Czechoslovak Mathematical Journal

Donghan Zhang

Neighbor sum distinguishing list total coloring of IC-planar graphs without 5-cycles

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 1, 111–124

Persistent URL: `http://dml.cz/dmlcz/149576`

Czechoslovak Mathematical Journal, 72 (147) (2022), 111–124

NEIGHBOR SUM DISTINGUISHING LIST TOTAL COLORING OF IC-PLANAR GRAPHS WITHOUT 5-CYCLES

Donghan Zhang, Xi'an

Received August 4, 2020. Published online November 15, 2021.

Abstract. Let $G=(V(G),E(G))$ be a simple graph and $E_G(v)$ denote the set of edges incident with a vertex v. A neighbor sum distinguishing (NSD) total coloring φ of G is a proper total coloring of G such that $\sum\limits_{z\in E_G(u)\cup\{u\}}\varphi(z)\neq\sum\limits_{z\in E_G(v)\cup\{v\}}\varphi(z)$ for each edge $uv\in E(G)$. Pilśniak and Woźniak asserted in 2015 that each graph with maximum degree Δ admits an NSD total $(\Delta+3)$-coloring. We prove that the list version of this conjecture holds for any IC-planar graph with $\Delta\geqslant 11$ but without 5-cycles by applying the Combinatorial Nullstellensatz.

Keywords: IC-planar graph; neighbor sum distinguishing list total coloring; Combinatorial Nullstellensatz; discharging method

MSC 2020: 05C10, 05C15

1. Introduction

We consider only simple graphs in this article. Any terms and notations not defined here can be found in [3].

Let $G=(V(G),E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u\in V(G)$, we use $E_G(u)$ to denote the set of edges incident with u. Let $d_G(u)$ and $N_G(u)$ denote the degree and the neighborhood of u, respectively. We use $\delta(G)$ and $\Delta=\Delta(G)$ to denote the minimum degree and the maximum degree of G, respectively.

Assume that k is a positive integer and $T(G)=V(G)\cup E(G)$. We call a mapping $\varphi\colon T(G)\to\{1,2,\ldots,k\}$ a *neighbor sum distinguishing* (for short NSD) total coloring of G if φ satisfies the following conditions

The research has been supported by Shangluo University Key Disciplines Project, Discipline name: Mathematics.

DOI: 10.21136/CMJ.2021.0333-20

(i) $\varphi(z_1) \neq \varphi(z_2)$ for any two adjacent or incident elements z_1, z_2 in $T(G)$,

(ii) $\sum\limits_{z \in E_G(u) \cup \{u\}} \varphi(z) \neq \sum\limits_{z \in E_G(v) \cup \{v\}} \varphi(z)$ for each edge $uv \in E(G)$.

The NSD *total chromatic number* of G, denoted by $\chi^t_\Sigma(G)$, is the smallest integer k such that G has an NSD k-total coloring. In 2015, Pilśniak and Woźniak in [4] stated an important conjecture about the NSD total coloring in the following.

Conjecture 1.1 ([4])**.** *For any graph G, $\chi^t_\Sigma(G) \leqslant \Delta(G) + 3$.*

Pilśniak and Woźniak in [4] proved that Conjecture 1.1 holds for some special graphs, such as complete graphs, bipartite graphs, cubic graphs and 2-degenerate graphs with $\Delta(G) \leqslant 3$. Yang et al. in [11] proved that any planar graph G with $\Delta(G) \geqslant 10$ satisfies this conjecture.

An IC-planar graph, put forward by Alberson in 2008 (see [1]), is a graph that can be drawn in a plane so that each edge is crossed at most once and two pairs of crossing edges share no common end vertex, i.e., two distinct crossings are independent.

There are also many results about IC-planar graphs which satisfy Conjecture 1.1, such as every IC-planar graph with $\Delta(G) \geqslant 13$ (see [6]), any triangle-free IC-planar graph with $\Delta(G) \geqslant 7$ (see [8]) and each IC-planar graph with $\Delta(G) \geqslant 10$ but without adjacent triangles, see [7].

A *k-list total assignment* of G is a mapping L that assigns to each member $z \in T(G)$ a set $L(z)$ of k integers. Given a list total assignment L of G, a mapping φ is called an NSD *total L-coloring* of G if it satisfies the following conditions

(i) φ is an NSD total coloring of G,

(ii) $\varphi(z) \in L(z)$ for each $z \in T(G)$.

The smallest integer k such that G has an NSD total L-coloring for any k-list total assignment L is called the NSD *total choice number* of G, denoted by $\mathrm{ch}^t_\Sigma(G)$. Clearly, $\chi^t_\Sigma(G) \leqslant \mathrm{ch}^t_\Sigma(G)$.

There are also many results about the list version of Conjecture 1.1.

Conjecture 1.2 ([4])**.** *For any graph G, $\mathrm{ch}^t_\Sigma(G) \leqslant \Delta(G) + 3$.*

Obviously, Conjecture 1.2 implies Conjecture 1.1. Qu et al. in [5] proved that this conjecture holds for any planar graph G with maximum degree $\Delta(G) \geqslant 13$. Wang et al. in [10] confirmed this conjecture for every planar graph G with maximum degree $\Delta(G) \geqslant 8$ but without adjacent triangles. Song et al. in [9] discussed any IC-planar graph G with maximum degree $\Delta(G) \geqslant 14$ and obtained the following.

Theorem 1.3 ([9])**.** *Let G be an IC-planar graph. Then*

$$\mathrm{ch}^t_\Sigma(G) \leqslant \max\{\Delta(G) + 3, 17\}.$$

In this paper, we obtain the following result.

Theorem 1.4. *Let G be an IC-planar graph without 5-cycles. Then*

$$\mathrm{ch}_{\Sigma}^{t}(G) \leqslant \max\{\Delta(G)+3, 14\}.$$

2. Preliminaries

In this section, we introduce some notions and two lemmas to show our results.

An l-vertex (l^+-vertex, l^--vertex) is a vertex of degree l (degree at least l, degree at most l). We use $n_G^l(v)$ ($n_G^{l^+}(v)$, $n_G^{l^-}(v)$) to denote the number of l-vertices (l^+-vertices, l^--vertices) adjacent to v.

A t-cycle (t^+-cycle, t^--cycle) is a cycle of length t (at least t, at most t). In particular, a 3-cycle with vertex set $\{v_1, v_2, v_3\}$ is called a $(d_G(v_1), d_G(v_2), d_G(v_3))$-*cycle* and is denoted by $[v_1v_2v_3]$ if $d_G(v_1) \leqslant d_G(v_2) \leqslant d_G(v_3)$.

Lemma 2.1 ([2]). *Suppose that $\mathbb{F}$ is an arbitrary field and $P \in \mathbb{F}[x_1, \ldots, x_n]$ with degree $\deg(P) = \sum\limits_{k=1}^{n} i_k$, where each i_k is a nonnegative integer number. If the coefficient $c_P(x_1^{i_1}, \ldots, x_n^{i_n})$ of the monomial $x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$ in P is nonzero, and if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_k| > i_k$, then there are $s_1 \in S_1, \ldots, s_n \in S_n$ such that $P(s_1, \ldots, s_n) \neq 0$.*

Let $m \geqslant 2$ be an integer number and $S_1, \ldots, S_m$ be m finite sets of real numbers. Define

$$\sum_{i=1}^{m} S_i = \{s_1 + \ldots + s_m \colon\ s_i \in S_i,\ s_i \neq s_j,\ \forall\, i \neq j\}.$$

Lemma 2.2 ([2]). *Assume that $m \geqslant 2$ is an integer number and $S_1, \ldots, S_m$ are m finite sets of real numbers, where $|S_i| = n_i$ and $n_1 \geqslant \ldots \geqslant n_m$. Define $n_1', \ldots, n_m'$ by*

$$n_1' = n_1 \quad \text{and} \quad n_i' = \min\{n_{i-1}' - 1, n_i\} \quad \text{for } 2 \leqslant i \leqslant m.$$

If $n_t' > 0$, then

$$\left|\sum_{i=1}^{m} S_i\right| \geqslant \sum_{i=1}^{m} n_i' - \frac{1}{2}m(m+1) + 1.$$

3. Proof of Theorem 1.4

Suppose that G is a counterexample to Theorem 1.4 with $E(G)$ being minimal. Let $k = \max\{\Delta(G) + 3, 14\}$ and L be a k-list total assignment of G. By the minimality of G, any subgraph G' of G has an NSD total L-coloring φ' for any k-list total assignment L. In the following, we will obtain an NSD total L-coloring φ of G from φ'. Then this contradicts the assumption that G is a counterexample to Theorem 1.4. Let $m(u) = \sum\limits_{z\in E_G(u)\cup\{u\}} \varphi(z)$. In the coloring φ', the definition of $m'(u)$ is the same as $m(u)$. Not stated otherwise, $\varphi(z) = \varphi'(z)$ for any $z \in T(G) \cap T(G')$. For any $z \in T(G)$, let $S(z)$ denote the set of the available colors for z.

Let v be a 4^--vertex. Since $|S(v)| \geqslant k - 2d_G(v) \geqslant 6 > d_G(v)$ for any k-list total assignment L if $T(G) \setminus \{v\}$ has a total coloring φ' satisfying the following conditions

(i) $\varphi'(z_1) \neq \varphi'(z_2)$ for any adjacent or incident elements $z_1, z_2 \in T(G) \setminus \{v\}$,

(ii) $\sum\limits_{z\in E_G(z_1)\cup\{z_1\}} \varphi'(z) \neq \sum\limits_{z\in E_G(z_2)\cup\{z_2\}} \varphi'(z)$ for any two adjacent vertices $z_1, z_2 \in V(G) \setminus \{v\}$,

(iii) $\varphi'(z) \in L(z)$ for each $z \in T(G) \setminus \{v\}$,

then there exists a color in $L(v)$ to color v such that the resulting coloring φ obtained from φ' is an NSD total L-coloring of G, a contradiction. For simplicity, we will omit the colors of all 4^--vertices in the following proof.

By Theorem 1.3, Claim 3.1 is immediate.

Claim 3.1. $\Delta(G) \leqslant 13$.

Claim 3.2. *Let v be a 5^--vertex of G. Then $n_G^{5^-}(v) = 0$.*

Proof. Suppose to the contrary that v has a neighbor v_1 with $d_G(v_1) \leqslant 5$. Without loss of generality, set $d_G(v) = d_G(v_1) = 5$. Let $G' = G - vv_1$ and φ' be an NSD total L-coloring of G'.

In order to obtain an NSD total L-coloring φ of G from φ', we first erase the colors of v and v_1 from φ'. Then $|S(v)| \geqslant 14-2(5-1) = 6$, $|S(vv_1)| \geqslant 14-(5-1)-(5-1) = 6$ and $|S(v_1)| \geqslant 14 - 2(5 - 1) = 6$.

Set $\varphi(v) = x_1$, $\varphi(vv_1) = x_2$, $\varphi(v_1) = x_3$ and $\varphi(z) = \varphi'(z)$ for each $z \in T(G') \setminus \{v, v_1\}$. Let

$$\begin{aligned} P &= P(x_1, x_2, x_3) \\ &= \prod_{w\in N_G(v)\setminus\{v_1\}} (m(v) - m'(w)) \\ &\quad \times \prod_{w\in N_G(v_1)\setminus\{v\}} (m(v_1) - m'(w))(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(m(v) - m(v_1)), \end{aligned}$$

where $m(v) = x_1+x_2+m'(v)-\varphi'(v)$ and $m(v_1) = x_2+x_3+m'(v_1)-\varphi'(v_1)$. Note that $\deg(P) = 12$. By the definition of NSD total L-coloring if there is a vector (c_1, c_2, c_3) in $(S(v), S(vv_1), S(v_1))$ such that $P(c_1, c_2, c_3) \neq 0$, then φ must be an NSD total L-coloring of G. By Lemma 2.1, we only need to find a monomial $P_0 = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ in P such that $c_P(P_0) \neq 0$ and $\deg(P_0) = \deg(P)$ with $a_i < 6$.

Let $P_0 = x_1^4x_2^4x_3^4$. Then we have $c_P(P_0) = 20 \neq 0$ via Mathematica. Thus, we can obtain an NSD total L-coloring φ of G from φ'. It is a contradiction. $\square$

Claim 3.3. *Let v be an l-vertex of G with $6 \leqslant l \leqslant 7$. Then $n_G^{4^-}(v) \leqslant l-6$.*

Proof. By contradiction, assume that v has $l-5$ neighbors $v_1, \ldots, v_{l-5}$ with $d_G(v_i) \leqslant 4$ for $1 \leqslant i \leqslant l-5$. Let $G' = G - \{vv_i\colon 1 \leqslant i \leqslant l-5\}$ and φ' be an NSD total L-coloring of G'.

In order to obtain an NSD total L-coloring φ of G from φ', we first erase the colors of v and v_i $(i = 1, \ldots, l-5)$ from φ'. Then $|S(v)| \geqslant 14 - 2(d_G(v) - (l-5)) = 4$ and $|S(vv_i)| \geqslant 14 - (d_G(v) - (l-5)) - (d_G(v_i) - 1) \geqslant 6$ for $1 \leqslant i \leqslant l-5$.

When $l = 6$, $l-5 = 1$. By Lemma 2.2, we have that

$$|S(v) + S(vv_1)| \geqslant 4 + 6 - \frac{1}{2}\cdot 2\cdot 3 + 1 = 8 > d_G(v) - 1.$$

When $l = 7$, $l-5 = 2$. By Lemma 2.2, we have that

$$|S(v) + S(vv_1) + S(vv_2)| \geqslant 4 + 5 + 6 - \frac{1}{2}\cdot 3\cdot 4 + 1 = 10 > d_G(v) - 2.$$

Under each of the above two cases, we can always find a color in $S(v)$ and a color in $S(vv_i)$ $(i = 1, \ldots, l-5)$ to color v and vv_i, such that the resulting coloring φ obtained from φ' satisfies $m(v) \neq m(z)$ for each $z \in N_G(v) \setminus \{v_1, \ldots, v_{l-5}\}$. Note that v_i $(i = 1, \ldots, l-5)$ is a 4^--vertex of G. Therefore, we can obtain an NSD total L-coloring φ of G from φ', a contradiction. $\square$

Claim 3.4. *Let v be an l-vertex of G with $8 \leqslant l \leqslant 9$. Then $n_G^{3^-}(v) \leqslant l-7$. Furthermore, $n_G^{4^-}(v) \leqslant l-7$ when $n_G^{3^-}(v) \geqslant 1$.*

Proof. Suppose to the contrary that v has $l-6$ neighbors $v_1, \ldots, v_{l-6}$ with $d_G(v_1) \leqslant 3$ and $d_G(v_i) \leqslant 4$ for $2 \leqslant i \leqslant l-6$. Let $G' = G - \{vv_i\colon 1 \leqslant i \leqslant l-6\}$ and φ' be an NSD total L-coloring of G'.

In order to obtain an NSD total L-coloring φ of G from φ', we first erase the colors of v and v_i $(i = 1, \ldots, l-6)$ from φ'. Then $|S(v)| \geqslant 14 - 2(d_G(v) - (l-6)) = 2$, $|S(vv_1)| \geqslant 14 - (d_G(v) - (l-6)) - (d_G(v_1) - 1) \geqslant 6$ and $|S(vv_i)| \geqslant 14 - (d_G(v) - (l-6)) - (d_G(v_i) - 1) \geqslant 5$ for $2 \leqslant i \leqslant l-6$.

When $l=8$, $l-6=2$. By Lemma 2.2, we have that

$$|S(v)+S(vv_1)+S(vv_2)| \geqslant 2+6+5-\frac{1}{2}\cdot 3\cdot 4+1 = 8 > d_G(v)-2.$$

When $l=9$, $l-6=3$. By Lemma 2.2, we have that

$$|S(v)+S(vv_1)+S(vv_2)+S(vv_3)| \geqslant 2+6+5+4-\frac{1}{2}\cdot 4\cdot 5+1 = 8 > d_G(v)-3.$$

Under each of the above two cases, we can always find a color in $S(v)$ and a color in $S(vv_i)$ $(i=1,\ldots,l-6)$ to color v and vv_i, such that the resulting coloring φ obtained from φ' satisfies $m(v)\neq m(z)$ for each $z\in N_G(v)\setminus\{v_1,\ldots,v_{l-6}\}$. Note that v_i $(i=1,\ldots,l-6)$ is a 4^--vertex of G. Therefore, we can obtain an NSD total L-coloring φ of G from φ', a contradiction. □

Claim 3.5. *Let v be a 10-vertex of G. Then $n_G^{3^-}(v)\leqslant 4$. Furthermore, $n_G^{4^-}(v)\leqslant 4$ when $n_G^{3^-}(v)\geqslant 1$.*

P r o o f. On the contrary, assume that v has five neighbors $v_1,\ldots,v_5$ with $d_G(v_1)\leqslant 3$ and $d_G(v_i)\leqslant 4$ for $2\leqslant i\leqslant 5$. Let $G'=G-\{vv_i\colon 1\leqslant i\leqslant 5\}$ and φ' be an NSD total L-coloring of G'.

In order to obtain an NSD total L-coloring φ of G from φ', we first erase the colors on v and v_i $(i=1,\ldots,5)$ from φ'. Then $|S(v)|\geqslant 14-2(10-5)=4$, $|S(vv_1)|\geqslant 14-(10-5)-(3-1)=7$ and $|S(vv_i)|\geqslant 14-(10-5)-(4-1)=6$ $(i=2,\ldots,5)$.

Set $\varphi(v)=x_1$, $\varphi(vv_i)=x_{i+1}$ $(i=1,\ldots,5)$ and $\varphi(z)=\varphi'(z)$ for each $z\in T(G')\setminus\{v,v_1,\ldots,v_5\}$. Let

$$P=P(x_1,\ldots,x_6)=\prod_{1\leqslant i<j\leqslant 6}(x_i-x_j)\prod_{w\in N_G(v)\setminus\{v_1,\ldots,v_5\}}(m(v)-m'(w)),$$

where $m(v)=\sum\limits_{i=1}^{6}x_i+m'(v)-\varphi'(v)$. Note that $\deg(P)=20$. By the definition of NSD total L-coloring if there is a vector $(c_1,\ldots,c_6)$ in $(S(v),S(vv_1),\ldots,S(vv_5))$, such that $P(c_1,\ldots,c_6)\neq 0$, then φ must be an NSD total L-coloring of G. By Lemma 2.1, we only need to find a monomial $P_0=x_1^{a_1}x_2^{a_2}\ldots x_6^{a_6}$ in P such that $c_P(P_0)\neq 0$ and $\deg(P_0)=\deg(P)$ with $a_1<4$, $a_2<7$ and $a_i<6$ $(i=3,\ldots,6)$.

Let $P_0=x_1^3x_2^6x_3^4x_4^5x_5^2$. Then we have $c_P(P_0)=1\neq 0$ by applying Mathematica. Thus, we can obtain an NSD total L-coloring φ of G from φ'. It is a contradiction. □

Claim 3.6. *Let v be an l-vertex of G with $11 \leqslant l \leqslant 13$. Then $n_G^{2^-}(v) \leqslant \lfloor \frac{3}{7}l \rfloor$. Furthermore, $n_G^{3^-}(v) \leqslant \lfloor \frac{3}{7}l \rfloor$ when $n_G^{2^-}(v) \geqslant 1$, and $n_G^{4^-}(v) \leqslant \lfloor \frac{3}{7}l \rfloor$ when $n_G^{2^-}(v) \geqslant 1$ and $n_G^{3^-}(v) \geqslant 2$.*

Proof. By contradiction, assume that v has $\lfloor \frac{3}{7}l \rfloor + 1$ neighbors $v_1, \ldots, v_{\lfloor \frac{3}{7}l \rfloor + 1}$ with $d_G(v_1) \leqslant 2$, $d_G(v_2) \leqslant 3$ and $d_G(v_i) \leqslant 4$ for $3 \leqslant i \leqslant \lfloor \frac{3}{7}l \rfloor + 1$. Let $G' = G - \{vv_i \colon 1 \leqslant i \leqslant \lfloor \frac{3}{7}l \rfloor + 1\}$ and φ' be an NSD total L-coloring of G'.

In order to obtain an NSD total L-coloring φ of G from φ', we first erase the colors of v and v_i ($i = 1, \ldots, \lfloor \frac{3}{7}l \rfloor + 1$) from φ'. Then $|S(v)| \geqslant l + 3 - 2(d_G(v) - (\lfloor \frac{3}{7}l \rfloor + 1)) = 2\lfloor \frac{3}{7}l \rfloor + 5 - l$, $|S(vv_1)| \geqslant l + 3 - (d_G(v) - (\lfloor \frac{3}{7}l \rfloor + 1)) - (d_G(v_1) - 1) \geqslant \lfloor \frac{3}{7}l \rfloor + 3$, $|S(vv_2)| \geqslant l + 3 - (d_G(v) - (\lfloor \frac{3}{7}l \rfloor + 1)) - (d_G(v_2) - 1) \geqslant \lfloor \frac{3}{7}l \rfloor + 2$ and $|S(vv_i)| \geqslant l + 3 - (d_G(v) - (\lfloor \frac{3}{7}l \rfloor + 1)) - (d_G(v_i) - 1) \geqslant \lfloor \frac{3}{7}l \rfloor + 1$ for $3 \leqslant i \leqslant \lfloor \frac{3}{7}l \rfloor + 1$.

For $l = 11$, we know that $\lfloor \frac{3}{7}l \rfloor + 1 = 5$, $|S(v)| \geqslant 2, |S(vv_1)| \geqslant 7$, $|S(vv_2)| \geqslant 6$ and $|S(vv_i)| \geqslant 5$ for $3 \leqslant i \leqslant 5$. By Lemma 2.2, we have that

$$|S(v) + S(vv_1) + \ldots + S(vv_5)| \geqslant \sum_{i=2}^{7} i - \frac{1}{2} \cdot 6 \cdot 7 + 1 = 7 > d_G(v) - 5.$$

For $l = 12$ or 13, we know that $\lfloor \frac{3}{7}l \rfloor + 1 = 6$, $|S(v)| \geqslant 2$, $|S(vv_1)| \geqslant 8$, $|S(vv_2)| \geqslant 7$ and $|S(vv_i)| \geqslant 6$ for $3 \leqslant i \leqslant 6$. By Lemma 2.2, we have that

$$|S(v) + S(vv_1) + \ldots + S(vv_6)| \geqslant \sum_{i=2}^{8} i - \frac{1}{2} \cdot 7 \cdot 8 + 1 = 8 > d_G(v) - 6.$$

Under each of the above cases, we can always find a color in $S(v)$ and a color in $S(vv_i)$ ($i = 1, \ldots, \lfloor \frac{3}{7}l \rfloor + 1$) to color v and vv_i such that the resulting coloring φ obtained from φ' satisfies $m(v) \neq m(z)$ for each $z \in N_G(v) \setminus \{v_1, \ldots, v_{\lfloor \frac{3}{7}l \rfloor + 1}\}$. Note that v_i ($i = 1, \ldots, \lfloor \frac{3}{7}l \rfloor + 1$) is a 4^--vertex of G. Therefore, we can obtain an NSD total L-coloring φ of G from φ', a contradiction. □

We delete all 2^--vertices from G and obtain the resulting graph H. Then $d_H(v) = d_G(v) - n_G^{2^-}(v)$ for each $v \in V(H)$. By Claims 3.2–3.6, the following Claims 3.7 and 3.8 are immediate.

Claim 3.7. *For the resulting graph H, each of the following results holds.*

(1) $\delta(H) \geqslant 3$,
(2) $d_H(v) = d_G(v)$ *if* $3 \leqslant d_G(v) \leqslant 6$,
(3) $d_H(v) \geqslant 6$ *if* $d_G(v) \geqslant 7$,
(4) $n_H^3(v) + n_H^4(v) \leqslant l - 6$ *when* $d_H(v) = l$ *with* $6 \leqslant l \leqslant 7$,
(5) $n_H^3(v) \leqslant l - 6$ *when* $d_H(v) = l$ *with* $8 \leqslant l \leqslant 10$.

Claim 3.8. *Each 3-cycle in H is either a $(3,7^+,7^+)$-cycle or a $(4,7^+,7^+)$-cycle or a $(5^+,6^+,6^+)$-cycle.*

For a planar graph, we call a face a *t-face* (or a *t^+-face*, a *t^--face*, an *(l_1,l_2,l_3)-face*) if its boundary is a t-cycle (or a t^+-cycle, a t^--cycle, an (l_1,l_2,l_3)-cycle, respectively), and use the boundary $[v_1v_2v_3]$ of a 3-face to represent the 3-face. A face is said to be *incident* with the vertices and edges in its boundary.

In the following, we always consider that the IC-planar graph G has been embedded into a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. We turn all crossings of G into new 4-vertices on the plane and obtain a plane graph $G^\times$ which is called the *associated plane graph* of G. For a vertex v in $G^\times$, we call it *false* if $v \in V(G^\times) \setminus V(G)$ and *real* otherwise. For a face f in $G^\times$, f is called a *false face* if it is incident with a false vertex and a real face otherwise. For convenience of discussion, a real l-vertex is still called an *l-vertex* in the following.

Let $H^\times$ be the associated plane graph of H. For a vertex $v \in V(H)$, let

$$f(v) = \text{the number of real 3-faces incident with } v, \text{ and}$$
$$f_{\mathrm{f}}(v) = \text{the number of false 3-faces incident with } v.$$

Note that each real vertex v is adjacent to at most a false 4-vertex and $d_{H^\times}(v) = d_H(v)$ in $H^\times$. Since G (and thus H) is an IC-planar graph without 5-cycles, we can directly obtain Claim 3.9 as follows.

Claim 3.9. *For each $v \in V(H)$ with $d_H(v) \geqslant 4$, each of the following results holds.*

(1) $0 \leqslant f_{\mathrm{f}}(v) \leqslant 2$.

(2) *When v is not adjacent to any false 4-vertex, $f(v) \leqslant \lfloor \frac{2}{3} d_{H^\times}(v) \rfloor$.*

(3) *When v is adjacent to a false 4-vertex and $0 \leqslant f_{\mathrm{f}}(v) \leqslant 1$, $f(v) \leqslant \lfloor \frac{2}{3} d_{H^\times}(v) \rfloor$ if $d_{H^\times}(v) \equiv 1 \pmod 3$ and $f(v) \leqslant \lfloor \frac{2}{3} d_{H^\times}(v) \rfloor - 1$ otherwise.*

(4) *When v is adjacent to a false 4-vertex and $f_{\mathrm{f}}(v) = 2$, $f(v) \leqslant \lfloor \frac{2}{3} d_{H^\times}(v) \rfloor - 1$ if $d_{H^\times}(v) \equiv 1 \pmod 3$ and $f(v) \leqslant \lfloor \frac{2}{3} d_{H^\times}(v) \rfloor - 2$ otherwise.*

In the following, we are ready to apply the discharging method on the associated plane graph $H^\times$ to prove that $H^\times$ (and thus H) does not exist. And so G does not exist. For each $z \in V(H^\times) \cup F(H^\times)$, we assign it a weight $\omega(z) = d_{H^\times}(z) - 4$. By Euler's formula, we have

$$\sum_{z \in V(H^\times) \cup F(H^\times)} \omega(z) = -8.$$

Next, we design some discharging rules to redistribute weights among vertices and faces, and keep the total weights unchanged. Note that a real l-vertex is still called an l-vertex. The discharging rules are as follows:

(R1) Suppose that $f = [v_1v_2v_3]$ is a real (l_1, l_2, l_3)-face in $H^\times$.

(R1.1) When $(l_1, l_2, l_3) \in \{(3, 7^+, 7^+), (4, 7^+, 7^+)\}$, f receives $\frac{1}{2}$ from v_i for $i = 2, 3$.

(R1.2) When $(l_1, l_2, l_3) = (5^+, 6^+, 6^+)$, f receives $\frac{1}{3}$ from v_i for $i = 1, 2, 3$.

(R2) Each 3-vertex receives $\frac{1}{3}$ from each neighbor.

(R3) Each false 3-face receives 1 from its incident false 4-vertex.

(R4) Suppose that z is a false 4-vertex in $H^\times$ and x a neighbor of z.

(R4.1) Let $d_{H^\times}(x) = 5$. Then z receives $\frac{2}{3}$ from x when $f_{\mathrm{f}}(x) = 2$ and $\frac{1}{3}$ otherwise.

(R4.2) Let $d_{H^\times}(x) = 6$. Then z receives $\frac{4}{3}$ from x when $f_{\mathrm{f}}(x) = 2$ and 1 otherwise.

(R4.3) Let $d_{H^\times}(x) = 7$. When $f_{\mathrm{f}}(x) \leqslant 1$, z receives 1 from x. When $f_{\mathrm{f}}(x) = 2$, z receives $\frac{4}{3}$ from x if x is adjacent to an l-vertex with $l \leqslant 4$ in $H^\times$ and $\frac{5}{3}$ otherwise.

(R4.4) Let $d_{H^\times}(x) \geqslant 8$. Then z receives $\frac{11}{6}$ from x when $f_{\mathrm{f}}(x) = 2$ and $\frac{4}{3}$ otherwise.

After applying the discharging rules, denote by $\omega'(z)$ the new weight for each $z \in V(H^\times) \cup F(H^\times)$. Since the total weights are not changed,

$$\sum_{z \in V(H^\times) \cup F(H^\times)} \omega'(z) = \sum_{z \in V(H^\times) \cup F(H^\times)} \omega(z) = -8 < 0.$$

Thus, there is at least one element $z_0 \in V(H^\times) \cup F(H^\times)$ satisfying

$$\omega'(z_0) < 0. \tag{3.1}$$

In the following, we check the new weight $\omega'(z)$ for each $z \in V(H^\times) \cup F(H^\times)$ to show that there is no z_0 satisfying $\omega'(z_0) < 0$, which is a contradiction to (3.1). Note that a real l-vertex is still called an l-vertex.

Since each false 3-face z is incident with a false 4-vertex, $\omega'(z) = 3 - 4 + 1 = 0$ by (R3). By Claim 3.8, we know that each real 3-face z is either a $(3, 7^+, 7^+)$-face or a $(4, 7^+, 7^+)$-face or a $(5^+, 6^+, 6^+)$-face. Thus, it is easy to verify that $\omega'(z) \geqslant 0$ by (R1) when z is a real 3-face. If z is a 4^+-face , then $\omega'(z) \geqslant 0$ since no rule is applied to it. Thus, $z_0 \notin F(H^\times)$.

Next, we show that $\omega'(z) \geqslant 0$ for each false 4-vertex $z \in V(H^\times) \setminus V(H)$. Pick arbitrarily a false 4-vertex z from $V(H^\times) \setminus V(H)$. Let $N_{H^\times}(z) = \{v_1, v_2, v_3, v_4\}$. Then, up to isomorphism, the configuration of the induced subgraph $H^\times[\{z\} \cup N_{H^\times}(z)]$ is one of the six configurations in Figure 1. Note that z is incident with at most four false 3-faces and adjacent to at most two l-vertices with $l \leqslant 4$ by Claims 3.2 and 3.7.

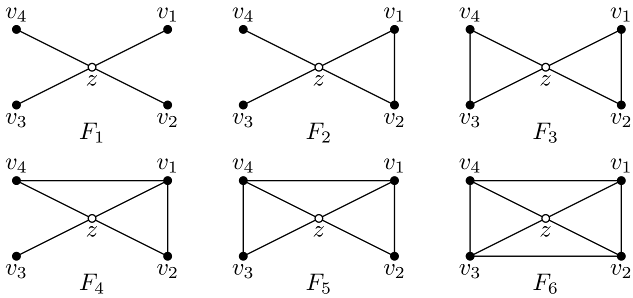

Figure 1. Six different configurations of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$

Case 1: Suppose that z is not adjacent to any 3-vertex.

Subcase 1.1: Let z be incident with at most two false 3-faces (see F_1–F_4 in Figure 1). Then $\omega'(z) \geqslant 4 - 2 + 2 \cdot 1 = 0$ by (R3)–(R4) since z is adjacent to at least two 6^+-vertices by Claims 3.2 and 3.7.

Subcase 1.2: Let z be incident with three false 3-faces (see F_5 in Figure 1). Note that $f_{\mathrm{f}}(v_i) = 2$ for $i = 1, 4$.

Subcase 1.2.1: Assume that z is not adjacent to any 4-vertex in $H^{\times}$. Then z is adjacent to at most two 5-vertices by Claims 3.2 and 3.7. If z is adjacent to at most one 5-vertex, then other neighbors of z are all 6^+-vertices. Thus, $\omega'(z) \geqslant 4 - 3 + \frac{1}{3} + 3 \cdot 1 = \frac{1}{3}$ by (R3)–(R4). If z is adjacent to two 5-vertices, then it must be $d_{H^{\times}}(v_2) = d_{H^{\times}}(v_3) = 5$ and $d_{H^{\times}}(v_i) \geqslant 6$ for $i = 1, 4$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 3 + 2 \cdot \frac{1}{3} + 2 \cdot \frac{4}{3} = \frac{1}{3}$ by (R3)–(R4).

Subcase 1.2.2: Assume that z is adjacent to exactly one 4-vertex in $H^{\times}$. Then z is adjacent to at most one 5-vertex by Claims 3.2 and 3.7. If z is not adjacent to any 5-vertex, then z has three neighbors which are all 6^+-vertices. Thus, $\omega'(z) \geqslant 4 - 3 + 3 \cdot 1 = 0$ by (R3)–(R4). If z is adjacent to exactly one 5-vertex, then it must be $d_{H^{\times}}(v_1) \geqslant 6$ and $d_{H^{\times}}(v_4) \geqslant 6$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 3 + \frac{1}{3} + 2 \cdot \frac{4}{3} = 0$ by (R3)–(R4).

Subcase 1.2.3: Assume that z is adjacent to two 4-vertices in $H^{\times}$. Then it must be $d_{H^{\times}}(v_2) = d_{H^{\times}}(v_3) = 4$ and $d_{H^{\times}}(v_i) \geqslant 8$ for $i = 1, 4$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 3 + 2 \cdot \frac{11}{6} = \frac{2}{3}$ by (R3)–(R4).

Subcase 1.3: Let z be incident with four false 3-faces (see F_6 in Figure 1). Then z has at least three neighbors which are 6^+-vertices by Claims 3.2 and 3.7. Note that $f_{\mathrm{f}}(v_i) = 2$ for $i = 1, 2, 3, 4$. Thus, $\omega'(z) \geqslant 4 - 4 + 3 \cdot \frac{4}{3} = 0$ by (R3)–(R4).

Case 2: Suppose that z is adjacent to exactly one 3-vertex.

Subcase 2.1: Let z be incident with at most one false 3-face (see F_1 and F_2 in Figure 1). Then $\omega'(z) \geqslant 4 - 1 - \frac{1}{3} + 2 \cdot 1 = \frac{2}{3}$ by (R2)–(R4) since z is adjacent to at least two 6^+-vertices by Claims 3.2 and 3.7.

Subcase 2.2: Let z be incident with two false 3-faces (see F_3 and F_4 in Figure 1). Then z is adjacent to at most one 4-vertex by Claims 3.2 and 3.7.

Subcase 2.2.1: Assume that the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ is F_3 in Figure 1. If z is not adjacent to any 4-vertex, then z is adjacent to one 5^+-vertex and two 7^+-vertices by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - \frac{1}{3} + 2 \cdot 1 + \frac{1}{3} = 0$ by (R2)–(R4). If z is adjacent to one 4-vertex, then z is adjacent to two 8^+-vertices by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - \frac{1}{3} + 2 \cdot \frac{4}{3} = \frac{1}{3}$ by (R2)–(R4).

Subcase 2.2.2: Assume that the configuration of $H^{\times}[\{z\} \cup N_{H^{\times}}(z)]$ is F_4 in Figure 1. Note that $f_{\mathrm{f}}(v_1) = 2$. If $d_{H^{\times}}(v_1) = 3$, then $d_{H^{\times}}(v_i) \geqslant 7$ for $i = 2, 3, 4$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - \frac{1}{3} + 2 \cdot 1 + \frac{4}{3} = 1$ by (R2)–(R4). If $d_{H^{\times}}(v_2) = 3$, then $d_{H^{\times}}(v_i) \geqslant 7$ for $i = 1, 4$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - \frac{1}{3} + 1 + \frac{4}{3} = 0$ by (R2)–(R4). Similarly, we can obtain $\omega'(z) \geqslant 0$ if $d_{H^{\times}}(v_4) = 3$. If $d_{H^{\times}}(v_3) = 3$, then $d_{H^{\times}}(v_1) \geqslant 7$ by Claims 3.2 and 3.7. If $d_{H^{\times}}(v_1) = 7$, then $d_{H^{\times}}(v_i) \geqslant 5$ for $i = 2, 4$ and $d_{H^{\times}}(v_2) + d_{H^{\times}}(v_4) \geqslant 11$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - \frac{1}{3} + \frac{4}{3} + 1 + \frac{1}{3} = \frac{1}{3}$ by (R2)–(R4). If $d_{H^{\times}}(v_1) \geqslant 8$, then $d_{H^{\times}}(v_i) \geqslant 4$ for $i = 2, 4$ and $d_{H^{\times}}(v_2) + d_{H^{\times}}(v_4) \geqslant 11$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - \frac{1}{3} + \frac{11}{6} + \frac{4}{3} = \frac{5}{6}$ by (R2)–(R4).

Subcase 2.3: Let z be incident with three false 3-faces (see F_5 in Figure 1). Then z is adjacent to at most one 4-vertex. Note that $f_{\mathrm{f}}(v_i) = 2$ for $i = 1, 4$. If $d_{H^{\times}}(v_1) = 3$, then $d_{H^{\times}}(v_i) \geqslant 7$ for $i = 2, 3, 4$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 3 - \frac{1}{3} + 2 \cdot 1 + \frac{4}{3} = 0$ by (R2)–(R4). Similarly, we can obtain $\omega'(z) \geqslant 0$ if $d_{H^{\times}}(v_4) = 3$. If $d_{H^{\times}}(v_2) = 3$, then $d_{H^{\times}}(v_i) \geqslant 7$ for $i = 1, 4$ by Claims 3.2 and 3.7. When $d_{H^{\times}}(v_4) = 7$, $d_{H^{\times}}(v_3) \geqslant 5$ and v_4 is not adjacent to any l-vertex with $3 \leqslant l \leqslant 4$ in $H^{\times}$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 3 - \frac{1}{3} + \frac{5}{3} + \frac{4}{3} + \frac{1}{3} = 0$ by (R2)–(R4). In the following, we assume that $d_{H^{\times}}(v_4) \geqslant 8$. If $d_{H^{\times}}(v_1) = 7$, then $d_{H^{\times}}(v_3) \geqslant 5$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 3 - \frac{1}{3} + \frac{11}{6} + \frac{4}{3} + \frac{1}{3} = \frac{1}{6}$ by (R2)–(R4). If $d_{H^{\times}}(v_1) \geqslant 8$, then $\omega'(z) \geqslant 4 - 3 - \frac{1}{3} + 2 \cdot \frac{11}{6} = \frac{1}{3}$ by (R2)–(R4). Similarly, we can obtain $\omega'(z) \geqslant 0$ if $d_{H^{\times}}(v_3) = 3$.

Subcase 2.4: Let z be incident with four false 3-faces (see F_6 in Figure 1). Note that $f_{\mathrm{f}}(v_i) = 2$ for $i = 1, 2, 3, 4$. If $d_{H^{\times}}(v_1) = 3$, then $d_{H^{\times}}(v_i) \geqslant 7$ for $i = 2, 3, 4$ by Claims 3.2 and 3.7. If $d_{H^{\times}}(v_3) = 7$, then v_3 is not adjacent to any l-vertex with $3 \leqslant l \leqslant 4$ in $H^{\times}$ by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 4 - \frac{1}{3} + \frac{5}{3} + 2 \cdot \frac{4}{3} = 0$ by (R2)–(R4). If $d_{H^{\times}}(v_3) \geqslant 8$, then $\omega'(z) \geqslant 4 - 4 - \frac{1}{3} + \frac{11}{6} + 2 \cdot \frac{4}{3} = \frac{1}{6}$ by (R2)–(R4). Similarly, we can obtain $\omega'(z) \geqslant 0$ if $d_{H^{\times}}(v_2) = 3$ or $d_{H^{\times}}(v_3) = 3$ or $d_{H^{\times}}(v_4) = 3$.

Case 3: Suppose that z is adjacent to two 3-vertices. Note that z is not incident with four false 3-faces since each 3-vertex is not adjacent to any 3-vertex by Claims 3.2 and 3.7.

Subcase 3.1: Let z be incident with at most one false 3-face (see F_1 and F_2 in Figure 1). Then $\omega'(z) \geqslant 4 - 1 - 2 \cdot \frac{1}{3} + 2 \cdot 1 = \frac{1}{3}$ by (R2)–(R4) since z is adjacent to two 7^+-vertices by Claims 3.2 and 3.7.

Subcase 3.2: Let z be incident with two false 3-faces (see F_3 and F_4 in Figure 1).

Subcase 3.2.1: Assume that the configuration of $H^\times[\{z\} \cup N_{H^\times}(z)]$ is F_3 in Figure 1. Then z is adjacent to two 8^+-vertices by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - 2 \cdot \frac{1}{3} + 2 \cdot \frac{4}{3} = 0$ by (R2)–(R4).

Subcase 3.2.2: Assume that the configuration of $H^\times[\{z\} \cup N_{H^\times}(z)]$ is F_4 in Figure 1. Then $d_{H^\times}(v_3) = 3$ and $d_{H^\times}(v_1) \geqslant 8$ by Claims 3.2 and 3.7. Note that $f_{\mathrm{f}}(v_1) = 2$. Since one of v_2 and v_4 is a 3-vertex, the other of v_2 and v_4 is a 7^+-vertex by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant 4 - 2 - 2 \cdot \frac{1}{3} + 1 + \frac{11}{6} = \frac{1}{6}$ by (R2)–(R4).

Subcase 3.3: Let z be incident with three false 3-faces (see F_5 in Figure 1). Then $d_{H^\times}(v_2) = d_{H^\times}(v_3) = 3$ and $d_{H^\times}(v_i) \geqslant 8$ for $i = 1, 4$ by Claims 3.2 and 3.7. Note that $f_{\mathrm{f}}(v_i) = 2$ for $i = 1, 4$. Thus, $\omega'(z) \geqslant 4 - 3 - 2 \cdot \frac{1}{3} + 2 \cdot \frac{11}{6} = 0$ by (R2)–(R4).

In summary, we know that z_0 is not a false 4-vertex.

Finally, we prove that $\omega'(z) \geqslant 0$ for each real vertex $z \in V(H)$. Pick arbitrarily a real vertex z from $V(H)$. Note that each real vertex gives no weight to any false face. Since $\delta(H) \geqslant 3$ by Claim 3.7, $\delta(H^\times) \geqslant 3$.

If z is a 3-vertex, then $\omega'(z) \geqslant 3 - 4 + 3 \cdot \frac{1}{3} = 0$ by (R2).

If z is a 4-vertex, then $\omega'(z) \geqslant 4 - 4 = 0$ since no rule is applied to it.

In the following, we assume that z is a 5^+-vertex. Note that $n^3_{H^\times}(z) \leqslant n^3_H(z)$ in $H^\times$.

Part 1: Suppose that z is not adjacent to any false 4-vertex. Then $f(z) \leqslant \lfloor \frac{2}{3} d_{H^\times}(z) \rfloor$ by Claim 3.9.

If $d_{H^\times}(z) = 5$, then $f(z) \leqslant 3$ by Claim 3.9 and $n^{4^-}_{H^\times}(z) = 0$ by Claims 3.2 and 3.7. Thus, z is not incident with any real 3-face containing a 4^--vertex. And so $\omega'(z) \geqslant 5 - 4 - 3 \cdot \frac{1}{3} = 0$ by (R1).

If $d_{H^\times}(z) = l$ with $6 \leqslant l \leqslant 10$, then $n^3_{H^\times}(z) \leqslant l - 6$ by Claim 3.7. Thus, $\omega'(z) \geqslant l - 4 - (l-6) \cdot \frac{1}{3} - \lfloor \frac{2}{3} l \rfloor \cdot \frac{1}{2} \geqslant 0$ by (R1)–(R2).

If $d_{H^\times}(z) = l$ with $11 \leqslant l \leqslant 13$, then $f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{2}{3} l \rfloor + 1$ when $l \equiv 1 \pmod 3$ and $f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{2}{3} l \rfloor$ otherwise since every real 3-face is incident with at most one 3-vertex by Claims 3.2 and 3.7. Thus, $\omega'(z) \geqslant l - 4 - \max\{f(z) \cdot \frac{1}{2} + n^3_{H^\times}(z) \cdot \frac{1}{3} : f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{2}{3} l \rfloor + 1 \text{ and } f(z) \leqslant \lfloor \frac{2}{3} l \rfloor\} \geqslant \frac{1}{9}(4l - 39) \geqslant \frac{5}{9}$ by (R1)–(R2).

Part 2: Suppose that z is adjacent to one false 4-vertex and $f_{\mathrm{f}}(z) \leqslant 1$. Then $f(z) \leqslant \lfloor \frac{2}{3} d_{H^\times}(z) \rfloor$ when $d_{H^\times}(z) \equiv 1 \pmod 3$ and $f(z) \leqslant \lfloor \frac{2}{3} d_{H^\times}(z) \rfloor - 1$ otherwise by Claim 3.9. Since every real 3-face is incident with at most one 3-vertex by Claims 3.2 and 3.7, $f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{2}{3} d_{H^\times}(z) \rfloor$.

If $d_{H^\times}(z) = 5$, then $f(z) \leqslant 2$ by Claim 3.9 and z is not adjacent to any l-vertex with $l \leqslant 4$ by Claims 3.2 and 3.7. Thus, z is not incident with any real 3-face containing a 4^--vertex. And so $\omega'(z) \geqslant 5 - 4 - 2 \cdot \frac{1}{3} - \frac{1}{3} = 0$ by (R1) and (R4).

If $d_{H^\times}(z)=6$, then $f(z)\leqslant 3$ by Claim 3.9 and z is not adjacent to any l-vertex with $l\leqslant 4$ by Claim 3.7. Thus, z is not incident with any real 3-face containing a 4^--vertex. And so $\omega'(z)\geqslant 6-4-3\cdot\frac{1}{3}-1=0$ by (R1) and (R4).

If $d_{H^\times}(z)=7$, then $f(z)\leqslant 4$ by Claim 3.9 and z is adjacent to at most one l-vertex with $l\leqslant 4$ by Claim 3.7. Thus, z is incident with at most two real 3-faces containing a 4^--vertex. And so $\omega'(z)\geqslant 7-4-2\cdot\frac{1}{3}-2\cdot\frac{1}{2}-\frac{1}{3}-1=0$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=8$, then $f(z)\leqslant 4$ by Claim 3.9 and $n^3_{H^\times}(z)\leqslant 2$ by Claim 3.7. Thus, $\omega'(z)\geqslant 8-4-4\cdot\frac{1}{2}-2\cdot\frac{1}{3}-\frac{4}{3}=0$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=9$, then $f(z)\leqslant 5$ by Claim 3.9 and $n^3_{H^\times}(z)\leqslant 3$ by Claim 3.7. Thus, $\omega'(z)\geqslant 9-4-5\cdot\frac{1}{2}-3\cdot\frac{1}{3}-\frac{4}{3}=\frac{1}{6}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=10$, then $f(z)\leqslant 6$ by Claim 3.9 and $n^3_{H^\times}(z)\leqslant 4$ by Claim 3.7. Thus, $\omega'(z)\geqslant 10-4-4\cdot\frac{1}{3}-6\cdot\frac{1}{2}-\frac{4}{3}=\frac{1}{3}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=11$, then $f(z)\leqslant 6$ by Claim 3.9 and $f(z)+n^3_{H^\times}(z)\leqslant 2\lfloor\frac{22}{3}\rfloor$. Thus, $\omega'(z)\geqslant 11-4-8\cdot\frac{1}{3}-6\cdot\frac{1}{2}-\frac{4}{3}=0$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=12$, then $f(z)\leqslant 7$ by Claim 3.9 and $f(z)+n^3_{H^\times}(z)\leqslant 2\lfloor\frac{24}{3}\rfloor$. Thus, $\omega'(z)\geqslant 12-4-9\cdot\frac{1}{3}-7\cdot\frac{1}{2}-\frac{4}{3}=\frac{1}{6}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=13$, then $f(z)\leqslant 8$ by Claim 3.9 and $f(z)+n^3_{H^\times}(z)\leqslant 2\lfloor\frac{26}{3}\rfloor$. Thus, $\omega'(z)\geqslant 13-4-8\cdot\frac{1}{3}-8\cdot\frac{1}{2}-\frac{4}{3}=1$ by (R1)–(R2) and (R4).

Part 3: Suppose that z is adjacent to one false 4-vertex and $f_{\mathrm{f}}(z)=2$. Then $f(z)\leqslant\lfloor\frac{2}{3}d_{H^\times}(z)\rfloor-1$ when $d_{H^\times}(z)\equiv 1\pmod 3$ and $f(z)\leqslant\lfloor\frac{2}{3}d_{H^\times}(z)\rfloor-2$ otherwise by Claim 3.9. Since every real 3-face is incident with at most one 3-vertex by Claims 3.2 and 3.7, $f(z)+n^3_{H^\times}(z)\leqslant 2\lfloor\frac{2}{3}d_{H^\times}(z)\rfloor-1$ when $d_{H^\times}(z)\equiv 1(\bmod 3)$ and $f(z)+n^3_{H^\times}(z)\leqslant 2\lfloor\frac{2}{3}d_{H^\times}(z)\rfloor-2$ otherwise.

If $d_{H^\times}(z)=5$, then $f(z)\leqslant 1$ by Claim 3.9 and z is not adjacent to any l-vertex with $l\leqslant 4$ by Claims 3.2 and 3.7. Thus, z is not incident with any real 3-face containing a 4^--vertex. And so $\omega'(z)\geqslant 5-4-\frac{2}{3}-\frac{1}{3}=0$ by (R1) and (R4).

If $d_{H^\times}(z)=6$, then $f(z)\leqslant 2$ by Claim 3.9 and z is not adjacent to any l-vertex with $l\leqslant 4$ by Claim 3.7. Thus, z is not incident with any real 3-face containing a 4^--vertex. And so $\omega'(z)\geqslant 6-4-\frac{4}{3}-2\cdot\frac{1}{3}=0$ by (R1) and (R4).

If $d_{H^\times}(z)=7$, then $f(z)\leqslant 3$ by Claim 3.9 and z is adjacent to at most one l-vertex with $l\leqslant 4$ by Claim 3.7. If z is not adjacent to any l-vertex with $l\leqslant 4$, then z is not incident with any real 3-face containing a 4^--vertex. Thus, $\omega'(z)\geqslant 7-4-3\cdot\frac{1}{3}-\frac{5}{3}=\frac{1}{3}$ by (R1) and (R4). If z is adjacent to one l-vertex with $l\leqslant 4$, then z is incident with at most two real 3-faces containing a 4^--vertex. Thus, $\omega'(z)\geqslant 7-4-2\cdot\frac{1}{2}-\frac{1}{3}-\frac{1}{3}-\frac{4}{3}=0$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=8$, then $f(z)\leqslant 3$ by Claim 3.9 and $n^3_{H^\times}(z)\leqslant 2$ by Claim 3.7. Thus, $\omega'(z)\geqslant 8-4-2\cdot\frac{1}{3}-3\cdot\frac{1}{2}-\frac{11}{6}=0$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z)=9$, then $f(z)\leqslant 4$ by Claim 3.9 and $n^3_{H^\times}(z)\leqslant 3$ by Claim 3.7. Thus, $\omega'(z)\geqslant 9-4-3\cdot\frac{1}{3}-4\cdot\frac{1}{2}-\frac{11}{6}=\frac{1}{6}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z) = 10$, then $f(z) \leqslant 5$ by Claim 3.9 and $n^3_{H^\times}(z) \leqslant 4$ by Claim 3.7. Thus, $\omega'(z) \geqslant 10 - 4 - 4 \cdot \frac{1}{3} - 5 \cdot \frac{1}{2} - \frac{11}{6} = \frac{1}{3}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z) = 11$, then $f(z) \leqslant 5$ by Claim 3.9 and $f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{22}{3} \rfloor - 2$. Thus, $\omega'(z) \geqslant 11 - 4 - 7 \cdot \frac{1}{3} - 5 \cdot \frac{1}{2} - \frac{11}{6} = \frac{1}{3}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z) = 12$, then $f(z) \leqslant 6$ by Claim 3.9 and $f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{24}{3} \rfloor - 2$. Thus, $\omega'(z) \geqslant 12 - 4 - 8 \cdot \frac{1}{3} - 6 \cdot \frac{1}{2} - \frac{11}{6} = \frac{1}{2}$ by (R1)–(R2) and (R4).

If $d_{H^\times}(z) = 13$, then $f(z) \leqslant 7$ by Claim 3.9 and $f(z) + n^3_{H^\times}(z) \leqslant 2\lfloor \frac{26}{3} \rfloor - 1$. Thus, $\omega'(z) \geqslant 13 - 4 - 8 \cdot \frac{1}{3} - 7 \cdot \frac{1}{2} - \frac{11}{6} = 1$ by (R1)–(R2) and (R4).

Therefore, $z_0 \notin V(H)$.

By the analysis above, there is no $z_0 \in V(H^\times) \cup F(H^\times)$ such that $\omega'(z_0) < 0$, which contradicts (3.1). The proof of Theorem 1.4 is completed. □

References

[1] *M. O. Albertson*: Chromatic number, independent ratio, and crossing number. Ars Math. Contemp. *1* (2008), 1–6. zbl MR doi

[2] *N. Alon*: Combinatorial Nullstellensatz. Comb. Probab. Comput. *8* (1999), 7–29. zbl MR doi

[3] *J. A. Bondy, U. S. R. Murty*: Graph Theory. Graduate Texts in Mathematics 244. Springer, Berlin, 2008. zbl MR doi

[4] *M. Pilśniak, M. Woźniak*: On the total-neighbor-distinguishing index by sums. Graphs Comb. *31* (2015), 771–782. zbl MR doi

[5] *C. Qu, G. Wang, G. Yan, X. Yu*: Neighbor sum distinguishing total choosability of planar graphs. J. Comb. Optim. *32* (2016), 906–916. zbl MR doi

[6] *C. Song, X. Jin, C. Xu*: Neighbor sum distinguishing total coloring of IC-planar graphs with short cycle restrictions. Discrete Appl. Math. *279* (2020), 202–209. zbl MR doi

[7] *C. Song, C. Xu*: Neighbor sum distinguishing total colorings of IC-planar graphs with maximum degree 13. J. Comb. Optim. *39* (2020), 293–303. zbl MR doi

[8] *W. Song, Y. Duan, L. Miao*: Neighbor sum distinguishing total coloring of triangle free IC-planar graphs. Acta Math. Sin., Engl. Ser. *36* (2020), 292–304. zbl MR doi

[9] *W. Song, L. Miao, Y. Duan*: Neighbor sum distinguishing total choosability of IC-planar graphs. Discuss. Math., Graph Theory *40* (2020), 331–344. zbl MR doi

[10] *J. Wang, J. Cai, B. Qiu*: Neighbor sum distinguishing total choosability of planar graphs without adjacent triangles. Theor. Comput. Sci. *661* (2017), 1–7. zbl MR doi

[11] *D. Yang, L. Sun, X. Yu, J. Wu, S. Zhou*: Neighbor sum distinguishing total chromatic number of planar graphs with maximum degree 10. Appl. Math. Comput. *314* (2017), 456–468. zbl MR doi

Author's address: Donghan Zhang, School of Mathematics and Statistics, Northwestern Polytechnical University, 1 Dongxiang Road, Chang'an District, Xi'an, Shaanxi 710129, P. R. China and School of Mathematics and Computer Application, Shangluo University, Shangluo, Shaanxi 726000, P. R. China, e-mail: `zhang_dh@mail.nwpu.edu.cn`.