Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 1, 1–38

Persistent URL: `http://dml.cz/dmlcz/149569`

Czechoslovak Mathematical Journal, 72 (147) (2022), 1–38

A CHARACTERIZATION OF SETS IN $\mathbb{R}^2$ WITH DC DISTANCE FUNCTION

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Received June 4, 2020. Published online July 15, 2021.

Abstract. We give a complete characterization of closed sets $F \subset \mathbb{R}^2$ whose distance function $d_F := \operatorname{dist}(\cdot, F)$ is DC (i.e., is the difference of two convex functions on $\mathbb{R}^2$). Using this characterization, a number of properties of such sets is proved.

Keywords: distance function; DC function; subset of $\mathbb{R}^2$

MSC 2020: 26B25

1. Introduction

The present article is a continuation of the article [10] which studies closed sets $F \subset \mathbb{R}^d$, whose distance function $d_F := \operatorname{dist}(\cdot, F)$ is DC (i.e., is the difference of two convex functions on $\mathbb{R}^d$). So we first briefly recall the motivation for our study and mention some results of [10].

It is well-known (see, e.g., [1], page 976) that, for a closed $F \subset \mathbb{R}^d$, the function $(d_F)^2$ is always DC but d_F need not be DC. The main motivation for the paper [10] was the question whether d_F is a DC function if $F \subset \mathbb{R}^d$ is a graph of a DC function $g\colon \mathbb{R}^{d-1} \to \mathbb{R}$. This question naturally arises in the theory of WDC sets (see [6], Question 2, page 829 and [5], Section 10.4.3). Let us note that WDC sets form a substantial generalization of Federer's sets with positive reach and still admit the definition of curvature measures (see [5] or [8]) and F as in the above question is a natural example of a WDC set in $\mathbb{R}^d$. The main result of [10] gives the affirmative answer to the above question in the case $d = 2$, but the case $d > 2$ remains open.

Following [10], we will use the following notation.

The research was supported by GAČR 18-11058S.

DOI: 10.21136/CMJ.2021.0228-20

Definition 1.1. For $d \in \mathbb{N}$, we set

$$\mathcal{D}_d := \{\emptyset\} \cup \{\emptyset \neq A \subset \mathbb{R}^d\colon\ A \text{ is closed and } d_A \text{ is DC}\}.$$

The elements of $\mathcal{D}_d$ will be called $\mathcal{D}_d$ *sets.*

Using this notation, the main result of [10] asserts that

$$\operatorname{graph} g \in \mathcal{D}_2, \text{ whenever } g\colon\ \mathbb{R} \to \mathbb{R} \text{ is DC.} \tag{1.1}$$

If $A \subset \mathbb{R}^d$ is a set with positive reach, then (see [10], Proposition 4.2) $A \in \mathcal{D}_d$ and also $\partial A \in \mathcal{D}_d$ and $\overline{\mathbb{R}^d \setminus A} \in \mathcal{D}_d$. It implies (see [10], Corollary 4.5) that $\operatorname{graph} g \in \mathcal{D}_2$ whenever $g\colon\ \mathbb{R}^{d-1} \to \mathbb{R}$ is a semiconcave function.

It is not known whether each WDC set $A \subset \mathbb{R}^d$ belongs to $\mathcal{D}_d$, but the statement is true for $d = 2$ by [11], Theorem 3.3.

Several results concerning general properties of classes $\mathcal{D}_d$ were obtained in [10], Section 4; we recall them in Subsection 2.3 below.

In the present article, we use the results of [10] to give complete characterizations of $\mathcal{D}_2$ sets. These characterizations are based on the notion of (s)-sets ("special $\mathcal{D}_2$-sets"), which have a formally simple definition (see Definition 3.1) but their structure can be rather complicated. The proofs of these characterizations are quite long and technical; they are contained in Sections 3 and 4.

Section 5 contains applications of our characterizations. For example, we prove (see Proposition 5.5) that each nowhere dense $\mathcal{D}_2$ set is a countable union of DC graphs (defined in Definition 2.14). Further, the system of all components of each $\mathcal{D}_2$ set is discrete (see Theorem 4.20) and each component is pathwise connected and locally connected (see Proposition 5.7). An important application is Theorem 5.12; its particular case asserts that if $F\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ is a bilipschitz bijection which is C^2 smooth (or, more generally, DC), then $F(M) \in \mathcal{D}_2$ for each $M \in \mathcal{D}_2$. It is an open question whether $\mathcal{D}_d$ has this stability property for $d > 2$.

2. Preliminaries

2.1. Basic notation. We denote by $B(x,r)$ ($U(x,r)$) the closed (open) ball with centre x and radius r. The boundary and the interior of a set M are denoted by ∂M and $\operatorname{int} M$, respectively. A mapping is called K-*Lipschitz* if it is Lipschitz with a (not necessarily minimal) constant $K \geqslant 0$. In any vector space V, we use the symbol 0 for the zero element and $\operatorname{span} M$ for the linear span of a set M.

In the Euclidean space $\mathbb{R}^d$, the origin is denoted by 0, the norm by $|\cdot|$ and the scalar product by $\langle \cdot, \cdot \rangle$. By S^{d-1} we denote the unit sphere in $\mathbb{R}^d$. $\operatorname{Tan}(A,a)$ denotes the tangent cone of $A \subset \mathbb{R}^d$ at $a \in \mathbb{R}^d$ ($u \in \operatorname{Tan}(A,a)$ if and only if $u = \lim_{i\to\infty} r_i(a_i - a)$ for some $r_i > 0$ and $a_i \in A$, $a_i \to a$).

If $x, y \in \mathbb{R}^d$, the symbol $\overline{x,y}$ denotes the closed segment (possibly degenerate). If also $x \neq y$, then $l(x,y)$ denotes the line joining x and y.

For $B \subset \mathbb{R}^2$ and $t \in \mathbb{R}$, we set $B_{[t]} := \{y \in \mathbb{R}\colon\ (t,y) \in B\}$. We also define $\pi_1\colon\ \mathbb{R}^2 \to \mathbb{R}$ by $\pi_1(x,y) = x$.

The distance function from a set $\emptyset \neq A \subset \mathbb{R}^d$ is $d_A := \operatorname{dist}(\cdot, A)$ and the metric projection of $z \in \mathbb{R}^d$ to A is $\Pi_A(z) := \{a \in A\colon\ \operatorname{dist}(z,A) = |z-a|\}$.

A system $\mathcal{A}$ of subsets of $\mathbb{R}^d$ is called *discrete*, if each $z \in \mathbb{R}^2$ has a neighbourhood which intersects at most one $A \in \mathcal{A}$. A set $D \subset \mathbb{R}^d$ is called *discrete*, if $\{\{d\}\colon\ d \in D\}$ is discrete.

By a rotation in $\mathbb{R}^2$ we always understand a rotation around the origin.

For $f\colon\ \mathbb{R}^d \to \mathbb{R}^k$ and $x, v \in \mathbb{R}^d$, the one-sided directional derivative of f at x in direction v is

$$f'_+(x,v) := \lim_{t\to 0+} \frac{f(x+tv) - f(x)}{t}.$$

2.2. DC functions. Let f be a real function defined on an open convex set $C \subset \mathbb{R}^d$. Then we say that f is a *DC function*, if it is the difference of two convex functions. We say that $F = (F_1, \dots, F_k)\colon\ C \to \mathbb{R}^k$ is a *DC mapping* if all components F_i of F are DC functions.

Semiconvex and semiconcave functions are special DC functions. Namely, f is a *semiconvex* (or, *semiconcave*) function, if there exist $a > 0$ and a convex function g on C such that

$$f(x) = g(x) - a|x|^2 \quad (\text{or, } f(x) = a|x|^2 - g(x)), \quad x \in C.$$

We will use the following well-known properties of DC functions and mappings.

Lemma 2.1. *Let C be an open convex subset of $\mathbb{R}^d$. Then the following assertions hold.*

(i) *If $f\colon\ C \to \mathbb{R}$ and $g\colon\ C \to \mathbb{R}$ are DC, then (for each $a \in \mathbb{R}$, $b \in \mathbb{R}$) the functions $|f|$, $af + bg$, $\max(f,g)$ and $\min(f,g)$ are DC.*

(ii) *Each locally DC mapping $f\colon\ C \to \mathbb{R}^k$ is DC.*

(iii) *Each DC function $f\colon\ C \to \mathbb{R}$ is Lipschitz on each compact convex set $Z \subset C$.*

(iv) *Let $G \subset \mathbb{R}^d$ and $H \subset \mathbb{R}^k$ be open sets. Let $f\colon\ G \to \mathbb{R}^k$ and $g\colon\ H \to \mathbb{R}^p$ be locally DC, and let $f(G) \subset H$. Then $g \circ f$ is locally DC on G.*

(v) *Let $G, H \subset \mathbb{R}^d$ be open sets, and let $f\colon\ G \to H$ be a locally bilipschitz and locally DC bijection. Then f^{-1} is locally DC on H.*

(vi) *Let* $f_i\colon C \to \mathbb{R}$, $i = 1, \ldots, k$, *be DC functions. Let* $f\colon C \to \mathbb{R}$ *be a continuous function such that* $f(x) \in \{f_1(x), \ldots, f_k(x)\}$ *for each* $x \in C$. *Then* f *is DC on* C.

(vii) *Each* C^2 *function* $f\colon C \to \mathbb{R}$ *is DC.*

Proof. Property (i) follows easily from definitions, see, e.g., [15], page 84. Property (ii) was proved in [7]. Property (iii) easily follows from the local Lipschitz continuity of convex functions. Assertion (iv) is "Hartman's superposition theorem" from [7]; for the proof see also [15] or [16], Theorem 4.2. Statement (v) follows from [16], Theorem 5.2. Assertion (vi) is a special case of [16], Lemma 4.8 ("Mixing lemma"). Property (vii) follows, e.g., from [16], Proposition 1.11 and (ii). $\square$

The following easy result (see [9], Lemma 2.3) is well-known.

Lemma 2.2. *Let* $F\colon (a,b) \to \mathbb{R}^d$ *be a DC mapping and* $x \in (a,b)$. *Then the one-sided derivatives* $F'_{\pm}(x)$ *exist. Moreover,*

$$\lim_{t\to x+} F'_{\pm}(t) = F'_{+}(x) \quad \textit{and} \quad \lim_{t\to x-} F'_{\pm}(t) = F'_{-}(x), \tag{2.1}$$

which implies that $F'_{+}(a)$ *is the strict right derivative of* F *at* x, *i.e.,*

$$\lim_{\substack{(y,z)\to(x,x)\\ y\neq z,\, y\geqslant x,\, z\geqslant x}} \frac{F(z)-F(y)}{z-y} = F'_{+}(x). \tag{2.2}$$

The notion of DC mappings between Euclidean spaces was generalized in [16] to the notion of DC mappings between Banach spaces using the notion of a "control function". We will use this notion only for real functions defined on open intervals $I \subset \mathbb{R}$. In this context we have (cf. [16], Definition 1.1) that a convex function $\varphi\colon I \to \mathbb{R}$ is a *control function* for a function $f\colon I \to \mathbb{R}$ if and only if both $\varphi + f$ and $\varphi - f$ are convex functions.

It is an easy fact (cf. [16], Lemma 1.6 (b)) that $f\colon I \to \mathbb{R}$ is DC if and only if it has a control function. We will use the following immediate consequence of [16], Proposition 1.13.

Lemma 2.3. *If* φ *is a control function for* f *on an open interval* I, *then*

$$\Bigl|\frac{f(z+k)-f(z)}{k} - \frac{f(z)-f(z-h)}{h}\Bigr| \leqslant \frac{\varphi(z+k)-\varphi(z)}{k} - \frac{\varphi(z)-\varphi(z-h)}{h},$$

whenever $k > 0$, $h > 0$, $z \in I$, $z + k \in I$ *and* $z - h \in I$.

For the origin of the following definition, see [12], page 28.

Definition 2.4. Let f be a function on $[a, b]$. For every partition $D = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a, b]$, we put

$$K(f, D) := \sum_{i=1}^{n-1} \left| \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right|.$$

If $n = 1$, we put $K(f, D) := 0$. Then the *convexity of f on* $[a, b]$ is defined as

$$K_a^b f := \sup K(f, D),$$

where the supremum is taken over all partitions D of $[a, b]$. If $K_a^b f < \infty$, we say that f *has bounded (or finite) convexity.*

The following fact is a consequence of [17], Theorem 3.1 (b).

Lemma 2.5. *If f is a DC function on (a, b) with a control function φ and $a < c < d < b$, then $K_c^d f \leqslant \varphi'_-(d) - \varphi'_+(c)$.*

Following [9], page 1617, we use the following terminology.

Definition 2.6. We will say that a function defined on a set $\emptyset \neq D \subset \mathbb{R}$ is a *DCR function*, if it is a restriction of a DC function defined on $\mathbb{R}$.

The following facts are well-known.

Lemma 2.7. *Let f be a continuous real function on $[a, b]$. Then the following conditions are equivalent:*
(i) *f is a DCR function.*
(ii) *f is the difference of two Lipschitz convex functions.*
(iii) *f has bounded convexity.*
(iv) *$f'_-(x)$ exists for each $x \in (a, b)$ and $V(f'_-, (a, b)) < \infty$.*
(v) *f is a restriction of a DC function defined on some $(u, v) \supset [a, b]$.*

Here $V(f'_-, (a, b))$ means the variation of f'_- over (a, b) in the usual sense; see, e.g., [17], page 322.

P r o o f. The implication (i) $\Rightarrow$ (ii) follows by Lemma 2.1 (iii), and (ii) $\Rightarrow$ (i) holds since each convex Lipschitz function on $[a, b]$ can be extended to a convex function on $\mathbb{R}$. The equivalence (ii) $\Leftrightarrow$ (iii) easily follows from [12], Theorem D, page 26 and (iii) $\Leftrightarrow$ (iv) is a particular case of [17], Proposition 3.4, page 382. The implication (i) $\Rightarrow$ (v) is trivial and (v) $\Rightarrow$ (iii) follows from Lemma 2.5. $\square$

We will need the following facts concerning DCR functions. They immediately follow from [17], Proposition 4.2 (or can be rather easily obtained using Lemma 2.1 (iv), (v)).

Lemma 2.8. *Let $\varphi\colon [a,b] \to [c,d]$ be a DCR increasing bilipschitz bijection and let $\omega\colon [c,d] \to \mathbb{R}$ be a DCR function. Then*
(i) *the function $\omega \circ \varphi$ is DCR on $[a,b]$ and*
(ii) *the function φ^{-1} is DCR on $[c,d]$.*

We will need also the following "DCR mixing lemma".

Lemma 2.9. *Let $I \subset \mathbb{R}$ be a closed interval and let $f\colon I \to \mathbb{R}$ be a continuous function. Let $f_i\colon I \to \mathbb{R}$, $i = 1,\dots,k$, be DCR functions such that $f(x) \in \{f_1(x),\dots,f_k(x)\}$ for each $x \in I$. Then f is DCR.*

Proof. Let $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ be a continuous extension of f which is locally constant on $\mathbb{R} \setminus I$ and let $\tilde{f}_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1,\dots,k$, be a DC extension of f_i. Then there are two constant (and so DC) functions $\tilde{f}_{k+1}$, $\tilde{f}_{k+2}$ on $\mathbb{R}$ such that $\tilde{f}(x) \in \{\tilde{f}_1(x),\dots,\tilde{f}_{k+2}(x)\}$, $x \in \mathbb{R}$. Consequently, $\tilde{f}$ is DC by Lemma 2.1 (vi), and so f is DCR. $\square$

The following lemma is a version of the "mixing lemma" [16], Lemma 4.8 (cf. Lemma 2.1 (vi)), which we need. Note that [16], Lemma 4.8 works even with DC mappings between Banach spaces, and Lemma 2.10 follows from its proof but not from its formulation.

Lemma 2.10. *Let F_i, $i = 1,\dots,k$, be DC functions on an open interval $J \subset \mathbb{R}$. Then there exists a convex function φ on J with the following property:*
(P) *If F is a continuous function on an open interval $I \subset J$ and*

$$F(x) \in \{F_1(x),\dots,F_k(x)\}, \quad x \in I,$$

then F is a DC function with the control function $\varphi|_I$.

Proof. Let f_i be a control function for F_i on J, $i = 1,\dots,k$. Set

$$\varphi := \sum_{i,j=1}^{k} h_{i,j}, \quad \text{where } h_{i,j} := f_i + f_j + \frac{1}{2}|F_i - F_j|.$$

The proof of Lemma 4.8 in [16], where φ is denoted by f, gives the assertion of property (P) for $I = J$. Observing that $f_i|_I$ is a control function for $F_i|_I$, property (P) follows. $\square$

We will need also the following easy "Lipschitz mixing lemma".

Lemma 2.11. *Let $K > 0$ and f_i, $i = 1, \dots, k$, be K-Lipschitz functions on an interval (of arbitrary type) $I \subset \mathbb{R}$. Let f be a continuous function on I such that $f(x) \in \{f_1(x), \dots, f_k(x)\}$, $x \in I$. Then f is K-Lipschitz on I.*

Proof. We will proceed by induction on k. The case $k = 1$ is trivial. Suppose that $k > 1$ and the lemma holds for "$k = k - 1$". To prove that f is K-Lipschitz, consider arbitrary points $a, b \in I$, $a < b$. Choose $1 \leqslant i_0 \leqslant k$ such that $f(a) = f_{i_0}(a)$ and set $c := \max\{a \leqslant x \leqslant b\colon\ f(x) = f_{i_0}(x)\}$.

If $c = b$, then $|f(b) - f(a)| = |f_{i_0}(b) - f_{i_0}(a)| \leqslant K(b - a)$.

If $c < b$, the induction hypothesis (applied to $f|_{(c,b]}$) implies that f is K-Lipschitz on $(c, b]$, and, consequently, also on $[c, b]$. Therefore,

$$|f(b) - f(a)| \leqslant |f_{i_0}(c) - f_{i_0}(a)| + |f(b) - f(c)| \leqslant K(c - a) + K(b - c) = K(b - a).$$

$\square$

By well-known properties of convex and concave functions, we easily obtain that each locally DC function f on an open set $U \subset \mathbb{R}^d$ has all one-sided directional derivatives finite and

$$(2.3) \quad g'_+(x, v) + g'_+(x, -v) \leqslant 0, \quad x \in U,\ v \in \mathbb{R}^d,\ \text{if } g \text{ is locally semiconcave on } U.$$

Recall that if $\emptyset \neq A \subset \mathbb{R}^d$ is closed, then d_A need not be DC; however (see, e.g., [2], Proposition 2.2.2),

$$(2.4) \qquad d_A \text{ is locally semiconcave (and so locally DC) on } \mathbb{R}^d \setminus A.$$

In [9] and [10] we worked with "DC hypersurfaces" in $\mathbb{R}^d$. Since we work here in $\mathbb{R}^2$ only, we use the following terminology.

Definition 2.12. We say that a set $A \subset \mathbb{R}^2$ is a 1-*dimensional DC surface* if there exist $v \in S^1$ and a Lipschitz DC function (i.e., the difference of two convex functions) g on $W := (\operatorname{span}\{v\})^\perp$ such that $A = \{w + g(w)v\colon\ w \in W\}$.

Remark 2.13. The notion of a 1-dimensional DC surface in $\mathbb{R}^2$ coincides with the notion of a DC hypersurface in $\mathbb{R}^2$ from [9] (but not with the notion of a DC hypersurface in $\mathbb{R}^2$ from [10], where the Lipschitz continuity of g is not required).

We also define, following [10], the notion of a *DC graph* in $\mathbb{R}^2$.

Definition 2.14. A set $P \subset \mathbb{R}^2$ will be called a *DC graph* if it is a rotated copy of $\operatorname{graph} f$ of a DCR function f on some compact (possibly degenerated) interval $\emptyset \neq I \subset \mathbb{R}$.

Note that P is a DC graph if and only if it is a nonempty connected compact subset of a 1-dimensional DC surface in $\mathbb{R}^2$.

We will need the following simple result which is possibly new and can be of some independent interest.

Proposition 2.15. *Let g be a continuous function on $[a,b]$ which is DC on (a,b) and let $P \subset [a,b]$ be a nowhere dense set. Then the set $g(P)$ is nowhere dense.*

P r o o f. We can suppose that P is closed. Suppose, to the contrary, that (the compact set) $g(P)$ is not nowhere dense and choose an open interval $I \subset g(P)$. Set

$$S := \{x \in (a,b) \colon\ g'_+(x) = 0 \text{ or } g'_-(x) = 0\}.$$

Then $g(S)$ is Lebesgue null; it follows, e.g., from [13], Theorem 4.5, page 271 (cf. [13], page 272, a note before Theorem 4.7). So we can choose a point $y_0 \in I \setminus (g(S) \cup \{g(a), g(b)\})$. Then the set $K := g^{-1}(\{y_0\}) \subset (a,b)$ is finite. Indeed, otherwise there exists a point $x \in K$ which is an accumulation point of the compact set K. Then clearly $x \in S$, which contadicts $y_0 \notin g(S)$. Let $K = \{x_1 < \ldots < x_p\}$. Lemma 2.2 implies that there exists $\delta > 0$ such that $a < x_1 - \delta < x_1 + \delta < x_2 - \delta < \ldots < x_p + \delta < b$ and g is strictly monotone both on $[x_i - \delta, x_i]$ and on $[x_i, x_i+\delta]$, $i = 1, \ldots, p$. Consequently, $Q := g\Big(P \cap \bigcup_{i=1}^{p} (x_i-\delta, x_i+\delta)\Big)$ is nowhere dense. Since $Z := g\Big([a,b] \setminus \bigcup_{i=1}^{p} (x_i-\delta, x_i+\delta)\Big)$ is compact and does not contain y_0, there exists $\sigma > 0$ such that $(y_0-\sigma, y_0+\sigma) \subset I \subset g(P)$ and $(y_0-\sigma, y_0+\sigma) \cap Z = \emptyset$. Consequently, $(y_0 - \sigma, y_0 + \sigma)$ is a subset of the nowhere dense set Q, which is a contradiction. $\square$

2.3. Known results concerning $\mathcal{D}_d$. In, [10], we proved several general results concerning systems $\mathcal{D}_d$. First recall that if $M \subset \mathbb{R}$ is closed, then

(2.5) M belongs to $\mathcal{D}_1$ if and only if the system of all components of $\mathcal{M}$ is locally finite.

It easily implies that $\mathcal{D}_1$ is closed with respect to both finite unions and finite intersections and that a closed $M \subset \mathbb{R}$ belongs to $\mathcal{D}_1$ if and only if $\partial M \in \mathcal{D}_1$.

However, the case $d > 1$ is different. It is easy to show that

(2.6) $\mathcal{D}_d$ is closed with respect to finite unions,

but [10], Example 4.1 shows that already $\mathcal{D}_2$ is not closed with respect to finite intersections. We observed that, for a closed set $M \subset \mathbb{R}^d$, $d \in \mathbb{N}$,

$$\partial M \in \mathcal{D}_d \Leftrightarrow (M \in \mathcal{D}_d \quad \text{and} \quad \overline{\mathbb{R}^d \setminus M} \in \mathcal{D}_d), \tag{2.7}$$

but [10], Example 4.1 provides an example of a set $M \in \mathcal{D}_2$ with $\partial M \notin \mathcal{D}_2$.

Important [10], Proposition 4.7 asserts that if $d \geqslant 2$ and $M \in \mathcal{D}_d$, then each bounded set $C \subset \partial M$ can be covered by finitely many "DC hypersurfaces".

In our terminology, we have in $\mathbb{R}^2$ the following result which has basic importance for the present article.

Proposition 2.16. *Let $M \in \mathcal{D}_2$. Then each bounded set $C \subset \partial M$ can be covered by finitely many* 1*-dimensional DC surfaces.*

Let us note that this result is not a particular case of [10], Proposition 4.7 (cf. Remark 2.13), but the proof of [10], Proposition 4.7 is based on [9], Corollary 5.4, and so [10], Proposition 4.7 holds also with the definition of DC hypersurfaces from [9] ("with Lipschitz continuity") and so Proposition 2.16 holds. (Moreover, it is easy to show that "Proposition 2.16 without Lipschitz continuity" implies "Proposition 2.16 with Lipschitz continuity".)

Proposition 2.16 easily implies that each nowhere dense $M \in \mathcal{D}_2$ can be covered by a locally finite system of DC graphs. On the other hand, (1.1) easily implies (see [10], Proposition 4.9) that

(2.8) if $M \subset \mathbb{R}^2$ is the union of a locally finite system of DC graphs, then $M \in \mathcal{D}_2$.

However, we have found an example (see [10], Example 4.10) of a nowhere dense set $M \in \mathcal{D}_2$ which is not the union of a locally finite system of DC graphs.

Let us note that we will prove in Proposition 5.5 that each nowhere dense $M \in \mathcal{D}_2$ is the union of a countable system of DC graphs.

We will also use the following easy facts which are not mentioned explicitly in [10].

Remark 2.17.

(i) If M is a $\mathcal{D}_2$ set and φ a similarity on $\mathbb{R}^2$, then $\varphi(M)$ is a $\mathcal{D}_2$ set.

(ii) The system $\mathcal{D}_2$ is closed with respect to locally finite unions.

Proof. The first part follows by Lemma 2.1 (i), (ii), (iv) and (vii) from the fact that $d_{\varphi(M)} = r d_M \circ \varphi^{-1}$, where $r > 0$ is the scaling ratio of φ.

To prove the second part, consider a locally finite system $\mathcal{M} \subset \mathcal{D}_2$ and denote $M := \bigcup \mathcal{M}$. We can suppose that $M \neq \emptyset$. If $z \in M$, choose $r > 0$ and $M_1, \dots, M_k \in \mathcal{M}$ such that $U(z,r) \cap M = U(z,r) \cap \bigcup\limits_{i=1}^{k} M_i$. Since $\widetilde{M} := \bigcup\limits_{i=1}^{k} M_i \in D_2$ by (2.6), and $d_{\widetilde{M}} = d_M$ on $U(z, \frac{1}{2}r)$, we have that d_M is DC on $U(z, \frac{1}{2}r)$.

Consequently, using also (2.4), we obtain that d_M is locally DC on $\mathbb{R}^2$, and so it is DC by Lemma 2.1 (ii). $\square$

3. Every (s)-set is a $\mathcal{D}_2$ set

In the next section, we will give a complete characterization of $\mathcal{D}_2$ sets using "special $\mathcal{D}_2$ sets" called *(s)-sets*. Their definition is formally rather simple, but it is not easy to prove that each (s)-set is a $\mathcal{D}_2$ set. In this section we will prove this important fact using the method of the proof of (1.1) from [10] together with (1.1) and several additional ideas.

Definition 3.1. Let $\emptyset \neq S \subset \mathbb{R}^2$ be a closed set. We say that S is an (s)-*set* if there exists $r > 0$ such that

$$S \subset \bigcup_{i=1}^{k} \operatorname{graph} f_i \text{ for some DCR functions } f_i\colon [0,r] \to \mathbb{R} \text{ with } (f_i)'_+(0) = f_i(0) = 0 \tag{3.1}$$

and

$$S = \bigcup_{h \in H} \operatorname{graph} h \text{ for a family } H \text{ of continuous functions on } [0,r]. \tag{3.2}$$

Remark 3.2. We will prove some nontrivial properties of (s)-sets in Section 5, here note only that each (s)-set is clearly nowhere dense and path connected.

Remark 3.3. Let S be an (s)-set with corresponding $r > 0$, functions $f_1, \dots, f_k$ and system H as in Definition 3.1. Pick $0 < \varrho < r$ and put $\widetilde{S} := S \cap ([0,\varrho] \times \mathbb{R})$. Then clearly $\widetilde{S}$ is an (s)-set (with corresponding functions $\tilde{f}_i = f_i|_{[0,\varrho]}$, $i = 1, \dots, k$, and system $\widetilde{H} = \{h|_{[0,\varrho]} \colon h \in H\}$).

Remark 3.4. Let us note that (3.1) gives that each (s)-set is a subset of a finite union of DC graphs. However, there are (s)-sets which do not equal to a finite union of DC graphs, see Example 5.6 below.

One of our technical tools is the following easy fact (see [10], Lemma 3.2).

Lemma 3.5. *Let V be a closed angle in $\mathbb{R}^2$ with vertex v and measure $0 < \alpha < \pi$. Then there exist an affine function S on $\mathbb{R}^2$ and a concave function ψ on $\mathbb{R}^2$ which is Lipschitz with constant $\sqrt{2}\tan(\alpha/2)$ such that $|z - v| + \psi(z) = S(z)$, $z \in V$.*

We will also use the following "concave mixing lemma" ([10], Lemma 3.1).

Lemma 3.6. *Let $U \subset \mathbb{R}^d$ be an open convex set and let $\gamma\colon U \to \mathbb{R}$ have finite one-sided directional derivatives $\gamma'_+(x,v)$, ($x \in U$, $v \in \mathbb{R}^d$). Suppose that*

$$\gamma'_+(x,v) + \gamma'_+(x,-v) \leqslant 0, \qquad x \in U,\ v \in \mathbb{R}^d, \tag{3.3}$$

and that

(3.4) graph γ *is covered by graphs of a finite number of concave functions defined on U.*

Then γ is a concave function.

The core of the present section is the proof of the following lemma, which easily implies that (s)-sets are $\mathcal{D}_2$ sets.

Lemma 3.7. *Let $f_1, \ldots, f_k$ be DC functions on $\mathbb{R}$ such that each f_i is constant on both $(-\infty, 0]$ and $[1, \infty)$. Let $\emptyset \neq M \subset \mathbb{R}^2$ be a closed set and H a system of continuous functions on $\mathbb{R}$ such that*

$$M = \bigcup_{h \in H} \operatorname{graph} h \subset \bigcup_{i=1}^{k} \operatorname{graph} f_i. \tag{3.5}$$

Then $M \in \mathcal{D}_2$.

Proof. First observe that each $h \in H$ is constant on both $(-\infty, 0]$ and $[1, \infty)$ by the continuity of h and (3.5).

We will proceed in two steps.

Step I: In the first step, we will prove that

(3.6) there exists a concave function Γ on $\mathbb{R}^2$ such that the function $d_M + \Gamma$ is locally concave on $\mathbb{R}^2 \setminus M$.

Observe that Lemma 2.1 (iii) implies that there exists $K > 0$ such that the functions $f_1, \ldots, f_k$ are K-Lipschitz on $[0, 1]$ and, consequently,

(3.7) each function $h \in H$ is K-Lipschitz on $[0, 1]$

by (3.5) and Lemma 2.11.

If $h \in H$ and $n \in \mathbb{N}$, denote by h_n the function on $\mathbb{R}$ for which $h_n(i/n) = h(i/n)$, $i = 0, \ldots, n$, which is affine on each interval $[(i-1)/n, i/n]$, $i = 1, \ldots, n$, and which is constant on both $(-\infty, 0]$ and $[1, \infty)$.

For $n \in \mathbb{N}$, set $H_n := \{h_n\colon\ h \in H\}$ and

$$M_n := \bigcup_{h \in H_n} \operatorname{graph} h. \tag{3.8}$$

Using (3.5), we obtain that each H_n is finite, and consequently each M_n is closed.

Obviously $M_n \cap ((-\infty, 0] \times \mathbb{R}) = M \cap ((-\infty, 0] \times \mathbb{R})$ and $M_n \cap ([1, \infty) \times \mathbb{R}) = M \cap ([1, \infty) \times \mathbb{R})$, $n \in \mathbb{N}$, and (3.7) easily implies that $M_n \cap ([0, 1] \times \mathbb{R}) \to M \cap ([0, 1] \times \mathbb{R})$ in the Hausdorff metric. Consequently, we easily obtain that

$$d_{M_n} \to d_M \quad \text{on } \mathbb{R}^2.$$

Now we will show that, to obtain (3.6), it is sufficient to find $D > 0$ and concave functions Ψ_n, $n \in \mathbb{N}$, such that

$$\Psi_n \text{ is } D\text{-Lipschitz on } \mathbb{R}^2 \text{ for each } n \in \mathbb{N} \tag{3.9}$$

and

$$d_{M_n} + \Psi_n \text{ is locally concave on } \mathbb{R}^2 \setminus M_n \text{ for each } n \in \mathbb{N}. \tag{3.10}$$

So suppose that such D and $\{\Psi_n\}$, $n \in \mathbb{N}$, are given and consider an arbitrary $z \in \mathbb{R}^2 \setminus M$. Then there exist $r > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geqslant n_0$, $B(z, r) \cap M_n = \emptyset$ and, consequently, (3.10) easily implies that $d_{M_n} + \Psi_n$ is concave on $B(z, r)$.

Further observe that we can suppose that $\Psi_n(0) = 0$, $n \in \mathbb{N}$. Then (3.9) gives that the sequence $\{\Psi_n\}$ is equicontinuous and pointwise bounded, and we can use a well-known version of the Arzelà-Ascoli theorem (see, e.g., [3], Theorem 4.44, page 137) to obtain a subsequence $\{\Psi_{n_p}\}$ converging to a D-Lipschitz concave function Γ on $\mathbb{R}^2$. Then $d_{M_{n_p}} + \Psi_{n_p} \to d_M + \Gamma$, consequently, $d_M + \Gamma$ is concave on $B(z, r)$, and so (3.6) holds.

To find $\{\Psi_n\}$ and D, we first define a finite subset A_n of M_n by

$$A_n := \Big\{ \Big(\frac{i}{n}, h\Big(\frac{i}{n}\Big) \Big)\colon\ 0 \leqslant i \leqslant n,\ h \in H \Big\}.$$

For each $n \in \mathbb{N}$ and $a = (a_1, a_2) \in A_n$, set

$$\begin{aligned}
s^+(n, a) &:= \max\{h'_+(a_1)\colon\ h \in H_n,\ h(a_1) = a_2\},\\
s_+(n, a) &:= \min\{h'_+(a_1)\colon\ h \in H_n,\ h(a_1) = a_2\},\\
s^-(n, a) &:= \max\{h'_-(a_1)\colon\ h \in H_n,\ h(a_1) = a_2\},\\
s_-(n, a) &:= \min\{h'_-(a_1)\colon\ h \in H_n,\ h(a_1) = a_2\}.
\end{aligned}$$

Now we will prove that there is a constant $C > 0$ such that, for each $n \in \mathbb{N}$,

$$\sum_{a \in A_n} |s^+(n,a) - s_-(n,a)| \leqslant C, \qquad \sum_{a \in A_n} |s^-(n,a) - s_+(n,a)| \leqslant C. \tag{3.11}$$

To this end, consider a convex function φ which corresponds to $J := \mathbb{R}$ and $F_i := f_i$, $i = 1, \dots, k$, by Lemma 2.10. Choose $L > 0$ such that φ is L-Lipschitz on $[-1, 2]$. Further consider an arbitrary $n \in \mathbb{N}$, $0 \leqslant i \leqslant n$ and $a = (a_1, a_2) \in A_n$ with $a_1 = i/n$. Now choose $\tilde{h} \in H$ and $\hat{h} \in H$ such that

$$\tilde{h}(i/n) = \hat{h}(i/n) = a_2, \quad (\tilde{h}_n)'_+(i/n) = s^+(n,a), \quad (\hat{h}_n)'_-(i/n) = s_-(n,a).$$

Set $g(x) := \tilde{h}(x)$ for $x \geqslant i/n$ and $g(x) := \hat{h}(x)$ for $x < i/n$. Then clearly

$$|s^+(n,a) - s_-(n,a)| = \Big| \frac{g((i+1)/n) - g(i/n)}{1/n} - \frac{g(i/n) - g((i-1)/n)}{1/n} \Big|.$$

Since $\operatorname{graph} g \subset \bigcup_{i=1}^{k} \operatorname{graph} f_i$, by its choice, φ is a control function for g and so Lemma 2.3 and the above equality imply

$$|s^+(n,a) - s_-(n,a)| \leqslant \frac{\varphi((i+1)/n) - \varphi(i/n)}{1/n} - \frac{\varphi(i/n) - \varphi((i-1)/n)}{1/n} \leqslant 2L. \tag{3.12}$$

Consequently,

$$\begin{aligned}
&\sum_{a \in A_n} |s^+(n,a) - s_-(n,a)| \\
&\qquad \leqslant \sum_{i=0}^{n} \sum_{\substack{(a_1,a_2) \in A_n, \\ a_1 = i/n}} \Big(\frac{\varphi((i+1)/n) - \varphi(i/n)}{1/n} - \frac{\varphi(i/n) - \varphi((i-1)/n)}{1/n} \Big) \\
&\qquad \leqslant k \sum_{i=0}^{n} \Big(\frac{\varphi((i+1)/n) - \varphi(i/n)}{1/n} - \frac{\varphi(i/n) - \varphi((i-1)/n)}{1/n} \Big) \\
&\qquad = k \Big(\frac{\varphi((n+1)/n) - \varphi(1)}{1/n} - \frac{\varphi(0) - \varphi(-1/n)}{1/n} \Big) \leqslant 2Lk =: C.
\end{aligned}$$

The second inequality of (3.11) follows quite analogously.

For each $n \in \mathbb{N}$ and $a = (a_1, a_2) \in A_n$, set

$$\begin{aligned}
p^+(n,a) &:= (a_1 + 1/n, a_2 + s^+(n,a)/n), \\
p_+(n,a) &:= (a_1 + 1/n, a_2 + s_+(n,a)/n), \\
p^-(n,a) &:= (a_1 - 1/n, a_2 - s^-(n,a)/n), \\
p_-(n,a) &:= (a_1 - 1/n, a_2 - s_-(n,a)/n), \\
A_n^1 &:= \{a \in A_n \colon\ s^+(n,a) - s_-(n,a) > 0\}, \\
A_n^2 &:= \{a \in A_n \colon\ s^-(n,a) - s_+(n,a) > 0\}.
\end{aligned}$$

Further set

$$V^1_{n,a} := \{z \in \mathbb{R}^2\colon \langle z-a, p^+(n,a)-a\rangle \leqslant 0,\ \langle z-a, p_-(n,a)-a\rangle \leqslant 0\} \quad \text{if } a \in A^1_n$$

and

$$V^2_{n,a} := \{z \in \mathbb{R}^2\colon \langle z-a, p_+(n,a)-a\rangle \leqslant 0,\ \langle z-a, p^-(n,a)-a\rangle \leqslant 0\} \quad \text{if } a \in A^2_n.$$

It is easy to see that each $V^1_{n,a}$ (or, $V^2_{n,a}$) is a closed angle with vertex a and measure

$$\alpha^1(n,a) := \arctan s^+(n,a) - \arctan s_-(n,a) \in (0,\pi),$$
$$(\text{or, } \alpha^2(n,a) := \arctan s^-(n,a) - \arctan s_+(n,a) \in (0,\pi)).$$

For $a \in A^1_n$ (or, $a \in A^2_n$) let $\psi^1_{n,a}$ and $S^1_{n,a}$ (or, $\psi^2_{n,a}$ and $S^2_{n,a}$) be the concave and affine functions on $\mathbb{R}^2$ which correspond to $V^1_{n,a}$ (or, $V^2_{n,a}$) by Lemma 3.5. If $a \in A_n \setminus A^1_n$ (or, $a \in A_n \setminus A^2_n$), put $\psi^1_{n,a}(z) := 0$ and $S^1_{n,a}(z) := 0$ (or, $\psi^2_{n,a}(z) := 0$ and $S^2_{n,a}(z) := 0$), $z \in \mathbb{R}^2$. Set

$$\Psi_n := \sum_{a \in A_n} (\psi^1_{n,a} + \psi^2_{n,a}).$$

Now fix an arbitrary $a \in A^1_n$. Using (3.12) we easily obtain

$$\alpha^1(n,a) \leqslant s^+(n,a) - s_-(n,a) \leqslant 2L.$$

Further, since the tangent function is convex on $[0, \frac{1}{2}\pi)$, the function $\omega(x) = \tan x/x$ is increasing on $(0, \frac{1}{2}\pi)$. These facts easily imply

$$\sqrt{2}\tan\Big(\frac{\alpha^1(n,a)}{2}\Big) \leqslant \sqrt{2}\cdot\frac{\alpha^1(n,a)}{2}\cdot\frac{L}{\arctan L} \leqslant (s^+(n,a)-s_-(n,a))\cdot\frac{L}{\sqrt{2}\arctan L}.$$

So, by the choice of $\psi^1_{n,a}$, we have that

$$\text{(3.13)} \qquad \psi^1_{n,a} \text{ is Lipschitz with constant } (s^+(n,a)-s_-(n,a))\cdot\frac{L}{\sqrt{2}\arctan L}.$$

Quite similarly we obtain that, for each $a \in A^2_n$,

$$\text{(3.14)} \qquad \psi^2_{n,a} \text{ is Lipschitz with constant } (s^-(n,a)-s_+(n,a))\cdot\frac{L}{\sqrt{2}\arctan L}.$$

Consequently, (3.13), (3.14) and (3.11) easily imply that there is a constant $D > 0$ such that (3.9) holds.

To prove (3.10), it is enough to prove that

(3.15) $d_{\operatorname{graph} h} + \Psi_n$ is locally concave on $\mathbb{R}^2 \setminus \operatorname{graph} h$ for each $n \in \mathbb{N}$ and $h \in H_n$.

Indeed, by (3.8) it is easy to see that

$$d_{M_n} + \Psi_n = \min_{h \in H_n} (d_{\operatorname{graph} h} + \Psi_n)$$

and it is enough to use (on each open ball $U \subset \mathbb{R}^2 \setminus M_n$) the fact that the minimum of a finite system of concave functions is a concave function.

To prove (3.15), fix an arbitrary $n \in \mathbb{N}$ and $h \in H_n$.

For $i = -1, \dots, n+1$ denote $z_i := (i/n, h(i/n))$ and

$$V_i := \{z \in \mathbb{R}^2 \colon \langle z - z_i, z_{i+1} - z_i \rangle \leqslant 0, \langle z - z_i, z_{i-1} - z_i \rangle \leqslant 0\}.$$

Now, for a fixed i, denote $a := z_i$. Then clearly $a \in A_n$ and, if the points z_{i-1}, z_i, z_{i+1} are not collinear, then

(3.16) $$\text{either } a \in A_n^1 \text{ and } V_i \subset V_{n,a}^1, \text{ or } a \in A_n^2 \text{ and } V_i \subset V_{n,a}^2.$$

Indeed, observe that

$$s_+(n,a) \leqslant \frac{h((i+1)/n) - h(i/n)}{1/n} \leqslant s^+(n,a),$$
$$s_-(n,a) \leqslant \frac{h(i/n) - h((i-1)/n)}{1/n} \leqslant s^-(n,a).$$

So, if

$$\frac{h((i+1)/n) - h(i/n)}{1/n} > \frac{h(i/n) - h((i-1)/n)}{1/n},$$

then $a \in A_n^1$ and an easy geometrical observation shows that $V_i \subset V_{n,a}^1$. Similarly, if

$$\frac{h((i+1)/n) - h(i/n)}{1/n} < \frac{h(i/n) - h((i-1)/n)}{1/n},$$

then $a \in A_n^2$ and $V_i \subset V_{n,a}^2$.

Denote $l_- := \mathbb{R} \times \{h(0)\}$, $l_+ := \mathbb{R} \times \{h(1)\}$ and for $n \in \mathbb{N}$ and $i = 0, \dots, n-1$ denote

$$\eta_i := d_{l(z_i, z_{i+1})} + \Psi_n$$

and

$$\eta_{-1} := d_{l_-} + \Psi_n, \quad \eta_n := d_{l_+} + \Psi_n.$$

We will prove that $\gamma := d_{\operatorname{graph} h} + \Psi_n$ is locally concave on $\mathbb{R}^2 \setminus \operatorname{graph} h$ using Lemma 3.6. So fix an arbitrary $b \in \mathbb{R}^2 \setminus \operatorname{graph} h$ and $\delta > 0$ such that $U := U(b, \delta) \subset \mathbb{R}^2 \setminus \operatorname{graph} h$. Then condition (3.3) of Lemma 3.6 holds by (2.3) and (2.4). To prove condition (3.4), consider an arbitrary $z \in U$ and choose $z^* \in \operatorname{graph} h$ such that $d_{\operatorname{graph} h}(z) = |z - z^*|$.

First note that if $z^* \notin \{z_0, \dots, z_n\}$ or $z^* = z_i$ for some $i \in \{0, 1, \dots, n\}$ and the points z_{i-1}, z_i, z_{i+1} are collinear, then clearly

$$\gamma(z) = d_{\operatorname{graph} h}(z) + \Psi_n(z) \in \bigcup_{i \in \{-1, \dots, n\}} \{\eta_i(z)\}. \tag{3.17}$$

Further assume that $z^* = z_i$ for some $i \in \{0, \dots, n\}$ and the points z_{i-1}, z_i, z_{i+1} are not collinear. Then clearly $z \in V_i$ and (3.16) holds. Consequently,

$$\text{either } |z - z^*| + \psi^1_{n,z_i}(z) = S^1_{n,z_i}(z), \text{ or } |z - z^*| + \psi^2_{n,z_i}(z) = S^2_{n,z_i}(z),$$

and therefore

$$\begin{aligned} \gamma(z) &= d_{\operatorname{graph} h}(z) + \Psi_n(z) \\ &\in \{S^1_{n,z_i}(z) + (\Psi_n - \psi^1_{n,z_i})(z), S^2_{n,z_i}(z) + (\Psi_n - \psi^2_{n,z_i})(z)\}. \end{aligned} \tag{3.18}$$

Since the graph of each function η_i, $-1 \leqslant i \leqslant n$, can be clearly covered by the graphs of two concave functions and the functions $\Psi_n - \psi^1_{n,z_i}$, $\Psi_n - \psi^2_{n,z_i}$ $(i = 0, \dots, n)$ are concave, (3.17) and (3.18) imply (3.4) and so γ is concave on U and therefore (3.15) holds, which completes the proof of (3.6).

Step II: In the second step, we first observe that by (1.1) there exist concave functions ω_i, $1 \leqslant i \leqslant k$, such that each function $d_{\operatorname{graph} f_i} + \omega_i$ is concave on $\mathbb{R}^2$. Set

$$\omega := \sum_{i=1}^{k} \omega_i \quad \text{and} \quad \sigma := \Gamma + \omega.$$

Then

$$d_{\operatorname{graph} f_i} + \sigma \text{ is concave}, \quad 1 \leqslant i \leqslant k, \tag{3.19}$$

and by (3.6)

$$d_M + \sigma \text{ is locally concave on } \mathbb{R}^2 \setminus M. \tag{3.20}$$

It is sufficient to prove that $d_M + \sigma$ is concave on $\mathbb{R}^2$.

For $i, j \in \{1, \dots, k\}$, $i \neq j$, denote by $P_{i,j}$ the set of all $x \in \mathbb{R}$ such that $f_i(x) = f_j(x)$ and such that for every $\varepsilon > 0$ there is $z \in (x-\varepsilon, x+\varepsilon)$ satisfying $f_i(z) \neq f_j(z)$. Obviously, each $P_{i,j}$ is a closed nowhere dense set and $P_{i,j} \subset [0,1]$.

Set

$$P_i := \bigcup \{P_{i,j}\colon\ 1 \leqslant j \leqslant k,\ j \neq i\}, \quad P := \bigcup_{i=1}^{k} P_i,$$

$$P_i^* := \{(x, f_i(x))\colon\ x \in P_i\}, \qquad P^* := \bigcup_{i=1}^{k} P_i^*.$$

Note that for every $z \in M \setminus P^*$, there is $i \in \{1, \dots, k\}$ and $\varrho > 0$ such that $M \cap U(z, \varrho) = \operatorname{graph} f_i \cap U(z, \varrho)$, and so

$$d_M(u) = d_{\operatorname{graph} f_i}(u), \quad u \in U\Big(z, \frac{\varrho}{2}\Big). \tag{3.21}$$

To prove the concavity of $d_M + \sigma$, it is clearly sufficient to prove that for each $p, q \in \mathbb{R}^2$ and $\varepsilon > 0$, there exists a line l which meets both $U(p, \varepsilon)$ and $U(q, \varepsilon)$ and $d_M + \sigma$ is concave on l. To this end, choose arbitrary p, q, ε. Further, using the notation

$$l(m, c) := \{(x, y)\colon\ y = mx + c\}, \quad m, c \in \mathbb{R},$$

we choose a line $l(m_0, c_0)$ which meets both $U(p, \varepsilon)$ and $U(q, \varepsilon)$.

Now observe that for each $c \in \mathbb{R}$ we have $l(m_0, c) \cap P_i^* \neq \emptyset$ if and only if there is $x \in P_i$ such that $m_0 x + c = f_i(x)$, i.e., $c \in g_i(P_i)$, where $g_i(x) := f_i(x) - m_0 x$, $x \in \mathbb{R}$. Since g_i is DC and $P_i \subset [0,1]$ is nowhere dense, Lemma 2.15 implies that $C_i := \{c \in \mathbb{R}\colon\ l(m_0, c) \cap P_i^* \neq \emptyset\}$ is nowhere dense. Consequently, we can choose $c_1 \in \mathbb{R}$ such that $l := l(m_0, c_1) \subset \mathbb{R}^2 \setminus P^*$ and l meets both $U(p, \varepsilon)$ and $U(q, \varepsilon)$. By (3.21), (3.19) and (3.20) we obtain that $d_M + \sigma$ is locally concave at each point of l, and thus concave on l. $\square$

Corollary 3.8. *Let $M \subset \mathbb{R}^2$ be as in Lemma 3.7. Then $\widetilde{M} := M \cap ([0,1] \times \mathbb{R}) \in \mathcal{D}_2$.*

P r o o f. Let $f_1, \dots, f_k$ and H be as in Lemma 3.7. First note that, by Lemmas 3.7 and (2.4), $d_{\widetilde{M}}$ is locally DC on

$$\mathbb{R}^2 \setminus ((M \cap (\{0\} \times \mathbb{R})) \cup (M \cap (\{1\} \times \mathbb{R}))) =: \mathbb{R}^2 \setminus (M_0 \cup M_1).$$

We prove that $d_{\widetilde{M}}$ is DC on some neighbourhood of each point in M_0. To do that, pick some $z \in M_0$. Let H_z be the system of all functions $h \in H$ such that $(0, h(0)) = z$ and put

$$f_z(x) = \max_{h \in H_z} h(x), \quad g_z(x) = \min_{h \in H_z} h(x), \quad x \in [0, \infty),$$
$$D_z := \{(x, y)\colon\ x \in [0, \infty),\ g_z(x) \leqslant y \leqslant f_z(x)\}.$$

Note that d_M is DC by Lemma 3.7. Since the functions $f_1, \dots, f_k$ are Lipschitz on $[0,1]$ by Lemma 2.1 (iii), they are Lipschitz on $[0,\infty)$. Consequently, both f_z and g_z are Lipschitz by (3.5) and Lemma 2.11, and therefore they are DCR by (3.5) and Lemma 2.9. Therefore, d_{D_z} is DC by (2.7) and (2.8), since ∂D_z is clearly the union of a locally finite system of DC graphs. It is easy to see (using the fact that the set M_0 has cardinality at most k and so, in particular, is finite) that for a sufficiently small $\varrho > 0$ we have $d_{\widetilde{M}}(w) \in \{d_{D_z}(w), d_M(w)\}$ whenever $w \in U(z,\varrho)$, and so d_M is DC on $U(z,\varrho)$ by Lemma 2.1 (vi).

Quite analogously one can prove that $d_{\widetilde{M}}$ is also DC on a neighbourhood of each point of M_1, and so $d_{\widetilde{M}}$ is locally DC, and therefore (by Lemma 2.1 (ii)) DC on $\mathbb{R}^2$ and $\widetilde{M} \in \mathcal{D}_2$. □

Corollary 3.9. *Every (s)-set $S \subset \mathbb{R}^2$ is a $\mathcal{D}_2$ set.*

P r o o f. Let S be an (s)-set and let $r > 0$, $f_1, \dots, f_k$ and H be as in Definition 3.1. We may assume (applying a suitable similarity and using Remark 2.17 (i) if necessary) that $r = 1$. For $h \in H$ define $\tilde{h}\colon \mathbb{R} \to \mathbb{R}$ by $\tilde{h} = h$ on $[0,1]$, $\tilde{h} = h(0)$ on $(-\infty, 0]$ and $\tilde{h} = h(1)$ on $[1,\infty)$. Set $\widetilde{H} := \{\tilde{h}\colon h \in H\}$. Similarly we extend functions f_i, calling the extensions $\tilde{f}_i$. Clearly (by Lemma 2.1 (vi)) each $\tilde{f}_i$ is a DC function on $\mathbb{R}$. Put $M := \bigcup\limits_{h \in \tilde{H}} \operatorname{graph} h$. Then

$$M = \bigcup_{h \in \tilde{H}} \operatorname{graph} h \subset \bigcup_{i=1}^{k} \operatorname{graph} \tilde{f}_i.$$

Since $S = M \cap ([0,1] \times \mathbb{R})$ and M satisfies the assumptions of Lemma 3.7, we obtain $S \in \mathcal{D}_2$ by Corollary 3.8. □

4. Complete characterizations of $\mathcal{D}_2$ sets

Lemma 4.1. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$ be sequences of real numbers such that*
(i) $0 < a_{n+1} \leqslant \frac{1}{3} a_n$, $n \in \mathbb{N}$, *and*
(ii) $\sum\limits_{n=1}^{\infty} |A_n|/a_n < \infty$.

Then the function

$$f(x) = \begin{cases} \dfrac{A_n - A_{n+1}}{a_n - a_{n+1}}(x - a_{n+1}) + A_{n+1} & \text{if } x \in (a_{n+1}, a_n], \\ 0 & \text{if } x = 0 \end{cases}$$

is a DCR function on $[0, a_1]$.

P r o o f. First note that condition (i) implies

$$a_n - a_{n+1} \geqslant a_n - \frac{a_n}{3} = \frac{2}{3}a_n \geqslant 2a_{n+1}. \tag{4.1}$$

Since $a_n \to 0$ by (i), we obtain $A_n = f(a_n) \to 0$ by (ii), and so f is continuous. Clearly $f'_-(x) = f'_-(a_n) = (A_n - A_{n+1})/(a_n - a_{n+1})$ for $x \in (a_{n+1}, a_n]$, $n \in \mathbb{N}$. Using (ii) and (4.1), we obtain

$$\sum_{n=1}^{\infty} |f'_-(a_n)| \leqslant \sum_{n=1}^{\infty} \Big(\Big| \frac{A_n}{a_n - a_{n+1}} \Big| + \Big| \frac{A_{n+1}}{a_n - a_{n+1}} \Big| \Big) \leqslant \sum_{n=1}^{\infty} \Big(\frac{|A_n|}{\frac{2}{3}a_n} + \frac{|A_{n+1}|}{2a_{n+1}} \Big) < \infty.$$

Therefore, we easily obtain

$$V(f'_-, (0, a_1)) = \sum_{n=1}^{\infty} |f'_-(a_n) - f'_-(a_{n+1})| < \infty,$$

and so f is a DCR function by Lemma 2.7. □

For $r > 0$, put

$$\begin{aligned} A_r^u &= \{(x,y)\colon\ 0 \leqslant x \leqslant r,\ |y| \leqslant ux\}, & A^u &= \{(x,y)\colon\ 0 \leqslant x,\ |y| \leqslant ux\}, \\ S_r^u &= \{(x,y)\colon\ |x| \leqslant r,\ |y| \leqslant u|x|\}, & S^u &= \{(x,y)\colon\ x \in \mathbb{R},\ |y| \leqslant u|x|\}. \end{aligned}$$

In the proof of Lemma 4.3 we will use the following geometrically obvious lemma.

Lemma 4.2. *Let $u > 0$. Then there exists $\alpha > 0$ such that $d_M(x,y) \geqslant \alpha \cdot \varrho$, whenever $M \subset \mathbb{R}^2$, $z = (a,b) \in \mathbb{R}^2$, $\varrho > 0$ and either*

$$M \cap (z + A_\varrho^{3u}) \subset \{z\}, \quad (x,y) \in (z + A_\varrho^{2u}), \quad x = a + \frac{\varrho}{2}, \tag{4.2}$$

or

$$M \cap (z - A_\varrho^{3u}) \subset \{z\}, \quad (x,y) \in (z - A_\varrho^{2u}), \quad x = a - \frac{\varrho}{2}. \tag{4.3}$$

Lemma 4.3. *Let $M \subset \mathbb{R}^2$ be closed, $z \in M$ and $u > 0$. Let there exist sequences $\{z_n\}$ in M and $\{\varrho_n\}$ in $(0, \infty)$ such that $z_n \to z$ and, for each $n \in \mathbb{N}$,*
(i) *$z_n \in (z + S^u) \setminus \{z\}$ and*
(ii) *either $(z_n + A_{\varrho_n}^{3u}) \cap M = \{z_n\}$ or $(z_n - A_{\varrho_n}^{3u}) \cap M = \{z_n\}$.*
Then $M \notin \mathcal{D}_2$.

Proof. Suppose, to the contrary, that d_M is DC.

We can suppose $z=0$. Let $z_n=(a_n,b_n)$. Without any loss of generality, we can suppose $a_1>a_2>\ldots>0$. To see this, we can pass to a subsequence and work with $M^s:=\{(x,y)\colon (-x,y)\in M\}$ (which belongs to $\mathcal{D}_2$ if and only if $M\in\mathcal{D}_2$) instead of M, if necessary. Further, we can assume (passing several times to a subsequence) that

$$a_{n+1}\leqslant \frac{a_n}{3}\quad \text{for each } n\in\mathbb{N}, \tag{4.4}$$

and for some $K\in[-u,u]$,

$$\frac{b_n}{a_n}\to K\quad\text{and}\quad \sum_{n=1}^{\infty}\Bigl|\frac{b_n}{a_n}-K\Bigr| < \infty. \tag{4.5}$$

Note that also (by $z_n\in A^u$ and (4.4))

$$\Bigl|\frac{b_n-b_{n+1}}{a_n-a_{n+1}}\Bigr|\leqslant \frac{a_nu+\frac{1}{3}a_nu}{\frac{2}{3}a_n}\leqslant 2u\quad\text{for each } n\in\mathbb{N}. \tag{4.6}$$

Set $A_n:=b_n-a_nK$. Then assumptions (i) and (ii) of Lemma 4.1 are satisfied by (4.4) and (4.5), and consequently we know that the function

$$f(x)=\begin{cases}\dfrac{A_n-A_{n+1}}{a_n-a_{n+1}}(x-a_{n+1})+A_{n+1} & \text{if } x\in(a_{n+1},a_n],\\ 0 & \text{if } x=0\end{cases}$$

is a DCR function on $[0,a_1]$ and Lemma 2.1 (i), (ii), (iv) easily imply that the functions $g(x)=f(x)+Kx$, $x\in[0,a_1]$, and $F(x)=d_M(x,g(x))$, $x\in[0,a_1]$, are also DCR functions. Observe that

$$g(a_n)=b_n,\quad n\in\mathbb{N}\text{ and } g \text{ is linear on each interval } [a_{n+1},a_n]. \tag{4.7}$$

Further, $F(0)=0$ and $F(a_n)=0$, $n\in\mathbb{N}$. So Lemma 2.2 implies that

$$0 \text{ is the right strict derivative of } F \text{at } 0. \tag{4.8}$$

Now consider $n>1$ and choose an $\alpha>0$ which corresponds to our u by Lemma 4.2.

If $(z_n+A^{3u}_{\varrho_n})\cap M=\{z_n\}$, choose $0<r_n<\varrho_n$ such that $a_n+r_n<a_{n-1}$ and set $x_n:=a_n+\frac{1}{2}r_n$, $y_n:=g(x_n)$. Since (4.7) and (4.6) imply $(x_n,y_n)\in z_n+A^{2u}_{r_n}$, we can apply Lemma 4.2 (with $z=0$, $\varrho=r_n$, $x=x_n$, $y=y_n$) and obtain $d_M(x_n,y_n)=F(x_n)\geqslant\alpha r_n$. Consequently,

$$\frac{F(x_n)-F(a_n)}{x_n-a_n}\geqslant 2\alpha. \tag{4.9}$$

If $(z_n - A^{3u}_{\varrho_n}) \cap M = \{z_n\}$, choose $0 < r_n < \varrho_n$ such that $a_n - r_n > a_{n+1}$ and set $x_n := a_n - \frac{1}{2} r_n$, $y_n := g(x_n)$. In the same way as in the first case we also obtain (4.9).

Now observe that, by the definition of the strict right derivative, (4.9) contradicts (4.8). □

Lemma 4.4. *Let $M \in \mathcal{D}_2$, $z = (x, y) \in M$, $s > 0$ and $u > 0$. Then the following assertions hold.*

(i) *If*

$$\partial M \cap (z + A^{3u}_s) \subset z + A^u_s, \tag{4.10}$$

then there is $s \geqslant r > 0$ such that either $M \cap (z + A^{3u}_r) = \{z\}$, or $\pi_1(M \cap (z + A^{3u}_r)) = [x, x + r]$.

(ii) *If*

$$\partial M \cap (z - A^{3u}_s) \subset z - A^u_s, \tag{4.11}$$

then there is $s \geqslant r > 0$ such that either $M \cap (z - A^{3u}_r) = \{z\}$, or $\pi_1(M \cap (z - A^{3u}_r)) = [x - r, x]$.

Proof. We will prove only assertion (i); the proof of (ii) is quite analogous.

Set $K := \pi_1(M \cap (z + A^{3u}_s))$ and observe that condition (4.10) implies that either $\pi_1(M \cap (z + A^{3u}_s)) = [x, x + s]$ or

$$M \cap (z + A^{3u}_s) \subset z + A^u_s. \tag{4.12}$$

So we can suppose that (4.12) holds.

Now suppose to the contrary that no $s \geqslant r > 0$ from the assertion of the lemma exists. Since K is compact, we can clearly find sequences of positive numbers $\{x_n\}$, $\{\varrho_n\}$ such that $x_n \to 0$,

$$x_n \in K \quad \text{and} \quad (x_n, x_n + \varrho_n) \cap K = \emptyset \quad \text{for each } n \in \mathbb{N}. \tag{4.13}$$

By the definition of K and (4.12), there exist y_n, $n \in \mathbb{N}$, such that $z_n := (x_n, y_n) \in M \cap (z + A^u_s)$. Since (4.13) clearly implies $(z_n + A^{3u}_{\varrho_n}) \cap M = \{z_n\}$, we obtain by Lemma 4.3 that $M \notin \mathcal{D}_2$, which is a contradiction. □

Lemma 4.5. *Let $M \in \mathcal{D}_2$ and $s, u > 0$. Suppose that $0 \in M$ and $f_1, \ldots, f_k\colon [0, s] \to \mathbb{R}$ are u-Lipschitz functions such that $f_i(0) = 0$, $i = 1, \ldots, k$, and*

$$\partial M \cap A^{3u}_s \subset \bigcup_{i=1}^{k} \operatorname{graph} f_i. \tag{4.14}$$

Let 0 *be an accumulation point of* $\partial M \cap A_s^u$. *Then there is some* $0 < \varrho < s$ *such that for every*

$$(x, y) \in M \cap \bigcup_{i=1}^{k} \operatorname{graph} f_i$$

with $x \in (0, \varrho)$, *there exist* $\delta > 0$ *and a* u*-Lipschitz function* $g\colon [x-\delta, x+\delta] \to \mathbb{R}$ *such that* $g(x) = y$ *and* $\operatorname{graph} g \subset M \cap \bigcup\limits_{i=1}^{k} \operatorname{graph} f_i$.

P r o o f. For the sake of brevity, we set $M^* := M \cap \bigcup\limits_{i=1}^{k} \operatorname{graph} f_i$ and observe that the properties of f_i imply $M^* \subset A_s^u$.

Now consider an arbitrary $z = (x, y) \in M^*$ with $x \in (0, s)$. Since $z \in A_s^u$, it is easy to see that we can assign to z a number $R_z > 0$ such that

$$z + S_{R_z}^{3u} \subset A_s^{3u} \tag{4.15}$$

and

$$\operatorname{graph} f_i \cap (z + S_{R_z}^{3u}) = \emptyset, \quad \text{whenever } 1 \leqslant i \leqslant k \text{ and } f_i(x) \neq y.$$

Then, using the u-Lipschitz continuity of all f_i and (4.14), we obtain

$$\partial M \cap (z + S_{R_z}^{3u}) \subset z + S_{R_z}^{u}. \tag{4.16}$$

So, by Lemma 4.4, we can choose $0 < r_z \leqslant R_z$ such that $\pi_1(M \cap (z + S_{r_z}^{3u}))$ is one of the following sets:

$$\{x\}, \quad [x - r_z, x], \quad [x, x + r_z], \quad [x - r_z, x + r_z].$$

Using Lemma 4.3, we easily obtain that there exists $s \geqslant \varrho > 0$ such that

$$\pi_1(M \cap (z + S_{r_z}^{3u})) = [x - r_z, x + r_z], \quad \text{whenever } x \in (0, \varrho). \tag{4.17}$$

We claim that even

$$\pi_1(M^* \cap (z + S_{r_z}^{3u})) = [x - r_z, x + r_z], \quad \text{whenever } x \in (0, \varrho). \tag{4.18}$$

Indeed, pick $t \in [x - r_z, x + r_z]$. To prove $t \in \pi_1(M^* \cap (z + S_{r_z}^{3u}))$, we distinguish two cases. If $(\partial M \cap (z + S_{r_z}^{3u}))_{[t]} = \emptyset$, we observe that $((z + S_{r_z}^{3u}) \setminus M)_{[t]} = \emptyset$, and so $(t, f_i(t)) \in M^* \cap (z + S_{r_z}^{3u})$, where i is chosen so that $z \in \operatorname{graph} f_i$. If $(\partial M \cap (z + S_{r_z}^{3u}))_{[t]} \neq \emptyset$, then (4.18) follows from (4.15) and (4.14). Now fix an arbitrary $z = (x, y) \in M^*$ with $x \in (0, \varrho)$ and denote $C := M^* \cap (z + S^{3u}_{r_z})$. So C is compact, and thus (4.18) implies that we can correctly define

$$g(t) = \min C_{[t]}, \quad t \in (x - r_z, x + r_z).$$

Then g is continuous on $(x - r_z, x + r_z) \cap (0, \varrho)$. Indeed, the compactness of C easily implies that g is lower semicontinuous on $(x - r_z, x + r_z) \cap (0, \varrho)$. To prove, moreover, the upper semicontinuity of g, consider $t \in (x - r_z, x + r_z) \cap (0, \varrho)$ and observe that (4.18) applied to $z^* := (t, g(t))$ implies that

$$g(\tau) - g(t) \leqslant 3u|\tau - t|, \quad \tau \in (t - r_{z^*}, t + r_{z^*}) \cap (x - r_z, x + r_z),$$

which implies that g is upper semicontinuous at t. By Lemma 2.11, we obtain that g is u-Lipschitz on $(x - r_z, x + r_z) \cap (0, \varrho)$.

Now, choosing $\delta > 0$ such that $[x - \delta, x + \delta] \subset (x - r_z, x + r_z) \cap (0, \varrho)$, we obtain the assertion of the lemma. □

Below, we will need some easy facts concerning 1-dimensional DC surfaces in $\mathbb{R}^2$, which are proved in [9], Remark 7.1 and Lemma 7.3. Let us note that in these observations from [9], the term "DC graph" has a different meaning than in [10] and the present article: it means there a 1-dimensional DC surface in $\mathbb{R}^2$.

Thus, in the present terminology, [9], Remark 7.1 gives the following.

Remark 4.6. Let $P \subset \mathbb{R}^2$ be a 1-dimensional DC surface in $\mathbb{R}^2$ and $a \in P$. Then

(i) $\mathrm{Tan}(P, a) \cap S^1$ is a two-point set, and
(ii) there exist 1-dimensional DC surfaces $P_1, P_2 \subset \mathbb{R}^2$ such that $P \subset P_1 \cup P_2$, $a \in P_1 \cap P_2$ and $\mathrm{Tan}(P_i, a)$ is a 1-dimensional space, $i = 1, 2$.

Further, [9], Lemma 7.3 corresponds to the following result.

Lemma 4.7. *Let P be a 1-dimensional DC surface in $\mathbb{R}^2$ and $0 \in P$. Suppose that $\mathrm{Tan}(P, 0)$ is a 1-dimensional space and $(0, 1) \notin \mathrm{Tan}(P, 0)$. Then there exists $\varrho^* > 0$ such that, for each $0 < \varrho < \varrho^*$, there exist $\alpha < 0 < \beta$ and a DCR function f on (α, β) such that $P \cap U(0, \varrho) = \mathrm{graph}\, f|_{(\alpha,\beta)}$.*

We will also need the following simple fact, which is a standard consequence of the Zorn lemma.

Lemma 4.8. *Let $L > 0$, $\varrho > 0$ and $F \subset [0, \varrho] \times \mathbb{R}$ be a closed set such that*

(i) *for each $(x, y) \in F$ with $0 < x < \varrho$, there exist $\delta > 0$ and an L-Lipschitz function g on $[x - \delta, x + \delta]$ such that $g(x) = y$ and $\mathrm{graph}\, g \subset F$;*
(ii) *for each $(x, y) \in F$ with $0 < x < \varrho$, there exists an L-Lipschitz function γ on $[0, \varrho]$ such that $\gamma(x) = y$ and $\mathrm{graph}\, \gamma \subset F$.*

P r o o f. To prove (ii), consider an arbitrary $(x, y) \in F$ with $0 < x < \varrho$.

Denote by P the set of all L-Lipschitz functions $f\colon (a_f, b_f) \to \mathbb{R}$ such that $(a_f, b_f) \subset (0, \varrho)$, $x \in (a_f, b_f)$, $f(x) = y$ and $\operatorname{graph} f \subset F$. By (i), we obtain $P \neq \emptyset$. Define a partial order on P by inclusion (i.e., $f_1 \leqslant f_2 \Leftrightarrow \operatorname{graph} f_1 \subset \operatorname{graph} f_2$). Let $\emptyset \neq T \subset P$ be a totally ordered set. Then $\bigcup\{\operatorname{graph} f\colon\ f \in T\}$ is clearly the graph of a function $g \in P$ which is an upper bound of T. Consequently, by the Zorn lemma, P contains a maximal element $f\colon (a_f, b_f) \to \mathbb{R}$ and we can extend f to an L-Lipschitz function γ on $[a_f, b_f]$. Observe that the points $(a_f, \gamma(a_f))$, $(b_f, \gamma(b_f))$ belong to F since the latter set is closed. We claim that $a_f = 0$ and $b_f = \varrho$. Indeed, otherwise we can use (i) (applied either to $(x, y) = (a_f, \gamma(a_f))$ or to $(x, y) = (b_f, \gamma(b_f))$) and easily obtain a contradiction with the maximality of f. Consequently, γ has all properties from (ii). $\square$

Lemma 4.9. *Let $M \in \mathcal{D}_2$ and $0 \in M$. Let $u > 0$ and $A^{4u} \cap \operatorname{Tan}(\partial M, 0) \cap S^1 = \{(1, 0)\}$. Then there exist $r > 0$ and an (s)-set S such that*

$$\partial M \cap A^u_r \subset S \subset M \cap A^u_r. \tag{4.19}$$

P r o o f. By Proposition 2.16, there exist $\eta > 0$ and 1-dimensional DC surfaces $P_1, \dots, P_n$ such that

$$\partial M \cap B(0, \eta) \subset P_1 \cup \ldots \cup P_n.$$

Diminishing η if necessary, we can suppose that $0 \in P_i$ for all i. Due to Remark 4.6 (ii) we can also suppose (changing n, if necessary) that $\operatorname{Tan}(P_i, 0)$ is a 1-dimensional linear space for every i. Put

$$I := \{1 \leqslant i \leqslant n\colon\ \operatorname{Tan}(P_i, 0) = \operatorname{span}\{(1, 0)\}\}. \tag{4.20}$$

By our assumptions, clearly $I \neq \emptyset$; we can suppose that $I = \{1, \dots, k\}$. By our assumptions and the definition of I, we can choose $t > 0$ such that

$$\partial M \cap A^{3u}_t \subset P_1 \cup \ldots \cup P_k.$$

Using Lemma 4.7, we obtain that for each $1 \leqslant i \leqslant k$ there exist $s_i \in (0, \infty)$ and a DCR function φ_i on $[0, s_i]$ such that $P_i \cap A^{3u}_{s_i} = \operatorname{graph} \varphi_i$. Note that (by (4.20)) $(\varphi_i)'_+(0) = 0$ and so, using Lemma 2.2, we obtain $0 < s \leqslant \min\{s_1, \dots, s_k\}$ such that, denoting $f_i := \varphi_i|_{[0,s]}$, we have that each f_i is u-Lipschitz on $[0, s]$ and

$$\partial M \cap A^{3u}_s \subset \bigcup_{i=1}^{k} \operatorname{graph} f_i. \tag{4.21}$$

Moreover, by the assumptions, 0 is an accumulation point of $\partial M \cap A^u_s$.

Thus the assumptions of Lemma 4.5 are satisfied. Let $0 < \varrho < s$ be the corresponding number from the assertion of Lemma 4.5. Set $r := \frac{1}{2}\varrho$ and

$$S := M \cap \bigcup_{i=1}^{k} \operatorname{graph} f_i \cap ([0, r] \times \mathbb{R}). \tag{4.22}$$

Using (4.21), it is easy to see that (4.19) holds. So it remains to prove that S is an (s)-set. Since $S \subset \bigcup_{i=1}^{k} \operatorname{graph} f_i$ and $S \setminus \{0\} \neq \emptyset$, it is sufficient to prove that for every $(x, y) \in S$ with $x \neq 0$ there exists a continuous function $h\colon [0, r] \to \mathbb{R}$ such that $h(x) = y$ and $\operatorname{graph} h \subset S$.

To construct h, observe that, by the choice of ϱ, the assertion of Lemma 4.5 holds. Consequently, for $F := M \cap \bigcup_{i=1}^{k} \operatorname{graph} f_i \cap ([0, \varrho] \times \mathbb{R})$ and $L := u$, the assumptions of Lemma 4.8 hold. Therefore, there exists a u-Lipschitz function γ on $[0, \varrho]$ such that $\gamma(x) = y$ and $\operatorname{graph} \gamma \subset F$.

Consequently, the function $h := \gamma|_{[0,r]}$ has the required properties. □

Lemma 4.10. *Let M and K be closed sets in $\mathbb{R}^2$ and let $x \in M$ and $\varrho > 0$ be such that $K \cap U(x, \varrho) \subset M$ and $\partial M \cap U(x, \varrho) \subset \partial K$. If d_K is DC on $U(x, \varrho)$, then d_M is DC on $U(x, \frac{1}{2}\varrho)$.*

P r o o f. Pick $z \in U(x, \frac{1}{2}\varrho)$. First note that if $z \in M$, then $d_M(z) = 0$. If $z \notin M$, then $d_M(z) = d_{\partial M}(z) \geqslant d_{\partial K}(z) \geqslant d_K(z)$ since $d_M(z) \leqslant |x - z|$, $\partial M \cap U(x, \varrho) \subset \partial K$ and $\partial K \subset K$. On the other hand, $K \cap U(x, \varrho) \subset M$ implies $d_M(z) \leqslant d_K(z)$ and so $d_M(z) = d_K(z)$. Consequently, $d_M(z) \in \{0, d_K(z)\}$, $z \in U(x, \frac{1}{2}\varrho)$, and so d_M is DC on $U(x, \frac{1}{2}\varrho)$ by Lemma 2.1 (vi). □

Now we can prove the following "local" characterization of $\mathcal{D}_2$ sets by (s)-sets.

Theorem 4.11. *Let $M \subset \mathbb{R}^2$ be a closed set and let P be the set of all isolated points of M. Then the following statements are equivalent:*

(i) $M \in \mathcal{D}_2$,

(ii) *for every $z \in \partial M \setminus P$, there are $\varrho > 0$, (s)-sets $S_1, \dots, S_m$ and pairwise different rotations $\gamma_1, \dots, \gamma_m$ such that*

$$\partial M \cap U(z, \varrho) \subset \bigcup_{i=1}^{m} (z + \gamma_i(S_i)) \subset M, \tag{4.23}$$

(iii) *for every $z \in \partial M \setminus P$, there are $\varrho > 0$, (s)-sets $S_1, \dots, S_m$ and rotations $\gamma_1, \dots, \gamma_m$ such that (4.23) holds.*

P r o o f. First we prove the implication (i) $\Rightarrow$ (ii). Let M be a $\mathcal{D}_2$ set and $z \in \partial M \setminus P$. Then z is not an isolated point of ∂M and we obtain by Proposition 2.16 and Remark 4.6 (i) that $T := \mathrm{Tan}(\partial M, z) \cap S^1$ is a nonempty finite set. Let $T = \{t_1, \dots, t_m\}$ and let γ_i be the rotation that maps $(1,0)$ to t_i, $i = 1, \dots, m$. Since T is finite, there is $u > 0$ such that

$$A^{4u} \cap (\mathrm{Tan}(\gamma_i^{-1}(\partial M - z), 0) \cap S^1) = \gamma_i^{-1}(t_i) = (1,0), \quad i = 1, \dots, m.$$

By Lemma 4.9, there is, for every $i = 1, \dots, m$, an $r_i > 0$ and an (s)-set S_i such that

$$\gamma_i^{-1}(\partial M - z) \cap A^u_{r_i} = \partial(\gamma_i^{-1}(M - z)) \cap A^u_{r_i} \subset S_i \subset \gamma_i^{-1}(M - z), \quad i = 1, \dots, m,$$

and consequently,

$$\partial M \cap (z + \gamma_i(A^u_{r_i})) \subset z + \gamma_i(S_i) \subset M. \tag{4.24}$$

By the definition of the tangent cone, there is some $\varrho > 0$ such that

$$\partial M \cap U(z, \varrho) \subset \bigcup_{i=1}^{m} (z + \gamma_i(A^u_{r_i})),$$

and so (4.24) implies that (4.23) holds, and the proof of the implication is finished.

The implication (ii) $\Rightarrow$ (iii) is clear and the implication (iii) $\Rightarrow$ (i) follows easily from Corollary 3.9, Lemma 4.10 (with $K = \bigcup_{i=1}^{m} (z + \gamma_i(A^u_\varrho)))$, (2.6) and Lemma 2.1 (ii), which concludes the proof of the theorem. $\square$

Remark 4.12. In (ii) (and in (iii)) we can demand that both ϱ and

$$\mathrm{diam}\Big(\bigcup_{i=1}^{m} (z + \gamma_i(S_i))\Big)$$

"are arbitrarily small" (i.e., smaller than any $\varepsilon > 0$ prescribed together with $z \in \partial M \setminus P$). To see this, choose a sufficiently small $0 < \varrho^* < \varrho$ and observe that (4.23) remains hold if we write ϱ^* instead of ϱ and $S_i^* := S_i \cap ([0, \varrho^*] \times \mathbb{R})$ (which is an (s)-set by Remark 3.3) instead of S_i, $i = 1, \dots, m$.

Corollary 4.13. *If $M \in \mathcal{D}_2$, then the set P of all isolated points of M is discrete.*

P r o o f. Suppose to the contrary that there exists a point $z \in \overline{P} \setminus P$. Then clearly $z \in \partial M \setminus P$ and we can choose $\varrho, S_1, \dots, S_m$ and $\gamma_1, \dots, \gamma_m$ as in Theorem 4.11 (ii); so (4.23) holds. Since $P \subset \partial M$, there exists $p \in P$ such that $p \in \bigcup_{i=1}^{m} (z + \gamma_i(S_i)) \subset M$, which contradicts the connectivity of $\bigcup_{i=1}^{m} (z + \gamma_i(S_i))$ (cf. Remark 3.2). $\square$

Next we prove Theorem 4.15, which gives a "global" characterization of general $\mathcal{D}_2$ sets by nowhere dense $\mathcal{D}_2$ sets. We will need the following simple observation.

Lemma 4.14. *Let K, M, $K \subset M$, be closed subsets of $\mathbb{R}^2$. Then the following conditions are equivalent:*

(i) *$M = K \cup C$, where C is the union of a system of components of $\mathbb{R}^2 \setminus K$,*

(ii) *$\partial M \subset \partial K$.*

If these conditions hold and $K \in \mathcal{D}_2$, then $M \in \mathcal{D}_2$.

Proof. Let (i) hold. Since $C \subset \operatorname{int} M$, we have $\partial M = M \setminus \operatorname{int} M \subset M \setminus C \subset K$. Obviously, $\partial M \subset \mathbb{R}^2 \setminus \operatorname{int} K$, and consequently $\partial M \subset K \setminus \operatorname{int} K = \partial K$. We have proved (i) $\Rightarrow$ (ii).

To prove (i) from (ii), it is sufficient to prove that if D is a component of $\mathbb{R}^2 \setminus K$, then either $D \subset M$ or $D \cap M = \emptyset$, but it follows from the fact that $\partial M \cap D \subset \partial K \cap D = \emptyset$, and so $M \cap D$ is both open and closed in the open connected set D.

To prove the last part of the lemma, it is sufficient to observe that if $K \in \mathcal{D}_2$ and $M \neq \emptyset$, then (ii) together with Lemma 4.10 and (2.4) imply that d_M is locally DC. Indeed, then d_M is DC on $\mathbb{R}^2$ by Lemma 2.1 (ii), and so $M \in \mathcal{D}_2$. $\square$

Theorem 4.15. *Let $M \subset \mathbb{R}^2$ be a closed set. Then the following conditions are equivalent:*

(i) *$M \in \mathcal{D}_2$,*

(ii) *there exists a nowhere dense $K \in \mathcal{D}_2$ such that $\partial M \subset K \subset M$,*

(iii) *there exists a nowhere dense $K \in \mathcal{D}_2$ such that $M = K \cup C$, where C is the union of a system of components of $\mathbb{R}^2 \setminus K$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 4.11 as follows.

Suppose that $M \in \mathcal{D}_2$ and let P be the set of all isolated points of M. By Corollary 4.13, P is discrete. We know that, for every $z \in \partial M \setminus P$, there are $\varrho^z > 0$, $m_z \in \mathbb{N}$, (s)-sets $S_1^z, \dots, S_{m_z}^z$ and rotations $\gamma_1^z, \dots, \gamma_{m_z}^z$ as in Theorem 4.11 (iii). By Remark 4.12, we can suppose $\varrho^z \leqslant 1$ and $\operatorname{diam} S_i^z \leqslant 1$, $i = 1, \dots, m_z$. The system $\{U(z, \varrho^z)\colon z \in \partial M \setminus P\}$ is an open cover of $\partial M \setminus P$. Hence, since $\partial M \setminus P$ is closed and locally compact, we can find $I \subset \mathbb{N}$ such that for every $n \in I$, there are $z_n \in \partial M \setminus P$, $\varrho_n > 0$, $k_n \in \mathbb{N}$, (s)-sets S_k^n, $k = 1, \dots, k_n$, and isometries γ_k^n, $k = 1, \dots, k_n$, such that

(a) $\partial M \setminus P \subset \bigcup\limits_{n \in I} U(z_n, \varrho_n)$,

(b) the system $\{U(z_n, \varrho_n)\colon n \in I\}$ is locally finite,

(c) $\operatorname{diam} S_k^n \leqslant 1$, $n \in I$, $k = 1, \dots, k_n$,

(d) $\partial M \cap U(z_n, \varrho_n) \subset \bigcup\limits_{k=1}^{k_n} (z_n + \gamma_k^n(S_k^n)) \subset M$ for every $n \in I$.

Put

$$K := P \cup \bigcup_{n\in I} \bigcup_{k=1}^{k_n} (z_n + \gamma_k^n(S_k^n)). \tag{4.25}$$

By (b) and (c) we obtain that the system $\{\bigcup_{k=1}^{k_n} (z_n + \gamma_k^n(S_k^n))\colon\ n \in I\}$ is a locally finite system of closed nowhere dense sets, and therefore K is closed nowhere dense. Moreover, $\partial M \subset K = \partial K$ by (a) and (d) and $K \subset M$ by (d). Finally, $K \in \mathcal{D}_2$ by Corollary 3.9, Remark 2.17 (ii) and Corollary 4.13.

The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow from Lemma 4.14. □

Remark 4.16.

(i) Lemma 4.14 shows that each nowhere dense $\mathcal{D}_2$ set K yields (via Lemma 4.14 (i)) some (sometimes infinitely many) $\mathcal{D}_2$ sets M with nonempty interior.

(ii) The problem whether a given closed set $M \subset \mathbb{R}^2$ belongs to $\mathcal{D}_2$ does not reduce (by our results) to the problem whether a corresponding nowhere dense set $K \subset \mathbb{R}^2$ belongs to $\mathcal{D}_2$, since there are usually many nowhere dense sets $K \subset M$ with $\partial M \subset K$. Note that these conditions hold for $K := \partial M$, but [10], Example 4.1 (or Example 5.9 below) gives an example of $M \in \mathcal{D}_2$ with $\partial M \notin \mathcal{D}_2$.

Finally, as a consequence of Theorem 4.11 and the proof of Theorem 4.15, we easily obtain the following characterizations of nowhere dense sets in $\mathcal{D}_2$:

Theorem 4.17. *Let $M \subset \mathbb{R}^2$ be a nowhere dense closed set and let P be the set of all isolated points of M. Then the following conditions are equivalent:*

(i) $M \in \mathcal{D}_2$,

(ii) *for every $z \in M \setminus P$, there are $\varrho > 0$, finitely many (s)-sets $S_1, \dots, S_m$ and pairwise different rotations $\gamma_1, \dots, \gamma_m$ such that*

$$M \cap U(z, \varrho) = \bigcup_{i=1}^{m} (z + \gamma_i(S_i)) \cap U(z, \varrho), \tag{4.26}$$

(iii) *for every $z \in M \setminus P$, there are $\varrho > 0$, finitely many (s)-sets $S_1, \dots, S_m$ and rotations $\gamma_1, \dots, \gamma_m$ such that (4.26) holds,*

(iv) *P is discrete and there exists a system $(S_\alpha)_{\alpha\in A}$ of (s)-sets and a system $(\gamma_\alpha)_{\alpha\in A}$ of isometries of $\mathbb{R}^2$ such that the system $(\gamma_\alpha(S_\alpha))_{\alpha\in A}$ is locally finite and such that*

$$M = P \cup \bigcup_{\alpha\in A} \gamma_\alpha(S_\alpha).$$

P r o o f. Denote by (ii)* (or, (iii)*) the condition which we obtain if we replace in (ii) (or, (iii)) equation (4.26) by the inclusions

$$M \cap U(z, \varrho) \subset \bigcup_{i=1}^{m} (z + \gamma_i(S_i)) \subset M. \tag{4.27}$$

Since $M = \partial M$, condition (4.27) is equivalent to (4.23), and so the equivalence of (i), (ii)* and (iii)* follows immediately from Theorem 4.11. Further, (4.27) clearly implies (4.26), and consequently (ii)* implies (ii) and (iii)* implies (iii).

Now we will show (iii) $\Rightarrow$ (iii)*. So suppose that (iii) holds and $x \in M \setminus P$ is given. Find $S_1, \dots, S_m$ and $\gamma_1, \dots, \gamma_m$ by (iii) and choose $\tilde{\varrho} > 0$ so small that $\widetilde{S}_i := S_i \cap ([0, \tilde{\varrho}] \times \mathbb{R}) \subset U(0, \varrho)$, $i = 1, \dots, m$. Then each $\widetilde{S}_i$ is an (s)-set by Remark 3.3 and clearly $M \cap U(z, \tilde{\varrho}) \subset \bigcup_{i=1}^{m} (z + \gamma_i(\widetilde{S}_i)) \subset M$, and thus we have proved (iii)*. The above argument proves also (ii) $\Rightarrow$ (ii)*.

Thus we obtain the equivalence of (i), (ii) and (iii).

To prove (i) $\Rightarrow$ (iv), suppose $M \in \mathcal{D}_2$. Then P is discrete by Corollary 4.13. Further observe that K from Theorem 4.15 equals to M (by Theorem 4.15 (ii)). So, chosing z_n, γ_k^n and S_k^n as in the proof of Theorem 4.15, we obtain that (4.25) holds and $M = K$. Since we know that the system of all sets of the form $z_n + \gamma_k^n(S_k^n)$ from (4.25) is locally finite, (iv) holds. Finally, the implication (iv) $\Rightarrow$ (i) follows from Corollary 3.9 and Remark 2.17. $\square$

Remark 4.18. In Theorem 4.17, $M = \partial M$ and so (4.23) implies (4.26). Consequently, Remark 4.12 shows that, in conditions (ii) and (iii) of Theorem 4.17, we can demand that both ϱ and $\operatorname{diam}(\bigcup_{i=1}^{m} (z + \gamma_i(S_i)))$ "are arbitrarily small".

An immediate consequence of Theorem 4.17 is the following result.

Corollary 4.19. *A nonempty nowhere dense perfect compact set is a $\mathcal{D}_2$ set if and only if it is a finite union of isometric copies of (s)-sets.*

The following result shows that, in some sense, it suffices to investigate connected $\mathcal{D}_2$ sets only.

Theorem 4.20. *A closed set $\emptyset \neq M \subset \mathbb{R}^2$ is a $\mathcal{D}_2$ set if and only if*

(i) *each component of M is a $\mathcal{D}_2$ set and*

(ii) *the system of all components of M is discrete.*

P r o o f. Suppose $M \in \mathcal{D}_2$ and consider an arbitrary $z \in M$ and the component C_z of M that contains z. To prove (ii), we will find $\varrho > 0$ such that

$$C_z \text{ is the only component of } M \text{ intersecting } U(z, \varrho). \tag{4.28}$$

The existence of ϱ is obvious if z is an isolated point of M. Otherwise we can find, by Theorem 4.11, $\varrho > 0$, (s)-sets $S_1, \dots, S_m$ and rotations $\gamma_1, \dots, \gamma_m$ such that (4.23) holds. Using Remark 3.2, we obtain

$$\bigcup_{i=1}^{m} (z + \gamma_i(S_i)) \subset C_z. \tag{4.29}$$

Let C be a component of M with $C \cap U(z, \varrho) \neq \emptyset$.

If $\partial C \cap U(z, \varrho) = \emptyset$, then $C \cap U(z, \varrho)$ is nonempty and both open and closed in $U(x, \varrho)$, so $U(z, \varrho) \subset C$, and thus $C = C_z$.

If $\partial C \cap U(z, \varrho) \neq \emptyset$, choose a point $q \in \partial C \cap U(z, \varrho)$ and observe that $q \in \partial C \subset C$. Further, since $q \in \partial C \subset \partial M$, by (4.23) and (4.29) we obtain $z \in C_z$. Thus $C = C_z$ and (ii) is proved.

To prove (i), consider a component C of M. To prove $C \in \mathcal{D}_2$, by Lemma 2.1 (ii) it is sufficient to show that d_C is locally DC on $\mathbb{R}^2$. Using (2.4), we see that it is sufficient to show that, for each $z \in C$, the function d_C is DC on a neigbourhood of z. By (ii) we can choose $\varrho > 0$ such that (4.28) holds. Then clearly $d_C = d_M$ on $U(z, \frac{1}{2}\varrho)$, and thus d_C is DC on $U(z, \frac{1}{2}\varrho)$.

Finally note that if (i) and (ii) hold, then $M \in \mathcal{D}_2$ by Remark 2.17 (ii). $\square$

5. Properties of $\mathcal{D}_2$ sets and images of $\mathcal{D}_2$ sets

First we will prove several properties of (s)-sets. Then, using our characterization theorems, we will obtain some results on general $\mathcal{D}_2$ sets. Finally we will prove Theorem 5.12 on the stability of $\mathcal{D}_2$ sets with respect to some deformations.

Recall that we already mentioned some simple properties of (s)-sets; see Remarks 3.2 and 3.3.

Further note that (3.1) easily implies that, for each (s)-set S,

$$\operatorname{Tan}(S, (0,0)) \cap S^1 = (1,0). \tag{5.1}$$

An easy consequence of the "mixing lemmas" is the following fact.

Lemma 5.1. *Let $S \subset \mathbb{R}^2$ be an (s)-set and $\pi_1(S) =: [0, r]$. Then there exists $K > 0$ such that each continuous $f\colon [0, r] \to \mathbb{R}$ with* $\operatorname{graph} f \subset S$ *is a K-Lipschitz DCR function.*

P r o o f. Let $f_1, \dots, f_k$ and H be as in Definition 3.1. By Lemma 2.7 (ii) we can choose $K > 0$ such that all f_i, $1 \leqslant i \leqslant k$, are K-Lipschitz functions. Let $f\colon [0, r] \to \mathbb{R}$ be a continuous function with $\operatorname{graph} f \subset S$. Then, using (3.1), we obtain that f is K-Lipschitz by Lemma 2.11 and DCR by Lemma 2.9. $\square$

Corollary 5.2. *If S is an (s)-set and H is as in (3.2), then all functions $h \in H$ are DCR functions, and they are equally Lipschitz.*

Lemma 5.3. *Let $S \subset \mathbb{R}^2$ be an (s)-set with $\pi_1(S) =: [0, r]$ and let H be as in (3.2). Then there exists a countable set $H^* \subset H$ such that $S = \bigcup_{h \in H^*} \operatorname{graph} h$.*

Proof. By (3.2), we have

$$S = \bigcup_{h \in H} \operatorname{graph} h \tag{5.2}$$

and, by Corollary 5.2, there exists $K > 0$ such that each function $h \in H$ is K-Lipschitz. Further choose $k \in \mathbb{N}$ by (3.1).

Now choose (using the definition of k and (5.2)), for each $t \in \mathbb{Q} \cap [0, r]$, functions $h_1^t, \dots, h_k^t \in H$ such that

$$S_{[t]} = \{h_1^t(t), \dots, h_k^t(t)\}$$

and set

$$H^* := \bigcup \{h_i^t \colon \ t \in \mathbb{Q} \cap [0, r],\ 1 \leqslant i \leqslant k\}.$$

Then H^* is countable. To prove

$$S = \bigcup_{h \in H^*} \operatorname{graph} h, \tag{5.3}$$

consider an arbitrary point $(x, y) \in S$. Since $S_{[x]}$ is finite, we can choose $\varepsilon > 0$ such that $S_{[x]} \cap (y - \varepsilon, y + \varepsilon) = \{y\}$. Further choose $x^* \in \mathbb{Q} \cap [0, r]$ such that $|x - x^*| < \varepsilon(2K)^{-1}$. By (5.2) there exists $h \in H$ with $h(x) = y$. Since $h(x^*) \in S_{[x^*]}$, by the definition of H^* there exists $h^* \in H^*$ with $h^*(x^*) = h(x^*)$. Since h, h^* are K-Lipschitz, we have

$$|h(x^*) - h(x)| \leqslant K|x - x^*| < \frac{\varepsilon}{2}, \quad |h(x^*) - h^*(x)| = |h^*(x^*) - h^*(x)| \leqslant K|x - x^*| < \frac{\varepsilon}{2},$$

and so $|h^*(x) - y| = |h^*(x) - h(x)| < \varepsilon$. Since $h^*(x) \in S_{[x]}$, we have $h^*(x) = y$, and thus (5.3) follows. □

Corollary 5.2 and Lemma 5.3 have the following immediate consequence.

Corollary 5.4. *Each (s)-set is a countable union of DC graphs.*

The following result easily follows.

Proposition 5.5. *Each nowhere dense $\mathcal{D}_2$ set M is a countable union of DC graphs.*

P r o o f. The statement follows from Theorem 4.17 (iv), Corollaries 4.13 and 5.4 and the easy fact that the image of a DC graph under an isometry of $\mathbb{R}^2$ is a DC graph. $\square$

The following example (which essentially coincides with [10], Example 4.10) shows a rather simple (s)-set which is not a finite union of DC graphs.

Example 5.6. Let $r := \frac{1}{2}$,

$$f_1(x) = x^5, \quad f_2(x) = -x^5, \quad x \in [0, r], \quad \text{and}$$
$$f_3(x) = x^5 \cos\frac{\pi}{x}, \quad x \in (0, r], \quad f_3(0) = 0.$$

Then Lemma 2.1 (vii) easily implies that f_1, f_2, f_3 are DCR functions. Let

$$A_k := \operatorname{graph}(f_3|_{((2k+1)^{-1},(2k)^{-1})}), \quad k = 1, 2, \ldots,$$
$$S := \operatorname{graph} f_1 \cup \operatorname{graph} f_2 \cup \bigcup_{k=1}^{\infty} A_k.$$

Since $f_3((2k)^{-1}) = f_1((2k)^{-1})$ and $f_3((2k+1)^{-1}) = f_2((2k+1)^{-1})$, $k = 1, 2, \ldots$, it is easy to see that S is an (s)-set. Further, it is easy to show (using, e.g., Remark 4.6 (i)) that every DC graph $B \subset S$ intersects at most one of the sets A_k. Consequently, S is not a finite union of DC graphs.

Proposition 5.7.

(i) *Each $\mathcal{D}_2$ set M is locally pathwise connected; in particular, it is locally connected.*

(ii) *Each connected $\mathcal{D}_2$ set M is pathwise connected. Moreover, any two points $x, y \in M$ can be connected by a rectifiable curve lying in M.*

P r o o f. Let $z \in M$, $r > 0$, and $U := U(z, r) \cap M$. To prove (i), it is sufficient to find a pathwise connected neigbhourhood $V \subset U$ of z in the subspace M. If z is an isolated or an interior point of M, the existence of V is obvious. Otherwise $z \in \partial M$ and we can find, by Remark 4.12, $\varrho \in (0, r)$, (s)-sets $S_1, \ldots, S_m$ and rotations $\gamma_1, \ldots, \gamma_m$ such that

$$\partial M \cap U(z, \varrho) \subset Z \subset M \quad \text{and} \quad Z \subset U(z, r), \tag{5.4}$$

where $Z := \bigcup_{i=1}^{m} (z + \gamma_i(S_i))$. Set

$$V := (M \cap U(z, \varrho)) \cup Z. \tag{5.5}$$

It is clearly sufficient to prove that V is pathwise connected. Note that Remark 3.2 implies that Z is pathwise connected and consider an arbitrary $y \in (M \cap U(z, \varrho)) \setminus Z$. Using (5.4), we obtain $y \in \operatorname{int} M \cap U(z, \varrho)$. It is easy to show that there exists a point $w \in \overline{y, z} \cap \partial M$ such that $\overline{y, w} \subset M$. Since clearly $\overline{y, w} \subset U(z, \varrho)$, we have $\overline{y, w} \subset V$. So y can be connected by a path in V with the point w which belongs to Z by (5.4). Consequently, V is pathwise connected.

The first part of (ii) holds since every connected, locally pathwise connected topological space is pathwise connected (see, e.g., [18], Theorem 27.5). To argue that the "moreover part" holds, we will say (for a while) that a set $A \subset \mathbb{R}^2$ is *r-path connected*, if any two points x and y in A can be connected in A by a rectifiable path. Corollary 5.2 implies that each (s)-set is r-path connected. Consequently, the argument in the proof of (i) gives that each V as in (5.5) is even r-path connected (and thus M is "locally r-pathwise connected"). So an obvious modification of the (standard easy) proof of [18], Theorem 27.5 gives that M is r-path connected. □

Remark 5.8. Using a straightforward easy (but not trivial) modification of the proof of [18], Theorem 27.5, we can obtain the following stronger result:

If M is a connected $\mathcal{D}_2$ set and $x \neq y \in M$, then there exist numbers $t_1 < t_2 < \ldots < t_m$ and a continuous injective $f\colon [t_1, t_m] \to M$ such that $f(t_1) = x$, $f(t_m) = y$ and each set $f([t_k, t_{k+1}])$, $k = 1, \ldots, m-1$, is a DC graph.

Note that this statement is equivalent to the assertion that every $x \neq y \in M$ can be connected in M by a simple curve of finite turn; for the notion of the turn see, e.g., [3].

Indeed, it is not difficult to see that each (s)-set and, consequently, also each V as in (5.5) has this connectivity property.

For each $\mathcal{D}_2$ set M, the system of all components of M is discrete (and so countable) by Theorem 4.20. In the following example, we show that the system of all components of ∂M can be uncountable.

Example 5.9. Let $C \subset [0, 1]$ be the classical Cantor ternary set and let $\{I_n\colon n \in \mathbb{N}\}$ be all bounded components of $\mathbb{R} \setminus C$. For each $n \in \mathbb{N}$, choose an interval $[u_n, v_n] \subset I_n$ and set $F := \overline{\bigcup_{n \in \mathbb{N}} [u_n, v_n]}$. Then $f := (d_F)^2$ is DC on $\mathbb{R}$ (see, e.g., [1], page 976), and so the set

$$K := \operatorname{graph} f \cup \operatorname{graph}(-f)$$

is a (nowhere dense) $\mathcal{D}_2$ set by (1.1) and (2.6). Put

$$M := \{(x, y)\colon\ y \geqslant f(x)\} \cup \{(x, y)\colon\ y \leqslant -f(x)\}.$$

Then M is a $\mathcal{D}_2$ set by Lemma 4.14. It is easy to see that $\pi_1(\partial M) = \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} (u_n, v_n)$ and so $\pi_1(\partial M)$ has uncountably many components. Consequently, ∂M has uncountably many components as well.

In particular, ∂M is not a $\mathcal{D}_2$ set by Theorem 4.20.

We already observed (see Corollary 4.13) that, for each $\mathcal{D}_2$ set M, the set of all isolated points of M is discrete. Now we prove a related result concerning exceptional points of $\mathcal{D}_2$ sets of another type.

Proposition 5.10. *Let M be a $\mathcal{D}_2$ set. Then the set*

$$E_M := \{z \in M\colon\ \mathrm{card}(\mathrm{Tan}(M, z) \cap S^1) = 1\} \tag{5.6}$$

is discrete.

P r o o f. First consider the case when M is an (s)-set; let $r > 0$, $f_1, \dots, f_k$ and H be as in Definition 3.1. Then

$$E_M \cap U(0, r) \subset \{0\}. \tag{5.7}$$

Indeed, if $z = (x, y) \in (M \cap U(0, r)) \setminus \{0\}$, then $0 < x < r$ and by (3.2) there exists $h \in H$ with $h(x) = y$. Since h is a DCR function by Corollary 5.2, we have $z \notin E_M$ (e.g., by Remark 4.6 (i)).

Further consider the case when M is a nowhere dense $\mathcal{D}_2$ set. Let P be the set of all isolated points of M. Obviously, for each $z \in (\mathbb{R}^2 \setminus M) \cup P$, there is an $\omega > 0$ such that $E_M \cap U(z, \omega) = \emptyset$. If $z \in M \setminus P$, let $\varrho > 0$, $S_1, \dots, S_m$ and $\gamma_1, \dots, \gamma_m$ be as in Theorem 4.17 (iii). Using (5.7) for $M = S_i$, $i = 1, \dots, m$, we easily obtain $\omega > 0$ such that $E_M \cap U(z, \omega) \subset \{z\}$ and conclude that E_M is a discrete set.

Finally consider the case of a general $\mathcal{D}_2$ set M. Let $M = K \cup C$ be the decomposition of M from Theorem 4.15 (iii). Since K is a nowhere dense $\mathcal{D}_2$ set, we know that E_K (defined as in (5.6)) is a discrete set. Thus it is sufficient to prove $E_M \subset E_K$. To this end, consider an arbitrary point $z \in E_M$. Then clearly $z \notin C$, and consequently $z \in K$. It is easy to see that z is not an isolated point of K, and therefore $\mathrm{card}(\mathrm{Tan}(K, z) \cap S^1) \geqslant 1$. Since $\mathrm{Tan}(K, z) \subset \mathrm{Tan}(M, z)$, we obtain $z \in E_K$, which completes the proof. $\square$

An important application of our characterizations of $\mathcal{D}_2$ sets is Theorem 5.12 below on images of $\mathcal{D}_2$ sets. First we prove a lemma on images of (s)-sets.

Lemma 5.11. *Let $0 \in G \subset \mathbb{R}^2$ be an open set, $c > 0$, and let $F\colon G \to \mathbb{R}^2$ be a locally DC mapping such that $F(0) = 0$ and $F'_+(0, (1, 0)) = (c, 0)$. Let $S \subset G$ be* *an (s)-set. Then there exist $a > 0$ and $b > 0$ such that*

$$S^* := F(S \cap ((-\infty, a] \times \mathbb{R})) \cap ((-\infty, b] \times \mathbb{R}) \tag{5.8}$$

is an (s)-set.

P r o o f. First note that F is locally Lipschitz on G by Lemma 2.1 (iii). Let $f_1, \dots, f_k$ be DCR functions on $[0, r]$ and H be a set of continuous functions on $[0, r]$ as in Definition 3.1. Without any loss of generality, we can suppose that graph $f_i \subset G$, $i = 1, \dots, k$. Indeed, otherwise we can diminish $r > 0$ and use Remark 3.3.

For $i = 1, \dots, k$, let

$$\varphi^i(x) = (\varphi_1^i(x), \varphi_2^i(x)) := F((x, f_i(x))), \quad x \in [0, r].$$

Since f_i is a DCR function, we can find $\varepsilon_i > 0$ and a DC extension $\tilde{f}_i \colon (-\varepsilon_i, r+\varepsilon_i) \to \mathbb{R}$ of f_i such that $(x, \tilde{f}_i(x)) \in G$, $x \in (-\varepsilon_i, r + \varepsilon_i)$. Then

$$\widetilde{\varphi}^i(x) = (\widetilde{\varphi}_1^i(x), \widetilde{\varphi}_2^i(x)) := F((x, \tilde{f}_i(x))), \quad x \in (-\varepsilon_i, r + \varepsilon_i),$$

is a DC mapping by Lemma 2.1 (ii), (iv). Consequently, φ_1^i and φ_2^i are DCR functions on $[0, r]$ by Lemma 2.7 ((i) $\Leftrightarrow$ (v)). Let $\eta_i(x) := (x, \tilde{f}_i(x))$, $x \in (-\varepsilon_i, r + \varepsilon_i)$. Then $(\eta_i)'_+(0) = (1, 0)$ and consequently, by the chain rule for one-sided directional derivatives (see, e.g., [14], Propositions 3.6 (i) and 3.5),

$$(\widetilde{\varphi}^i)'_+(0) = (c, 0), \quad (\varphi_1^i)'_+(0) = (\widetilde{\varphi}_1^i)'_+(0) = c, \quad (\varphi_2^i)'_+(0) = (\widetilde{\varphi}_2^i)'_+(0) = 0.$$

Consequently, Lemma 2.2 gives that c is the strict right derivative of $\widetilde{\varphi}_1^i$ at 0, which easily implies that there exist $0 < r_i < r$ and $0 < \varrho_i$ such that $\psi_i := \varphi_1^i|_{[0,r_i]}$ is an increasing DCR function and $\psi_i \colon [0, r_i] \to [0, \varrho_i]$ is a bilipschitz bijection. Then Lemma 2.8 implies that $h_i := \varphi_2^i \circ (\psi_i)^{-1}$ is a DCR function on $[0, \varrho_i]$ and it is easy to see that

$$F(\operatorname{graph}(f_i|_{[0,r_i]})) = \operatorname{graph} h_i \quad \text{and} \quad (h_i)'_+(0) = 0.$$

Set

$$a := \min(r_1, \dots, r_k), \quad b := \min(\varphi_1^1(a), \dots, \varphi_1^k(a)) \quad \text{and} \quad f_i^* := h_i|_{[0,b]}, \quad i = 1, \dots, k.$$

Then clearly the set S^* from (5.8) satisfies $S^* \subset \bigcup_{i=1}^k \operatorname{graph} f_i^*$.

For each $h \in H$, put

$$E_h := F(\operatorname{graph} h|_{[0,a]}) \cap ((-\infty, b] \times \mathbb{R}).$$

Since $S^* = \bigcup_{h \in H} E_h$, to prove that S^* is an (s)-set it suffices to show that, for each $h \in H$, the set E_h is a graph of a continuous function h^* on $[0, b]$. Since E_h is compact (and each function with compact graph is continuous), it is sufficient to prove that

$$E_h \text{ is a graph of a function } h^* \text{ on } [0,b]. \tag{5.9}$$

Set $\omega(x) := \pi_1(F((x,h(x))))$, $x \in [0,a]$. Then ω is continuous and, consequently, $\omega([0,a])$ is a closed interval. Since $\omega(0) = 0$ and, for some $1 \leqslant i \leqslant k$, $\omega(a) = \varphi_1^i(a) \geqslant b$, we obtain $[0,b] \subset \omega([0,a])$. So, to prove (5.9), it is sufficient to show that ω is injective. Suppose, to the contrary, that there exist $0 \leqslant x_1 < x_2 \leqslant a$ such that $\omega(x_1) = \omega(x_2)$. Set $u_0 := x_1$. Further observe that there exists $1 \leqslant i_0 \leqslant k$ such that $h(x_1) = f_{i_0}(x_1)$ and $u_1 := \max\{u \in [x_1,x_2]\colon\ h(u) = f_{i_0}(u)\} > u_0$. Then clearly either $u_1 = x_2$ or we can choose $1 \leqslant i_1 \leqslant k$ such that $h(u_1) = f_{i_1}(u_1)$ and $u_2 := \max\{u \in [x_1,x_2]\colon\ h(u) = f_{i_1}(u)\} > u_1$. Proceeding in this way, we obtain numbers $x_1 = u_0 < u_1 < \ldots < u_q = x_2$ with $1 \leqslant q \leqslant k$ and pairwise different indices $i_0, i_1, \ldots, i_{q-1}$ such that $h(u_k) = f_{i_k}(u_k)$ and $h(u_{k+1}) = f_{i_k}(u_{k+1})$ for each $0 \leqslant k \leqslant q-1$. Then $\omega(u_k) = \varphi_1^{i_k}(u_k) < \varphi_1^{i_k}(u_{k+1}) = \omega(u_{k+1})$, and therefore $\omega(x_1) = \omega(u_0) < \omega(u_1) < \ldots < \omega(u_q) = \omega(x_2)$, a contradiction, which completes the proof. □

Theorem 5.12. *Let $G \subset \mathbb{R}^2$, $G^* \subset \mathbb{R}^2$ be open sets and let $F\colon G \to G^*$ be a bijection which is locally bilipschitz and locally DC. Let $M \subset G$ be a $\mathcal{D}_2$ set such that $F(M)$ is a closed set. Then $F(M)$ is a $\mathcal{D}_2$ set.*

P r o o f. First consider the case when M is nowhere dense.

To prove that $M^* := F(M) \in \mathcal{D}_2$, we will verify the validity of condition (iii) of Theorem 4.17 for M^*. To this end, consider an arbitrary point $z^* \in M^*$ which is not an isolated point of M^* and set $z := F^{-1}(z^*)$. Since $M \in \mathcal{D}_2$, by Theorem 4.17 (iii), there exist $\varrho > 0$, (s)-sets $S_1, \ldots, S_m$ and rotations $\gamma_1, \ldots, \gamma_m$ such that

$$M \cap U(z,\varrho) = \bigcup_{i=1}^{m} (z + \gamma_i(S_i)) \cap U(z,\varrho). \tag{5.10}$$

Remark 4.18 shows that we can suppose that

$$U(z,\varrho) \subset G \quad \text{and} \quad z + \gamma_i(S_i) \subset G, \quad i = 1, \ldots, m. \tag{5.11}$$

For each $i = 1, \ldots, m$, we will apply Lemma 5.11 in the following way. Set $v_i := \gamma_i((1,0))$ and $w_i := F'_+(z, v_i)$. Since F is locally bilipschitz, we have $w_i \neq 0$, and consequently, we can choose a rotation γ_i^* and $c_i > 0$ such that $\gamma_i^*((c_i,0)) = w_i$. Now, for each $i = 1, \ldots, m$, set $\beta_i(u) := z + \gamma_i(u)$, $\beta_i^*(u) := z^* + \gamma_i^*(u)$, $u \in \mathbb{R}^2$, and

$$F^i := (\beta_i^*)^{-1} \circ F \circ \beta_i. \tag{5.12}$$

Then F^i is a locally bilipschitz and locally DC bijection from $G_i := (\beta_i)^{-1}(G)$ onto $G_i^* := (\beta_i^*)^{-1}(G^*)$, $F^i(0) = 0$ and $(F^i)'(0,(1,0)) = (c_i, 0)$. Since $S_i \subset G_i$ by (5.11), Lemma 5.11 implies that there exist $a_i > 0$ and $b_i > 0$ such that

$$S_i^* := F^i(S_i \cap ((-\infty, a_i] \times \mathbb{R})) \cap ((-\infty, b_i] \times \mathbb{R})$$

is an (s)-set. Now choose $\varrho^* > 0$ so small that

$$\varrho^* < \min(b_1, \dots, b_m) \quad \text{and} \quad \operatorname{diam} F^{-1}(U(z^*, \varrho^*)) < \min(\varrho, a_1, \dots, a_m). \tag{5.13}$$

Set $V := F^{-1}(U(z^*, \varrho^*))$. Then clearly, for each i,

$$(\beta_i)^{-1}(V) \subset (-\infty, a_i) \times \mathbb{R} \quad \text{and} \quad (\beta_i^*)^{-1}(U(z^*, \varrho^*)) \subset (-\infty, b_i) \times \mathbb{R}$$

and, consequently,

$$F(\beta_i(S_i) \cap V) = \beta_i^*(S_i^*) \cap U(z^*, \varrho^*). \tag{5.14}$$

Since $V \subset U(z, \varrho)$ by (5.13), using (5.10) we obtain $M \cap V = \bigcup_{i=1}^{m} (\beta_i(S_i) \cap V)$. Consequently, (5.14) implies

$$M^* \cap U(z^*, \varrho^*) = F(M \cap V) = \bigcup_{i=1}^{m} \beta_i^*(S_i^*) \cap U(z^*, \varrho^*),$$

and thus condition Theorem 4.17 (iii) holds for M^*.

To finish the proof, consider an arbitrary $\mathcal{D}_2$ set $M \subset G$. By Theorem 4.15 there exists a nowhere dense $\mathcal{D}_2$ set K such that $\partial M \subset K \subset M$. Observe that $K^* := F(K)$ is closed, since it is clearly closed in G^* and $\overline{K^*} \subset M^* := F(M) \subset G^*$. Consequently, $\partial M^* = F(\partial M) \subset F(K) = K^*$, and so $\partial M^* \subset K^* \subset M^*$. Since K^* is clearly nowhere dense and $K^* \in \mathcal{D}_2$ by the first part of the proof, Theorem 4.15 implies that $M^* \in \mathcal{D}_2$. $\square$

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