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SIGNED GRAPHS WITH AT MOST THREE EIGENVALUES

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Abstract. We investigate signed graphs with just 2 or 3 distinct eigenvalues, mostly in the context of vertex-deleted subgraphs, the join of two signed graphs or association schemes.

Keywords: signed graph; join; adjacency matrix; main eigenvalue; net-degree; association scheme

MSC 2020: 05C22, 05C50

1. INTRODUCTION

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is a simple unsigned graph, called the *underlying graph*, and $\sigma\colon\ E \to \{1, -1\}$ is the *sign function* or the *signature*. The number of vertices of $\dot{G}$ is called the *order* and is denoted by n. The edge set of $\dot{G}$ is composed of subsets of positive and negative edges. The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1's which correspond to negative edges. The *eigenvalues* of $\dot{G}$ are identified as the eigenvalues of $A_{\dot{G}}$, and they form the *spectrum* of $A_{\dot{G}}$.

Throughout the paper we interpret a graph as a signed graph with all the edges being positive and, where no confusion arises, we write '$\dot{G}$ has k eigenvalues' to mean that $\dot{G}$ has exactly k distinct eigenvalues.

Unsigned graphs with 2 eigenvalues are well-known, while those that are connected and have 3 eigenvalues are either strongly regular or nonregular. Moreover, every noncomplete connected strongly regular graph has 3 eigenvalues and these graphs have received a great deal of attention in the last 70 years. Nonregular

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graphs with 3 eigenvalues are less studied, but some progress in this direction is reported in [5], [6], [17], [18], [25] for example. In the case of connected signed graphs with 2 or 3 eigenvalues we have a much more complicated situation. First, those with 2 eigenvalues are neither determined nor fully described; they are known to be regular, moreover strongly regular in the sense of [23]. Signed graphs with 3 eigenvalues may or may not be regular, and if they are, then they may or may not be strongly regular. Some results concerning signed graphs with 2 eigenvalues can be found in [11], [12], [13], [14], [23], [24], while those that are regular and have 3 eigenvalues are considered in [1]. Some open problems related to signed graphs with a comparatively small number of eigenvalues are given in [2].

The purpose of this paper is to give some properties and constructions of signed graphs with 2 or 3 eigenvalues, with special attention to those that are nonregular and have 3 eigenvalues, one of which is simple. Our results are based on the join operation and the examination of vertex-deleted subgraphs or cones over prescribed signed graphs. Similar techniques in the framework of graphs can be found in [5], [17], [25]. Some new constructions of regular signed graphs with 3 eigenvalues and bipartite signed graphs with 2 or 3 eigenvalues are also provided.

Additional terminology and notation are given in Section 2. Our results are reported in the remaining sections. In particular, in Section 3 we compute the characteristic polynomial of the join of two signed graphs in terms of their eigenvalues and main angles. This result is an extension of a known result formulated in a particular case of unsigned graphs. Some basic results are given in Section 4. The main results and aforementioned new constructions are separated in Sections 5 and 6.

2. Preliminaries

We write I, O, J, $\mathbf{0}$ and $\mathbf{j}$ for the identity matrix, the all-0 matrix, the all-1 matrix, the all-0 vector and the all-1 vector, respectively.

We say that a signed graph $\dot{G}$ is connected, complete, regular or bipartite if the same holds for its underlying graph G. The vertex degree of a vertex in $\dot{G}$ is transferred from G as well. The *net-degree* of a vertex is the difference between the number of positive and the number of negative edges incident with it. In particular, a signed graph is said to be *net-regular* if the net-degree is constant on the vertex set.

A signed graph is said to be *homogeneous* if all its edges have the same sign (in particular, if its edge set is empty). Otherwise, it is said to be *inhomogeneous*. The negation $-\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$. We write $\dot{G}^+$ for the subgraph of $\dot{G}$ determined by the positive edges.

The *join* of (disjoint) signed graphs $\dot{G}$ and $\dot{H}$ is the signed graph $\dot{G} \nabla \dot{H}$ obtained from $\dot{G}$ and $\dot{H}$ by adding a positive edge between each vertex of $\dot{G}$ and each vertex of $\dot{H}$. We call $K_1 \nabla \dot{H}$ the *cone* over $\dot{H}$.

We say that $\dot{G}$ and $\dot{H}$ are *isomorphic* if there is a permutation matrix P such that $A_{\dot{H}} = P^{-1} A_{\dot{G}} P$. In this case we write $\dot{G} \cong \dot{H}$. We say that $\dot{G}$ and $\dot{H}$ are *switching equivalent* if there is a vertex subset $S \subseteq V(\dot{G})$ such that $\dot{H}$ is obtained by reversing the sign of every edge with one end in S and the other in $V(\dot{G}) \setminus S$. In this case we write $\dot{G} \simeq \dot{H}$. Evidently, the underlying graphs of switching equivalent signed graphs are isomorphic. If the vertex labelling is transferred from the common underlying graph, then $\dot{G} \simeq \dot{H}$ holds if and only if there is a diagonal matrix D with ± 1 on diagonal such that $A_{\dot{H}} = D^{-1} A_{\dot{G}} D$. Clearly, isomorphism and switching equivalence preserve the spectrum.

It is known that $\mathbf{j}$ is an eigenvector of $\dot{G}$ if and only if $\dot{G}$ is net-regular, and then $\mathbf{j}$ belongs to the eigenspace of the net-degree, see [28]. The largest absolute value of eigenvalues of $\dot{G}$ is called the *spectral radius* and it does not exceed the spectral radius of the underlying graph G with equality if and only if is $\dot{G} \simeq G$ or $-\dot{G} \simeq G$, see [22].

If $\lambda_1, \lambda_2, \ldots, \lambda_l$ are the (distinct) eigenvalues of a signed graph $\dot{G}$, let P_i be the matrix representing the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\lambda_i)$ with respect to the canonical basis. The spectral decomposition of $A_{\dot{G}}$ is given by $A_{\dot{G}} = \sum\limits_{i=1}^{l} \lambda_i P_i$. The numbers $\beta_i = \|P_i \mathbf{j}\| / \sqrt{n}$ are called the *main angles* of $\dot{G}$. An eigenvalue of $\dot{G}$ is called *main* if the corresponding main angle is nonzero. Equivalently, it is main if there is an associated eigenvector not orthogonal to the main direction $\mathbf{j}$. It is clear that every signed graph has at least one main eigenvalue. Moreover, it has exactly one main eigenvalue if and only if it is net-regular (and then the net-degree is the unique main eigenvalue). Spectral decomposition, main angles and main eigenvalues are defined for every real symmetric matrix.

For basic notions and notation on graphs not given here we refer the reader to [8], [9]. More details on signed graphs can be found in Zaslavsky's papers, see [27], [28].

3. The characteristic polynomial of the join of signed graphs

In this section we express the characteristic polynomial of the join of two signed graphs in terms of their eigenvalues and corresponding main angles. A similar result in the context of graphs can be found in [9].

We start with a more general consideration. Let A be a real symmetric $n_1 \times n_1$ matrix with spectral decomposition $A = \mu_1 P_1 + \mu_2 P_2 + \ldots + \mu_m P_m$, and let B be a real symmetric $n_2 \times n_2$ matrix with spectral decomposition $B = \lambda_1 Q_1 + \lambda_1 Q_2 + \ldots + \lambda_l Q_l$.

We write

$$A^* = J - A, \quad B^* = J - B, \quad q_1(x) = n_1 \sum_{i=1}^{m} \frac{\beta_i^2}{x - \mu_i} \quad \text{and} \quad q_2(x) = n_2 \sum_{j=1}^{l} \frac{\gamma_j^2}{x - \lambda_j},$$

where $\beta_i = \|P_i\mathbf{j}\|/\sqrt{n_1}$ and $\gamma_j = \|Q_j\mathbf{j}\|/\sqrt{n_2}$. Thus, the β_i and λ_j are the main angles of μ_i and λ_j, respectively. For the characteristic polynomial of A^* we have

$$\begin{aligned} P_{A^*}(x) &= \det(xI + A - J) = \det(xI + A) - \mathbf{j}^\top \mathrm{adj(xI + A)}\mathbf{j} \\ &= \det(xI + A) - \mathbf{j}^\top \det(xI + A)(xI + A)^{-1}\mathbf{j} \\ &= \det(xI + A)\left(1 - \sum_{i=1}^{m} \frac{\|P_i\mathbf{j}\|^2}{x + \mu_i}\right). \end{aligned}$$

Similarly,

$$P_{B^*}(x) = \det(xI + B)\left(1 - \sum_{j=1}^{l} \frac{\|Q_j\mathbf{j}\|^2}{x + \lambda_j}\right).$$

Hence,

$$P_{A^*}(x) = (-1)^{n_1} P_A(-x)(1 + q_1(-x)) \quad \text{and} \quad P_{B^*}(x) = (-1)^{n_1} P_B(-x)(1 + q_2(-x)).$$

Note that the matrix $\begin{pmatrix} A & O \\ O & B \end{pmatrix}^*$ has characteristic polynomial

$$\begin{aligned} &(-1)^{n_1+n_2} P_A(-x)P_B(-x)(1 + q_1(-x) + q_2(-x)) \\ &\quad = (-1)^{n_1+n_2}(-P_A(-x)P_B(-x) + P_B(-x)(-1)^{n_1} P_{A^*}(x) + P_A(-x)(-1)^{n_2} P_{B^*}(x)). \end{aligned}$$

Now suppose that A, B are the adjacency matrices of signed graphs $\dot{G}$, $\dot{H}$, and let $M = \begin{pmatrix} A^* & O \\ O & B^* \end{pmatrix}$. Then the adjacency matrix of $\dot{G}\nabla\dot{H}$ is M^*. Replacing A with A^* and B with B^* in the previous equality, we obtain

$$\begin{aligned} &(-1)^{n_1+n_2} P_{M^*}(x) \\ &\quad = -P_{A^*}(-x)P_{B^*}(-x) + P_{B^*}(-x)(-1)^{n_1} P_A(x) + P_{A^*}(-x)(-1)^{n_2} P_B(x) \\ &\quad = -(-1)^{n_1} P_A(x)(1 + q_1(x))(-1)^{n_2} P_B(x)(1 + q_2(x)) \\ &\qquad + (-1)^{n_1+n_2} P_B(x)(1 + q_2(x))P_A(x) + (-1)^{n_1+n_2} P_A(x)(1 + q_1(x))P_B(x) \\ &\quad = (-1)^{n_1+n_2} P_A(x)P_B(x)(-(1 + q_1(x))(1 + q_2(x)) + 1 + q_1(x) + 1 + q_2(x)). \end{aligned}$$

Hence, we arrive at the following result.

Theorem 3.1. *Let $\dot{G}$ be a signed graph of order n_1 with distinct eigenvalues μ_1, $\mu_2, \ldots, \mu_m$ and corresponding main angles $\beta_1, \beta_2, \ldots, \beta_m$ and $\dot{H}$ be a signed graph of order n_2 with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_l$ and corresponding main angles $\gamma_1, \gamma_2, \ldots, \gamma_l$. Then*

$$P_{\dot{G}\nabla\dot{H}}(x) = P_{\dot{G}}(x)P_{\dot{H}}(x)\left(1 - n_1n_2\sum_{i=1}^{m}\sum_{j=1}^{l}\frac{\beta_i^2\gamma_j^2}{(x-\mu_i)(x-\gamma_j)}\right).$$

The following result is a direct consequence of the previous theorem.

Corollary 3.2. *The cone over a signed graph $\dot{H}$ of order n has characteristic polynomial*

$$P_{K_1\nabla\dot{H}}(x) = P_{\dot{H}}(x)\left(x - \sum_{i=1}^{m}\frac{n\beta_i^2}{x-\mu_i}\right),$$

where $\mu_1, \mu_2, \ldots, \mu_m$ are the distinct eigenvalues of $\dot{H}$ and $\beta_1, \beta_2, \ldots, \beta_m$ are the corresponding main angles.

4. Basic results

This section contains several basic results on signed graphs with 2 eigenvalues and their vertex-deleted subgraphs, together with results on signed graphs with 3 eigenvalues, one of which is simple. We start by transferring a well-known result concerning unsigned graphs.

Lemma 4.1. *If $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ are the eigenvalues (with possible repetitions) of a signed graph $\dot{G}$, then $\lambda_2 \leqslant 0$ if and only if $\dot{G}$ is switching equivalent to a complete multipartite graph with possible isolated vertices. In particular, $\lambda_2 < 0$ holds if and only if $\dot{G}$ is switching equivalent to the complete graph.*

P r o o f. Observe that if $\dot{G}$ is not switching equivalent to its underlying graph G, then $\dot{G}$ contains at least one negative cycle as an induced subgraph, see [27], [28]. Since the second largest eigenvalue of every negative cycle is greater than 0, see [20], Section 4, we conclude (using eigenvalue interlacing) that $\lambda_2 > 0$. Thus, if $\lambda_2 \leqslant 0$, then $\dot{G}$ is switching equivalent to G, and the first part of the proof follows by the well-known result which states that the second largest eigenvalue of a graph G is nonpositive if and only if G is complete multipartite up to isolated vertices, see [8], Theorem 6.7.

In the particular case when $\lambda_2 < 0$ we see immediately that $\dot{G}$ is connected, and then it must be complete; for otherwise, it would contain $2K_1$ as an induced subgraph (with $\lambda_2 \geqslant \lambda_2(2K_1) = 0$). The opposite implication follows directly. $\square$

We proceed with some consequences of the previous result.

Corollary 4.2. *A connected signed graph $\dot{G}$ has 2 eigenvalues, one of which is simple, if and only if $\dot{G}$ is switching equivalent to a complete graph or its negation.*

Proof. This result is a direct consequence of the latter part of Lemma 4.1. □

In the following two corollaries we consider vertex-deleted subgraphs of signed graphs with 2 eigenvalues.

Corollary 4.3. *Let $\dot{G}$ be a connected signed graph of order $n \geqslant 3$, and let $\dot{H}$ be a vertex-deleted subgraph of $\dot{G}$. Then $\dot{G}$ and $\dot{H}$ both have 2 eigenvalues if and only if either $\dot{G} \simeq K_n$ or $\dot{G} \simeq -K_n$.*

Proof. If $\dot{G}$ is switching equivalent to K_n or $-K_n$, then $\dot{G}$ has 2 eigenvalues, and every vertex-deleted subgraph is switching equivalent to K_{n-1} or $-K_{n-1}$. Hence $\dot{H}$ also has 2 eigenvalues.

For the converse, suppose that $\dot{G}$ is not switching equivalent to K_n or $-K_n$. By Corollary 4.2, the eigenvalues of $\dot{G}$, say μ and λ, are nonsimple, and therefore the same eigenvalues appear in the spectrum of $\dot{H}$. Moreover, since both have 2 eigenvalues, $\dot{G}$ and $\dot{H}$ are regular and have a common vertex degree (equal to $-\mu\lambda$, which follows by considering their minimal polynomials), which is impossible. □

Corollary 4.4. *Every vertex-deleted subgraph of a connected signed graph $\dot{G}$ with 2 eigenvalues has 3 eigenvalues if and only if $\dot{G}$ is not switching equivalent to K_n or $-K_n$.*

Proof. By eigenvalue interlacing, a vertex-deleted subgraph has at most 3 eigenvalues, and so the assertion follows from Corollary 4.3. □

Clearly, every vertex-deleted subgraph from the previous corollary is nonregular. Accordingly, this corollary provides an easy first construction of nonregular signed graphs with 3 eigenvalues.

In the following two results we consider specified signed graphs with 3 eigenvalues. Similar results in the framework of graphs can be found in Van Dam's paper, see [25].

Lemma 4.5. *If $\dot{G}$ is connected with 3 eigenvalues, one of which is simple and nonintegral, then $\dot{G}$ is switching equivalent to a complete bipartite graph.*

Proof. Denote by θ a simple nonintegral eigenvalue and by μ, λ the remaining two. Without loss of generality, we may assume that $\theta \geqslant 0$, since otherwise (along with the observation that a complete bipartite graph is switching equivalent to its negation) we can consider $-\dot{G}$. At least one of μ, λ, say λ, is also simple and nonintegral. If μ is simple, we have $\dot{G} \simeq K_{1,2}$, and so we assume further that μ is nonsimple. We have $\theta, \lambda = \frac{1}{2}(a \pm \sqrt{b})$, for $a, b \in \mathbb{Z}$ and $b \neq 0$.

If $a = 0$, then $\mu = 0$, and by Lemma 4.1, $\dot{G}$ is switching equivalent to a complete multipartite graph, moreover since λ is simple, to a complete bipartite graph.

In what follows, we eliminate the possibility that $a \neq 0$. We consider only $a > 0$, since the other case is analogous. So, for $a > 0$, since $\operatorname{tr}(A_{\dot{G}}) = 0$, we have $a = (2 - n)\mu$, where n is the number of vertices of $\dot{G}$. It follows that μ is negative and integral.

For $\mu = -1$ we have $\lambda = \frac{1}{2}(n - 2 - \sqrt{b})$. If $\lambda < -1$, we have $n < \sqrt{b}$, i.e., $\theta > n - 1$, a contradiction. If $\lambda > -1$, then $\dot{G}$ is complete, for otherwise it contains the path with 3 vertices as an induced subgraph (which is impossible since then the least eigenvalue of $\dot{G}$ would be at most $-\sqrt{2}$). At the same time, since μ has multiplicity $n - 2$, $\dot{G}$ contains a subgraph with 2 vertices that avoids -1 in the spectrum, necessarily $2K_1$ – a contradiction. When $\mu = -2$, we know from [15] that $\dot{G}$ has at most 4 vertices, and we resolve this situation by inspection. For $\mu \leqslant -3$ we have $a \geqslant 3(n - 2)$ giving $\theta > n$, the final contradiction. □

Lemma 4.6. *Let $\dot{G}$ be a nonintegral signed graph with spectrum $[\theta, \mu^m, \lambda^l]$, where $1 + m + l = n$. If $\dot{G}$ is not switching equivalent to a complete bipartite graph, then $m = l$ and one of the following holds:*

(i) $\theta = 0$, $\mu, \lambda = \pm\sqrt{b}/2$ *for* $b \equiv 0 \pmod 4$,

(ii) $\theta = \frac{1}{2}(1 - n)$, $\mu, \lambda = \frac{1}{2}(1 \pm \sqrt{b})$ *for* $b \equiv 1 \pmod 4$ *or*

(iii) $\theta = \frac{1}{2}(n - 1)$, $\mu, \lambda = \frac{1}{2}(-1 \pm \sqrt{b})$ *for* $b \equiv 1 \pmod 4$.

Proof. Since $\dot{G}$ is not switching equivalent to a complete bipartite graph, θ is integral by Lemma 4.5; then $m = l = \frac{1}{2}(n-1)$ and $\mu, \lambda = \frac{1}{2}(a \pm \sqrt{b})$ for $a, b \in \mathbb{Z}$ and $b \neq 0$. It follows that $\theta + \frac{1}{2}(n - 1)a = 0$, which gives $a \in \{0, 1, -1\}$, since otherwise we would have $|\theta| \geqslant n - 1$, which is impossible.

For $a = 0$ we obtain (i), where $b \equiv 0 \pmod 4$ since $\mu\lambda$ is integral (as we see by considering the coefficients of the minimal polynomial of $A_{\dot{G}}$). In a very similar way, we find that the possibilities $a = 1$ and $a = -1$ lead to (ii) and (iii), respectively. □

5. Main results

We first consider the join of two connected net-regular signed graphs both with 2 nonsimple eigenvalues. We show that the join cannot have 2 eigenvalues and determine when it has 3 eigenvalues.

Theorem 5.1. *Let $\dot{G}$ and $\dot{H}$ be connected net-regular signed graphs such that neither of them is switching equivalent to a complete graph or its negation. Assume* *that $\dot G$ and $\dot H$ have spectra $[\varrho^r, \mu^m]$ and $[\nu^v, \lambda^l]$, respectively (with $r+m=n_1$ and $v+l=n_2$), and let ϱ, ν be the main eigenvalues.*

(i) *$\dot G \nabla \dot H$ cannot have 2 eigenvalues.*

(ii) *If $\dot G$ and $\dot H$ either do not have a common eigenvalue or share the same set of eigenvalues, then $\dot G \nabla \dot H$ cannot have 3 eigenvalues.*

(iii) *If $\dot G$ and $\dot H$ have exactly one eigenvalue in common, then $\dot G \nabla \dot H$ has 3 eigenvalues if and only if $\varrho=\nu$, $2\varrho=\mu+\lambda$ and $\mu\lambda=\varrho^2-n_1n_2$. In this situation, the spectrum of $\dot G \nabla \dot H$ is $[((\mu+\lambda)/2)^{r+v-2},\ \mu^{m+1},\ \lambda^{l+1}]$.*

Proof. By Theorem 3.1, we have

$$
\begin{aligned}
(5.1)\qquad P_{\dot G\nabla\dot H}(x) &= (x-\varrho)^r(x-\mu)^m(x-\nu)^v(x-\lambda)^l\Big(1-n_1n_2\frac{1}{(x-\varrho)(x-\nu)}\Big)\\
&= (x-\varrho)^{r-1}(x-\mu)^m(x-\nu)^{v-1}(x-\lambda)^l((x-\varrho)(x-\nu)-n_1n_2).
\end{aligned}
$$

By Corollary 4.2, we have $r,m,v,l \geqslant 2$, and so none of the first four terms of the right-hand side vanishes.

For (i) we have either $\varrho=\nu$, $\mu=\lambda$ or $\varrho=\lambda$, $\mu=\nu$. In the former case we also have $(x-\varrho)^2-n_1n_2=(x-\varrho)(x-\mu)$ or $(x-\varrho)^2-n_1n_2=(x-\mu)^2$. In both situations, by equating coefficients of x, we find that $\varrho=\mu$, a contradiction. In the latter case we also have $(x-\varrho)(x-\nu)-n_1n_2=(x-\varrho)^2$ or $(x-\varrho)(x-\nu)-n_1n_2=(x-\nu)^2$, which both lead to the forbidden scenario $\varrho=\nu$, and we are done.

For (ii), if $\dot G$ and $\dot H$ do not have a common eigenvalue, we see immediately that $\dot G \nabla \dot H$ has at least 4 eigenvalues, namely ϱ, μ, ν and λ.

Assume now that they share the same eigenvalue set. As in the proof of (i), we have either $\varrho=\nu$, $\mu=\lambda$ or $\varrho=\lambda$, $mu=\nu$.

In the former case we also have $(x-\varrho)^2-n_1n_2=(x-\mu)(x-\theta)$ or $(x-\varrho)^2-n_1n_2=(x-\varrho)(x-\theta)$ for some $\theta\notin\{\varrho,\mu\}$. The second equality immediately gives $\theta=\varrho$. The first equality implies $2\varrho=\theta+\mu$ (coefficients of x) and $\theta\mu=\varrho^2-n_1n_2$ (constant terms). The last two equalities give

$$(5.2)\qquad (\varrho-\mu)^2=n_1n_2,$$

but since $\dot G$ and $\dot H$ are not switching equivalent to complete graphs or their negations, we have $|\varrho|, |\mu| \geqslant 2$ (which follows easily by observing that the net-degree ϱ is integral, and then μ is also integral). This implies that $\varrho-\mu \leqslant -\varrho\mu$. Since $-\varrho\mu$ is the vertex degree of both $\dot G$ and $\dot H$, we also have $-\varrho\mu<n_i$, $i\in\{1,2\}$, which contradicts (5.2).

In the latter case we also have $(x-\varrho)(x-\nu)-n_1n_2=(x-\varrho)(x-\theta)$ or $(x-\varrho)(x-\nu)-n_1n_2=(x-\nu)(x-\theta)$ for some $\theta\notin\{\varrho,\nu\}$, but the last restriction on θ makes both equalities impossible.

For (iii), the possibility $\varrho = \lambda$ is eliminated as in the previous part of the proof (i.e., by equating coefficients of x and those of x^0).

By setting $\varrho = \nu$, we arrive at either $(x-\varrho)^2 - n_1n_2 = (x-\varrho)(x-\mu)$, $(x-\varrho)^2 - n_1n_2 = (x-\varrho)(x-\lambda)$ or $(x-\varrho)^2 - n_1n_2 = (x-\mu)(x-\lambda)$. The first two possibilities are eliminated as before. The last one gives $2\varrho = \mu+\lambda$ and $\mu\lambda = \varrho^2 - n_1n_2$.

Conversely, if $\varrho = \nu$, $2\varrho = \mu+\lambda$ and $\mu\lambda = \varrho^2 - n_1n_2$, then equality (5.1) reduces to $P_{\dot{G}\nabla\dot{H}}(x) = (x-\varrho)^{r+v-2}(x-\mu)^{m+1}(x-\lambda)^{l+1}$, which completes the proof. □

Observe that in part (iii) of the previous theorem, $\dot{G} \nabla \dot{H}$ has vertex degrees $n_1 - \varrho\lambda$ and $n_2 - \varrho\mu$, and is regular if $n_1 - n_2 = \varrho(\lambda-\mu)$. It is net-regular for $n_1 = n_2$ (with net-degree $\varrho + n_1$). At this moment we are unable to give an example, notwithstanding several numerical experiments.

Using Theorem 3.1, we find easily that if $\dot{H}$ is a connected signed graph with 2 non-simple eigenvalues, then $(n_1K_1) \nabla \dot{H}$ has more than 2 eigenvalues. In what follows we determine whether $(n_1K_1) \nabla \dot{H}$ has 3 eigenvalues.

Theorem 5.2. *Let $\dot{H}$ be a connected signed graph with spectrum $[\nu^v, \lambda^l]$ such that $v, l \geqslant 2$ and $v + l = n_2$.*

(i) *If $n_1 \geqslant 2$, then $(n_1K_1) \nabla \dot{H}$ has more than 3 eigenvalues.*

(ii) *$K_1 \nabla \dot{H}$ has 3 eigenvalues if and only if $\dot{H}$ has exactly one main eigenvalue and if this eigenvalue is ν, then $n_2 = \lambda(\lambda-\nu)$. In this situation, the spectrum of $K_1 \nabla \dot{H}$ is $[\nu-\lambda,\ \nu^{v-1},\ \lambda^{l+1}]$.*

Proof. By Theorem 3.1, we have

$$(5.3)\quad P_{(n_1K_1)\nabla\dot{H}}(x) = x^{n_1-1}(x-\nu)^{v-1}(x-\lambda)^{l-1} \times (x(x-\nu)(x-\lambda) - n_1n_2(\gamma^2(x-\lambda) + (1-\gamma^2)(x-\nu))),$$

where γ is the main angle of ν.

For (i), suppose by way of contradiction that $(n_1K_1)\nabla\dot{H}$ has at most 3 eigenvalues. Equality (5.3) shows that the roots of

$$(5.4)\quad x(x-\nu)(x-\lambda) - n_1n_2(\gamma^2(x-\lambda) + (1-\gamma^2)(x-\nu)) = 0$$

must belong to $\{0, \nu, \lambda\}$. As in the previous theorem, after straightforward algebraic computation, we easily eliminate all the possibilities that arise.

Consider (ii) and assume first that $K_1 \nabla \dot{H}$ has exactly 3 eigenvalues. It follows that exactly one root of (5.4) (with $n_1 = 1$) differs from ν, λ. (There cannot be two such roots, by eigenvalue interlacing.) If ν is the unique main eigenvalue of $\dot{H}$, then $\gamma^2 = 1$, and (5.4) reduces to a quadratic equation. If its roots are θ and ν, we obtain $n_2 = 0$, which is impossible. If they are θ and λ, then we have $\theta = \nu - \lambda$ and $-\theta\lambda = n_2$, which leads to $\lambda(\lambda - \nu) = n_2$.

If now both eigenvalues of $\dot{H}$ are main, then the case in which the roots of (5.4) are θ, ν and λ is eliminated as before (it leads to $n_2 = 0$). The case in which θ and λ (of multiplicity 2) are the roots in question leads to the system

$$\theta + 2\lambda = \lambda + \nu, \quad \lambda(2\theta + \lambda) = \lambda\nu - n_2, \quad -\theta\lambda^2 = n_2(\gamma^2\lambda + (1 - \gamma^2)\nu),$$

which arises by equating the coefficients of x^2, x and x^0. Solving this system, we obtain $\theta = \nu - \lambda$, $n_2 = \lambda(\lambda - \nu)$ and $\gamma^2 = 1$. The last equality means in fact that λ is nonmain, and reduces this case to the case with one main eigenvalue. Of course, the case in which θ and ν (of multiplicity 2) are the corresponding roots is analogous.

Conversely, if ν is the unique main eigenvalue of $\dot{H}$ and $n_2 = \lambda(\lambda - \nu)$, by inserting $n_1 = 1$, $\gamma^2 = 1$ and $n_2 = \lambda(\lambda - \nu)$ into (5.3), we get

$$P_{K_1 \nabla \dot{H}}(x) = (x - \nu + \lambda)(x - \nu)^{v-1}(x - \lambda)^{l+1},$$

and we are done. □

Evidently, $K_1 \nabla \dot{H}$ is neither regular or net-regular. Here is a simple corollary.

Corollary 5.3. *Under the assumptions on $\dot{H}$ formulated in Theorem 5.2, if $K_1 \nabla \dot{H}$ has 3 eigenvalues, then the eigenvalues of $\dot{H}$ and the eigenvalues of $K_1 \nabla \dot{H}$ are integers.*

Proof. By Theorem 5.2 (ii), $\dot{H}$ has exactly one main eigenvalue; then this is the net-degree, hence an integer. Evidently, the other eigenvalue must also be integral. Consequently, the eigenvalues of $K_1 \nabla \dot{H}$ (expressed in terms of those of $\dot{H}$ as in the corresponding theorem) are also integral. □

We continue with an example, in fact an infinite family of signed graphs $\dot{H}$ with 2 eigenvalues, such that $K_1 \nabla \dot{H}$ has 3 eigenvalues.

Example 5.4. It has been proved in [23] that an inhomogeneous net-regular complete bipartite signed graph $\dot{H}$ with $2n$ vertices has 2 eigenvalues if and only if $A_{\dot{H}^+}$ is the incidence matrix of a symmetric balanced incomplete design with parameters $(4(r - l), r, l)$. Now, it is a matter of routine (but the reader can consult the same reference once again) to see that the eigenvalues of $\dot{H}$ are $\pm\sqrt{r - l}$. Further, the parameters of a Menon design (given by $(4s^2, s(2s + 1), s(s + 1))$ for $s \in \mathbb{Z} \setminus \{0\}$) have the required form, and moreover, the eigenvalues of $\dot{H}$ ($\pm 2s$) satisfy the assumption of Theorem 5.2 (ii). Therefore, in this case $K_1 \nabla \dot{H}$ has the spectrum $[4s, (2s)^{n-1}, (-2s)^{n+1}]$. For $s = 1$ we obtain the cone over the signed graph obtained by reversing the sign of every edge in a perfect matching of $K_{4,4}$.

We proceed with cones over net-regular signed graphs with 3 eigenvalues, such that the net-degree is a simple eigenvalue.

Theorem 5.5. *Let $\dot{H}$ be a net-regular signed graph with spectrum $[\nu, \mu^m, \lambda^l]$ such that $m, l \geqslant 2$, $1 + m + l = n$, ν is equal to the net-degree and $\mu > \lambda$.*

(i) *If $\nu \notin (\lambda, \mu)$, then $K_1 \nabla \dot{H}$ has at least 3 eigenvalues and if their number is 3, then either $\mu(\mu - \nu) = n$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n$ and $\nu > \mu > 0 > \lambda$.*

(ii) *If $\nu \in (\lambda, \mu)$ and $K_1 \nabla \dot{H}$ has at most 3 eigenvalues, then $\mu(\mu - \lambda) = n$ or $\lambda(\lambda - \mu) = n$, with 2 eigenvalues precisely if both equalities are satisfied, equivalently $\nu = \mu + \lambda$.*

Conversely, if $\mu(\mu - \nu) = n$ or $\lambda(\lambda - \nu) = n$, then $K_1 \nabla \dot{H}$ has 2 eigenvalues if $\nu = \mu + \lambda$, and 3 eigenvalues otherwise.

P r o o f. Since ν is the unique main eigenvalue of $\dot{H}$, by Corollary 3.2, we have

$$P_{K_1 \nabla \dot{H}}(x) = (x - \mu)^m (x - \lambda)^l (x^2 - \nu x - n). \tag{5.5}$$

Consider (i). If $K_1 \nabla \dot{H}$ has 2 eigenvalues, then $x^2 - \nu x - n = (x - \mu)(x - \lambda)$, giving $\nu = \mu + \lambda$, which is impossible under the assumption on the location of ν.

If $K_1 \nabla \dot{H}$ has 3 eigenvalues, then either $x^2 - \nu x - n = (x - \theta)(x - \mu)$ or $x^2 - \nu x - n = (x - \theta)(x - \lambda)$ for $\theta \notin \{\mu, \lambda\}$. We consider the former case, while the latter one follows analogously. We have $\theta + \mu = \nu$ and $\theta\mu = -n$, which yields $\mu(\mu - \nu) = n$, and it remains to prove the given chain of inequalities. We have 2 possibilities: $\nu < \lambda$ or $\nu > \mu$. For $\nu < \lambda$ we immediately get $\mu > 0$ (because $\theta = \nu - \mu < 0$ and $\theta\mu = -n$).

When $\lambda \geqslant 0$, $K_1 \nabla \dot{H}$ has exactly one negative eigenvalue (namely θ), and by Lemma 4.1 it is switching equivalent to the negation of a complete multipartite graph with 3 eigenvalues. This is impossible since a complete multipartite graph with 3 eigenvalues is not a cone unless it is a star (see [9], page 47), but then $m = 1$. For $\nu > \mu$, in a similar way we obtain $\mu < 0$, but this is impossible as before by the latter part of Lemma 4.1.

Now consider (ii). As before, from $x^2 - \nu x - n = (x - \theta)(x - \mu)$ we obtain $\theta = \nu - \mu$ and $\mu(\mu - \nu) = n$. If $K_1 \nabla \dot{H}$ has 2 eigenvalues, then $\theta = \mu$ or $\theta = \lambda$. For the first possibility we have $\nu = 2\mu$, which contradicts the location of ν. For the second possibility we find $\nu = \mu + \lambda$. The case $x^2 - \nu x - n = (x - \theta)(x - \lambda)$ is again analogous, and the case of 2 eigenvalues again implies $\nu = \mu + \lambda$, which completes this part of the proof.

Conversely, if $\mu(\mu - \nu) = n$, then $x^2 - \nu x - n = (x - (\nu - \mu))(x - \mu)$, which means that $K_1 \nabla \dot{H}$ has at most 3 eigenvalues with 2 eigenvalues exactly if $\nu - \mu = \lambda$, and similarly for $\lambda(\lambda - \nu) = n$. $\square$

In the previous theorem, the spectrum of $K_1 \nabla \dot{H}$ depends on which of the two possible equalities is attained, but it can easily be computed from (5.5). Evidently, $K_1 \nabla \dot{H}$ is regular if and only if $\dot{H}$ is complete. Since $\dot{H}$ is net-regular, $K_1 \nabla \dot{H}$ is never net-regular.

Example 5.6. Examples of signed graphs for Theorem 5.4 with $\nu = \mu + \lambda$ can easily be found. Namely, if $\dot{H}$ is a complete signed graph with n vertices whose negative edges induce a strongly regular graph with parameters $(n, 2f, e, f)$, then the eigenvalues of $\dot{H}$ (which are the Seidel eigenvalues of the corresponding strongly regular graph) are $\nu = n - 2f - 1$ and $\mu, \lambda = f - e \pm \sqrt{(e-f)^2 + 4f} - 1$. Observe that ν, μ and λ are distinct, which follows by examining the parameters, but also by Corollary 4.3. Using the relation between the parameters of a strongly regular graph, which, for example, can be found in [21], Theorem 3.4.6, we confirm that in our case $\nu = \mu + \lambda$ holds. Thus, by Theorem 5.4 (ii), $K_1 \nabla \dot{H}$ has 2 eigenvalues. The reader will recognize that this construction produces signed graphs that arise from the so-called regular two-graph equivalence class, see [19].

6. More examples

In the first of the following two subsections we propose a definition of semi-complements of signed graphs. As an immediate application, we use them to construct signed graphs with a small number of eigenvalues which arise from 3-class association schemes. In the second subsection, we consider bipartite signed graphs with at most 3 eigenvalues. We believe that the reader is familiar with association schemes and related definitions. In fact, all terminology and undefined notions can be found in [7].

6.1. Semi-complements and 3-class association schemes. Let $\dot{G}$ be a signed graph of order n with adjacency matrix $A = (a_{ij})$, and let π be the cyclic permutation $-1 \mapsto 1 \mapsto 0 \mapsto -1$. We define the *semi-complements* of $\dot{G}$ as the signed graphs $\dot{G}'$, $\dot{G}''$ with adjacency matrices $B = (b_{ij})$, $C = (c_{ij})$, respectively, where

$$b_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \pi(a_{ij}) & \text{if } i \neq j \end{cases} \quad \text{and} \quad c_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \pi(b_{ij}) & \text{if } i \neq j. \end{cases}$$

Accordingly, $A = A_1 - A_2$, $B = A_2 - A_3$ and $C = A_3 - A_1$, where A_i is the adjacency matrix of a graph G_i $(1 \leqslant i \leqslant 3)$, $A_1 + A_2 + A_3 = J - I$ and

$$E(K_n) = E(G_1) \dot{\cup} E(G_2) \dot{\cup} E(G_3). \tag{6.1}$$

It is convenient to write $\dot{G}_{1,2}$, $\dot{G}_{2,3}$, $\dot{G}_{3,1}$ for $\dot{G}$, $\dot{G}'$, $\dot{G}''$, respectively. If $G_1 \cong G_2 \cong G_3$, then the tripartition (6.1) is said to be a 3-decomposition of K_n, as in [9], Section 9.4.

We note that G_1, G_2, G_3 determine a 3-class association scheme if and only if each G_i is regular of positive degree and each A_iA_j is a linear combination of I, A_1, A_2, A_3. In this situation we see (by taking transposes) that $A_iA_j = A_jA_i$ and so $\langle I, A_1, A_2, A_3\rangle$ is a 4-dimensional commutative $\mathbb{R}$-algebra $\mathcal{A}$ (called a *Bose-Mesner algebra*). For any matrix $M \in \mathcal{A}$, the matrices I, M, M^2, M^3, M^4 are linearly dependent, and so M has at most 4 eigenvalues. In particular, this is true for the adjacency matrices of signed graphs $\dot{G}_{i,j}$, and below we give examples with 3 and 4 eigenvalues.

It is known that if G_1, G_2, G_3 are strongly regular, then they determine a 3-class association scheme (see [26], page 78) and all are of Latin square or negative Latin square type (see [26], page 76). Previously it was shown in [16] that if G is a strongly regular graph with parameters (n, r, e, f) and $G_i \cong G$ $(1 \leqslant i \leqslant 3)$, then there exists $k \in \mathbb{N}$ such that (with a consistent choice of sign)

$$n = (3k \pm 1)^2, \quad r = 3k^2 \pm 2k, \quad e = k^2 - 1, \quad f = k^2 \pm k.$$

Example 6.1. Consider the 3-decomposition $E(K_7) = E(G_1)\dot{\cup}E(G_2)\dot{\cup}E(G_3)$, where G_1, G_2, G_3 are the 7-cycles 12345671, 13572461, 14736251, respectively (see [9], Remark 9.4.3). It is straightforward to check that this 3-decompositon gives rise to an association scheme by expressing each A_iA_j as a linear combination of I, A_1, A_2, A_3. One can find the eigenvalues of $G_{1,2}$ explicitly using the eigenvalues of the 7-cycles, but to demonstrate that there are 4 distinct eigenvalues one can make use of the relations $A_1^2 = A_2 + 2I$, $A_2^2 = A_3 + 2I$, $A_1A_2 = A_1 + A_3$ to show that A $(= A_1 - A_2)$ satisfies $x(x^3 - 7x + 7) = 0$. Since $A\mathbf{j} = \mathbf{0}$ and $x^3 - 7x + 7$ is irreducible, the eigenvalues of $G_{1,2}$ are 0 and the three (conjugate) roots of $x^3 - 7x + 7$.

If instead we take $E(K_7) = E(G_1)\dot{\cup}E(G_2)\dot{\cup}E(G_3)$, where G_1, G_2, G_3 are the 7-cycles 12345671, 14275361, 13746251, then $A_2A_3 \neq A_3A_2$ (as noted in [9], Remark 9.4.3), and so in this case the 3-decomposition does not determine an association scheme.

Example 6.2. Here we consider the 3-decomposition of K_n into strongly regular graphs with parameters

$$n = (3k + 1)^2, \quad r = 3k^2 + 2k, \quad e = k^2 - 1, \quad f = k^2 + k$$

(see [16] or [9], Example 9.4.4). The nonmain eigenvalues of G_i are $\varrho_1 = k$ and $\varrho_2 = -2k - 1$. From [9], Remark 9.4.3 we have

$$\mathcal{E}_{A_1}(\varrho_2) = \mathcal{E}_{A_2}(\varrho_1) \cap \mathcal{E}_{A_3}(\varrho_1), \quad \mathcal{E}_{A_2}(\varrho_2) = \mathcal{E}_{A_3}(\varrho_1) \cap \mathcal{E}_{A_1}(\varrho_1),$$
$$\mathcal{E}_{A_3}(\varrho_2) = \mathcal{E}_{A_1}(\varrho_1) \cap \mathcal{E}_{A_2}(\varrho_1) \quad \text{and} \quad \mathbb{R}^n = \langle \mathbf{j}\rangle \oplus \mathcal{E}_{A_1}(\varrho_2) \oplus \mathcal{E}_{A_2}(\varrho_2) \oplus \mathcal{E}_{A_3}(\varrho_2).$$

We know that $(A_1 - A_2)\mathbf{j} = \mathbf{0}$. Now suppose that θ is a nonzero eigenvalue of $A_1 - A_2$ with an eigenvector $\mathbf{x}$ orthogonal to $\mathbf{j}$. Then $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3$, where $\mathbf{x}_i \in \mathcal{E}_{A_i}(\varrho_2)$ $(1 \leqslant i \leqslant 3)$. We have

$$A_1\mathbf{x}_1 = \varrho_2\mathbf{x}_1, \quad A_1\mathbf{x}_2 = \varrho_1\mathbf{x}_2, \quad A_1\mathbf{x}_3 = \varrho_1\mathbf{x}_3$$

and

$$A_2\mathbf{x}_1 = \varrho_1\mathbf{x}_1, \quad A_2\mathbf{x}_2 = \varrho_2\mathbf{x}_2, \quad A_2\mathbf{x}_3 = \varrho_1\mathbf{x}_3.$$

Hence, $\theta\mathbf{x}_1 + \theta\mathbf{x}_2 + \theta\mathbf{x}_3 = (\varrho_2 - \varrho_1)\mathbf{x}_1 + (\varrho_1 - \varrho_2)\mathbf{x}_2 + 0\mathbf{x}_3$, and so either $\mathbf{x}_3 = \mathbf{0}$, $\mathbf{x}_2 = \mathbf{0}$, $\theta = \varrho_2 - \varrho_1$ or $\mathbf{x}_3 = \mathbf{0}$, $\mathbf{x}_1 = \mathbf{0}$, $\theta = \varrho_1 - \varrho_2$. Each eigenspace $\mathcal{E}_{A_i}(\varrho_2)$ $(1 \leqslant i \leqslant 3)$ has dimension $\frac{1}{3}(n-1) = 3k^2 + 2k$, and so the eigenvalues of $\dot{G}_{1,2}$ are 0, $\varrho_1 - \varrho_2 = 3k+1$ and $\varrho_2 - \varrho_1 = -3k-1$ with multiplicities $3k^2+2k+1$, $3k^2+2k$ and $3k^2+2k$, respectively. When $k=1$, the corresponding tripartition is the well-known 3-decomposition of K_{16} into three copies of the Clebsch graph.

Motivated by the notion of a self-complementary graph, we investigate the situation in which $\dot{G}_{1,2} \cong \dot{G}_{2,3}$. In this case there exists a permutation matrix P such that $P^{-1}A_1P = A_2$ and $P^{-1}A_2P = A_3$. Then

$$\begin{aligned} P^{-1}A_3P = P^{-1}(J - I - A_1 - A_2)P &= J - I - P^{-1}A_1P - P^{-1}A_2P \\ &= J - I - A_2 - A_3 = A_1. \end{aligned}$$

Hence, $P^{-1}(A_3 - A_1)P = A_1 - A_2$, and so $\dot{G}_{1,2} \cong \dot{G}_{2,3} \cong \dot{G}_{3,1}$. We say that the isomorphic signed graphs $\dot{G}_{1,2}$, $\dot{G}_{2,3}$, $\dot{G}_{3,1}$ are *semi-complementary.* For a simple example we may take

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

An infinite family of examples arises in [9], Example 9.4.3 as a particular case of Example 6.2 above. These are illustrations of the following result.

Theorem 6.3. *Suppose that K_n has a 3-decomposition into (isomorphic) connected regular graphs G_1, G_2, G_3 which determine a 3-class association scheme. Then each of the signed graphs $\dot{G}_{1,2}$, $\dot{G}_{2,3}$, $\dot{G}_{3,1}$ has just 3 or 4 eigenvalues. If $\dot{G}_{1,2}$, $\dot{G}_{2,3}$, $\dot{G}_{3,1}$ are semi-complementary with just 3 eigenvalues, then*

- ▷ *these eigenvalues are 0, λ, $-\lambda$ for some $\lambda \in \mathbb{R}$, and*
- ▷ *the graphs G_1, G_2, G_3 are strongly regular of Latin square or negative Latin square type.*

P r o o f. Let G_i have the adjacency matrix A_i $(1 \leqslant i \leqslant 3)$, and let $A = A_1 - A_2$. We have already noted that A has at most 4 eigenvalues. Since one of them is 0, A has at least 3 eigenvalues.

Now suppose that $\dot{G}_{1,2}$, $\dot{G}_{2,3}$, $\dot{G}_{3,1}$ are semi-complementary, with just 3 eigenvalues 0, λ, μ. Let r be the vertex degree in each G_i. In view of the simultaneous diagonalizability of A_1, A_2, A_3, we may write

$$A_1\mathbf{x}_i = \lambda_i\mathbf{x}_i, \quad A_2\mathbf{x}_i = \mu_i\mathbf{x}_i, \quad A_3\mathbf{x}_i = \nu_i\mathbf{x}_i \quad (1 \leqslant i \leqslant n),$$

where the common eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are linearly independent and we take $\mathbf{x}_1 = \mathbf{j}$, $\lambda_1 = \mu_1 = \nu_1 = r$. Note that $\lambda_i + \mu_i + \nu_i = -1$ $(i \geqslant 2)$, the μ_j $(j > 1)$ are the λ_i $(i > 1)$ in some order and the ν_k $(k > 1)$ are the λ_i $(i > 1)$ in some order.

Suppose by way of contradiction that each G_i has 4 distinct eigenvalues r, a, b, c. If $a = \lambda_i$, then $i \neq 1$ and since $\lambda_i - \mu_i$, $\mu_i - \nu_i \in \{0, \lambda, \mu\}$, there are initially 9 possibilities for $(\lambda_i, \mu_i, \nu_i)$, namely:

$$\begin{array}{llll@{\qquad}llll}
\text{(a)} & a & a & a, & \text{(f)} & a & a-\mu & a-\mu-\lambda, \\
\text{(b)} & a & a & a-\lambda, & \text{(g)} & a & a-\lambda & a-\lambda-\mu, \\
\text{(c)} & a & a-\lambda & a-\lambda, & \text{(h)} & a & a-\lambda & a-2\lambda, \\
\text{(d)} & a & a & a-\mu, & \text{(j)} & a & a-\mu & a-2\mu. \\
\text{(e)} & a & a-\mu & a-\mu, & & & &
\end{array}$$

Cases (a), (h) and (j) are ruled out immediately because there $3a = -1$ or $3(a-\lambda) = -1$ or $3(a-\mu) = -1$, contradicting the fact that a, $a-\lambda$ and $a-\mu$ are algebraic integers. In cases (b)–(e) we have $\lambda + \mu = 0$, because $\nu_i - \lambda_i \in \{0, \lambda, \mu\}$. If $\lambda + \mu \neq 0$, then cases (f) and (g) remain and yield $-\lambda - \mu \in \{\lambda, \mu\}$, whence $2\lambda + \mu = 0$ or $\lambda + 2\mu = 0$.

We now distinguish two possibilities for $\lambda + \mu$.

The case $\lambda + \mu = 0$. In cases (b) and (c), we have $3a - \lambda = -1$ and $3a - 2\lambda = -1$, respectively. We may take $a - \lambda = b = \lambda_j$. The possibilities for $(\lambda_j, \mu_j, \nu_j)$ are:

$$\begin{array}{llll@{\qquad}llll}
\text{(i)} & b & b & b-\lambda, & \text{(iii)} & b & b & b-\mu, \\
\text{(ii)} & b & b-\lambda & b-\lambda, & \text{(iv)} & b & b-\mu & b-\mu.
\end{array}$$

In subcase (i) we have $3a - 4\lambda = -1$, leading to the contradiction $\lambda = 0$. In subcase (ii) we have $3a - 5\lambda = -1$, leading to the same contradiction. In subcase (iii) we have the consistent triples

$$\begin{array}{ccc}
a & a-\lambda & a-\lambda \\
b & b & b-\mu
\end{array}$$

and we consider the fourth eigenvalue. If $c = \lambda_k$, then the possibilities for $(\lambda_k, \mu_k, \mu_k)$ are:

$$\begin{array}{cccccc} c & c & c-\lambda & c & c & c-\mu \\ c & c-\lambda & c-\lambda & c & c-\mu & c-\mu \end{array}$$

In the first case we have $3c - \lambda = -1$, leading to the contradiction $3(2c - a) = -1$. In the second case we have $3c - 2\lambda = -1$, leading to the contradiction $a = c$. In the third case we have $3c - \mu = -1$, leading to the contradiction $b = c$. In the fourth case we have $3c - 2\mu = -1$, leading to the contradiction $3(2b - c) = -1$. In subcase (iv) we have the consistent triples

$$\begin{array}{ccc} a & a & a-\lambda \\ b & b-\mu & b-\mu \end{array}$$

and we consider the fourth eigenvalue as before. In turn, the contradictions are $a = c$, $3(2a - c) = -1$, $3(2c - b) = -1$, $b = c$.

We can deal similarly with cases (d) and (e) by interchanging λ and μ.

The case $\lambda + \mu \neq 0$. It remains to consider cases (f) and (g), where again we write $b = a - \lambda$. In case (f) we have $3a - 2\mu - \lambda = -1$, and in case (g) we have $3a - 2\lambda - \mu = -1$. If $b = \lambda_j$, then the possibilities for $(\lambda_j, \mu_j, \nu_j)$ are:

(i) $b \quad b-\mu \quad b-\mu-\lambda$, (ii) $b \quad b-\lambda \quad b-\lambda-\mu$.

For the first possibility we have $3b - 2\mu - \lambda = -1$ and $3a - \mu - 2\lambda = -1$, since $a \neq b$. Equivalently, $3a - 2\mu - 4\lambda = -1$ and $3a - \mu - 2\lambda = -1$, from which we obtain the contradiction $3a = -1$. For the second possibility we have $3b - 2\lambda - \mu = -1$ and $3a - \lambda - 2\mu = -1$, equivalently $3a - 5\lambda - \mu = -1$ and $3a - \lambda - 2\mu = -1$. We deduce that $\mu = 4\lambda$, and this is a contradiction because λ and μ must have different signs.

It follows that G_1 has at most 3 eigenvalues and is strongly regular. Since G_1 is not complete, it is of Latin square type or negative Latin square type (see [26], Theorem 4.1). Finally, we have $\lambda + \mu = 0$, for otherwise we obtain a contradiction as in the case $\lambda + \mu \neq 0$ above. $\square$

There arises a question in relation to a possible converse: If K_n admits a 3-decomposition into strongly regular graphs G_1, G_2, G_3 of Latin square type or negative Latin square type, are the signed graphs $\dot{G}_{1,2}$, $\dot{G}_{2,3}$, $\dot{G}_{3,1}$ semi-complementary?

6.2. Bipartite signed graphs. If $\dot{G}$ is a bipartite signed graph, then its adjacency matrix can be written in the form

$$A_{\dot{G}} = \begin{pmatrix} O & N^\intercal \\ N & O \end{pmatrix}, \tag{6.2}$$

which gives

$$A_{\dot{G}}^2 = \begin{pmatrix} N^\intercal N & O \\ O & NN^\intercal \end{pmatrix}.$$

Thus, if $\dot{G}$ has 2 eigenvalues, then by considering the minimal polynomial, we see that N is a square matrix satisfying $N^\intercal N = NN^\intercal = rI$, in other words, N is a weighing matrix of weight r. (We recall that a weighing matrix M of weight $r > 0$ is a square $(0, 1, -1)$-matrix satisfying $M^\intercal M = rI$; then we also have $MM^\intercal = rI$.) Conversely, if N is a weighing matrix of weight r, then the eigenvalues of $\dot{G}$ are $\pm\sqrt{r}$. We record this as the following lemma.

Lemma 6.4. *A signed graph with adjacency matrix* (6.2) *has* 2 *eigenvalues if and only if N is a weighing matrix.*

It is conjectured by Seberry (cf. [4]) that if n is a multiple of 4, then a weighing matrix exists for all r ($1 \leqslant r \leqslant n$); there are plenty of examples (not listed here). It is also known that for $n \equiv 2 \pmod 4$, every weighing matrix of order n satisfies $r \leqslant n-1$, where r must be the sum of two squares. Finally, every weighing matrix of odd order is such that r is a square and $(n-k)^2 + n - k + 1 \geqslant n$. More details and a classification of weighing matrices of orders up to 15 and order 17 can be found in [10].

In the particular case where N is a $(1, -1)$-matrix (i.e., N is a Hadamard matrix), the existence of a bipartite net-regular signed graph with 2 eigenvalues is resolved in terms of balanced incomplete block designs, as we already mentioned in Example 5.4.

Assume further that N is a $p \times q$ matrix with $p \leqslant q$. If $N^\intercal N$ has 2 eigenvalues such that one of them is zero, then $A_{\dot{G}}^2$ has the same eigenvalues, which means that $\dot{G}$ has at most 3 eigenvalues. Moreover, by Lemma 6.2, it cannot have 2 eigenvalues, so it has exactly 3. Thus, we have the following.

Lemma 6.5. *If N is a $p \times q$ $(0, 1, -1)$-matrix (with $p \leqslant q$) and $N^\intercal N$ has exactly* 2 *eigenvalues, one of them being zero, then the signed graph with adjacency matrix* (6.2) *has* 3 *eigenvalues.*

Observe that for $p < q$, zero is an eigenvalue of $N^\intercal N$, since $N^\intercal N$ and $NN^\intercal$ share the same nonzero eigenvalues (together with their multiplicities) and the latter matrix has smaller order. For $p = q$, by setting N to be the adjacency matrix of a signed graph with 3 eigenvalues $\pm\lambda$ and 0, we see that $\dot{G}$ has 3 eigenvalues.

We conclude the subsection with another construction. Let $\dot{H}$ be a signed multigraph in which two vertices are either nonadjacent, adjacent by a positive or a negative edge or adjacent by one positive and one negative edge (such a pair of edges form the so-called *negative digon*). We introduce the vertex-edge orientation η: $V(\dot{H}) \times E(\dot{H}) \to \{0, 1, -1\}$ formed by obeying the following rules: (1) $\eta(i, jk) = 0$

if $i \notin \{j,k\}$, (2) $\eta(i,ij)=1$ or $\eta(i,ij)=-1$ and (3) $\eta(i,ij)\eta(j,ij)=-\sigma(ij)$. The vertex-edge incidence matrix B_η is the matrix whose rows and columns are indexed by $V(\dot{G})$ and $E(\dot{G})$, respectively, such that its (i,e)-entry is equal to $\eta(i,e)$.

Now, if $\dot{H}$ is connected and $|V(\dot{H})| \leqslant |E(\dot{H})|$, then we deduce from [24], Theorem 4.3 that $B_\eta^\intercal B_\eta$ has 2 eigenvalues, one of them being zero, if and only if $\dot{H}$ is switching equivalent to either the complete graph or the signed multigraph obtained by inserting a negative parallel edge between every pair of adjacent vertices of a regular graph. By the same result, for $|V(\dot{H})| > |E(\dot{H})|$, $B_\eta B_\eta^\intercal$ never has such eigenvalues. This leads to the following result.

Proposition 6.6. *If N is the vertex-edge incidence matrix of a connected signed multigraph $\dot{H}$ described above, then $\dot{G}$ with adjacency matrix* (6.2) *has* 3 *eigenvalues if and only if $\dot{H}$ is switching equivalent to either the complete graph or the signed graph obtained by inserting a negative parallel edge between every pair of adjacent vertices of a regular graph.*

We note that $B_\eta^\intercal B_\eta - 2I$ is the adjacency matrix of the signed line graph of $\dot{H}$ (in the sense of [3], [24]) and $B_\eta B_\eta^\intercal$ is the Laplacian matrix of $\dot{H}$.

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