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ON THE MINIMAXNESS AND COATOMICNESS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R and M an R-module. We wish to investigate the relation between vanishing, finiteness, Artinianness, minimaxness and $\mathcal{C}$-minimaxness of local cohomology modules. We show that if M is a minimax R-module, then the local-global principle is valid for minimaxness of local cohomology modules. This implies that if n is a nonnegative integer such that $(H_I^i(M))_{\mathfrak{m}}$ is a minimax $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \operatorname{Max}(R)$ and for all $i < n$, then the set $\operatorname{Ass}_R(H_I^n(M))$ is finite. Also, if $H_I^i(M)$ is minimax for all $i \geqslant n \geqslant 1$, then $H_I^i(M)$ is Artinian for $i \geqslant n$. It is shown that if M is a $\mathcal{C}$-minimax module over a local ring such that $H_I^i(M)$ are $\mathcal{C}$-minimax modules for all $i < n$ (or $i \geqslant n$), where $n \geqslant 1$, then they must be minimax. Consequently, a vanishing theorem is proved for local cohomology modules.

Keywords: local cohomology module; minimax module; coatomic module; Artinian module; local-global principle

MSC 2020: 13D45, 13E05, 13C05

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and I is an ideal of R. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H_I^i(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}_R^i(R/I^n, M).$$

For more details about the local cohomology, we refer the reader to [5].

In [23], Zöschinger introduced the interesting class of minimax modules and in [23] and [24] he gave equivalent conditions for a module to be minimax. An R-module M is called *minimax* if there is a finitely generated submodule F of M such that M/F is Artinian. The class of minimax modules includes all finitely generated and all

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Artinian modules. Based on [3], Lemma 2.1, the class of minimax modules is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules. Using the fact that finitely generated modules and Artinian modules both have finitely many associated primes and the properties of associated primes on short exact sequences, one can deduce that minimax modules have finitely many associated primes.

In [22], Zöschinger defined and investigated coatomic modules over commutative Noetherian rings. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. The class of coatomic modules is a Serre subcategory of the category of R-modules. Moreover, it is clear that every finitely generated R-module is coatomic and that every coatomic Artinian module has finite length. In [19], Rudlof characterized modules which are extensions of coatomic modules by Artinian modules. In the light of [21], Theorem 2.3, this class of R-modules is a Serre subcategory of the category of R-modules. Rezaei in [17], called these modules $\mathcal{C}$-minimax and proved that there exists a close relation between $\mathcal{C}$-minimaxness and minimaxness of local homology and local cohomology modules. Note that the class of coatomic (or $\mathcal{C}$-minimax) modules is strictly larger than the class of finitely generated (or minimax) modules, see Example 2.4.

In this paper we obtain some results for the vanishing, finiteness, Artinianness, minimaxness and $\mathcal{C}$-minimaxness of local cohomology modules. Here, we extend several main results of [2] and [17] to the class of minimax and $\mathcal{C}$-minimax R-modules.

In Section 2, we review and prove some of the standard facts on minimax and $\mathcal{C}$-minimax modules which are needed in our proofs. In Section 3, our main results are stated and proved.

In [9], Hartshorne introduced the notion of I-cofinite modules. An R-module M is I-cofinite if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is finitely generated for each i. In Theorem 3.6 we show that if M is a minimax R-module, then the local-global principle is valid for minimaxness of local cohomology modules. This result generalizes [2], Theorem 2.8. Moreover, if M is minimax such that $(0 :_M I)$ is finitely generated and $H^i_I(M)$ are minimax for all $i \leqslant n$, where $n \geqslant 0$, then they are I-cofinite minimax.

An important problem in commutative algebra is determining when the set of associated primes of the ith local cohomology modules $H^i_I(M)$ is finite, see [10], Problem 4. As a corollary of Theorem 3.6, we generalize the main result of Brodmann and Lashgari, see [3], Theorem 2.5, [4] and [11], Corollary 2.3.

Let M be a $\mathcal{C}$-minimax module over a local ring R and n be a positive integer. We prove in Theorem 3.9 (or 3.13) that the local cohomology modules $H^i_I(M)$ are $\mathcal{C}$-minimax modules for all $i < n$ (or $i \geqslant n$) if and only if they are minimax for all $i < n$ (or $i \geqslant n$). These results are the generalizations of the results proved by Rezaei, see [17]. It is also shown in Theorem 3.16 that under the above assumptions, if the local cohomology modules $H^i_I(M)$ are coatomic for all $i \geqslant n$, then they are actually zero in this range. This result is a generalization of [2], Theorem 3.9 and [20], Proposition 3.1.

Let $\mathfrak{p}$ be a prime ideal of R. A nonzero R-module M is called $\mathfrak{p}$-*secondary* if its multiplication map by any element r of R is either surjective or nilpotent and $\mathfrak{p} = \sqrt{(0 :_R M)}$. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is *representable*. If M is representable and $M = S_1 + S_2 + \ldots + S_n$, where S_i is $\mathfrak{p}_i$-secondary for all $1 \leqslant i \leqslant n$, is a minimal secondary representation of M, then the n-element set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is called the *set of attached prime ideals* of M and is denoted by $\mathrm{Att}_R(M)$. Also, we let

$$\mathrm{cd}(I, M) = \sup\{n \geqslant 0\colon\ H^n_I(M) \neq 0\}$$

and

$$\mathrm{q}(I, M) = \sup\{n \geqslant 0\colon\ H^n_I(M) \text{ is not Artinian}\}.$$

We prove in Theorem 3.3 that if I is an ideal of a local ring $(R, \mathfrak{m})$ and M is a $\mathcal{C}$-minimax R-module with $\dim M = d \geqslant 1$, then

$$\mathrm{Att}_R(H^d_I(M)) \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon\ \mathrm{cd}(I, R/\mathfrak{p}) = d\}.$$

Also, it is shown in Proposition 3.5 that if M is a $\mathcal{C}$-minimax R-module and N is an arbitrary R-module such that $\mathrm{Supp}_R(N) \subseteq \mathrm{Supp}_R(M)$, then $\mathrm{cd}(I, N) \leqslant \mathrm{cd}(I, M)$. This generalizes [2], Proposition 3.7, [7], Theorem 1.4 and [8], Theorem 2.2.

Throughout this paper, we assume that R is a commutative Noetherian ring with nonzero identity, I is an ideal of R, $V(I)$ is the set of all prime ideals of R containing I, $\mathrm{Spec}(R)$ is the set of all prime ideals and $\mathrm{Max}(R)$ is the set of all maximal ideals of R. For any unexplained notation and terminology we refer the reader to [5] and [12].

2. Preliminaries

In this section, we review some properties of minimax and $\mathcal{C}$-minimax modules which are crucial in our proofs.

Lemma 2.1. *Let I be an ideal of R and M a minimax R-module. Then the following statements hold:*

(i) *If $\mathfrak{p}$ is a non-maximal prime ideal of R, then $M_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module.*

(ii) *If $\mathrm{Supp}_R(M) \subseteq \mathrm{Max}(R)$, then M is Artinian.*

(iii) *There exists a long exact sequence*

$$0 \to H_I^0(F) \to H_I^0(M) \to H_I^0(A) \xrightarrow{f} H_I^1(F) \to H_I^1(M) \to 0 \tag{2.1}$$

and the isomorphism

$$H_I^i(F) \cong H_I^i(M) \tag{2.2}$$

for all $i \geqslant 2$, where F is finitely generated and A is Artinian.

Proof. By definition of minimax modules there exists an exact sequence

$$0 \to F \to M \to A \to 0, \tag{2.3}$$

where F is finitely generated and A is Artinian. The proof of (i) and (ii) is straightforward. The last assertion follows by applying the functor $\Gamma_I(-)$ to the above exact sequence and Theorem 6.1.2 of [5]. $\square$

Recall that a class of R-modules is a *Serre subcategory* of the category of R-modules when it is closed under taking submodules, quotients and extensions, for example, the classes of Noetherian modules, Artinian modules and minimax modules are Serre subcategories, see [3], Lemma 2.1. As in standard notation, we let $\mathcal{S}$ stand for a Serre subcategory of the category of R-modules.

Remark 2.2. Based on [6], Theorem 1.2.5 if M is a finitely generated R-module and $M \neq IM$, then $\min\{i\colon \operatorname{Ext}_R^i(R/I, M) \neq 0\}$ is equal to the common length of the maximal M-sequences in I, which is called the *grade* of I on M and denoted by $\operatorname{grade}_I(M)$. In the case, where $\mathcal{S}$ is a Serre subcategory of the category of R-modules consisting of R-modules with finitely many associated prime ideals and M an R-module belonging to $\mathcal{S}$, by the same method as in the proof of Theorem 1.2.5 of [6], it is easy to check that this assertion holds. Therefore, if M is a minimax R-module and $M \neq IM$, then

$$\operatorname{grade}_I(M) = \min\{i\colon \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

The following lemma, which is an extension of [5], Theorem 6.2.7, is useful in the proof of Lemma 3.4.

Lemma 2.3. *Let M be a minimax R-module such that $IM \neq M$. Then $\operatorname{grade}_I(M)$ is the least integer i such that $H_I^i(M) \neq 0$.*

Proof. By [1], Lemma 2.5, $\Gamma_I(M) = 0$ if and only if I contains a nonzero divisor on M. Hence, the assertion follows by the same method as in the proof of Theorem 6.2.7 of [5], using Remark 2.2. $\square$

The modules which are an extension of a coatomic module by an Artinian module were studied by Rudlof, see [19]. Rezaei in [17] introduced these modules as $\mathcal{C}$-minimax modules and proved some results about the $\mathcal{C}$-minimaxness of local homology and local cohomology modules. An R-module M is said to be $\mathcal{C}$-minimax if there is a coatomic submodule N of M such that M/N is Artinian. Clearly, this class of R-modules includes coatomic modules and Artinian modules. Moreover, since every finitely generated R-module is coatomic, the class of $\mathcal{C}$-minimax modules contains all minimax R-modules. Also, one can use the fact that the class of coatomic modules over a local ring has finitely many associated primes (see [22], Folgerung 2) together with the properties of associated primes on short exact sequences to deduce that the set of associated primes of any $\mathcal{C}$-minimax module over a local ring is finite. In [18], [19], Rudlof has given many equivalent conditions for a module to be $\mathcal{C}$-minimax. First, we give an example to show that the class of coatomic (or $\mathcal{C}$-minimax) modules is strictly larger than the class of finitely generated (or minimax) modules.

Example 2.4. We claim that $M := (R/\mathfrak{m})^{(\mathbb{N})}$ is a coatomic (and so a $\mathcal{C}$-minimax) R-module, which is not minimax. Obviously, every semisimple module is coatomic. So, M is a coatomic module, which is neither finitely generated nor Artinian. Let K be an arbitrary R-submodule of M. Then there is a subset I of $\mathbb{N}$ such that $M = K \oplus ((R/\mathfrak{m})^{(I)})$. If K is finitely generated, then I is an infinite set and so M/K is not Artinian. This implies that M is not a minimax R-module.

The following lemma states a characterization of coatomic and $\mathcal{C}$-minimax modules.

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module. Then*

(i) *M is a coatomic module if and only if there exists an integer $s \geqslant 1$ such that $\mathfrak{m}^s M$ is a finitely generated module,*

(ii) *M is a $\mathcal{C}$-minimax module if and only if there exists an integer $s \geqslant 1$ such that $\mathfrak{m}^s M$ is a minimax module.*

Proof. See [19], Theorem 3.3 and [22], Satz 2.4. $\square$

Lemma 2.6. *Let I be an ideal of R and M an R-module. If $\mathcal{S}$ is a Serre subcategory of the category of R-modules, then IM belongs to $\mathcal{S}$ if and only if $M/(0 :_M I)$ belongs to $\mathcal{S}$.*

Proof. The assertion follows by the same method as in [2], Lemma 3.1. $\square$

Lemma 2.7. *Let $(R,\mathfrak{m})$ be a local ring, I an ideal of R and M a $\mathcal{C}$-minimax R-module. Then there is an integer $s \geqslant 1$ such that*

(i) *$M/(0:{}_M\mathfrak{m}^s)$ is minimax,*

(ii) *$\operatorname{Supp}_R(M) = \operatorname{Supp}_R(M/(0:{}_M\mathfrak{m}^s))$,*

(iii) *$H_I^i(M) \cong H_I^i(M/(0:{}_M\mathfrak{m}^s))$ for all $i \geqslant 1$.*

Proof. Statement (i) follows by Lemmas 2.5 (ii) and 2.6. For (ii), it is enough to show that $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(M/(0:{}_M\mathfrak{m}^s))$. Let $\mathfrak{m} \neq \mathfrak{p} \in \operatorname{Supp}_R(M)$. Then there is a nonzero element $x \in M$ such that $(0:{}_R x) \subseteq \mathfrak{p}$. It is easy to check that $x \notin (0:{}_M\mathfrak{m}^s)$. So, $\bar{x} = x + (0:{}_M\mathfrak{m}^s)$ is a nonzero element of $M/(0:{}_M\mathfrak{m}^s)$ such that $(0:{}_R\bar{x}) \subseteq \mathfrak{p}$. Therefore, $\mathfrak{p} \in \operatorname{Supp}_R(M/(0:{}_M\mathfrak{m}^s))$.

On the other hand, $(0:{}_M\mathfrak{m}^s)$ is an $\mathfrak{m}$-torsion R-module and so it is a I-torsion. Hence, $H_I^i((0:{}_M\mathfrak{m}^s)) = 0$ for all $i \geqslant 1$. Now, considering the exact sequence

$$0 \to (0:{}_M\mathfrak{m}^s) \to M \to M/(0:{}_M\mathfrak{m}^s) \to 0, \tag{2.4}$$

we have

$$H_I^i(M) \cong H_I^i(M/(0:{}_M\mathfrak{m}^s)) \tag{2.5}$$

for all $i \geqslant 1$, and (iii) is proved. □

3. Main results

In this section we prove some properties of local cohomology modules $H_I^i(M)$ when M is a minimax or $\mathcal{C}$-minimax R-module. In this way we generalize some of the main results of [2] and [17]. The following lemma is a generalization of [2], Lemma 3.3.

Lemma 3.1. *Let I be an ideal of R and M a minimax R-module with $\dim M = d < \infty$. Then*

(i) *$\dim H_I^{d-i}(M) \leqslant i$ for all i,*

(ii) *if R is semi local, then $\operatorname{Supp}_R(H_I^{d-1}(M))$ is a finite set consisting of all prime ideals $\mathfrak{p}$ of R such that $\dim R/\mathfrak{p} \leqslant 1$.*

Proof. (i) Let $\mathfrak{p} \in \operatorname{Supp}_R(H_I^{d-i}(M))$. Then $(H_I^{d-i}(M))_\mathfrak{p} = H_{IR_\mathfrak{p}}^{d-i}(M_\mathfrak{p}) \neq 0$. Thus, it follows from [5], Theorem 6.1.2, that $\dim_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant d-i$ and so

$$\dim R/\mathfrak{p} \leqslant d - \dim_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant i.$$

(ii) If $d=1$, then $H_I^{d-1}(M) = \Gamma_I(M)$ and $\dim \Gamma_I(M) \leqslant 1$. Hence,

$$\operatorname{Supp}_R(H_I^{d-1}(M)) \subseteq \operatorname{Ass}_R(M) \cup \operatorname{Max} R.$$

Therefore, $\operatorname{Supp}_R(H_I^{d-1}(M))$ is a finite set by assumption. Now, suppose that $d > 1$. Since M is a minimax R-module, by Lemma 2.1 there exists a finitely generated R-module F with $\dim F \leqslant d$ such that the sequence

$$H_I^1(F) \to H_I^1(M) \to 0$$

is exact and

$$H_I^i(F) \cong H_I^i(M)$$

for all $i \geqslant 2$. Thus, $\operatorname{Supp}_R(H_I^i(M)) \subseteq \operatorname{Supp}_R(H_I^i(F))$ for all $i \geqslant 1$. So, considering part (i), it is enough to show that $\operatorname{Supp}_R(H_I^{d-1}(F))$ is a finite set. For this, if $\dim F < d$, then in the light of [5], Theorem 6.1.2, and [5], Exercise 7.1.7, $H_I^{d-1}(F)$ is Artinian and so $\operatorname{Supp}_R(H_I^{d-1}(F))$ is finite. Now, assume that $\dim F = d$ and $\mathfrak{m} \in \max(R)$. Then according to [2], Lemma 3.3 (b), and what was mentioned in the previous case ($\dim F < d$), $\operatorname{Supp}_{R_\mathfrak{m}}(H_{\mathfrak{a}R_\mathfrak{m}}^{d-1}(F_\mathfrak{m}))$ is finite. Therefore, $\operatorname{Supp}_R(H_I^{d-1}(F))$ is a finite set by assumption, as desired. $\square$

Lemma 3.2. *Let I be an ideal of R and M a minimax R-module with* $\dim M = d \geqslant 1$. *Then $H_I^d(M)$ is an Artinian and I-cofinite R-module.*

Proof. In view of Theorem 3.4 of [1], it is enough to show that $H_I^d(M)$ is I-cofinite. By Lemma 2.1 and [13], Proposition 5.1, it remains to exclude the case when $d = 1$. Now, considering the exact sequence (2.1) we obtain the exact sequence

$$0 \to \operatorname{Im} f \to H_I^1(F) \to H_I^1(M) \to 0.$$

Since $\dim F \leqslant \dim M = 1$, by [5], Theorem 6.1.2 and [13], Proposition 5.1, $H_I^1(F)$ is I-cofinite. So, from the exact sequence

$$0 \to \operatorname{Hom}_R(R/I, \operatorname{Im} f) \to \operatorname{Hom}_R(R/I, H_I^1(F))$$

we deduce that $\operatorname{Hom}_R(R/I, \operatorname{Im} f)$ is finitely generated. Thus, Proposition 4.1 of [13] implies that $\operatorname{Im} f$ is I-cofinite, since it is an Artinian I-torsion R-module. Therefore, $H_I^1(M)$ is I-cofinite. $\square$

Theorem 3.3. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M be a $\mathcal{C}$-minimax R-module with* $\dim M = d \geqslant 1$. *Then*

(i) *$H_I^d(M)$ is an Artinian and I-cofinite R-module,*
(ii) $\operatorname{Att}_R(H_I^d(M)) \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R(M)\colon \operatorname{cd}(I, R/\mathfrak{p}) = d\}$,
(iii) *$\operatorname{Supp}_R(H_I^{d-1}(M))$ is a finite set consisting of all prime ideals $\mathfrak{p}$ of R such that* $\dim R/\mathfrak{p} \leqslant 1$.

P r o o f. (i) Since M is $\mathcal{C}$-minimax, by Lemma 2.7 there is an integer $s \geqslant 1$ such that $M/(0 :_M \mathfrak{m}^s)$ is minimax, $\dim_R(M) = \dim_R(M/(0 :_M \mathfrak{m}^s))$ and $H^i_I(M) \cong H^i_I(M/(0 :_M \mathfrak{m}^s))$ for all $i \geqslant 1$. Therefore, $H^d_I(M)$ is Artinian and I-cofinite by Lemma 3.2.

(ii) From Lemma 2.7 and [1], Theorem 3.6 we conclude that

$$\operatorname{Att}_R(H^d_I(M)) \subseteq \{\mathfrak{p} \in \operatorname{Supp}_R(M) \colon \operatorname{cd}(I, R/\mathfrak{p}) = d\}.$$

Let $\mathfrak{p} \in \operatorname{Supp}_R(M) \setminus \operatorname{Ass}_R(M)$ such that $\operatorname{cd}(I, R/\mathfrak{p}) = d$. Then there is $\mathfrak{q} \in \operatorname{Ass}_R(M)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$. Thus, $\dim R/\mathfrak{p} < \dim R/\mathfrak{q} \leqslant d$. Therefore, $H^d_I(R/\mathfrak{p}) = 0$ by Grothendieck's Vanishing Theorem (see [5], Theorem 6.1.2) a contradiction. This completes the proof.

(iii) It is an immediate consequence of isomorphism (2.5) and Lemma 3.1 (ii). $\square$

Lemma 3.4. *Let M be a nonzero $\mathcal{C}$-minimax R-module. Then M is an I-torsion if and only if* $\operatorname{cd}(I, M) = 0$.

P r o o f. The forward direction is clear. For the other direction, in the light of [22], Section 1, Folgerung, there is no loss of generality in assuming $(R, \mathfrak{m})$ is local. Also, by Lemma 2.7 we can assume that M is minimax. If $\operatorname{Supp}_R(M) \nsubseteq V(I)$, then the module $\overline{M} = M/\Gamma_I(M)$ is nonzero and $\Gamma_I(\overline{M}) = 0$. Therefore, $t = \operatorname{grade}_I(\overline{M}) > 0$ by [1], Lemma 2.5 and so $H^t_I(M) \cong H^t_I(\overline{M}) \neq 0$ by Lemma 2.3, a contradiction. $\square$

We next generalize [2], Proposition 3.7, [7], Theorem 1.4 and [8], Theorem 2.2.

Proposition 3.5. *Let I be an ideal of R and M a $\mathcal{C}$-minimax R-module. If N is an arbitrary R-module such that* $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$, *then* $\operatorname{cd}(I, N) \leqslant \operatorname{cd}(I, M)$.

P r o o f. Without loss of generality we may assume that $(R, \mathfrak{m})$ is local and $N \neq 0$. If $\operatorname{cd}(I, M) = 0$, then by Lemma 3.4, $\operatorname{Supp}_R(M) \subseteq V(I)$. Thus, by assumption, $\operatorname{Supp}_R(N) \subseteq V(I)$ and therefore, $H^i_I(N) = 0$ for all $i > 0$. Hence, $\operatorname{cd}(I, N) = 0$.

Now, suppose that $\operatorname{cd}(I, M) \geqslant 1$. From Lemma 2.7 it follows that there is an integer $s \geqslant 1$ such that $M/(0 :_M \mathfrak{m}^s)$ is minimax, $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(M/(0 :_M \mathfrak{m}^s))$ and $\operatorname{cd}(I, M) = \operatorname{cd}(I, M/(0 :_M \mathfrak{m}^s))$. So, we may assume that M is minimax. Therefore, there exists a finitely generated submodule F of M such that M/F is Artinian. Set $A := M/F$. This implies that

$$\operatorname{cd}(I, M) \leqslant \max\{\operatorname{cd}(I, F), \operatorname{cd}(I, A)\} = \operatorname{cd}(I, F)$$

and $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(F)$. As in Lemma 2.1, we have the exact sequence

$$\ldots \to H^1_I(F) \to H^1_I(M) \to 0,$$

and the isomorphism $H^i_I(F) \cong H^i_I(M)$ for all $i \geqslant 2$. Hence, it is easy to see that $\operatorname{cd}(I, M) = \operatorname{cd}(I, F)$. Now the assertion follows by [7], Theorem 1.4. $\square$

The following theorem, which is one of our main results, states that the local-global principle is valid for minimax local cohomology modules. This result generalizes [2], Theorem 2.8.

Theorem 3.6. *Let I be an ideal of R and M a minimax R-module. If n is a nonnegative integer, then the following statements are equivalent:*
(i) *$H_I^i(M)$ is a minimax R-module for all $i \leqslant n$,*
(ii) *$(H_I^i(M))_{\mathfrak{m}}$ is a minimax $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \operatorname{Max}(R)$ and for all $i \leqslant n$.*

Moreover, if $(0 :_M I)$ is finitely generated, then the above assertions are equivalent to the following:
(iii) *$H_I^i(M)$ is an I-cofinite minimax R-module for all $i \leqslant n$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are straightforward. It is sufficient to prove (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii). In the case when $n = 0$, the R-module $\Gamma_I(M)$ is minimax by the assumption. Also, if $(0 :_M I)$ is finitely generated, then $\Gamma_I(M)$ is I-cofinite by [13], Proposition 4.3 and the fact that

$$\operatorname{Hom}_R(R/I, \Gamma_I(M)) = \operatorname{Hom}_R(R/I, M) = (0 :_M I).$$

So, the assertion holds in this case. Let $\mathfrak{m} \in \operatorname{Max}(R)$. Since M is minimax, by Lemma 2.1 we have the exact sequence

$$0 \to (H_I^0(F))_{\mathfrak{m}} \to (H_I^0(M))_{\mathfrak{m}} \to (H_I^0(A))_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} (H_I^1(F))_{\mathfrak{m}} \to (H_I^1(M))_{\mathfrak{m}} \to 0,$$

and the isomorphism

$$(H_I^i(F))_{\mathfrak{m}} \cong (H_I^i(M))_{\mathfrak{m}}$$

for all $i \geqslant 2$, where F is finitely generated and A is Artinian. Therefore, in the light of [2], Theorem 2.8, it is enough to show the assertion in the case when $n = 1$. By assumption and the above exact sequence, $(H_I^i(F))_{\mathfrak{m}}$ is minimax for $i = 0, 1$. Thus, $H_I^i(F)$ is minimax and I-cofinite for $i \leqslant 1$ by [2], Theorem 2.8. Now, considering the exact sequence (2.1) we obtain that $H_I^i(M)$ is minimax for $i = 0, 1$, and (ii) $\Rightarrow$ (i) is proved. On the other hand, the exact sequence

$$0 \to \operatorname{Im} f \to H_I^1(F) \to H_I^1(M) \to 0$$

induces the exact sequence

$$0 \to \operatorname{Hom}_R(R/I, \operatorname{Im} f) \to \operatorname{Hom}_R(R/I, H_I^1(F)),$$

which implies that $\operatorname{Hom}_R(R/I, \operatorname{Im} f)$ is finitely generated. Therefore, by Artinianness of $\operatorname{Im} f$ and Proposition 4.1 of [13] we conclude that $\operatorname{Im} f$ is I-cofinite. Consequently, $H_I^1(M)$ is I-cofinite, as required. $\square$

The following result provides a generalization of the main result of Brodmann and Lashgari, see [3], Theorem 2.5, [4] and [11], Corollary 2.3.

Corollary 3.7. *Let I be an ideal of R and M a minimax R-module. If n is a nonnegative integer such that $(H_I^i(M))_{\mathfrak{m}}$ is a minimax $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \operatorname{Max}(R)$ and for all $i < n$, then for any minimax submodule N of $H_I^n(M)$, the R-module $\operatorname{Hom}_R(R/I, H_I^n(M)/N)$ is minimax. In particular, the set $\operatorname{Ass}_R(H_I^n(M)/N)$ is finite.*

P r o o f. The assertion follows from Theorem 3.6 and [11], Corollary 2.3. □

Theorem 3.8. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M a minimax R-module. If n is a nonnegative integer, then the following statements are equivalent:*
(i) *$H_I^i(M)$ is a minimax R-module for all $i \leqslant n$,*
(ii) *$H_I^i(M)$ is a $\mathcal{C}$-minimax R-module for all $i \leqslant n$.*

P r o o f. Since the class of $\mathcal{C}$-minimax modules contains all minimax modules, it suffices to show (ii) $\Rightarrow$ (i). As in Lemma 2.1, we have the exact sequence

$$0 \to H_I^0(F) \to H_I^0(M) \to H_I^0(A) \to H_I^1(F) \to H_I^1(M) \to 0,$$

and the isomorphism

$$H_I^i(F) \cong H_I^i(M)$$

for all $i \geqslant 2$, where F is finitely generated and A is Artinian. So by the hypothesis and the fact that the category of $\mathcal{C}$-minimax R-modules is a Serre subcategory consisting of all Artinian and all finitely generated R-modules, $H_I^i(F)$ is $\mathcal{C}$-minimax for all $i \leqslant n$. Therefore, Theorem 2.23 of [17] implies that $H_I^i(F)$ is minimax for all $i \leqslant n$. This completes the proof. □

Theorem 3.9. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M a $\mathcal{C}$-minimax R-module. If n is a nonnegative integer, then the following statements are equivalent:*
(i) *$H_I^i(M)$ is a minimax R-module for all $i \leqslant n$,*
(ii) *$H_I^i(M)$ is a $\mathcal{C}$-minimax R-module for all $i \leqslant n$.*

P r o o f. This follows by Lemma 2.7 and Theorem 3.8. □

Corollary 3.10. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M a $\mathcal{C}$-minimax R-module. Then*

$$\inf\{i \in \mathbb{N} \colon H_I^i(M) \text{ is not } \mathcal{C}\text{-minimax}\} = \inf\{i \in \mathbb{N} \colon H_I^i(M) \text{ is not minimax}\}.$$

The following theorem is a generalization of [2], Theorem 2.3 and [14], Theorem 4.9.

Theorem 3.11. *Let I be an ideal of R and M a minimax R-module. If n is a positive integer, then the following statements are equivalent:*
(i) *$H^i_I(M)$ is an Artinian R-module for all $i \geqslant n$,*
(ii) *$H^i_I(M)$ is a minimax R-module for all $i \geqslant n$.*

Proof. We only need to show that (ii) $\Rightarrow$ (i). Let $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \operatorname{Max}(R)$. Then by assumption and Lemma 2.1 we conclude that $M_{\mathfrak{p}}$ and $(H^i_I(M))_{\mathfrak{p}} \cong H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})$ are finitely generated for all $i \geqslant n$. Therefore, Theorem 3.9 of [2] implies that $(H^i_I(M))_{\mathfrak{p}} = 0$ for all $i \geqslant n$. Thus, $\operatorname{Supp}_R(H^i_I(M)) \subseteq \operatorname{Max}(R)$, which yields that $H^i_I(M)$ is Artinian for all $i \geqslant n$ by Lemma 2.1 (ii), as desired. $\square$

The following corollary is an extension of [2], Corollary 2.4.

Corollary 3.12. *Let I be an ideal of R and M a minimax R-module. If $\operatorname{q}(I, M) \geqslant 1$, then $H^{\operatorname{q}(I,M)}_I(M)$ is not minimax. In particular, it is not finitely generated.*

Theorem 3.13. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M a $\mathcal{C}$-minimax R-module. If n is a positive integer, then the following statements are equivalent:*
(i) *$H^i_I(M)$ is Artinian for all $i \geqslant n$,*
(ii) *$H^i_I(M)$ is minimax for all $i \geqslant n$,*
(iii) *$H^i_I(M)$ is $\mathcal{C}$-minimax for all $i \geqslant n$.*

Proof. Since M is $\mathcal{C}$-minimax, by Lemma 2.5 there is an integer $s \geqslant 1$ such that $M/(0 :_M \mathfrak{m}^s)$ is minimax and $H^i_I(M) \cong H^i_I(M/(0 :_M \mathfrak{m}^s))$ for all $i \geqslant 1$. We therefore suppose henceforth in this proof that M is a minimax R-module. Thus, by Lemma 2.1 there exists a finitely generated R-module F such that the sequence

$$H^1_I(F) \to H^1_I(M) \to 0$$

is exact and

$$H^i_I(F) \cong H^i_I(M)$$

for all $i \geqslant 2$. Now, the assertion follows from [17], Theorem 2.24. Note that a careful study of the proof of Theorem 2.24 of [17] implies that the assertion of that theorem holds for all $i \geqslant n$. $\square$

The following result is a generalization of Corollary 3.12 in the case when R is a local ring.

Corollary 3.14. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M a $\mathcal{C}$-minimax R-module. If $\mathrm{q}(I, M) \geqslant 1$, then $H_I^{\mathrm{q}(I,M)}(M)$ is not $\mathcal{C}$-minimax. In particular, if M is a minimax module over the local ring R, then $H_I^{\mathrm{q}(I,M)}(M)$ is not coatomic.*

Theorem 3.15. *Let I be an ideal of a local ring $(R, \mathfrak{m})$ and M a $\mathcal{C}$-minimax R-module. Assume that n is a positive integer such that $\mathrm{Supp}_R(H_I^i(M)) \subseteq \{\mathfrak{m}\}$ for all $i < n$. Then $H_I^i(M)$ is Artinian for all $0 < i < n$.*

Proof. Since M is $\mathcal{C}$-minimax, by Lemma 2.7 there is an integer $s \geqslant 1$ such that $\overline{M} := M/(0 :_M \mathfrak{m}^s)$ is minimax and $H_I^i(M) \cong H_I^i(\overline{M})$ for all $i \geqslant 1$. So, $\mathrm{Supp}_R(H_I^i(\overline{M})) \subseteq \{\mathfrak{m}\}$ for all $0 < i < n$, by the hypothesis. On the other hand, since $(0 :_M \mathfrak{m}^s)$ is I-torsion, we have the exact sequence

$$0 \to H_I^0((0 :_M \mathfrak{m}^s)) \to H_I^0(M) \to H_I^0(\overline{M}) \to 0,$$

which implies that

$$\mathrm{Supp}_R(H_I^0(\overline{M})) \subseteq \mathrm{Supp}_R(H_I^0(M)) \subseteq \{\mathfrak{m}\}.$$

Therefore, Theorem 2.8 of [16] yields that $H_I^i(\overline{M})$ is Artinian for all $i < n$. This completes the proof. $\square$

The following results are generalizations of [2], Theorem 3.9 and [20], Proposition 3.1. See also [15], Theorem 2.6 and Corollaries 2.7–2.8.

Theorem 3.16. *Let I be an ideal of R and M a $\mathcal{C}$-minimax R-module. If n is a positive integer, then the following statements are equivalent:*
(i) *$H_I^i(M) = 0$ for all $i \geqslant n$,*
(ii) *$H_I^i(M)$ is finitely generated for all $i \geqslant n$,*
(iii) *$H_I^i(M)$ is coatomic for all $i \geqslant n$.*

Proof. Implication (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): By [22], Section 1, Folgerung we may assume that $(R, \mathfrak{m})$ is a local ring. We proceed by induction on $d := \dim M$. If $d = 0$, then $H_I^i(M) = 0$ for all $i \geqslant 1$ by [5], Theorem 6.1.2. Let $d > 0$. It follows from [5], Corollary 2.1.7 that $H_I^i(M) \cong H_I^i(M/\Gamma_I(M))$ for all $i \geqslant 1$. Since $M/\Gamma_I(M)$ is a torsion free $\mathcal{C}$-minimax R-module, without loss of generality, we can assume that M is torsion free. Hence, there exists an element $x \in I \subseteq \mathfrak{m}$ which is regular on M. Now, the short exact sequence

$$0 \to M \xrightarrow{.x} M \to M/xM \to 0$$

induces the long exact sequence

$$\ldots \to H_I^i(M) \xrightarrow{.x} H_I^i(M) \to H_I^i(M/xM) \to H_I^{i+1}(M) \to \ldots,$$

where $\dim M/xM < \dim M$. By the assumption, $H_I^i(M/xM)$ is coatomic for all $i \geqslant n$. Then it follows from induction hypothesis that $H_I^i(M/xM) = 0$ for all $i \geqslant n$. Now the above long exact sequence yields that $H_I^i(M) = xH_I^i(M)$ for all $i \geqslant n$. Note that in the light of Lemma 2.5, the coatomic modules satisfy Nakayama's Lemma. Thus, $H_I^i(M) = 0$ for all $i \geqslant n$, as desired. □

Corollary 3.17. *Let I be an ideal of R and M a $\mathcal{C}$-minimax R-module. If* $\operatorname{cd}(I, M) \geqslant 1$, *then $H_I^{\operatorname{cd}(I,M)}(M)$ is not coatomic. In particular, it is not finitely generated.*

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