Archivum Mathematicum

Marco Antonio Lázaro Velásquez

A half-space type property in the Euclidean sphere

*Archivum Mathematicum*, Vol. 58 (2022), No. 1, 49–63

Persistent URL: `http://dml.cz/dmlcz/149446`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 58 (2022), 49–63

# A HALF-SPACE TYPE PROPERTY IN THE EUCLIDEAN SPHERE

Marco Antonio Lázaro Velásquez

*Dedicated to Manfredo P. do Carmo, in memory*

Abstract. We study the notion of strong $r$-stability for the context of closed hypersurfaces $\Sigma^n$ ($n \geq 3$) with constant $(r+1)$-th mean curvature $H_{r+1}$ immersed into the Euclidean sphere $\mathbb{S}^{n+1}$, where $r \in \{1, \dots, n-2\}$. In this setting, under a suitable restriction on the $r$-th mean curvature $H_r$, we establish that there are no $r$-strongly stable closed hypersurfaces immersed in a certain region of $\mathbb{S}^{n+1}$, a region that is determined by a totally umbilical sphere of $\mathbb{S}^{n+1}$. We also provide a rigidity result for such hypersurfaces.

## 1. Introduction and statements of the results

The notion of *stability* concerning closed hypersurfaces of constant mean curvature in Riemannian manifolds was first studied by Barbosa and do Carmo in [8], and Barbosa, do Carmo and Eschenburg in [9], where they proved that geodesic spheres are the only stable critical points in a simply connected space form of the area functional for volume-preserving variations. On the other hand, with respect to the notion of *strong stability* related to constant mean curvature closed hypersurfaces (that is, for all variations, not necessarily volume-preserving variations), it is well known that *there are no strongly stable closed hypersurfaces with constant mean curvature in the Euclidean sphere* $\mathbb{S}^{n+1}$ (for instance, see [3, Section 2]). Following the same direction, the author together with Aquino, de Lima and dos Santos obtained in [6] an extension of this result when the space form is either the Euclidean space $\mathbb{R}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$. More precisely, they proved that there does not exist a strongly stable closed hypersurface with constant mean curvature $H$ immersed in either $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ ($n \geq 3$) and such that its total umbilicity operator $\Phi$ satisfies the condition

$$|\Phi| \leq \frac{2\sqrt{n(n-1)}(H^2+c)}{(n-2)|H|},$$

2020 *Mathematics Subject Classification*: primary 53C42; secondary 53C21.
*Key words and phrases*: Euclidean sphere, closed hypersurfaces, $(r+1)$-th mean curvature, strong $r$-stability, geodesic spheres, upper (lower) domain enclosed by a geodesic sphere.
Received March 7, 2021. Editor J. Slovák.
DOI: 10.5817/AM2022-1-49

where $c = 0$ or $c = -1$ according to the space form be $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$, respectively. When $n = 2$ they also showed that there does not exist strongly stable closed surface with constant mean curvature immersed in either $\mathbb{R}^3$ or $\mathbb{H}^3$.

In [1], Alencar, do Carmo and Colares extended the results of [8] and [9] to the context of closed hypersurfaces with constant scalar curvature in a space form. More specifically, they showed that closed hypersurfaces with constant scalar curvature of a space form are the critical points of the so-called 1-area functional for volume-preserving variations and, for the case $\mathbb{S}^{n+1}$ and $\mathbb{R}^{n+1}$, they also proved that a closed hypersurface with constant scalar curvature is stable if and only if it is a geodesic sphere. More recently Alías, Brasil and Sousa [4] and Cheng [12] have studied the notion of strong stability of closed hypersurfaces with constant (normalized) scalar curvature $R$ immersed into $\mathbb{S}^{n+1}$, where they obtained characterizations of the Clifford torus via some estimates of the first eigenvalue of stability when $R = 1$ and $R > 1$, respectively.

The natural generalization of mean and scalar curvatures for an $n$-dimensional hypersurface of space forms are the $r$-th mean curvatures $H_r$, for $r \in \{0, \ldots, n\}$, where $H_0$ is identically equal to 1 by definition. In fact, $H_1$ is just the mean curvature $H$ and $H_2$ defines a geometric quantity which is related to the scalar curvature.

In [7], Barbosa and Colares studied the notion of *r-stability* (see item $(a)$ of Remark 2 to understand this concept) for closed hypersurfaces immersed with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{0, \ldots, n-2\}$, in space forms. In this setting, they showed that such hypersurfaces in a simply connected space form are $r$-stable if and only if they are geodesic spheres. Moreover, in [14], the author and de Lima were able to establish another characterization result concerning $r$-stability through the analysis of the first eigenvalue of an operator naturally attached to the $r$-th mean curvature.

Motivated by all the work described above, a question appears naturally:

> *Are there closed hypersurfaces which are strongly $r$-stable with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{1, \ldots, n-2\}$?*

With the intention of addressing this issue and seeking a possible answer (affirmative or not), we can slightly change our question and propose the new question:

> *On what conditions is it possible to guarantee the existence (or nonexistence) of hypersupefaces with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{1, \ldots, n-2\}$, that are strongly $r$-stable?*

Our proposal here is to investigate the strong $r$-stability concerning closed hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{1, \ldots, n-2\}$, immersed into the $(n+1)$-dimensional Euclidean sphere $\mathbb{S}^{n+1}$, with $n \geq 3$ (see Definition 1). For this, in Section 2 we recorded some main facts about the hypersurfaces immersed in $\mathbb{S}^{n+1}$ and in Section 3 we describe the variational problem that gives rise to the notion of strong $r$-stability. Next, initially we prove that geodesic spheres of $\mathbb{S}^{n+1}$ are strongly $r$-stable (see Proposition 2), which provides an affirmative answer to our first question. Afterwards, to achieve our goals, we make use of the Riemannian warped product $(0,\pi)\times_{\sin\tau}\mathbb{S}^n$, $\tau\in(0,\pi)$, which models a certain open region $\Omega^{n+1}$ of $\mathbb{S}^{n+1}$ (see equations (4.1), (4.2) and (4.3)) and, in Proposition 3, we calculate the differential operator $L_r$ (associated with the variational problem that defines the notion of strong $r$-stability) acting on an support function $\xi$ (see equation (4.9)) naturally attached to a hypersurface $\psi:\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r\in\{1,\dots,n-2\}$, immersed in $\Omega^{n+1}$. Then, under a suitable restriction on $H_r$ and $H_{r+1}$, we use the formula of $L_r(\xi)$ to show that if a closed hypersurface $\psi:\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r\in\{1,\dots,n-2\}$, in $\mathbb{S}^{n+1}$ is strongly $r$-stable, then it must be a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1}\subset\mathbb{S}^{n+1}$ of level $\tau_0=\frac{\pi}{4}$ (for a better understanding of this region, we recommend the reader to see Definition 2), which provides a partial converse of Proposition 2. More specifically, we have established the following rigidity result for strongly $r$-stable hypersurfaces in $\mathbb{S}^{n+1}$:

**Theorem 1.** *Let $\psi\colon\Sigma^n\looparrowright\Omega^{n+1}\subset\mathbb{S}^{n+1}$ $(n\geq 3)$ be a strongly $r$-stable closed hypersurface with constant $(r+1)$-th mean curvature $H_{r+1}$, $r\in\{1,\dots,n-2\}$. If the $r$-th mean curvature $H_r$ of $\psi:\Sigma^n\looparrowright\Omega^{n+1}$ obeys the condition*

$$H_{r+1}\geq H_r\geq 1\quad\text{on }\Sigma^n, \tag{1.1}$$

*then $\psi(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1}\subset\mathbb{S}^{n+1}$ of level $\tau_0=\pi/4$.*

The motivation to assume the hypothesis (1.1) in Theorem 1 is described in Remark 3, while the restrictions $r\neq\{0,n-1,n\}$ are explained in item $(b)$ of Remark 2. As an immediate consequence of this result, we establish a result of nonexistence for strongly $r$-stable closed hypersurfaces immersed in $\mathbb{S}^{n+1}$, which can be understood as an answer to our second question.

**Theorem 2.** *There is no strongly $r$-stable closed hypersurface $\Sigma^n$ $(n\geq 3)$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r\in\{1,\dots,r+2\}$, immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1}\subset\mathbb{S}^{n+1}$ of level $\tau_0=\pi/4$, with $r$-th mean curvature $H_r$ satisfying the inequality $H_{r+1}\geq H_r\geq 1$ on $\Sigma^n$.*

From our results listed above we can conclude that the region of $\mathbb{S}^{n+1}$ that contains the set of closed hypersurfaces $\psi:\Sigma^n\looparrowright\mathbb{S}^{n+1}$ $(n\geq 3)$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r\in\{1,\dots,n-2\}$, which are strongly $r$-stable and whose $r$-th mean curvature $H_r$ satisfies the condition (1.1), is small. It is in this configuration that our results can be understood as a half-space type property of strongly $r$-stable closed hypersurfaces in the Euclidean sphere $\mathbb{S}^{n+1}$ (cf. Remark 4).

Finally, in Corollary 1 and 2 we write Theorems 1 and 2 for the case of closed hypersurfaces immersed into $\mathbb{S}^{n+1}$ with constant (normalized) scalar curvature $R$. The proofs of the main results of this work is carried out in Section 4.

## 2. Background

Unless stated otherwise, all manifold considered on this work will be connected, while *closed* means compact without boundary. Let $\mathbb{S}^{n+1}$ be the $(n+1)$-dimensional

Euclidean sphere. We will consider immersions $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ of closed orientable hypersurfaces $\Sigma^n$ in $\mathbb{S}^{n+1}$. In this setting, we denote by $d\Sigma$ the volume element with respect to the metric induced by $\psi$, $C^\infty(\Sigma^n)$ the ring of real functions of class $C^\infty$ defined on $\Sigma^n$ and by $\mathfrak{X}(\Sigma^n)$ the $C^\infty(\Sigma^n)$-module of vector fields of class $C^\infty$ on $\Sigma^n$. Since $\Sigma^n$ is orientable, one can choose a globally defined unit normal vector field $N$ on $\Sigma^n$. Let

$$\begin{array}{rcccl} A & : & \mathfrak{X}(\Sigma^n) & \rightarrow & \mathfrak{X}(\Sigma^n) \\ & & Y & \mapsto & A(Y) = -\overline{\nabla}_Y N\,. \end{array} \tag{2.1}$$

denote the shape operator with respect to $N$, so that, at each $q \in \Sigma^n$, $A$ restricts to a self-adjoint linear map $A_q : T_q\Sigma \rightarrow T_q\Sigma$.

According to the ideas established by Reilly [16], for $1 \leq r \leq n$, if we let $S_r(q)$ denote the $r$-th *elementary symmetric function* on the eigenvalues of $A_q$, we get $n$ functions $S_r \in C^\infty(\Sigma^n)$ such that

$$\det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r\, t^{n-r}\,,$$

where $I : \mathfrak{X}(\Sigma^n) \rightarrow \mathfrak{X}(\Sigma^n)$ is the identity operator and $S_0 = 1$ by definition. If $q \in \Sigma^n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_q\Sigma$ formed by eigenvectors of $A_q$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n)\,, \tag{2.2}$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the $r$-th elementary symmetric polynomial on the indeterminates $X_1, \ldots, X_n$.

For $1 \leq r \leq n$, one defines the *r-th mean curvature* $H_r$ (also called *higher order mean curvature*) of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ by

$$\binom{n}{r} H_r = S_r = S_r(\lambda_1, \ldots, \lambda_n)\,. \tag{2.3}$$

In particular, for $r = 1$,

$$H_1 = \frac{1}{n} \sum_{k=1}^{n} \lambda_k = H$$

is the *mean curvature* of the hypersurface $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, which is the main extrinsic curvature. When $r = 2$, $H_2$ defines a geometric quantity which is related to the (intrinsic) *normalized scalar curvature* $R$ of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$. More precisely, it follows from the Gauss equation of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ that

$$R = 1 + H_2. \tag{2.4}$$

We can also define (cf. [16, Section 1]), for $0 \leq r \leq n$, the so-called *r-th Newton transformation* $P_r : \mathfrak{X}(\Sigma^n) \rightarrow \mathfrak{X}(\Sigma^n)$ by setting $P_0 = I$ and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = S_r I - A P_{r-1}\,.$$

A trivial induction shows that

$$P_r = S_r I - S_{r-1} A + S_{r-2} A^2 - \cdots + (-1)^r A^r,$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since $P_r$ is a polynomial in $A$ for every $r$, it is also self-adjoint and commutes with $A$. Therefore, all bases of $T_p(\Sigma^n)$ diagonalizing $A$ at $p \in \Sigma^n$ also diagonalize all of the $P_r$ at $p$. Let $\{e_1, \ldots, e_n\}$ be such a basis. Denoting by $A_i$ the restriction of $A$ to $\langle e_i \rangle^\perp \subset T_p(\Sigma^n)$, it is easy to see that

$$\det(tI - A_i) = \sum_{j=0}^{n-1} (-1)^j S_j(A_i)\, t^{n-1-j},$$

where

$$S_j(A_i) = \sum_{\substack{1 \le j_1 < \ldots < j_m \le n \\ j_1, \ldots, j_m \neq i}} \lambda_{j_1} \cdots \lambda_{j_m}\,. \tag{2.5}$$

With the above notations, it is also immediate to check that

$$P_r(e_i) = S_r(A_i) e_i\,, \tag{2.6}$$

and hence (cf. [7, Lemma 2.1])

$$\begin{cases} \operatorname{tr}(P_r) = (n-r)S_r = b_r H_r\,; \\ \operatorname{tr}(AP_r) = (r+1)S_{r+1} = b_r H_{r+1}\,; \\ \operatorname{tr}(A^2 P_r) = S_1 S_{r+1} - (r+2) S_{r+2} = n\frac{b_r}{r+1} H H_{r+1} - b_{r+1} H_{r+2}\,, \end{cases} \tag{2.7}$$

where $b_r = (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$.

Associated to each Newton transformation $P_r$ one has the second order linear differential operator $L_r \colon C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$, given by

$$L_r(f) = \operatorname{tr}\left(P_r \operatorname{Hess} f\right)\,. \tag{2.8}$$

We observed that $L_0 = \Delta$, the Laplacian operator on $\Sigma^n$, and $L_1 = \square$, the Yau's square operator on $\Sigma^n$ (cf. [13, Equation (1.7)]).

## 3. The variational problem

For a closed orientable hypersurface $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ as in the previous section, a *variation* of it is a smooth mapping $X \colon (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{RP}^{n+1}$ such that, for every $t \in (-\epsilon, \epsilon)$, the map

$$\begin{array}{rcccl} X_t : \Sigma^n & \looparrowright & \mathbb{S}^{n+1} & & \\ q & \mapsto & X_t(q) & = & X(t,q) \end{array} \tag{3.1}$$

is an immersion, with $X_0 = x$. In what follows, we let $d\Sigma_t$ denote the volume element of the metric induced on $\Sigma^n$ by $X_t$, and $N_t$ will stand for the unit normal vector field along $X_t$.

The *variational field* associated to the variation $X \colon (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{S}^{n+1}$ is $\frac{\partial X}{\partial t}|_{t=0} \in \mathfrak{X}(X((-\epsilon, \epsilon) \times \Sigma^n))$. Letting

$$f_t = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle, \tag{3.2}$$

we get

$$\frac{\partial X}{\partial t} = f_t N_t + \left(\frac{\partial X}{\partial t}\right)^{\top},$$

where $(\,\cdot\,)^{\top}$ stands for the tangential component.

The *balance of volume* of the variation $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is the functional

$$\begin{array}{rcl} \mathcal{V} : (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{V}(t) \;=\; \displaystyle\int_{\Sigma^n\times[0,t]} X^*(dV)\,, \end{array}$$

and we say that $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is a *volume-preserving* variation for $x\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ if $\mathcal{V}(t) = \mathcal{V}(0) = 0$, for all $t \in (-\epsilon,\epsilon)$. Moreover, following [7], we define the *$r$-th area functional*

$$\begin{array}{rcl} \mathcal{A}_r : (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{A}_r(t) \;=\; \displaystyle\int_{\Sigma^n} F_r\left(S_1(t), S_2(t), \ldots, S_r(t)\right) d\Sigma_t\,, \end{array}$$

where $S_r(t) = S_r(t,\cdot)$ is the $r$-th elementary symmetric fuunction of $\Sigma^n$ via the immersion (3.1) and $F_r$ is recursively defined by setting $F_0 = 1$, $F_1 = S_1(t)$ and, for $2 \le r \le n-1$,

$$F_r = S_r(t) + \frac{(n-r+1)}{r-1}\,F_{r-2}\,.$$

The following lemma is well known and can be found in [7].

**Lemma 1.** *Let $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a closed hypersurface. If $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is a variation of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ then*

(a) $\frac{d}{dt}\mathcal{V}(t) = \int_{\Sigma^n} f_t d\Sigma_t$, *where $f_t$ is the function defined in* (3.2). *In particular, $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ is a volume-preserving variation for $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ if and only if $\int_{\Sigma^n} f_t d\Sigma_t = 0$ for all $t \in (-\epsilon,\epsilon)$.*

(b) $\frac{d}{dt}\mathcal{A}_r(t) = -b_r\int_{\Sigma^n} H_{r+1}(t) f_t d\Sigma_t$, *where $b_r = (r+1)\binom{n}{r+1}$ and $H_{r+1}(t) = H_{r+1}(t,\cdot)$ is the $(r+1)$-th mean curvature of $\Sigma^n$ via the immersion* (3.1).

**Remark 1.** From [9, Lemma 2.2], given a closed hypersurface $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, if $f \in C^\infty(\Sigma^n)$ is such that

$$\int_{\Sigma^n} f d\Sigma = 0\,, \tag{3.3}$$

then there exists a volume-preserving variation $X\colon (-\epsilon,\epsilon)\times\Sigma^n \to \mathbb{S}^{n+1}$ for $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ whose variational field is just $\frac{\partial X}{\partial t}|_{t=0} = fN$.

In order to characterize hypersurfaces of $\mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature, we will consider the variational problem of minimizing the $r$-th area functional $\mathcal{A}_r$ for all volume-preserving variations of the closed hypersurface $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$.

The *Jacobi functional* $\mathcal{J}_r$ associated to the problem is given by

$$\begin{array}{rcl} \mathcal{J}_r : (-\epsilon,\epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathcal{J}_r(t) \;=\; \mathcal{A}_r(t) + \varrho\mathcal{V}(t)\,, \end{array}$$

where $\varrho$ is a constant to be determined. As an immediate consequence of Lemma 1 we get

$$\frac{d}{dt}\mathcal{J}_r(t) = \int_{\Sigma^n} \{-b_r H_{r+1}(t) + \varrho\} f_t d\Sigma_t \,,$$

where $f_t$ is the function defined in (3.2) and $b_r = (r+1)\binom{n}{r+1}$ and $H_{r+1}(t) = H_{r+1}(t,\cdot)$ is the $(r+1)$-th mean curvature of $\Sigma^n$ via the immersion (3.1). In order to choose $\varrho$, let

$$\overline{\mathcal{H}} = \frac{1}{\mathrm{Area}(\Sigma^n)} \int_{\Sigma^n} H_{r+1} d\Sigma$$

be a integral mean of the function $H_{r+1}$ along the $\Sigma^n$. We call the attention to the fact that, in the case that $H_{r+1}$ is constant, one has

$$\overline{\mathcal{H}} = H_{r+1}\,, \tag{3.4}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\varrho = b_r\overline{\mathcal{H}}$, we arrive at

$$\frac{d}{dt}\mathcal{J}_r(t) = b_r \int_{\Sigma^n} \left\{-H_{r+1}(t) + \overline{\mathcal{H}}\right\} f_t d\Sigma_t \,.$$

In particular,

$$\frac{d}{dt}\mathcal{J}_r(t)\Big|_{t=0} = b_r \int_{\Sigma^n} \left\{-H_{r+1} + \overline{\mathcal{H}}\right\} f_0 d\Sigma. \tag{3.5}$$

Now, following the same ideas of [8, Proposition 2.7], from (3.5), (3.4) and Remark 1 we can establish the following result, which characterizes all the critical points of the variational problem described above.

**Proposition 1.** *Let $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be a closed hypersurface. The following statements are equivalent:*

(a) *$\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ has constant $(r+1)$-th mean curvature functions $H_{r+1}$;*

(b) *we have $\delta_f\mathcal{A}_r = \frac{d}{dt}\mathcal{A}_r(t)|_{t=0} = 0$ for all volume-preserving variations of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$;*

(c) *we have $\delta_f\mathcal{J}_r = \frac{d}{dt}\mathcal{J}_r(t)|_{t=0} = 0$ for all variations of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$.*

Motivated by the ideas established in [4], [2] and [12], we exchanged our studying problem and now we wish to detect hypersurfaces $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ which minimize the Jacobi functional $\mathcal{J}_r$ for all variations of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$. Then, Proposition 1 shows that the critical points for this new variational problem coincide with those of the first variational problem, namely, are the closed hypersurfaces $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ with constant $(r+1)$-th mean curvature $H_{r+1}$. Currently, geodesic spheres of $\mathbb{S}^{n+1}$ and Clifford hypersurfaces of $\mathbb{S}^{n+1}$ are examples for these critical points. So, for such a critical point, we need computing the second variation $\delta^2_f\mathcal{J}_r = \frac{d^2}{dt^2}\mathcal{J}_r(t)|_{t=0}$ of the Jacobi functional $\mathcal{J}_r$. This will motivate us to establish the following notion of stability.

**Definition 1.** Let $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ $(n \geq 3)$ be a closed hypersurface with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{1,\dots,n-2\}$. We say that $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ is strongly $r$-stable if $\delta^2_f\mathcal{J}_r \geq 0$ for all $f \in C^\infty(\Sigma^n)$.

From [7, Proposition 4.4] we get that the sought formula for the second variation $\delta_f^2\mathcal{J}_r$ of $\mathcal{J}_r$ is given by

$$\delta_f^2\mathcal{J}_r = -(r+1)\int_{\Sigma^n} f\mathcal{L}(f)d\Sigma\,, \tag{3.6}$$

where

$$\mathcal{L} = L_r + \frac{nb_r}{r+1}HH_{r+1} - b_{r+1}H_{r+2} + b_rH_r \tag{3.7}$$

is the *Jacobi differential operator* associated with our variational problem. Here, $L_r$ is the differential operator defined in (2.8), $H$, $H_r$, $H_{r+1}$ and $H_{r+2}$ are the mean curvature, the $r$-th mean curvature, the $(r+1)$-th mean curvature and the $(r+2)$-th mean curvature of $\psi\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, respectively, and $b_k = (k+1)\binom{n}{k+1}$ for $k \in \{r, r+1\}$.

**Remark 2.** Regarding our definition of strong stability, we note that:

(a) From a geometrical point of view, the notion of $r$-stability, namely, when $\delta_f^2\mathcal{A}_r \geq 0$ for all $f \in C^\infty(\Sigma^n)$ satisfying the condition (3.3), is more natural than the notion the strong $r$-stability. However, from an analytical point of view, the strong $r$-stability is more natural and easier to use. The analytical interest is due to its possible applications to Geometric Analysis such as: the approach of bifurcation techniques related to our variational problem, the study of evolution problems related to the differential operator of Jacobi $\mathcal{L}$, problems of eigenvalue of $\mathcal{L}$, the search for notions of parabolicity for $\mathcal{L}$, uniqueness (or multiliqueness) of solutions to problems of initial value involving $\mathcal{L}$, among others.

(b) In Definition 1, we put the restriction $r \neq 0$ due to the fact that there are no strongly stable constant mean curvature closed hypersurfaces in $\mathbb{S}^{n+1}$ (cf. [3, Section 2]), whereas the constraint $r \neq \{n+1, n\}$ is due to the explicit expression that admits $\delta_f^2\mathcal{J}_r$ (see equations (3.6) and (3.7)).

In [7, Proposition 5.1] was established that the geodesic spheres of $\mathbb{S}^{n+1}$ are $r$-stable. We note that the proof of this result can be used to affirm that the geodesic spheres of $\mathbb{S}^{n+1}$ are also strongly $r$-stable. Here, for completeness of content, we present a proof.

**Proposition 2.** *For any $r \in \{1, \dots, n-2\}$, the geodesic spheres of $\mathbb{S}^{n+1}$ $(n \geq 3)$ are strongly $r$-stable.*

**Proof.** Let $\Sigma^n$ be a geodesic sphere of $\mathbb{S}^{n+1}$ and let $\iota\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ be its inclusion map into $\mathbb{S}^{n+1}$. Since $\Sigma^n$ is totally umbilical then its principal curvatures are all equal to a certain constant $\lambda$. By choosing the normal vector we may assume that $\lambda \geq 0$. Thus, from (2.2), (2.3) and (2.5), respectively, we have for $r \in \{1, \dots, n-2\}$ that

$$S_r = \binom{n}{r}\lambda^r = \text{constant}\,, \qquad H_r = \lambda^r = \text{constant}$$

and

$$S_r(A_i) = \binom{n-1}{r}\lambda^r = \text{constant}. \tag{3.8}$$

Next, if $e_1, \dots, e_n$ are the principal directions of $\Sigma^n$, from (2.8), (2.6) and (3.8), we get

$$\begin{aligned} L_r(f) &= \sum_{i=1}^{n} \langle \mathrm{Hess}(f)(e_i), P_r(e_i)\rangle \\ &= \binom{n-1}{r}\lambda^r \sum_{i=1}^{n} \langle \mathrm{Hess}(f)(e_i), e_i\rangle = \binom{n-1}{r}\lambda^r \Delta f\,, \end{aligned}$$

for all $f \in C^\infty(\Sigma^n)$.

Then, from (3.6), (3.7) and (2.7), we obtain

$$\begin{aligned} \delta_f^2 \mathcal{J}_r &= -\int_{\Sigma^n} \Big\{ \binom{n-1}{r}\lambda^r \Delta f + b_r H_r f \\ &\quad + \Big( n\frac{b_r}{r+1} H H_{r+1} - b_{r+1} H_{r+2}\Big) f \Big\} f d\Sigma \\ &= -\int_{\Sigma^n} \Big\{ \binom{n-1}{r}\lambda^r f\Delta f + (n-r)\binom{n}{r}\lambda^r f^2 \\ &\quad + \Big[ n\binom{n}{r+1}\lambda^{r+2} - (n-r-1)\binom{n}{r+1}\lambda^{r+2}\Big] f^2 \Big\} d\Sigma \\ &= -\binom{n-1}{r}\lambda^r \int_{\Sigma^n} \left\{ f\Delta f + n f^2 + n\lambda^2 f^2 \right\} d\Sigma \\ &= \binom{n-1}{j}\lambda^r \int_{\Sigma^n} \left\{ -f\Delta f - n(1+\lambda^2) f^2 \right\} d\Sigma\,. \end{aligned} \tag{3.9}$$

Now, let $\eta_1$ be the first eigenvalue of the Laplacian $\Delta$ of $\iota\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$, which admits the following min-max characterization (cf. [11, Section 1.5])

$$\eta_1 = \min \Big\{ -\int_{\Sigma^n} f\Delta f d\Sigma \Big/ \int_{\Sigma^n} f^{\,2}\, d\Sigma \ :\ f \in C^\infty(\Sigma^n)\ , f \neq 0 \Big\}. \tag{3.10}$$

Since $\lambda \geq 0$, from (3.9) and (3.10) we get

$$\delta_f^2\, \mathcal{J}_r \geq \binom{n-1}{r}\lambda^r \int_{\Sigma^n} \left\{ \eta_1 - n(1+\lambda^2)\right\} f^2 d\Sigma\,,$$

for all $f \in C^\infty(\Sigma^n)$. But, since $\iota(\Sigma^n)$ is isometric to an $n$-dimensional Euclidean sphere with constant sectional curvature equal to $\lambda^2+1$, we have that $\eta_1 = n(\lambda^2+1)$. Hence, for every $f \in C^\infty(\Sigma^n)$ we get

$$\delta_f^2 \mathcal{J}_r \geq \binom{n-1}{r}\lambda^r \int_{\Sigma^n} \left\{ \eta_1 - n(1+\lambda^2)\right\} f^2 d\Sigma = 0\,.$$

Therefore, according to Definition 1, $\iota\colon \Sigma^n \looparrowright \mathbb{S}^{n+1}$ must be strongly $r$-stable. $\square$

## 4. Proof of the main results

In order to obtain a rigidity result concerning to strongly $r$-stable closed hypersurfaces immersed into $(n+1)$-dimensional unit Euclidean sphere $\mathbb{S}^{n+1}$, we need to describe a Riemannian warped product that models a certain region of $\mathbb{S}^{n+1}$.

Let $\mathbf{P}$ be the *north pole* of $\mathbb{S}^{n+1}$ and $\mathbb{S}^n$ be the *equator* orthogonal to $\mathbf{P}$. From [15, Example 2], the open region

$$\Omega^{n+1} := \mathbb{S}^{n+1} \setminus \{\mathbf{P}, -\mathbf{P}\} \tag{4.1}$$

is isometric to the Riemannian warped product

$$(0,\pi) \times_{\sin\tau} \mathbb{S}^n, \quad \tau \in (0,\pi) . \tag{4.2}$$

At the moment, making $\mathbf{P} = (0,\dots,0,1) \in \mathbb{S}^{n+1}$ and identifying the point $q = (q_1,\dots,q_{n+1}) \in \mathbb{S}^n$ with $q = (q_1,\dots,q_{n+1},0) \in \mathbb{S}^{n+1}$, we have that the correspondence

$$\begin{array}{rcccl} \Psi : (0,\pi)\times_{\sin\tau}\mathbb{S}^n & \to & \Omega^{n+1}\subset\mathbb{S}^{n+1} & & \\ (\tau,q) & \mapsto & \Psi(\tau,q) & = & (\cos\tau)\,q + (\sin\tau)\,\mathbf{P}, \end{array} \tag{4.3}$$

defines an isometry between (4.2) and (4.1). We denote by

$$\Phi : \Omega^{n+1} \subset \mathbb{S}^{n+1} \to (0,\pi)\times_{\sin\tau}\mathbb{S}^n \tag{4.4}$$

as being the inverse of $\Psi$.

If $d\tau^2$ and $d\sigma^2$ denote the metrics of $(0,\pi)$ and $\mathbb{S}^n$, respectively, then

$$\langle\,,\rangle = (\pi_I)^*\left(d\tau^2\right) + (\sin\tau)^2(\pi_{\mathbb{S}^n})^*\left(d\sigma^2\right) ,$$

is the tensor metric of the Riemannian warped product (4.2), where $\pi_I$ and $\pi_{\mathbb{S}^n}$ denote the projections onto the $(0,\pi)$ and $\mathbb{S}^n$, respectively. In this context, the vector field

$$(\sin\tau)\frac{\partial}{\partial\tau} \in \mathfrak{X}\left((0,\pi)\times_{\sin\tau}\mathbb{S}^n\right)$$

is a *conformal* and *closed* one (in the sense that its dual 1-form is closed), with conformal factor $\cos\tau$. Moreover, from [15, Proposition 1], for each $\tau_0 \in (0,\pi)$, the *slice* $\{\tau_0\}\times\mathbb{S}^n$ of the *foliation*

$$(0,\pi) \ni \tau_0 \longmapsto \{\tau_0\}\times\mathbb{S}^n$$

is a $n$-dimensional geodesic sphere of $\mathbb{S}^{n+1}$, parallel to the equator $\mathbb{S}^n$, with shape operator (see (2.1)) $A_{\tau_0}$ given by

$$\begin{array}{rcccl} A_{\tau_0} : \mathfrak{X}(\{\tau_0\}\times\mathbb{S}^n) & \to & \mathfrak{X}(\{\tau_0\}\times\mathbb{S}^n) & & \\ Y & \mapsto & A_{\tau_0}(Y) & = & -\overline{\nabla}_Y(-\partial_\tau) = \dfrac{(\cos\tau_0)}{(\sin\tau_0)}\,Y \end{array} \tag{4.5}$$

with respect to the orientation given by $-\frac{\partial}{\partial\tau}$. Thus, from (2.2), (2.3) and (4.5), we get for $r\in\{0,\dots,n\}$ that the $r$-th elementary symmetric function $\mathcal{S}_r$ and the $r$-th mean curvature $\mathcal{H}_r$ of each slice $\{\tau_0\}\times\mathbb{S}^n$ are

$$\mathcal{S}_r = \binom{n}{r}(\cot\tau_0)^r \quad \text{and} \quad \mathcal{H}_r = (\cot\tau_0)^r , \tag{4.6}$$

respectively. We note that $\mathcal{S}_r$ and $\mathcal{H}_r$ are constant on $\{\tau_0\}\times\mathbb{S}^n$.

In order to facilitate the understanding of certain regions in the Euclidean sphere, we have established the following notions.

**Definition 2.** Fixed $\tau_0 \in (0,\pi)$, the region

$$\Phi^{-1}\left( (0,\tau_0)\times_{\sin\tau}\mathbb{S}^n \right) = \{\, q\in\mathbb{S}^{n+1} \,:\, \Phi(q)\in(0,\tau_0)\times_{\sin\tau}\mathbb{S}^n \,\}$$

of $\mathbb{S}^{n+1}$ that corresponds to

$$(0,\tau_0)\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n$$

will be called of upper domain enclosed by the geodesic sphere of $\Omega^{n+1}$ of level $\tau_0$. Similarly, the region

$$\Phi^{-1}\left( (\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \right) = \{\, q\in\mathbb{S}^{n+1} \,:\, \Phi(q)\in(\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \,\}$$

of $\mathbb{S}^{n+1}$ that corresponds to

$$(\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n$$

will be called of lower domain enclosed by the geodesic sphere of $\Omega^{n+1}$ of level $\tau_0$. In turn, the regions

$$\Phi^{-1}\left( (0,\tau_0]\times_{\sin\tau}\mathbb{S}^n \right) = \{\, q\in\mathbb{S}^{n+1} \,:\, \Phi(q)\in(0,\tau_0]\times_{\sin\tau}\mathbb{S}^n \,\}$$

and

$$\Phi^{-1}\left( [\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \right) = \{\, q\in\mathbb{S}^{n+1} \,:\, \Phi(q)\in[\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \,\}$$

of $\mathbb{S}^{n+1}$ that corresponds to

$$(0,\tau_0]\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n$$

and

$$[\tau_0,\pi)\times_{\sin\tau}\mathbb{S}^n \subset (0,\pi)\times_{\sin\tau}\mathbb{S}^n,$$

respectively, will be called of closure of the upper domain and closure of the lower domain enclosed by the geodesic sphere of $\Omega^{n+1}$ of level $\tau_0$, where $\Phi$ is the isometry given in (4.4).

For example, from Definition 2 we have that the upper domain enclosed by the geodesic sphere of $\Omega^{n+1}$ of level $\tau=\pi/2$ is the open upper hemisphere (minus the north pole $\mathbf{P}$) of $\mathbb{S}^{n+1}$, which is isometric to the Riemannian warped product

$$\left(0,\frac{\pi}{2}\right)\times_{\sin\tau}\mathbb{S}^n, \quad \tau\in(0,\pi/2)$$

According to the ideas established in [5, Section 5], we will consider that the orientable hypersurfaces $\psi\colon\Sigma^n \looparrowright \Omega^{n+1}\subset\mathbb{S}^{n+1}$ for which their Gauss map $N$ satisfies

$$-1 \le \left\langle \Phi_*(N(q)), \frac{\partial}{\partial\tau} \right\rangle_{\Phi(\psi(q))} < 0$$

for all $q\in\Sigma^n$. In this setting, for such a hypersurface $\psi\colon\Sigma^n \looparrowright \Omega^{n+1}\subset\mathbb{S}^{n+1}$ we define the *normal angle* $\theta$ as being the smooth function

$$\begin{array}{rcccl} \theta: \Sigma^n & \to & \left[0,\dfrac{\pi}{2}\right) & & \\ q & \mapsto & \theta(q) & = & \arccos\left(-\left\langle \Phi_*(N(q)), \dfrac{\partial}{\partial\tau}\right\rangle_{\Phi(\psi(q))}\right). \end{array} \tag{4.7}$$

Thus, on $\Sigma^n$ the normal angle $\theta$ verifies

$$0 < \cos\theta = -\Big\langle \Phi_*(N), \frac{\partial}{\partial\tau}\Big\rangle \leq 1\,. \tag{4.8}$$

Moreover, since the orientation of the slice $\{\tau_0\}\times\mathbb{S}^n$ is given by $-\frac{\partial}{\partial\tau}$, the normal angle $\theta$ of $\{\tau_0\}\times\mathbb{S}^n$ is such that $\cos\theta = 1$.

We need the following result, whose proof is a consequence of a suitable formula due to Barros and Sousa [10].

**Proposition 3.** *Let $\psi\colon \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \geq 2)$ be an orientable hypersurface with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{0,\dots,n-2\}$. If*

$$\begin{array}{rcccl} \xi : \Sigma^n & \to & \mathbb{R} & & \\ q & \mapsto & \xi(q) & = & -\sin\tau\cos\theta(q), \end{array} \tag{4.9}$$

*where $\theta$ is the normal angle of $\Sigma^n$ defined in* (4.7)*, then the formula of the differential operator $L_r$ defined in* (2.8) *acting on $\xi$ is given by*

$$\begin{aligned} L_r(\xi) = &-\left(\frac{nb_r}{r+1}\,HH_{r+1} - b_{r+1}H_{r+2} + b_rH_r\right)\xi \\ &- b_rH_r\sin\tau\cos\theta - b_rH_{r+1}\cos\tau. \end{aligned} \tag{4.10}$$

*where $H$, $H_r$, $H_{r+1}$ and $H_{r+2}$ are the mean curvature, $r$-th mean curvature, $(r+1)$-th mean curvature and $(r+2)$-th mean curvature of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, respectively, and $b_k = (k+1)\binom{n}{k+1}$ for $k \in \{r, r+1\}$. Here, for simplicity we are adopting the abbreviated notations $H_j = H_j \circ \psi^{-1} \circ \Phi^{-1}$, $j \in \{1, r, r+1, r+2\}$, where $\Phi$ is the isometry described in* (4.4)*.*

**Proof.** From Theorem 2 of [10],

$$\begin{aligned} L_r(\xi) = &-\Big(\frac{nb_r}{r+1}\,HH_{r+1} - b_{r+1}H_{r+2} + b_rH_r\Big)\xi \\ &-\ b_rH_r\,\Phi_*(N)(\cos\tau)(\cos\tau) - b_rH_{r+1}\cos\tau\,. \end{aligned} \tag{4.11}$$

Observing that

$$\overline{\nabla}\cos\tau = \Big\langle \overline{\nabla}\cos\tau, \frac{\partial}{\partial\tau}\Big\rangle\frac{\partial}{\partial\tau} = (\cos\tau)'\frac{\partial}{\partial\tau} \;=\; -\sinh\tau\frac{\partial}{\partial\tau}\,,$$

from (4.8) we have that

$$\begin{aligned} \Phi_*(N)(\cos\tau) &= \langle\overline{\nabla}\cos\tau, \Phi_*(N)\rangle \\ &= -\Big\langle \frac{\partial}{\partial\tau}, \Phi_*(N)\Big\rangle\sin\tau \;=\; \sin\tau\cos\theta. \end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.11) we obtain (4.10). □

**Remark 3.** For $1 \leq r \leq n-1$, from (4.6) we can observe that the $(r+1)$-th mean curvature $\mathcal{H}_{r+1}$, of slice the $\{\tau_0\}\times\mathbb{S}^n$, with $\tau_0 \in (0, \frac{\pi}{4})$, of the Riemannian warped product $(0,\pi)\times_{\sin\tau}\mathbb{S}^n$ verify the inequalities

$$\mathcal{H}^{r+1} = \mathcal{H}_{r+1} > \mathcal{H}_r > \cdots > \mathcal{H}_2 > \mathcal{H} > 1.$$

Taking into account this situation, we established in Theorem 1 a rigidity result for strongly $r$-stable closed hypersurfaces immersed into $\mathbb{S}^{n+1}$.

**Proof of Theorem 1.** Since the hypersurface

$$\Phi \circ \psi\colon \Sigma^n \looparrowright (0,\pi) \times_{\sin\tau} \mathbb{S}^n \tag{4.13}$$

is strongly $r$-stable, where $\Phi$ is the isometry described in (4.4), from (3.6) and (3.7) following Definition 1 we get

$$0 \leq -\int_{\Phi(x(\Sigma^n))} \left\{ L_r(f) + \left( \frac{nb_r}{r+1} HH_{r+1} - b_{r+1}H_{r+2} + b_r H_r \right) f \right\} f d\Phi(\Sigma)$$

for all $f \in C^\infty(\Sigma^n)$, where $L_r$ is the differential operator defined in (2.8), $d\Phi(\Sigma)$ denotes the volume element of $\Sigma^n$ induced by (4.13), $b_k = (k+1)\binom{n}{k+1}$ for $k \in \{r, r+1\}$ and, for simplicity, we use the notations $H_j = H_j \circ \psi^{-1} \circ \Phi^{-1}$, $j \in \{1, r, r{+}1, r{+}2\}$. In particular, considering the smooth function $\xi = -\sin\tau\cos\theta$ defined in (4.9), from Proposition 3 we obtain

$$\begin{aligned} 0 &\leq b_r \int_{\Phi(\psi(\Sigma^n))} (-H_r \sin\tau\cos\theta - H_{r+1}\cos\tau) \sin\tau\cos\theta \, d\Phi(\Sigma) \\ &\leq b_r \int_{\Phi(\psi(\Sigma^n))} (H_r\cos\theta - H_{r+1}) \cos\tau\sin\tau\cos\theta \, d\Phi(\Sigma) \\ &\leq b_r \int_{\Phi(\psi(\Sigma^n))} (\cos\theta - 1)\, H_r \cos\tau\sin\tau\cos\theta \, d\Phi(\Sigma) \end{aligned} \tag{4.14}$$

where in the last inequality we use the condition (1.1). Now, since $H_r \geq 1$ on $\Sigma^n$, the normal angle $\theta$ of $\Sigma^n$ verifies the inequalities established in(4.8), and $\cos\tau$ and $\sin\tau$ are positive values when $\tau \in (0, \pi/4]$, then from the (4.14) we obtain

$$0 \leq b_r \int_{\Phi(\psi(\Sigma^n))} (\cos\theta - 1)\, H_r \cos\tau\sin\tau\theta \, d\Phi(\Sigma) \leq 0\,.$$

Therefore, $\cos\theta = 1$ on $\Sigma^n$ and, consequently, there is $\tau_0 \in (0, \pi/4]$ such that $\Phi(\psi(\Sigma^n)) = \{\tau_0\} \times \mathbb{S}^n$. □

With respect to the notion of strong stability related to closed hypersurfaces with constant mean curvature immersed into Euclidean sphere $\mathbb{S}^{n+1}$, it is well known that *there are no strongly stable closed hypersurfaces with constant mean curvature in* $\mathbb{S}^{n+1}$ (cf. [3, Section 2]). In the context of the higher order mean curvatures, from Theorem 1 we can establish a nonexistent result to strongly $r$-stable closed hypersurfaces immersed in $\mathbb{S}^{n+1}$ (see Theorem 2).

**Proof of Theorem 2.** Assuming that there is a strongly $r$-stable closed hypersurface $\psi\colon \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \geq 3)$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{1, \ldots, r+2\}$, immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$ and with $r$-th mean curvature $H_r$ satisfying $H_{r+1} \geq H_r \geq 1$ on $\Sigma^n$, from Theorem 1 we get that $\psi(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$, obtaining a contradiction. □

**Remark 4.** Consider all closed hypersurfaces $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$ $(n \geq 3)$ with constant $(r+1)$-th mean curvature $H_{r+1}$, $r \in \{1, \ldots, n-2\}$, which are strongly $r$-stable and that satisfy the condition $H_{r+1} \geq H_r \geq 1$, where $H_r$ is the $r$-th mean curvature of $\psi : \Sigma^n \looparrowright \mathbb{S}^{n+1}$, from Theorems 1 and 2 we can conclude that the region of the Euclidean sphere $\mathbb{S}^{n+1}$ that contains all these hypersurfaces is small when compared to the set of closed hypersurfaces of $\mathbb{S}^{n+1}$ that do not verify all these assumptions. It is in this context that our results can be understood as a half-space type property for this class of hypersurfaces of $\mathbb{S}^{n+1}$.

For the case $r = 1$, taking into account (2.4), we can exchange the second mean curvature $H_2$ for the normalized scalar curvature $R$ in equation (3.5) and then rewrite our Definition 1 in terms of $R$. In this context, an immediate application of Theorem 1 and Theorem 2 gives the following results.

**Corollary 1.** *Let $\psi : \Sigma^n \looparrowright \Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \geq 3)$ be a strongly 1-stable closed hypersurface with constant normalized scalar curvature $R$. If the mean curvature $H$ of $\psi : \Sigma^n \looparrowright \Omega^{n+1}$ obeys the condition $R - 1 \geq H \geq 1$ on $\Sigma^n$, then $\psi(\Sigma^n)$ is isometric to a geodesic sphere contained in the closure of the upper domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ of level $\tau_0 = \pi/4$.*

**Corollary 2.** *There is no strongly 1-stable closed hypersurface $\Sigma^n$ $(n \geq 3)$ with constant normalized scalar curvature $R$ immersed into the lower domain enclosed by the geodesic sphere of $\Omega^{n+1} \subset \mathbb{S}^{n+1}$ $(n \geq 3)$ of level $\tau_0 = \pi/4$, with mean curvature $H$ satisfying the condition $R - 1 \geq H \geq 1$ on $\Sigma^n$.*

**Acknowledgement.** The author would like to thank the referees for their comments and suggestions, which enabled him to reach at a considerable improvement of the original version of this work. The author is partially supported by CNPq, Brazil, grant 311224/2018-0.

## References

[1] Alencar, H., do Carmo, M., Colares, A.G., *Stable hypersurfaces with constant scalar curvature*, Math. Z. **213** (1993), 117–131.

[2] Alías, L.J., Barros, A., Brasil Jr., A., *A spectral characterization of the $H(r)$-torus by the first stability eigenvalue*, Proc. Amer. Math. Soc. **133** (2005), 875–884.

[3] Alías, L.J., Brasil Jr., A., Perdomo, O., *On the stability index of hypersurfaceswith constant mean curvature in spheres*, Proc. Amer. Math. Soc. **135** (2007), 3685–3693.

[4] Alías, L.J., Brasil Jr., A., Sousa Jr., L., *A characterization of Clifford tori with constant scalar curvatrue one by the first stability eigenvalue*, Bull. Braz. Math. Soc. **35** (2004), 165–175.

[5] Aquino, C.P., de Lima, H., *On the rigidity of constant mean curvature complete vertical graphs in warped products*, Differential Geom. Appl. **29** (2011), 590–596.

[6] Aquino, C.P., de Lima, H.F., dos Santos, Fábio R., Velásquez, Marco A.L., *On the first stability eigenvalue of hypersurfaces in the Euclidean and hyperbolic spaces*, Quaest. Math. **40** (2017), 605–616.

[7] Barbosa, J.L.M., Colares, A.G., *Stability of hypersurfaces with constant $r$-mean curvature*, Ann. Global Anal. Geom. **15** (1997), 277–297.

[8] Barbosa, J.L.M., do Carmo, M., *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), 339–353.

[9] Barbosa, J.L.M., do Carmo, M., Eschenburg, J., *Stability of hypersurfaces with constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 1123–138.

[10] Barros, A., Sousa, P., *Compact graphs over a sphere of constant second order mean curvature*, Proc. Amer. Math. Soc. **137** (2009), 3105–3114.

[11] Chavel, I., *Eigenvalues in Riemannian Geometry*, Academic Press, Inc., 1984.

[12] Cheng, Q., *First eigenvalue of a Jacobi operator of hypersurfaces with a constant scalar curvature*, Proc. Amer. Math. Soc. **136** (2008), 3309–3318.

[13] Cheng, S.Y., Yau, S.T., *Hypersurfaces with constant scalar curvature*, Math. Ann. **225** (1977), 195–204.

[14] de Lima, H.F., Velásquez, Marco A.L., *A new characterization of $r$-stable hypersurfaces in space forms*, Arch. Math. (Brno) **47** (2011), 119–131.

[15] Montiel, S., *Unicity of constant mean curvature hypersurfaces in some Riemannian manifolds*, Indiana Univ. Math. J. **48** (1999), 711–748.

[16] Reilly, R.C., *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geom. **8** (1973), 465–477.

Departamento de Matemática,
Universidade Federal de Campina Grande,
58.409-970 Campina Grande, Paraíba, Brazil
*E-mail*: `marco.velasquez@mat.ufcg.edu.br`