x_6x_7x_8x_9x_{10}x_{12}, x_2x_3x_5x_7x_8x_9x_{10}x_{12}, x_2x_3x_4x_5x_7x_9x_{10}x_{12},\\ &x_1x_3x_4x_5x_6x_7x_{10}x_{12}, x_1x_2x_3x_4x_5x_7x_{10}x_{12}, x_1x_3x_4x_6x_8x_9x_{10}x_{11},\\ &x_1x_2x_4x_5x_7x_8x_{10}x_{11}, x_1x_3x_4x_5x_6x_8x_{10}x_{11}, x_1x_2x_4x_6x_7x_8x_9x_{11},\\ &x_1x_2x_3x_4x_6x_8x_9x_{11}).\end{aligned}$$

It has already been shown in [9] that F does not satisfy the persistence property (and hence does not satisfy the strong persistence property either). One can show, using Macaulay2 (see [5]), that in the polynomial ring $R = K[x_1, \ldots, x_{12}]$ over a field K, $\mathfrak{m} = (x_1, \ldots, x_{12}) \in \mathrm{Ass}_R(R/F^3) \setminus \mathrm{Ass}_R(R/F^4)$ and $(F^4 \colon {}_R F) \neq F^3$. Now, we construct the clutter $\mathcal{C}$ on $\{x_1, \ldots, x_{12}\}$ whose edge ideal is F, that is, $F = I(\mathcal{C})$, as follows:

$$\begin{aligned}\mathcal{C} := \{&\{x_1, x_3, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}\}, \{x_2, x_4, x_5, x_7, x_9, x_{10}, x_{11}, x_{12}\},\\ &\{x_1, x_2, x_4, x_5, x_7, x_{10}, x_{11}, x_{12}\}, \{x_2, x_3, x_5, x_6, x_8, x_9, x_{11}, x_{12}\},\\ &\{x_1, x_2, x_3, x_6, x_8, x_9, x_{11}, x_{12}\}, \{x_2, x_4, x_5, x_6, x_7, x_9, x_{11}, x_{12}\},\\ &\{x_1, x_3, x_6, x_7, x_8, x_9, x_{10}, x_{12}\}, \{x_2, x_3, x_5, x_7, x_8, x_9, x_{10}, x_{12}\},\\ &\{x_2, x_3, x_4, x_5, x_7, x_9, x_{10}, x_{12}\}, \{x_1, x_3, x_4, x_5, x_6, x_7, x_{10}, x_{12}\},\\ &\{x_1, x_2, x_3, x_4, x_5, x_7, x_{10}, x_{12}\}, \{x_1, x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}\},\\ &\{x_1, x_2, x_4, x_5, x_7, x_8, x_{10}, x_{11}\}, \{x_1, x_3, x_4, x_5, x_6, x_8, x_{10}, x_{11}\},\\ &\{x_1, x_2, x_4, x_6, x_7, x_8, x_9, x_{11}\}, \{x_1, x_2, x_3, x_4, x_6, x_8, x_9, x_{11}\}\}.\end{aligned}$$

Then $F = I(\mathcal{C})$ does not satisfy the strong persistence property. On the other hand, one can deduce from the definition that the complement clutter of $\mathcal{C}$, that is $\mathcal{C}^c$, is as follows:

$$\begin{aligned}\mathcal{C}^c := \{&\{x_2,x_4,x_5,x_7\},\{x_1,x_3,x_6,x_8\},\{x_3,x_6,x_8,x_9\},\{x_1,x_4,x_7,x_{10}\},\\ &\{x_4,x_5,x_7,x_{10}\},\{x_1,x_3,x_8,x_{10}\},\{x_2,x_4,x_5,x_{11}\},\{x_1,x_4,x_6,x_{11}\},\\ &\{x_1,x_6,x_8,x_{11}\},\{x_2,x_8,x_9,x_{11}\},\{x_6,x_8,x_9,x_{11}\},\{x_2,x_5,x_7,x_{12}\},\\ &\{x_3,x_6,x_9,x_{12}\},\{x_2,x_7,x_9,x_{12}\},\{x_3,x_5,x_{10},x_{12}\},\{x_5,x_7,x_{10},x_{12}\}\}.\end{aligned}$$

This implies that the edge ideal of $I(\mathcal{C}^c)$ is given by

$$\begin{aligned}I(\mathcal{C}^c) = (&x_2x_4x_5x_7, x_1x_3x_6x_8, x_3x_6x_8x_9, x_1x_4x_7x_{10}, x_4x_5x_7x_{10}, x_1x_3x_8x_{10},\\ &x_2x_4x_5x_{11}, x_1x_4x_6x_{11}, x_1x_6x_8x_{11}, x_2x_8x_9x_{11}, x_6x_8x_9x_{11}, x_2x_5x_7x_{12},\\ &x_3x_6x_9x_{12}, x_2x_7x_9x_{12}, x_3x_5x_{10}x_{12}, x_5x_7x_{10}x_{12}).\end{aligned}$$

By using Macaulay2 (see [5]), one can check that $(I(\mathcal{C}^c)^4 :_R I(\mathcal{C}^c)) \neq I(\mathcal{C}^c)^3$ and $\mathfrak{m} = (x_1,\ldots,x_{12}) \in \mathrm{Ass}_R(R/I(\mathcal{C}^c)^3) \setminus \mathrm{Ass}_R(R/I(\mathcal{C}^c)^4)$, that is, $I(\mathcal{C}^c)$ does not satisfy the strong persistence property and the persistence property.

Question 6.4. *Let $R = K[x_1,\ldots,x_n]$ be a polynomial ring over a field K and $\mathrm{SPP}_+(n)$ (or $\mathrm{SPP}_-(n)$) be the set of square-free monomial ideals in R such that they satisfy (or do not satisfy) the strong persistence property. Then, does the following limit exist? Can it be zero?*

$$\lim_{n\to\infty} \frac{|\mathrm{SPP}_-(n)|}{|\mathrm{SPP}_+(n)|},$$

where $|A|$ denotes the cardinality of A.

To realize Question 6.5, we first recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal as it has been introduced in [7]. Let I be a monomial ideal in a polynomial ring $R = K[x_1,\ldots,x_n]$ over a field K. We also denote by $V^*(I)$ the set of monomial prime ideals containing I. Let $\mathfrak{p} = (x_{i_1},\ldots,x_{i_r})$ be a monomial prime ideal with $\mathfrak{p} \in V^*(I)$. The *monomial localization* of I with respect to $\mathfrak{p}$, denoted by $I(\mathfrak{p})$, is the ideal in the polynomial ring $R(\mathfrak{p}) = K[x_{i_1},\ldots,x_{i_r}]$ which is obtained from I by applying the K-algebra homomorphism $R \to R(\mathfrak{p})$ with $x_j \mapsto 1$ for all $x_j \notin \{x_{i_1},\ldots,x_{i_r}\}$.

Question 6.5. *Let I be a monomial ideal in a polynomial ring $R = K[x_1,\ldots,x_n]$ over a field K. If I has the symbolic strong persistence property, then does $I(\mathfrak{p})$ have the symbolic strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$?*

To state the next question, one has to recall the definition of monomial ideals of clutter type.

Definition 6.1. Let I be a non-square-free monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \ldots, u_r\}$. We say that I is of *clutter type* if $\sqrt{u_i} \nmid \sqrt{u_j}$ (or equivalently, $\mathrm{supp}(u_i) \nsubseteq \mathrm{supp}(u_j)$) for each $1 \leqslant i \neq j \leqslant r$.

Example 6.1. Let $I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2)$ be a monomial ideal in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ over a field K. Then one can rapidly see that I is of clutter type. Note that I does not satisfy both the persistence property and strong persistence property since $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}_R(R/I) \setminus \mathrm{Ass}_R(R/I^2)$ and $(I^2 \colon_R I) \neq I$.

Question 6.6. *Does every non-square-free monomial ideal of clutter type have the symbolic strong persistence property?*

Question 6.7. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K with $\mathcal{G}(I) = G_1 \cup \ldots \cup G_r$ such that for each $1 \leqslant i \neq j \leqslant r$,*

$$\{x_s \colon x_s \mid u \text{ for some } u \in G_i\} \cap \{x_t \colon x_t \mid u \text{ for some } u \in G_j\} = \emptyset.$$

Then does I have the symbolic strong persistence property if and only if (G_i) has the symbolic strong persistence property for some $1 \leqslant i \leqslant r$?

Question 6.8. *Let I be an ideal in a commutative Noetherian ring R. Then does I have the symbolic strong persistence property if and only if $I_\mathfrak{p}$ has the strong persistence property for all $\mathfrak{p} \in \mathrm{Min}(I)$, where $I_\mathfrak{p}$ denotes the localization of I at $\mathfrak{p}$?*

Question 6.9. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, and w a weight over R. Then does I have the symbolic strong persistence property if and only if I_w has the symbolic strong persistence property, where I_w denotes the weight of I?*

Question 6.10. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, and $1 \leqslant i \leqslant n$. If I has the symbolic strong persistence property, then does $I_{\setminus x_i}$ have the symbolic strong persistence property, where $I_{\setminus x_i}$ denotes the contracted of I at x_i?*

Question 6.11. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Then does I have the symbolic strong persistence property if and only if I^* have the symbolic strong persistence property, where I^* denotes the expansion of I?*

Question 6.12. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K, and h be a monomial in R. Also, let $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$. Then does I has the symbolic strong persistence property if and only if hI have the symbolic strong persistence property?*

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