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On the gaps between q-binomial coefficients

Florian Luca, Sylvester Manganye

Abstract. In this note, we estimate the distance between two q-nomial coefficients $\left|\binom{n}{k}_q - \binom{n'}{k'}_q\right|$, where $(n,k) \neq (n',k')$ and $q \geq 2$ is an integer.

1 Introduction

In this paper, $q \geq 2$ is an integer and for $n > k \geq 1$,

$$\binom{n}{k}_q := \frac{(q^{n-k+1}-1)(q^{n-k+2}-1)\cdots(q^n-1)}{(q-1)(q^2-1)\cdots(q^k-1)}$$

is the q-binomial coefficient. We are interested in the distinct values of $\binom{n}{k}_q$. Since $\binom{n}{k}_q = \binom{n}{n-k}_q$, we assume that $n \geq 2k$. It was shown in [1] that under these conditions

$$\binom{n}{k}_q \neq \binom{n'}{k'}_q \quad \text{for} \quad (n,k) \neq (n',k'), \quad n \geq 2k, \ n' \geq 2k'.$$

The proof is an easy application of the primitive divisor theorem for members of Lucas sequences. Thus, taking

$$\mathcal{B}_q := \left\{ \binom{n}{k}_q : n \geq 2k \geq 2 \right\},$$

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the elements from $\mathcal{B}_q$ are distinct. Assume $\mathcal{B}_q = \{b_1, b_2, \ldots\}$, where the elements b_i are listed increasingly. We are interested in a lower bound for $b_{i+1} - b_i$. We have the following theorem:

Theorem 1. *The inequality*

$$b_{N+1} - b_N \geq \exp\left((\log b_N)^{1/3}\right)$$

holds for all $q \geq 2$ and all $N \geq 163{,}000$.

Corollary 1. *The inequality $b_{N+1} - b_N > 100$ always holds except when $N \leq 8$ for $q = 2$ or $N \leq 4$ for $q \in \{3, 4, 5, 6, 7, 8, 9, 10\}$.*

2 Some auxiliary results

We put $m := k(n-k)$.

Lemma 1. *We have*

$$\frac{q^m}{4} < \binom{n}{k}_q < 4q^m$$

for all $q \geq 2$ and $n \geq 2k$.

Proof. We have

$$\binom{n}{k}_q = \frac{q^{n-(k-1)+n-(k-2)+\cdots+n}}{q^{k+k-1+\cdots+1}} \left(\prod_{1\leq j\leq k}\left(1 - \frac{1}{q^{n-j+1}}\right)\right)\left(\prod_{j=1}^{k}\left(1 - \frac{1}{q^j}\right)\right)^{-1}.$$

The first factor in the right–hand side above is q^m. As for the others, the inequality

$$\frac{1}{4} < 0.288 < \prod_{j\geq 1}\left(1 - \frac{1}{2^j}\right) \leq \prod_{a\leq j\leq b}\left(1 - \frac{1}{q^j}\right) < 1$$

holds for all positive integers $a < b$ and $q \geq 2$. Taking $(a, b) = (n-k+1, k)$, or $(a, b) = (1, k)$, respectively, we get that

$$\frac{1}{4} < \left(\prod_{j=1}^{k}\left(1 - \frac{1}{q^{n-j+1}}\right)\right)\left(\prod_{j=1}^{k}\left(1 - \frac{1}{q^j}\right)\right)^{-1} < 4,$$

which finishes the proof. □

From now on, $(n, k) \neq (n', k')$ are such that $n \geq 2k$, $n' \geq 2k'$. For a positive integer ℓ we write

$$\Phi_\ell(X) = \prod_{\substack{1\leq j\leq \ell \\ \gcd(j,\ell,)=1}} (X - e^{2\pi i j/\ell}) \in \mathbb{Z}[X]$$

for the ℓth cyclotomic polynomial.

Lemma 2. *Assume that* $[n-k+1, n] \cap [n'-k'+1, n'] \neq \emptyset$. *Then*

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \Phi_\ell(q), \quad \textit{where} \quad \ell \in [n-k+1, n] \cap [n'-k'+1, n'].$$

Proof. Since $q^\ell - 1 = \prod_{d \mid \ell} \Phi_d(q)$, it follows that

$$\binom{n}{k}_q = \prod_{d \in \mathcal{D}(n,k)} \Phi_d(q)^{\alpha(d,n,k)},$$

where

$$\mathcal{D}(n,k) = \bigcup_{j \in [1,k]} \{d \geq 1 : d \mid n-j+1 \text{ or } d \mid j\},$$

and $\alpha(d,h,k)$ are some integers. Since $\binom{n}{k}_q$ is a rational function in q which is an integer for all $q \geq 2$, it follows that $\alpha(d,n,k) \geq 0$ for all $d \in \mathcal{D}(n,k)$. Further, it is easy to see that $d = n-j+1$ has $\alpha(d,n,k) \geq 1$ for all $j \in [1,k]$, since $\Phi_{n-j+1}(q) \mid q^{n-j+1} - 1$ and $\Phi_{n-j+1}(q)$ is not a factor of $\prod_{i=1}^k (q^i - 1)$ because $n-j+1 \geq n-k+1 > k$. Thus, if $\ell \in [n-k+1, n] \cap [n'-k'+1, n']$, then $\Phi_\ell(q)$ is a factor of both $\binom{n}{k}_q$ and $\binom{n'}{k'}_q$. Thus, their difference is nonzero and a multiple of $\Phi_\ell(q)$, which finishes the proof of the lemma. □

Lemma 3. *Assume that* $[n-k+1, n] \cap [n'-k'+1, n'] = \emptyset$. *Put again* $m := k(n-k)$, $m' := k'(n-k')$. *Then:*

(i) If $m' < m$, *then*

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \frac{1}{7} \binom{n'}{k'}_q.$$

(ii) If $m' = m$ *and* $k' < k$, *then*

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| \geq \frac{2}{q^{n+1}} \binom{n'}{k'}_q.$$

Proof. From the arguments from the proof of Lemma 1, we have

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right| = \left| q^m \left(\frac{\prod_{j=1}^k (1 - 1/q^{n-j+1})}{\prod_{j=1}^k (1 - 1/q^j)} \right) - q^{m'} \left(\frac{\prod_{j=1}^{k'} (1 - 1/q^{n'-j+1})}{\prod_{j=1}^{k'} (1 - 1/q^j)} \right) \right|.$$

We analyze the two cases.

(i) In this case,

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right|$$
$$= \binom{n'}{k'}_q \left| q^{m-m'} \left(\frac{\prod_{j=1}^k (1 - 1/q^{n-j+1})}{\prod_{j=1}^k (1 - 1/q^j)} \right) \left(\frac{\prod_{j=1}^{k'} (1 - 1/q^{n'-j+1})}{\prod_{j=1}^{k'} (1 - 1/q^j)} \right)^{-1} - 1 \right|.$$

In the right, the coefficient of $q^{m-m'}$ is $(P/Q)(Q'/P')$, where

$$P = \prod_{j=1}^{k}(1 - 1/q^{n-j+1}) \qquad Q = \prod_{j=1}^{k}(1 - 1/q^{j}),$$

and P', Q' are obtained from P, Q by changing (k, n) to (k', n'), respectively. All of P, Q, P', Q' are smaller than 1. We have the following lemma:

Lemma 4. *The inequality*

$$\prod_{j=a}^{b}(1 - 1/q^{j}) \geq q^{-1/3} \tag{1}$$

holds for all $q \geq 2$ and $a \geq 1$ and any $b \geq a$ except for possibly

$$(a, q) = (1, 2),\ (1, 3),\ (2, 2),\ (3, 2).$$

Proof. Taking logarithms, the desired inequality becomes

$$\sum_{j=a}^{b} \log\left(1 - \frac{1}{q^j}\right) > -\frac{\log q}{3}.$$

The inequality $\log(1 - x) > -2x$ holds for all $x \in (0, 1/2)$. So, using this with $x = 1/q^j$ for $j \in [a, b]$, it suffices to show that

$$-\sum_{j=a}^{b} \frac{2}{q^j} > -\frac{\log q}{3},$$

which is equivalent to

$$\sum_{j=a}^{b} \frac{1}{q^j} < \frac{\log q}{6}.$$

Taking the sum on the left to infinity, it is a geometrical progression whose sum is $1/(q^{a-1}(q - 1))$. Thus, it suffices that

$$q^{a-1}(q - 1) \geq \frac{6}{\log q}.$$

The above inequality holds for all $a \geq 1$ and $q \geq 5$. It also holds for $a \geq 5$ and any $q \geq 2$. So, it remains to check the given inequality for (a, q) with $a \in [1, 4]$ and $q \in [2, 4]$, and we get the list of exceptions. □

To apply the above lemma, notice that $(P/Q)(P'/Q')^{-1} = PQ'(QP')^{-1}$, and $(QP')^{-1} > 1$. Furthermore, P is a product as the one appearing in (1) with $a = n - k + 1 \geq k + 1 \geq 2$, while Q' is a product like the one appearing in (1) but with $a = 1$. Thus, by Lemma 4, we have that the inequality

$$\min\{P, Q'\} \geq q^{-1/3}$$

holds unless $q \in \{2,3\}$. So, unless $q \in \{2,3\}$, we have that

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |q^{m-m'-2/3} - 1| \geq |q^{1/3} - 1| \geq |2^{1/3} - 1| > 1/4.$$

Assume next that $q = 2,3$. If $q = 3$, then

$$\min\{P, Q'\} \geq \prod_{j=1}^{\infty}(1 - 1/3^j) > 0.56, \quad \max\{P, Q'\} \geq \prod_{j\geq 2}(1 - 1/3^j) > 0.84,$$

so

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |3 \times 0.56 \times 0.84 - 1| > 0.4 > 1/4.$$

It remains to treat the case $q = 2$. If $k' \leq k$, then $P/Q(P'/Q')^{-1} = P(Q/Q')^{-1}P'^{-1}$ and both $Q/Q' \leq 1$, $P' < 1$. Furthermore, P is a product like in (1) starting at $n - k + 1$. Thus, if $n - k + 1 \geq 4$, then

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |2^{m-m'-1/3} - 1| \geq |2^{2/3} - 1| > 1/2.$$

If $m - m' \geq 2$, then since

$$P \geq \prod_{j\geq 1}(1 - 1/2^j) > 0.288,$$

we get

$$|q^{m-m'}(P/Q)(P'/Q')^{-1} - 1| \geq |2^2 \times 0.288 - 1| > 1/7.$$

Thus, we only need to analyze the situation $n - k + 1 \leq 3$ and $m' = m - 1$. Since $n - k \geq k$, this gives $k \leq 2$ and then $n \leq k + 2 \leq 4$. Thus, $(n,k) = (2,1),\ (3,1),\ (4,1),\ (4,2)$. Further, $m = nk - k^2 = k(n-k) \leq 4$. Since $m' < m$, we get $m' = k'(n' - k') < 4$, so $(n',k') = (2,1),\ (3,1),\ (4,1)$. Now we compute

$$\left| \binom{n}{k}_q - \binom{n'}{k'}_q \right|$$

over all such possibilities (n,k,n',k') and $q = 2$, and conclude that the desired inequality holds in these cases as well.

This was if $k' \leq k$. Assume next that $k' > k$. Then

$$(P/Q)(P'/Q')^{-1} = P(Q'/Q)P'^{-1}$$

and Q'/Q is a product as in (1) starting at $a = k' + 1 \geq 3$. Thus, if $\min\{n - k + 1, k' + 1\} \geq 4$, then (1) holds and so

$$|q^{m-m'}P(Q'/Q)P'^{-1} - 1| \geq |2^{1/3} - 1| > 1/4.$$

Thus, we treat the case $\min\{n - k + 1, k' + 1\} \leq 3$. Since $n - k + 1 \geq k + 1$ and $k' > k$, it follows that

$$k + 1 = \min\{k + 1, k' + 1\} \leq \min\{n - k + 1, k' + 1\} \leq 3\,,$$

so $k \in \{1,2\}$. Thus,

$$\min\{n-1, k'+1\} \le \min\{n-k+1, k'+1\} \le 3\,,$$

so either $n \le 4$ or $(k',k) = (2,1)$. If $m - m' \ge 2$, then since

$$\prod_{j\ge 2}(1-1/2^j) \ge 0.57,$$

it follows that

$$|q^{m-m'}P(Q'/P)P'^{-1} - 1| \ge |4 \times (0.57)^2 - 1| > 1/4.$$

Thus, it remains to treat the case $m' = m-1$. If $n \le 4$, then

$$k'^2 \le k'(n'-k') = m' = m - 1 = k(n-k) - 1 \le 3\,,$$

so $k' = 1$, contradicting the fact that $k' > k$. Thus, $(k',k) = (2,1)$ so Q'/Q is a product like in (1) starting at $k'+1 = 3$. If also $n - k + 1 \ge 3$, then since

$$\prod_{j\ge 3}(1-1/2^j) > 0.77,$$

it follows that

$$|q^{m-m'}P(Q'/Q)P'^{-1} - 1| \ge |2 \times (0.77)^2 - 1| > 1/6.$$

Hence, it remains to treat the case when $n-k+1 = 2$, so $(n,k) = (2,1)$, so $m = 1$ and then $m' = m - 1 = 0$, a contradiction. This takes care of (i).

(ii). In this case, since $k(n-k) = k'(n'-k')$ and $k' < k$, it follows that $n' - k' > n - k$ and since $[n-k+1, n]$ and $[n'-k'+1, n']$ are disjoint, it follows that $n' - k' \ge n$. With the notations from part (i), we have

$$\left|\binom{n}{k}_q - \binom{n'}{k'}_q\right| = q^m|(P/Q) - (P'/Q')| = \binom{n'}{k'}_q |(P/(Q/Q')P'^{-1} - 1|.$$

Now

$$P/(Q/Q')P'^{-1} = \prod_{j=1}^{k'}\left(\frac{1-1/q^{n-k+j}}{1-1/q^{n'-k'+j}}\right)\prod_{j=k'}^{k-1}\left(\frac{1-1/q^{n-(k-j)+1}}{1-1/q^{j+1}}\right). \tag{2}$$

Let us notice the following order

$$k'+1 \le \cdots \le k < n-k+1 \le \cdots \le n < n'-k'+1 < \cdots < n'.$$

Using the inequalities

$$1 - 1/q^{\ell} > \exp\left(-\frac{2}{q^{\ell}}\right) \quad \text{and} \quad 1 - 1/q^{\ell} < \exp\left(-\frac{1}{q^{\ell}}\right),$$

for ℓ an index participating in the numerator, respectively, denominator of the right–hand side of (2), we get to get that

$$P/(Q/Q')P'^{-1} > \exp\left(\frac{1}{q^{k'+1}} + \cdots + \frac{1}{q^k} - \frac{2}{q^{n-k+1}} - \cdots - \frac{2}{q^n} + \frac{1}{q^{n'-k'+1}} + \cdots + \frac{1}{q^{n'}}\right).$$

Now

$$\frac{2}{q^{n-k+1}} + \cdots + \frac{2}{q^n} < 2\left(\sum_{j \geq n-k+1} \frac{1}{q^j}\right) - \frac{2}{q^{n+1}} = \frac{2}{q^{n-k}(q-1)} - \frac{2}{q^{n+1}}.$$

Hence,

$$P/(Q/Q')P'^{-1} > \exp\left(\frac{1}{q^{k'+1}} + \cdots + \frac{1}{q^k} - \frac{2}{q^{n-k}(q-1)} + \frac{2}{q^{n+1}} + \frac{1}{q^{n'-k'+1}} + \cdots + \frac{1}{q^{n'}}\right). \tag{3}$$

If $q \geq 3$, then since $n - k \geq k$, it follows that

$$\frac{1}{q^{k'+1}} + \cdots + \frac{1}{q^k} - \frac{2}{q^{n-k}(q-1)} \geq \frac{1}{q^k} - \frac{1}{q^{n-k}} \geq 0,$$

so the amount under the exponential in the right–hand side of (3) is at least $2/q^{n+1}$. Since $e^x - 1 > x$ for positive x, it follows that in these cases

$$|P/(Q/Q')P'^{-1} - 1| > \frac{2}{q^{n+1}}.$$

The same conclusion holds if $q = 2$ and either $k < n - k$, or $k' < k - 1$. But if $q = 2$, $k = n - k$ and $k' = k - 1$, then

$$m = k(n-k) = k^2 = m' = (k-1)(n' - (k-1))\,.$$

Thus, $k - 1$ divides k^2, which is possible only for $k = 2$. Hence, $(k, n) = (2, 4)$, and then $k' = 1$ and

$$4 = m = m' = n' - k' = n - 1\,,$$

so $n' = 5$. In this case,

$$|P/(Q/Q')P'^{-1} - 1| = \left|\frac{(1 - 1/2^3)(1 - 1/2^4)}{(1 - 1/2^2)(1 - 1/2^5)} - 1\right| > 0.12 > \frac{2}{q^{n+1}}.$$

Hence,

$$\left|\binom{n}{k}_q - \binom{n'}{k'}_q\right| > \frac{2}{q^{n+1}}\binom{n'}{k'}_q,$$

holds in all cases, which completes the proof of this lemma. □

3 The proof of Theorem 1

We are now ready to do some estimates. We distinguish several cases.

3.1 The case of Lemma 3 (i)

In this case, putting $b_{N'} = \binom{n'}{m'}_q$, we need to decide when the inequality

$$\frac{1}{7} b_{N'} \geq \exp((\log b_{N'})^{1/3})$$

holds. This is equivalent to

$$\log b_{N'} \geq \log 7 + (\log b_{N'})^{1/3}.$$

Using also Lemma 1, it is enough to show that

$$m' \log q - \log 4 \geq \log 7 + (m' \log q + \log 4)^{1/3}.$$

Dividing by $\log q$ and using the fact that $q \geq 2$, it is enough that

$$m' \geq \frac{\log 28}{\log 2} + \left(\frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3}\right)^{1/3},$$

an inequality which holds for all $m' \geq 8$.

3.2 The case of Lemma 3 (ii)

In this case, $m' = m$ and we want that

$$\log b_{N'} + \log 2 - (n+1) \log q \geq (\log b_{N'})^{1/3}.$$

Using again Lemma 1, it suffices that

$$m' \log q - \log 4 + \log 2 \geq (n+1) \log q + (m' \log q + \log 4)^{1/3}.$$

We have $k(n-k) = m'$, so $n+1 = m'/k + k + 1$ and $k > k'$. Thus, $k \in [2, \sqrt{m'}]$. The function $x \mapsto m'/x + x + 1$ in the interval $[2, \sqrt{m'}]$ since its derivative is $-m'/x^2 + 1 \leq 0$. Thus, $n+1 \leq m'/2 + 3$. Hence, it suffices that the inequality

$$m' \log q \geq \log 2 + (m'/2 + 3) \log q + (m' \log q + \log 4)^{1/3}$$

holds. Dividing by $\log q$ and using the fact that $q \geq 2$, it suffices that

$$\frac{m'}{2} - 3 \geq 1 + \left(\frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3}\right)^{1/3},$$

an inequality which holds for all $m' \geq 15$.

3.3 The case of Lemma 2 and $q \geq 3$

We assume $\ell \geq 3$. In this case,

$$\Phi_\ell(q) = \prod_{\substack{1 \leq j \leq \ell \\ \gcd(j,\ell)=1}} |q - e^{2\pi i j/\ell}| \geq (q-1)^{\phi(\ell)} = \exp(\phi(\ell)\log(q-1)). \qquad (4)$$

So, we need to show that

$$\phi(\ell)\log(q-1) \geq (\log b_{N'})^{1/3},$$

or, using again Lemma 1, that

$$\phi(\ell)\log(q-1) \geq (m'\log q + \log 4)^{1/3}.$$

By Theorem 15 in [2], we have

$$\phi(\ell) > \frac{\ell}{1.8\log\log\ell + 2.6/\log\ell} \qquad \text{for all} \qquad \ell \geq 3.$$

Thus, dividing also by $\log(q-1)$, it suffices to show that

$$\frac{\ell}{1.8\log\log\ell + 2.6/\log\ell} \geq \left(\frac{m'\log q}{(\log(q-1))^3} + \frac{\log 4}{(\log(q-1))^3}\right)^{1/3}.$$

The functions

$$x \mapsto \frac{x}{1.8\log\log x + 2.6/\log x} \qquad \text{and} \qquad x \mapsto \frac{\log x}{(\log(x-1))^3}$$

have the property that the first one is increasing and the second one is decreasing for $x \geq 3$, as it can be confirmed by computing their derivatives. Since $\ell \geq n' - k' + 1 \geq \sqrt{m'} + 1$, it suffices that

$$\frac{\sqrt{m'}+1}{1.8\log\log(\sqrt{m'}+1) + 2.6/\log(\sqrt{m'}+1)} \geq \left(\frac{m'\log 3}{(\log 2)^3} + \frac{\log 4}{(\log 2)^3}\right)^{1/3},$$

an inequality which holds for $m' > 15{,}300$.

3.4 The case of Lemma 2 and $q = 2$

Here, $q - 1 = 1$, so inequality (4) is useless. Instead we use the formula

$$\Phi_\ell(2) = \prod_{d|n} (2^{n/d} - 1)^{\mu(d)},$$

where μ is the Möbius function. Factoring out the "main" terms, we get

$$\Phi_\ell(2) \geq 2^{\sum_{d|\ell} \mu(d)\ell/d} \prod_{j \geq 1} (1 - 1/2^j) > 2^{\phi(\ell)-2}.$$

Thus, we get that

$$\Phi_\ell(2) \geq \exp((\phi(\ell)-2)\log 2).$$

Thus, in order to prove the desired inequality it suffices, again via Lemma 1, to show that

$$\phi(\ell) - 2 \geq \left(\frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3}\right)$$

for some $\ell \in [n'-k'+1, n']$. The argument from Subsection 3.3 shows that this inequality holds provided that

$$\frac{\sqrt{m'}+1}{1.8\log\log(\sqrt{m'}+1) + 2.6/\log(\sqrt{m'}+1)} - 2 \geq \left(\frac{m'}{(\log 2)^2} + \frac{\log 4}{(\log 2)^3}\right)^{1/3},$$

an inequality which holds for $m' > 8100$.

To summarize, we proved:

Lemma 5. *If $m \geq m' \geq 15{,}300$, then*

$$\left|\binom{n}{k}_q - \binom{n'}{k'}_q\right| \geq \exp\left(\left(\log\binom{n'}{k'}_q\right)^{1/3}\right).$$

Thus, the inequality in the theorem may fail only if $b_{N'} = \binom{n'}{k'}_q$ for some $m' \leq M' := 15{,}300$. Since $m' = k'(n'-k')$, it follows that for a fixed m', the number of pairs (n', k') with $m' = k'(n'-k')$ is at most $\tau(m')$, where $\tau(s)$ is the number of divisors of s (in fact, it is smaller than that since $k' \leq n'-k'$, but we will not get into such details). Thus, those N' can be at most the first

$$\begin{aligned}\sum_{m' \leq M'} \tau(m') &= \sum_{m' \leq M'} \sum_{d' \mid m'} 1 \leq \sum_{d' \leq M'} \sum_{\substack{m' \leq M' \\ m' \equiv 0 \pmod{d)'}}} 1 \\ &= \sum_{d' \leq M'} \left\lfloor \frac{M'}{d'} \right\rfloor \leq M' \sum_{d' \leq M'} \frac{1}{d'} \\ &\leq M'\left(1 + \int_1^{M'} \frac{dt}{t}\right) \leq M'(1+\log M') < 163{,}000,\end{aligned}$$

which finishes the proof.

4 The proof of the Corollary 1

We follow the previous steps of the proof of Theorem 1. For the situation treated in Subsection 3.1, we need $b_{N'} > 700$. By Lemma 1, this gives $q^{m'} > 700$, which is satisfied for $m' \geq 9$. Since $m' = k'(n'-k') \geq n'/2$, it follows that the last inequality is satisfied for $n' \geq 18$. Thus, it remains to study the case $n' < 17$. In this case, $m' \leq (n'/2)^2$, so $m' \leq 72$. If $m \geq 83$, then $m - m' \geq 11$, so by Lemma 1, we have

$$\binom{n}{k}_q \geq \frac{q^m}{4} \geq 2^7(4q^{m'}) > 2^7\binom{n'}{k'}_q,$$

so

$$\binom{n}{k}_q - \binom{n'}{k'}_q \geq (2^7 - 1)\binom{n'}{k'}_q > 100.$$

Thus, it suffices to consider the case $m \leq 82$, leading to $n/2 \leq k(n-k) \leq m$, so $n \leq 164$. Thus, for Subsection 3.1, it suffices to check in the range $\max\{n, n'\} \leq 164$. A similar argument works for the situation treated in Subsection 3.2. Namely, here we need that $2b_{N'}/q^{n+1} > 100$. Together with Lemma 1, this is satisfied for $q^{m'-n-1} > 200$, which in turn holds if $m' - n \geq 9$. Now $m' = k(n-k)$, where $k > k'$ so either $k = 2$, or $k \geq 3$. When $k = 2$, we have

$$9 \leq m' - n = 2(n-2) - n = n - 4$$

so the desired inequality is satisfied for $n \geq 13$. When $k \geq 3$, we have that

$$m' - n = k(n-k) - n \geq 3n/2 - n = n/2$$

and so the desired inequality holds for $n \geq 18$. Thus, it suffices to assume that $n \leq 17$, leading to $m \leq (17/)^2$, so $m \leq 72$. Since in this case we have $m = m'$, we get that $n'/2 \leq m' = m \leq 72$, so $n' \leq 144$. Thus, in this case it suffices to check in the range $\max\{n, n'\} \leq 144$. For Subsection 3.3, all we need is that $2^{\phi(\ell)} \geq 100$, so $\phi(\ell) > 6$ for some $\ell \in [n-k+1, n] \cap [n'-k'+1, n']$. Now $\phi(\ell) > 6$ for $\ell > 18$, so the desired inequality is satisfied provided that $n - k + 1 \geq 19$. Since $n - k \geq n/2$, the last inequality holds for $n \geq 36$. Thus, it suffices to check it for $n \leq 35$ and since $[n' - k' + 1, n']$ intersects nontrivially $[n - k + 1, n]$, we get that $n' - k' + 1 \leq n \leq 35$. Thus, $n'/2 \leq n' - k' \leq 34$, so $n' \leq 68$. Thus, in this case it suffices check the range $\max\{n, n'\} \leq 68$. Finally, for Subsection 3.4, we want $\Phi_n(2) > 100$ and we checked that this is so for all $n \geq 19$. To do so, we use a consequence of the Primitive Divisor Theorem to the effect that $\Phi_n(2)$ is divisible by a prime $p \equiv 1 \pmod{n}$ for all $n > 6$ (this is a primitive prime factor of $2^n - 1$). In particular, $\Phi_n(2) > 100$ if $n > 100$, so we only needed to check the values of $\Phi_n(2)$ for $n \leq 100$ and got that the largest n with $\Phi_n(2) \leq 100$ is $n = 18$. Thus, it suffices to consider the case $n \leq 18$, and since $n' - k' + 1 \leq n \leq 18$, we get that $n' \leq 34$. Thus, in all cases $\max\{n, n'\} \leq 200$. Putting everything together, we conclude that $b_{N+1} - b_N > 100$ unless both b_N, b_{N+1} correspond to q-nomial coefficients $\binom{n}{k}_q$ or $\binom{n'}{k'}_q$ with $\max\{n, n'\} \leq 200$. Further, unless $q = 2$, we are in the cases from Subsections 3.1, 3.2, 3.3, respectively, and in these there cases, invoking Lemma 1, the lower bounds on $b_{N+1} - b_N$ are $q^m/28$, $q^{m-(n+1)}/2$, $(q-1)^{\phi(\ell)}$, respectively. In the first case we have the exponent $m \geq 2$, while in the other two cases the exponents are $m - (n+1) \geq 1$, $\phi(\ell) \geq 1$. Thus, the inequality $b_{N+1} - b_N > 100$ is satisfied if $q > 201$ independently on (n, k, n', k'). Hence, we only need to check the situations $q \leq 201$ and $\max\{n, n'\} \leq 200$. A computation in this range finishes the job.

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