Juan Bory-Reyes

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A note on the solvability of homogeneous Riemann boundary problem with infinity index

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Abstract. In this note we establish a necessary and sufficient condition for solvability of the homogeneous Riemann boundary problem with infinity index on a rectifiable open curve. The index of the problem we deal with considers the influence of the requirement of the solutions of the problem, the degree of non-smoothness of the curve at the endpoints as well as the behavior of the coefficient at these points.

1 Introduction

The theory of Riemann boundary problem for analytic functions of one complex variable was extensively studied by F.D. Gakhov in [1] and partially by Lu Jian-Ke in [2]. After these well-known monographs, there have appeared in the literature deeper discussions aimed at the constructing of an analogous theory for this problem with different backgrounds and classes of analytic functions and curves. Striking applications of the results in mathematical physics and engineering include, for instance, elasticity theory, hydro and aerodynamics, shell theory, quantum mechanics, theory of orthogonal polynomials, and so on.

The classical Riemann boundary problem for on open rectifiable Jordan curve may be formulated as follows: Let $\gamma = \widetilde{a_1 a_2}$ be an oriented open rectifiable Jordan curve in $\mathbb{C}$ with endpoints a_1, a_2, $\hat{\gamma} = \gamma \setminus \{a_1, a_2\}$. Let G be a given continuous function on γ such that $G(t) \neq 0$ for $t \in \gamma$.

We consider the following problem: Find function $\Phi(z)$ analytic in $\mathbb{C} \setminus \gamma$ continuous up to $\hat{\gamma}$ from the left and the right with finite order at infinity (i.e., exists $r > 0$, and $n \in \mathbb{Z}$ such that for $|z| > r$ we have $\Phi(z) = \sum_{m=-\infty}^{n} c_m z^m$) and

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Affiliation:
SEPI-ESIME-Zacatenco. Instituto Politécnico Nacional. Av. IPN s/n, Col. San Pedro Zacatenco, 07738 Gustavo A. Madero, CDMX, México.
E-mail: juanboryreyes@yahoo.com

satisfying the inequality

$$|\Phi(z)| \leq c|z-a_k|^{-\nu_k}, \quad c = \text{const}, 0 < \nu_k < 1, \tag{1}$$

in a neighborhood of each endpoints a_k, $k = 1, 2$. Moreover, the boundary values of the solution $\Phi(z)$ on the curve γ, denoted by $\Phi^{\pm}$ respectively, are required to satisfy the following conjugation condition

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \hat{\gamma}. \tag{2}$$

For abbreviation, we will denote by $B_n(\gamma)$ the set of analytic functions in $\mathbb{C} \setminus \gamma$ continuous up to $\hat{\gamma}$ from the left and the right with order $n := \max\{m \in \mathbb{Z} : c_m \neq 0\}$ at infinity.

In [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15] the authors constructs examples of open curves in the complex plane for which the number of linearly independent solutions of the problem (2) depends on the character of the behavior of the curve in the neighborhood of its ends.

Particularly, solvability conditions and the general form of the solution of problem (2) is considered in [10], [11], [12] and the dimension of the solution space of the problem is expressed in term of a concept of index of the problem, which takes into account the influence of the requirement of the solutions of the problem, the degree of non-smothness of the curve at the endpoints as well as the behavior of the coefficient at these points.

The motivation for writting this paper comes from the much excellent applicability of the classical Phragmèn Lindelöf principle [16], [17] (see also [18], [19]) to the solution of the Riemann boundary problem with infinity index for analytic functions on the complex plane. N.V. Govorov is recognized as the first to have applied such idea, see [20], but some others authors further draw their attention to this subject, see [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33] and the references given there.

We have attempted to present the material in as self-contained a way as feasible. The end of a proof will be indicated by the symbol $\Box$.

2 Fundamental terminology and results

In this introductory section, we repeat the relevant material from [10], [11], [12], which form the basis of our study.

Definition 1. We call a pair $\prec \gamma, G \succ$ compatible if the function

$$\Gamma(z) := \frac{1}{2\pi i}\int_{\gamma} \frac{\ln G(x)}{x-z}dx, z \in \mathbb{C} \setminus \gamma, \tag{3}$$

is an analytic functions in $\mathbb{C} \setminus \gamma$ continuous up to $\hat{\gamma}$ from the left and the right, $\ln G(x)$ represents an arbitrary continuous branch of the logarithm function and exists $q_k \in \mathbb{Z}, k = 1, 2$ such that for z in a neighborhood of a_k it follows that

$$(z-a_k)^{q_k} \exp \Re\{\Gamma(z)\} = O(|z-a_k|^{-\nu_k}). \tag{4}$$

Definition 2. Let $\prec\gamma, G\succ$ compatible. The characteristic $\lambda_k, k = 1, 2$ of the pair $\prec\gamma, G\succ$ in the point a_k is defined to be the infimum of $q \in \mathbb{Z}$ such that (4) holds.

Definition 3. We define the index of the problem (2) to be

$$\kappa := \begin{cases} -\infty, \text{ if } \prec\gamma, G\succ \text{ is not compatible,} \\ -\lambda_1 - \lambda_2, \text{ if } \prec\gamma, G\succ \text{ is compatible and } |\lambda_k| < \infty, k = 1, 2, \\ \infty, \text{ if } \prec\gamma, G\succ \text{ is compatible and at least one } \lambda_k = -\infty. \end{cases} \tag{5}$$

The compatibility condition of the pair $\prec\gamma, G\succ$ describe the very close connection between γ and G to ensure the solvability of problem (2). The works [10], [11], [12] give examples of compatible pairs.

Theorem 1. *[11, Theorem 1] Let $\prec\gamma, G\succ$ compatible. Then the following are true*

- *If at list one $\lambda_k = -\infty$. Then, for all n integer, the problem (2) has infinity number of linearly independent solutions in $B_n(\gamma)$,*
- *If the index κ of the problem (2) is finite. Then the number of linearly independent solutions in $B_n(\gamma)$ is given by $\max\{\kappa + n + 1, 0\}$.*

Definition 4. A pair $\prec\gamma, G\succ$ is said to be skew-compatible if the Cauchy type integral (3) is merely an analytic functions in $\mathbb{C} \setminus \gamma$ continuous up to $\hat{\gamma}$ from the left and the right.

Skew-compatibility condition of a pair $\prec\gamma, G\succ$ will be assumed through the paper in an strict sense, i.e., an skew-compatible pair is not compatible and so the index of the corresponded problem (2) will be always $\kappa = -\infty$.

The boundary behavior of (3) over rectifiable curves, which is directly connected to the skew-compatibility of the pair $\prec\gamma, G\succ$, was one of the subjects of investigations in [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15].

The following result give precisely what is meant by a skew-compatible pair.

Proposition 1. *Every solution $\Phi \in B_n(\gamma)$ of problem (2) admits the representation*

$$\Phi(z) = \Psi(z) \exp \Gamma(z),\ z \in \mathbb{C} \setminus \gamma, \tag{6}$$

where Ψ is analytic in $\mathbb{C} \setminus \{a_1, a_2\}$ with order n at infinity.

Proof. Skew-compatibility condition of the pair $\prec\gamma, G\succ$ yields that $Y(z) := \exp \Gamma(z)$ is analytic in $\mathbb{C} \setminus \gamma$, with boundary value limits from the left and the right $\lim_{z\to t\pm} \exp \Gamma(z) =: Y^{\pm}(t) \neq 0$ for $t \in \hat{\gamma}$.

It follows by Privalov'theorem, see [34], that for almost all $t \in \hat{\gamma}$ the following equality holds

$$\Gamma^{+}(t) - \Gamma^{-}(t) = \ln G(t). \tag{7}$$

Then, by continuity (7) holds everywhere in $\hat{\gamma}$. If we combine this with the fact $Y^{\pm}(t) \neq 0$, it is clear that

$$\Gamma(t) = \frac{Y^{+}(t)}{Y^{-}(t)},\ t \in \hat{\gamma} \tag{8}$$

and hence

$$\frac{\Phi^+(t)}{Y^+(t)} = \frac{\Phi^-(t)}{Y^-(t)}, \; t \in \hat{\gamma}. \tag{9}$$

Therefore, classical Painleve'theorem, see [35], shows that there exists and analytic extension of $\dfrac{\Phi(z)}{Y(z)}$ through $\hat{\gamma}$, that is the desired function $\Psi(z)$, which has clearly the same order at infinity as $\Phi(z)$. □

Proposition 2. *Let $\prec\gamma, G\succ$ compatible. Then the general solution of the problem (2) in $B_n(\gamma)$ is given by (6) if and only if $\Psi(z)$ be meromorphic in $\mathbb{C} \setminus \{a_1, a_2\}$ possessing singularities in $a_k, k = 1, 2$ of finite order greater than the order of λ_k respectively.*

Proof. The proof is based on the following observation. Given a function $\Psi(z)$ analytic in $\mathbb{C} \setminus \{a_1, a_2\}$ so its singularities are only of polar type with order greater than $\lambda_k, k = 1, 2$ respectively, then the function

$$\Phi(z) = \Psi(z) \exp \Gamma(z), \quad z \in \mathbb{C} \setminus \gamma, \tag{10}$$

is solution of the problem (2) in $B_n(\gamma)$, where $\lambda_k, k = 1, 2$ are the characteristics of the pair $\prec\gamma, G\succ$ at the endpoints $a_k, k = 1, 2$ respectively. □

3 Main result

Without loss of generality we can assume that $\gamma = \widetilde{a_1 a_2}$ be an oriented open rectifiable Jordan curve in $\mathbb{C}$ with endpoints a_1, a_2 such that $G(a_2) = 1$ and $\ln G(a_2) = 0$, so the analysis in this section is reduced to the case a_1.

Definition 5. Let $\prec \gamma, G \succ$ be skew-compatible. The order of $\prec \gamma, G \succ$ is, by definition, the number

$$\rho_\gamma(G) := \inf\{\nu > 0 : \lim_{\mathbb{C}\setminus\gamma \ni z \to a_1} |z - a_1|^{-\nu} |\Re\, \Gamma(z)|\}.$$

Theorem 2. *Let $\prec\gamma, G\succ$ be skew-compatible (so $\kappa = -\infty$). For $\Phi \in B_n(\gamma)$ being solution of the problem (2) is necessary and for $\rho_\gamma(G) < \frac{1}{2}$ sufficient that*

$$\Phi(z) = \Psi\Big(\frac{1}{z - a_1}\Big) Y(z), \quad z \in \mathbb{C} \setminus \gamma, \tag{11}$$

where $\Psi(w)$ is an entire function of order ρ_Φ such that $\rho_\Phi \le \rho_\gamma(G)$, for which the asymptotic inequality

$$\ln\Big|\Psi\Big(\frac{1}{z - a_1}\Big)\Big| \le -\max\{\Re\, \Gamma^+(z), \Re\, \Gamma^-(t)\} - \nu_1 \ln|z - a_1|, \quad \hat{\gamma} \ni t \to a_1,$$

holds. Here $\rho_\Phi = \lim_{r\to\infty} \dfrac{\ln\ln M_\Phi(r)}{\ln r}$, $M_\Phi(r) = \max_{|w|=r} |\Phi(w)|$, $\nu_1 \in [0, 1)$.

Proof. Necessity
Let $\Phi \in B_n(\gamma)$ be a solution of the problem (2). On account of Proposition 1 we have

$$\ln\left|\Psi\left(\frac{1}{t-a_1}\right)\right| = \ln|\Phi^{\pm}(t)| - \ln|Y^{\pm}(t)|, \tag{12}$$

hence

$$\ln\left|\Psi\left(\frac{1}{t-a_1}\right)\right| \leq -\ln|Y^{\pm}(t)| - \nu_1 \ln|t-a_1|. \tag{13}$$

By the skew-compatibility of the pair $\prec \gamma, G \succ$ the Sojostski-Plemelj formula for (3) holds, then

$$\ln|Y^{\pm}(t)| \leq \max\{\Re\,\Gamma^{+}(z), \Re\,\Gamma^{-}(t)\}. \tag{14}$$

Combining (12) with (14) yields (11).

On the other hand, we have

$$|Y(z)|^{-1} \leq \exp\ b|z-a_1|^{-\rho_\Phi}, \quad b>0, \tag{15}$$

which give $\rho_\Phi \leq \rho_\gamma(G)$, and the necessity follows.
Sufficiency

Assume $\rho_\Phi \leq \rho_\gamma(G)$ and suppose (11) to hold. From this last and (14) we can assert that

$$|t-a_k|^{\nu_k}|\Phi^{\pm}(t)| \leq c, \quad t \in \hat{\gamma}. \tag{16}$$

Applying $\rho_\Phi \leq \rho_\gamma(G)$ and the hypothesis $\rho_\gamma(G) < \dfrac{1}{2}$, hence for every ρ_1 such that $\rho_\gamma(G) < \rho_1 < \dfrac{1}{2}$ and r sufficiently great we conclude that Assume $\rho_\Phi \leq \rho_\gamma(G)$ and suppose (11) to hold. From this last and (14) we can assert that

$$\max_{|z-a_1|=r}|\Phi(z)| = \max_{|z-a_1|=r}\left|\Psi\left(\frac{1}{z-a_1}\right)Y(z)\right| \leq \exp r^{\rho_1}. \tag{17}$$

Since $\rho_1 < \dfrac{1}{2}$, (16) and (17) show that $(z-a_1)^{\nu_k}\Phi(z)$ is bounded, by Phragmèn-Lindelöf principle [36, pag. 357]. Therefore, it is clear that $\Phi(z)$ is solution of problem (2) and the proof is complete. □

We direct the reader's attention to the fact that expression (11) in the general solution of the problem, the function $\Psi(w)$ is an arbitrary entire function of order ρ_Φ under the following conditions:

$$\rho_\Phi \leq \rho_\gamma(G), \tag{18}$$

and

$$\ln\left|\Psi\left(\frac{1}{z-a_1}\right)\right| \leq -\max\{\Re\,\Gamma^{+}(z), \Re\,\Gamma^{-}(t)\} - \nu_1 \ln|z-a_1|, \quad \hat{\gamma} \ni t \to a_1. \tag{19}$$

Taking into account that due to the inequality $\rho_\Phi < 1$ the function $\Phi(w)$ admits a representation in terms of canonical product, which is clear from Hadamard's theorem (see [37, pag. 24]).

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