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OPTION VALUATION UNDER THE VG PROCESS BY A DG METHOD

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Abstract. The paper presents a discontinuous Galerkin method for solving partial integro-differential equations arising from the European as well as American option pricing when the underlying asset follows an exponential variance gamma process. For practical purposes of numerical solving we introduce the modified option pricing problem resulting from a localization to a bounded domain and an approximation of small jumps, and we discuss the related error estimates. Then we employ a robust numerical procedure based on piecewise polynomial generally discontinuous approximations in the spatial domain. This technique enables a simple treatment of the American early exercise constraint by a direct encompassing it as an additional nonlinear source term to the governing equation. Special attention is paid to the proper discretization of non-local jump integral components, which is based on splitting integrals with respect to the domain according to the size of the jumps. Moreover, to preserve sparsity of resulting linear algebraic systems the pricing equation is integrated in the temporal variable by a semi-implicit Euler scheme. Finally, the numerical results demonstrate the capability of the numerical scheme presented within the reference benchmarks.

Keywords: option pricing; variance gamma process; integro-differential equation; American style options; discontinuous Galerkin method; semi-implicit discretization

MSC 2020: 65M60, 35Q91, 65M15, 91G60, 91G80

1. Introduction

The basic ideas of option pricing methods date back to [4] and [31]. These authors proposed relatively simple techniques based on no-arbitrage arguments and leading to the risk-neutral world, i.e., if one can constitute a hedged portfolio of option and its underlying asset, it should have a riskless return and the actual risk attitude of

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market participants should not matter. Notwithstanding, the crucial assumption is the existence of complete markets and Gaussian distribution of log-returns of the underlying asset prices. However, empirical observations show that the returns of financial asset prices are not normally distributed—instead, the fat tails and asymmetry are present, which might lead to market incompleteness.

In order to cope with these facts, one can consider either stochastic volatility, probability distribution with more parameters, or jumps (or their combination), see, e.g., [7] for a review and discussion of various empirical facts, such as large and sudden movements in the price (jumps), heavy tails (kurtosis), risk asymmetry (skewness) and market incompleteness (perfect hedging is sometimes impossible). Obviously, incorporating additional properties into the model commonly increases its complexity and using numerical approaches can be inevitable.

In this contribution we focus on the variance gamma (VG) model, which dates back to the pioneering papers [29] as concerns the symmetric case and [27], [28] as concerns the asymmetric case. Because of its nice interpretations, either as a subordinating Brownian motion depending on the market activity or a difference of two gamma processes measuring the arrival of either positive or negative information, the model has become quite popular in financial modelling including option pricing, despite that it adds an integral operator to governing equations of original option pricing models. As a result, the unknown pricing function is characterized as a solution of a relevant partial integro-differential equation (PIDE) and its valuation leads to new theoretical and numerical issues.

The objective of the paper is to provide the readers the comprehensive methodological concept that forms and improves the option valuation under the VG process, taking into account its infinite activity and finite variation as well. From the previous research, let us quote at least the Monte Carlo (MC) approach [14] and multinomial method [5] that are based on a stochastic concept of option prices. In contrast, the most flexible and efficient way, which is able to resolve many details of realistic option pricing, is a PIDE approach. There is a vast literature focused on finite difference (FD) schemes related to the VG option pricing model, e.g., [2], [8] and [17], but relatively little can be found about variational techniques, such as finite element based methods that provide solutions in the entire computational domain and are also suitable for complex geometries compared to FD methods.

The presented work is the follow up to the results from [20] where a jump process of a finite activity is assumed. In a similar way we transform the original governing equation to the initial-boundary value problem that reflects the properties of the VG process. Next, a proper numerical method is employed. We focus on a discontinuous Galerkin (DG) method, for a complete overview see book [11]. The DG approach is based on piecewise polynomial generally discontinuous approximations in the spatial domain. As a result, sensitivity measures are resolved more properly and a simple treatment of the American early exercise constraint is possible by direct encompassing a penalty term to the governing equation. Further, mention a weak imposition of boundary conditions and a simple adaptivity implementation. Moreover, incorporating an upwind stabilization makes the numerical scheme sufficiently robust with respect to various VG model parameters. Last but not least, the sophisticated treatment of the non-local jump integral operator according to the size of jumps improves the scheme as well. All the above attributes make this method a suitable candidate for solving the problem of option pricing under the VG process. Apart from our recent results [18], [19] and [20], let us mention some of the few other applications of the DG approach in option pricing problems, e.g., [24] and [32].

The paper is organized as follows. In Section 2 we introduce the option pricing problem under the VG model, including relevant partial integro-differential equations. Next, in Section 3 the discretization of the problem is provided. Finally, in Section 4 numerical experiments with the European and American option pricing are evaluated.

2. Option pricing problem under VG model

An option is a special type of financial derivative giving its holder the right to trade an underlying asset S and it is limited by the maturity time T. The simplest forms of this right are the right to buy it (call option) and the right to sell it (put option) for the prespecified price $\mathcal{K}$, usually called the strike price. If the trade is executed, it is said that the option is exercised.

Various types of options are categorized by families of options, related to the determination of the payoff and dates on which the option may be exercised. According to the structure of the payoff, these contracts are classified as vanilla options or as exotic ones, whose payoff results from additional conditions enforced on options. From the second aspect, the way of exercise, we distinguish two major option styles, namely European and American options that allow option exercising at the maturity and before or at the maturity, respectively. This early exercise feature increases the complexity of option pricing models and poses challenging problems in valuation.

In the rest of the paper, we consider both the European and American styles for the plain vanilla option contract, i.e., option whose payoff function $p(S)$ depends only on a difference between the actual underlying asset price S and the strike price $\mathcal{K}$, specifically

$$p(S) = \begin{cases} \max(S - \mathcal{K}, 0) & \text{for a call}, \\ \max(\mathcal{K} - S, 0) & \text{for a put}. \end{cases} \tag{2.1}$$

Despite the piecewise linear character of (2.1), this payoff makes the pricing procedure more challenging than in the case of forwards or swaps.

The forthcoming section is organized as follows. We start with a brief introduction into gamma and variance gamma stochastic processes, followed by a description of the model dynamics and the derivation of the corresponding pricing equation for European style options. Then, we present a localization and a modification of the governing equation that allow us to formulate the resulting option pricing problem as an initial-boundary value one. Finally, we extend the whole pricing procedure to American style options.

2.1. Gamma and variance gamma processes. In order to present the construction of the variance gamma process, it is first necessary to recall the definition of the gamma process itself. Let $t \geqslant 0$ be the actual time and $\{\gamma_{\mu,\nu}(t)\}_{t\geqslant 0}$ denote the gamma process with the mean rate $\mu \in \mathbb{R}$ and variance rate $\nu > 0$. By definition, $\gamma_{\mu,\nu}(t)$ is an increasing, time-continuous stochastic process with independent increments $\gamma_{\mu,\nu}(t+\Delta t) - \gamma_{\mu,\nu}(t)$ over disjoint intervals $(t, t+\Delta t)$ that follow a gamma distribution with the mean $\mu\Delta t$ and variance $\nu\Delta t$, given by the probability density function

$$g_{a,d}(x) = \frac{d^a}{\Gamma(a)} x^{a-1} \exp(-dx), \quad x > 0, \tag{2.2}$$

where $a = \mu^2\Delta t/\nu > 0$ and $d = \mu/\nu > 0$ are the shape-rate parameters, and $\Gamma(\cdot)$ denotes the gamma function, see [1].

Further, we present a brief introduction into the construction of VG processes and their properties. The history of the VG process dates back to the pioneering paper of Madan and Seneta [29], where this new stochastic process was introduced in two ways—as a time changed Brownian motion or as a difference of two gamma processes. For the purpose of a robust approach based on the concept of a stochastic subordination [7], we recall the first case. Let

$$\theta t + \sigma W(t) \tag{2.3}$$

be a drifted Brownian motion, where $\theta \in \mathbb{R}$ denotes the drift, $\sigma > 0$ is the volatility and $\{W(t)\}_{t\geqslant 0}$ means the standard Brownian motion.

Then the VG process is the three-parameter stochastic process $\{X(t)\}_{t\geqslant 0}$, obtained by evaluating (2.3) at stochastic times given by a gamma process with the unit mean rate, i.e.,

$$X(t) \equiv X_{\sigma,\nu,\theta}(t) = \theta\gamma_{1,\nu}(t) + \sigma W(\gamma_{1,\nu}(t)). \tag{2.4}$$

This process belongs to the family of Lévy processes [7], specifically to the class of pure jump processes, since (2.4) does not have any continuous component. The dynamics of (2.4) is best explained by describing its simulation in terms of (σ,ν,θ) for the corresponding Lévy measure $\overline{\nu}(\mathrm{d}x)$ (see [27]), given by

$$\overline{\nu}(\mathrm{d}x) = k_{\mathrm{VG}}(x)\,\mathrm{d}x = \frac{1}{\nu|x|}\exp\Big(\frac{\theta x}{\sigma^2}\Big)\exp\Big(-\frac{|x|}{\sigma}\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}\Big)\,\mathrm{d}x. \tag{2.5}$$

Taking into account the properties of the function k_{VG}, specifically

$$\int_{\mathbb{R}} k_{\mathrm{VG}}(x)\,\mathrm{d}x = \infty \quad \text{and} \quad \int_{\mathbb{R}} \min(1,|x|)k_{\mathrm{VG}}(x)\,\mathrm{d}x < \infty, \tag{2.6}$$

we speak of processes with infinite activity and finite variation, respectively. In other words, almost all paths of $X(t)$ have an infinite number of jumps on every compact interval and they have bounded variation. Both properties are inherited from the gamma process, see the detailed explanation in [27]. In contrast to the BS framework, the property of bounded variation agrees more tightly with economics theories.

The Lévy measure for the VG process (2.5) has an equivalent expression obtainable through a direct manipulation of (2.5). It is closely related to the decomposition of the VG process as a difference of two gamma processes. Hence, the function k_{VG} can be rewritten as

$$k_{\mathrm{VG}}(x) = \frac{1}{\nu}\Big(\frac{\mathrm{e}^{-\lambda_{\mathrm{n}}|x|}}{|x|}\mathbb{1}_{x<0} + \frac{\mathrm{e}^{-\lambda_{\mathrm{p}}x}}{x}\mathbb{1}_{x>0}\Big), \tag{2.7}$$

where $\mathbb{1}_{x<0}$ and $\mathbb{1}_{x>0}$ are indicator functions of subsets $(-\infty,0)$ and $(0,\infty)$, respectively. The positive parameters λ_{n} and λ_{p} provide a way to control decreases and increases in the VG process and take the form

$$\lambda_{\mathrm{n}} = \sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}} + \frac{\theta}{\sigma^2} > 0 \quad \text{and} \quad \lambda_{\mathrm{p}} = \sqrt{\frac{\theta^2}{\sigma^4}+\frac{2}{\sigma^2\nu}} - \frac{\theta}{\sigma^2} > 1. \tag{2.8}$$

In the rest of the paper we prefer the specification of the Lévy measure of the VG process via the set of three parameters $(\nu,\lambda_{\mathrm{n}},\lambda_{\mathrm{p}})$ through which one can indirectly control tails, skewness and kurtosis of the distribution. In particular, heavy tails, higher kurtosis and negative skewness are important for the approximation of a real asset price process, cf. the normal Gaussian distribution in the BS framework [4] and [31].

2.2. Market model and governing equations. Consider the underlying asset $S(t)$ paying a constant dividend yield q with a constant interest rate r. Further, we assume that the movement of the asset prices $\{S(t)\}_{t\geqslant 0}$ is driven by the VG process. The fact that this price process allows for jumps makes the market incomplete, i.e., there exists a large set of equivalent martingale measures consistent with the condition of the absence of arbitrage, see [22].

Following [10], one can assume the existence of the risk-neutral probability measure Q, i.e., some equivalent martingale measure such that the discounted process $\{\mathrm{e}^{-(r-q)t}S(t)\}_{t\geqslant 0}$ is a Q-martingale. Under Q, the asset price process is described by the exponential dynamics

$$S(t) = S(0)\exp(L(t)), \tag{2.9}$$

where the Lévy process $\{L(t)\}_{t\geqslant 0}$ is defined as a drifted VG process (2.4), see [27]. Since the VG process is of finite variation, the log-price process $L(t)$ inherits this property and its Lévy-Itô decomposition can be written in the simplified superposition as

$$L(t) = bt + \int_{\mathbb{R}} x\overline{\mu}^L(t,\,\mathrm{d}x), \tag{2.10}$$

where b is the drift of the logarithmic price of the asset and the integral of a Poisson random measure $\overline{\mu}^L$ represents the VG process (2.4). For the detailed information, we refer the reader to semimartingale theory [23].

On the other hand, since (2.5) and (2.7) result to $\int_{\mathbb{R}}|x|\overline{\nu}(\mathrm{d}x) < \infty$, the Lévy process $L(t)$ has the finite first moment and thus the big jumps can be compensated by $\overline{\nu}^L(t,\,\mathrm{d}x) = t\cdot\overline{\nu}(\mathrm{d}x)$, called the compensator of $\overline{\mu}^L$. More precisely, the Lévy-Itô decomposition (2.10) takes a more general form

$$L(t) = \Big(b + \int_{|x|<1} x\overline{\nu}(\mathrm{d}x)\Big)t + \int_{|x|\geqslant 1} x\overline{\mu}^L(t,\,\mathrm{d}x) + \int_{|x|<1} x(\overline{\mu}^L - \overline{\nu}^L)(t,\,\mathrm{d}x). \tag{2.11}$$

Taking into account the martingale property of the discounted process $\mathrm{e}^{-(r-q)t}S(t)$, the drift term in (2.11) has to satisfy the relation (see [12])

$$b + \int_{|x|<1} x\overline{\nu}(\mathrm{d}x) = r - q - \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x\mathbb{1}_{|x|<1})\overline{\nu}(\mathrm{d}x). \tag{2.12}$$

From (2.12) we get the particular value of the drift term of the Lévy process $L(t)$

$$b = r - q + \int_{\mathbb{R}} (1-\mathrm{e}^x)\overline{\nu}(\mathrm{d}x) = r - q + \omega, \quad \omega\in\mathbb{R}. \tag{2.13}$$

Referring to [2], one can express

$$\omega = \int_{\mathbb{R}} (1-\mathrm{e}^x)k_{\mathrm{VG}}(x)\,\mathrm{d}x = \frac{1}{\nu}\Big(\ln\Big(1+\frac{1}{\lambda_{\mathrm{n}}}\Big) + \ln\Big(1-\frac{1}{\lambda_{\mathrm{p}}}\Big)\Big), \tag{2.14}$$

which plays the crutial role of a compensation constant that renders the process $\{S(0)\exp(\omega t + X(t))\}_{t\geqslant 0}$ into a Q-martingale. Further, putting (2.10) and (2.12) together, we get the risk-neutral process of the underlying asset log-price under the VG dynamics in the form

$$L(t) = (r - q + \omega)t + X(t). \tag{2.15}$$

For a better view of the asset dynamics given by (2.9) and (2.15), Figure 1 shows sample paths of an underlying asset with the particular settings.

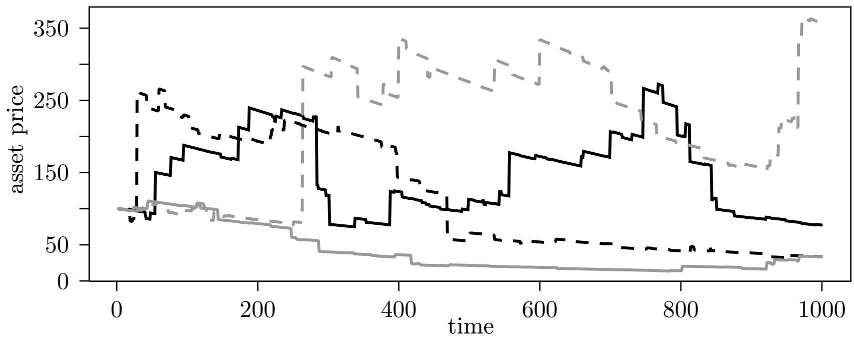

Figure 1. Sample paths of an asset driven by the VG process with $S(0) = 100$, $r = 0.1$, $q = 0.0$, $\nu = 0.5$, $\lambda_{\rm n} = 2.8504$ and $\lambda_{\rm p} = 2.4948$.

In what follows, we consider a European option contract written on the underlying asset (2.9) with the terminal payoff (2.1). Following the steps from [7], Chapter 12, the value of such option, denoted by $V(S,t)$, can be expressed as the discounted conditional expectation of the payoff function under the risk-neutral probability measure Q, which has the following form, taking into account the Markov property,

$$V(S,t) = \mathrm{e}^{-r(T-t)} E_Q[p(S(T)) \mid S(t) = S]. \tag{2.16}$$

By making the change of variables $x = \ln(S/\mathcal{K})$, $\hat{t} = T - t$ and noting that

$$w_0(x) = p(\mathcal{K}\mathrm{e}^x)/\mathcal{K}, \quad w(x,\hat{t}) = V(\mathcal{K}\mathrm{e}^x, T-\hat{t})/\mathcal{K}, \tag{2.17}$$

we obtain from (2.16) and (2.9) the expression for $w(x,\hat{t})$ through the following steps:

$$\begin{aligned} w(x,\hat{t}) &= V(\mathcal{K}\mathrm{e}^x, T-\hat{t})/\mathcal{K} = V(S,t)/\mathcal{K} = \mathrm{e}^{-r(T-t)} E_Q[p(S(T)) \mid S(t) = S]/\mathcal{K} \\ &= \mathrm{e}^{-r(T-t)} E_Q[p(S(t)\exp(L(T-t)))]/\mathcal{K} = \mathrm{e}^{-r\hat{t}} E_Q[p(\mathcal{K}\mathrm{e}^x \exp(L(\hat{t})))/\mathcal{K}] \\ &= \mathrm{e}^{-r\hat{t}} E_Q[w_0(x + L(\hat{t}))]. \end{aligned} \tag{2.18}$$

Further, the infinitesimal generator $\mathcal{L}^L$ (see [7] for the definition) of the process $L(\hat{t})$, applied to function $f(x,\hat{t})$, has the form

$$\begin{aligned} \mathcal{L}^L f(x,\hat{t}) &= \left(b + \int_{|y|<1} y\overline{\nu}(\mathrm{d}y)\right)\frac{\partial f}{\partial x}(x,\hat{t}) \\ &\quad + \int_{\mathbb{R}} (f(x+y,\hat{t}) - f(x,\hat{t}) - y\mathbb{1}_{|y|<1}\frac{\partial f}{\partial x}(x,\hat{t}))\overline{\nu}(\mathrm{d}y) \\ &= (r - q + \omega)\frac{\partial f}{\partial x}(x,\hat{t}) + \int_{\mathbb{R}} (f(x+y,\hat{t}) - f(x,\hat{t}))\overline{\nu}(\mathrm{d}y). \end{aligned} \tag{2.19}$$

Assume that $w\colon \mathbb{R}\times[0,T]\to\mathbb{R}_0^+$ defined by (2.18) is once continuously differentiable in both arguments on the domain $\mathbb{R}\times[\hat{t}_0,T]$ for some $\hat{t}_0>0$. By martingale pricing theory [7], one can show that

$$\frac{\partial}{\partial\hat{t}}(\mathrm{e}^{r\hat{t}}w(x,\hat{t}))=\mathcal{L}^L(\mathrm{e}^{r\hat{t}}w(x,\hat{t}));\tag{2.20}$$

in other words, as the fundamental result, the price function w satisfies the resulting PIDE

$$\frac{\partial w}{\partial\hat{t}}\underbrace{-(r-q+\omega)\frac{\partial w}{\partial x}+rw}_{\mathcal{D}(w)}=\underbrace{\int_{\mathbb{R}}(w(x+y,\hat{t})-w(x,\hat{t}))k_{\mathrm{VG}}(y)\,\mathrm{d}y}_{\mathcal{I}(w)}.\tag{2.21}$$

The typical feature of the VG model as well as a general pure jump model with finite variation is that the corresponding pricing equation (2.21) is formally of the first order, since there is no diffusion term in the differential operator $\mathcal{D}$ and the compensation constant ω contributes directly to the convection term of $\mathcal{D}$.

Since the smoothness condition $w\in\mathcal{C}^1(\mathbb{R}\times[0,T])$ does not hold in the VG model (see a counterexample in [9]), the classical solution is not available for the problem (2.21) for all $(x,\hat{t})\in\mathbb{R}\times(0,T)$, subject to the initial condition

$$w(0,\hat{t})=w_0(x)=\begin{cases}\max(\mathrm{e}^x-1,0) & \text{for a call,}\\ \max(1-\mathrm{e}^x,0) & \text{for a put.}\end{cases}\tag{2.22}$$

Therefore, under weakened regularity assumptions on the price function w on can show that w defined by (2.16)–(2.18) is a generalized (viscosity) solution of the Cauchy problem (2.21)–(2.22). The detailed description of the concept of viscosity solutions for option pricing problems can be found in [9].

However, if we replace the Cauchy problem (2.21)–(2.22) with a new problem, the solution of which in a certain sense approximates the original option contract, it is even possible to introduce the notion of a weak solution in Sobolev spaces, see Section 2.3.

2.3. Localized and modified problem. For a later numerical approach to solving the pricing PIDE it is necessary to deal with the unbounded spatial domain $\mathbb{R}$, the singularity of the integral operator $\mathcal{I}$ from (2.21) and the non-local character of the term $w(x+y,\hat{t})$. Therefore, in this section, we introduce a new modified pricing problem that reflects these issues, and discuss for the localization and approximation errors. The inspiring ideas for the approach presented come from [7], [8] and [17].

At first, we truncate the spatial domain $\mathbb{R}$ to a bounded interval $\Omega = (x_{\min}, x_{\max})$, where $x_{\min} < 0$ and $x_{\max} > 0$ stand for the minimal and maximal scaled logarithmic asset price, respectively. Without loss of generality, we assume that $x_{\min} = -x_{\max}$ in the rest of the paper.

Next, we modify the approach from [17], split the integral term $\mathcal{I}$ into three parts and evaluate or approximate each of them separately. Let $\varepsilon > 0$. Then one can write

$$\begin{aligned}(2.23)\qquad \mathcal{I}(w)(x,\hat{t}) = &\underbrace{\int_{|y|\leqslant\varepsilon}(w(x+y,\hat{t}) - w(x,\hat{t}))k_{\mathrm{VG}}(y)\,\mathrm{d}y}_{I_1(w)} \\ &- \underbrace{\int_{|y|>\varepsilon} w(x,\hat{t})k_{\mathrm{VG}}(y)\,\mathrm{d}y}_{I_2(w)} + \underbrace{\int_{|y|>\varepsilon} w(x+y,\hat{t})k_{\mathrm{VG}}(y)\,\mathrm{d}y}_{I_3(w)}.\end{aligned}$$

The term I_1 represents the contribution of small jumps and it can be approximated with the aid of the Taylor formula up to the second order (under appropriate regularity assumptions),

$$(2.24)\qquad w(x+y,\hat{t}) = w(x,\hat{t}) + y\frac{\partial w}{\partial x}(x,\hat{t}) + \frac{y^2}{2}\frac{\partial^2 w}{\partial x^2}(x,\hat{t}) + \mathcal{O}(\varepsilon^3), \quad y \in [-\varepsilon,\varepsilon].$$

Then, we have

$$(2.25)\qquad I_1(w) \approx \frac{\partial w}{\partial x}(x,\hat{t})\underbrace{\int_{|y|\leqslant\varepsilon} yk_{\mathrm{VG}}(y)\,\mathrm{d}y}_{\beta(\varepsilon)} + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(x,\hat{t})\underbrace{\int_{|y|\leqslant\varepsilon} y^2k_{\mathrm{VG}}(y)\,\mathrm{d}y}_{\sigma^2(\varepsilon)}$$

and an easy calculation leads to

$$(2.26)\qquad \beta(\varepsilon) = \frac{1}{\nu\lambda_{\mathrm{p}}}(1-\mathrm{e}^{-\lambda_{\mathrm{p}}\varepsilon}) - \frac{1}{\nu\lambda_{\mathrm{n}}}(1-\mathrm{e}^{-\lambda_{\mathrm{n}}\varepsilon}),$$

$$(2.27)\qquad \sigma^2(\varepsilon) = \frac{1}{\nu\lambda_{\mathrm{p}}^2}(1-\mathrm{e}^{-\lambda_{\mathrm{p}}\varepsilon} - \lambda_{\mathrm{p}}\varepsilon\mathrm{e}^{-\lambda_{\mathrm{p}}\varepsilon}) + \frac{1}{\nu\lambda_{\mathrm{n}}^2}(1-\mathrm{e}^{-\lambda_{\mathrm{n}}\varepsilon} - \lambda_{\mathrm{n}}\varepsilon\mathrm{e}^{-\lambda_{\mathrm{n}}\varepsilon}).$$

In other words, the approximate evaluation of (2.25) leads to the additional convection as well as diffusion terms. In particular, the nonzero diffusion coefficient (2.27) completely changes the character and the order of the differential operator; and thanks to its newly acquired ellipticity, standard numerical techniques can be conveniently used.

R e m a r k 2.1. Note that one can intuitively use the Taylor expansion of only the first order in (2.24) due to the natural assumptions of $\mathcal{C}^1$-smoothness of w, see (2.21). In this case we simply put $\sigma^2(\varepsilon) = 0$, but this treatment does not allow us to derive estimates based on the approximation of the process (2.15) by an appropriate finite activity one, see [7].

For the term I_2 we may simply use direct calculation and obtain the additional reaction term. More precisely,

$$I_2(w) = w(x,\hat{t}) \int_{|y|>\varepsilon} k_{\mathrm{VG}}(y)\,\mathrm{d}y = \lambda(\varepsilon) w(x,\hat{t}), \tag{2.28}$$

where

$$\lambda(\varepsilon) = \frac{1}{\nu}\left(\int_{\varepsilon}^{\infty} \frac{\mathrm{e}^{-\lambda_{\mathrm{p}} y}}{y}\,\mathrm{d}y - \int_{-\infty}^{-\varepsilon} \frac{\mathrm{e}^{\lambda_{\mathrm{n}} y}}{y}\,\mathrm{d}y\right) = \frac{1}{\nu}(E_1(\lambda_{\mathrm{p}}\varepsilon) + E_1(\lambda_{\mathrm{n}}\varepsilon)). \tag{2.29}$$

In the last relation in (2.29) we use the definition of the exponential integral

$$E_1(s) = \int_s^{\infty} \frac{\mathrm{e}^{-\zeta}}{\zeta}\,\mathrm{d}\zeta, \quad s > 0;$$

for details see [1].

Finally, the last term I_3 has already a smooth kernel, but due to presence of the term $w(x+y,\hat{t})$ it is non-local. In the first instance we split I_3 in two parts, related to the left and right tails of the VG distribution, and make the change of variables $z = x + y$, then it can be written as

$$I_3(w) = \underbrace{\int_{-\infty}^{x-\varepsilon} w(z,\hat{t}) k_{\mathrm{VG}}(z-x)\,\mathrm{d}z}_{I_3^-(w)} + \underbrace{\int_{x+\varepsilon}^{\infty} w(z,\hat{t}) k_{\mathrm{VG}}(z-x)\,\mathrm{d}z}_{I_3^+(w)}. \tag{2.30}$$

In order to evaluate the contributions of large jumps as $|z| \to \infty$, we require that the pricing function w satisfies the so-called knock-out condition, defined as the discounted and shifted payoff

$$w(z,\hat{t}) = \mathrm{e}^{-r\hat{t}} w_0(z + (r-q)\hat{t}) \quad \text{for } z \in \mathbb{R} \setminus \Omega, \tag{2.31}$$

to reflect asymptotic values of European option prices as $S \to 0+$ and $S \to \infty$, see [15]. In logarithmic prices, the asymptotic behaviour is

$$\lim_{x\to-\infty} w(x,\hat{t}) = 0, \quad \lim_{x\to\infty} \{w(x,\hat{t}) - (\mathrm{e}^{x-q\hat{t}} - \mathrm{e}^{-r\hat{t}})\} = 0, \quad \hat{t} > 0, \quad \text{(call)} \tag{2.32}$$

$$\lim_{x\to-\infty} \{w(x,\hat{t}) - (\mathrm{e}^{-r\hat{t}} - \mathrm{e}^{x-q\hat{t}})\} = 0, \quad \lim_{x\to\infty} w(x,\hat{t}) = 0, \quad \hat{t} > 0. \quad \text{(put)} \tag{2.33}$$

From the financial point of view, the condition (2.31) can be interpreted as a payment of an artificial rebate, see [13].

In what follows, suppose $x_{\max} \geqslant |r-q|T$ (as usually holds in practice) and the case of a call option, then the knock-out condition (2.31) results into

$$w(z,\hat{t}) = \begin{cases} 0 & \text{for } z \leqslant -x_{\max}, \\ e^{z-q\hat{t}} - e^{-r\hat{t}} & \text{for } z \geqslant x_{\max}. \end{cases} \tag{2.34}$$

Thus, the contribution of large jumps from the left tail I_3^- (i.e., with the negative argument of the kernel k_{VG}) can be represented by the non-local truncated integral operator $\mathcal{I}_\varepsilon^-$, defined on the bounded domain for $x \in \Omega$ as

$$\mathcal{I}_\varepsilon^-(w)(x,\hat{t}) = \begin{cases} 0 & \text{if } x-\varepsilon \leqslant -x_{\max}, \\ \displaystyle\int_{-x_{\max}}^{x-\varepsilon} w(z,\hat{t})k_{\mathrm{VG}}(z-x)\,\mathrm{d}z & \text{if } x-\varepsilon > -x_{\max}. \end{cases} \tag{2.35}$$

Analogously, in the case of the right tail we get $I_3^+(w) = \mathcal{I}_\varepsilon^+(w) + \mathcal{R}^\varepsilon$, where the integral operator $\mathcal{I}_\varepsilon^+$ is defined in a similar manner as (2.35), i.e.,

$$\mathcal{I}_\varepsilon^+(w)(x,\hat{t}) = \begin{cases} 0 & \text{if } x+\varepsilon \geqslant x_{\max}, \\ \displaystyle\int_{x+\varepsilon}^{x_{\max}} w(z,\hat{t})k_{\mathrm{VG}}(z-x)\,\mathrm{d}z & \text{if } x+\varepsilon < x_{\max}, \end{cases} \tag{2.36}$$

and now the source term $\mathcal{R}^\varepsilon$ results from the nonzero condition (2.34) as

$$\mathcal{R}^\varepsilon(x,\hat{t}) = \begin{cases} \displaystyle\int_{x+\varepsilon}^{\infty} (e^{z-q\hat{t}} - e^{-r\hat{t}})k_{\mathrm{VG}}(z-x)\,\mathrm{d}z & \text{if } x+\varepsilon \geqslant x_{\max}, \\ \displaystyle\int_{x_{\max}}^{\infty} (e^{z-q\hat{t}} - e^{-r\hat{t}})k_{\mathrm{VG}}(z-x)\,\mathrm{d}z & \text{if } x+\varepsilon < x_{\max}. \end{cases} \tag{2.37}$$

Using the back transformation $y = z - x$ and definition of the exponential integral, (2.37) can be expressed as

$$\mathcal{R}^\varepsilon(x,\hat{t}) = \begin{cases} \dfrac{1}{\nu}(e^{x-q\hat{t}}E_1((\lambda_{\mathrm{p}}-1)\varepsilon) - e^{-r\hat{t}}E_1(\lambda_{\mathrm{p}}\varepsilon)), & x+\varepsilon \geqslant x_{\max}, \\ \dfrac{1}{\nu}(e^{x-q\hat{t}}E_1((\lambda_{\mathrm{p}}-1)(x_{\max}-x)) - e^{-r\hat{t}}E_1(\lambda_{\mathrm{p}}(x_{\max}-x))), & \text{otherwise.} \end{cases} \tag{2.38}$$

A similar approach for put options leads to the expression

$$\mathcal{R}^\varepsilon(x,\hat{t}) = \begin{cases} \dfrac{1}{\nu}(e^{-r\hat{t}}E_1(\lambda_{\mathrm{n}}\varepsilon) - e^{x-q\hat{t}}E_1((\lambda_{\mathrm{n}}+1)\varepsilon)), & x-\varepsilon \leqslant -x_{\max}, \\ \dfrac{1}{\nu}(e^{-r\hat{t}}E_1(\lambda_{\mathrm{n}}(x_{\max}+x)) - e^{x-q\hat{t}}E_1((\lambda_{\mathrm{n}}+1)(x_{\max}+x))), & \text{otherwise.} \end{cases} \tag{2.39}$$

Actually, within the aforementioned localization and approximation of the integral operator we follow the idea from [7], where the finite activity process

$$L^{\varepsilon}(t) = (r - q + \omega(\varepsilon))t + X^{\varepsilon}(t) \tag{2.40}$$

approximates process (2.15) to avoid the singularity of the kernel k_{VG} near to zero.

Therefore, to ensure the probabilistic representation of the solution of the new modified pricing equation it is necessary not to violate the martingale property of (2.40). This condition is essential in order that we are able to establish a relation between solutions of the original and the modified pricing problems, see Theorem 2.1. As in (2.15) the martingale property can be preserved by a suitable choice of the compensation constant $\omega(\varepsilon)$ in (2.40), specifically

$$\omega(\varepsilon) = \int_{|y|>\varepsilon} (1 - \mathrm{e}^{y}) k_{\mathrm{VG}}(y)\,\mathrm{d}y = \lambda(\varepsilon) - \kappa(\varepsilon), \tag{2.41}$$

where $\kappa(\varepsilon) = \nu^{-1}(E_1((\lambda_{\mathrm{p}} - 1)\varepsilon) + E_1((\lambda_{\mathrm{n}} + 1)\varepsilon))$.

In the next paragraph, we describe in detail the relationship between ω and $\omega(\varepsilon)$. Using the Taylor series of e^{y} about $y = 0$, we have

$$\omega - \omega(\varepsilon) = \int_{|y|\leqslant\varepsilon} (1 - \mathrm{e}^{y}) k_{\mathrm{VG}}(y)\,\mathrm{d}y = -\beta(\varepsilon) - \frac{\sigma^2(\varepsilon)}{2} - \int_{|y|\leqslant\varepsilon} \sum_{i=3}^{\infty} \frac{y^i}{i!} k_{\mathrm{VG}}(y)\,\mathrm{d}y \tag{2.42}$$

and the Taylor remainder, expressed in the integral form, can be easily estimated as

$$\begin{aligned} \left|\int_{|y|\leqslant\varepsilon} \sum_{i=3}^{\infty} \frac{y^i}{i!} k_{\mathrm{VG}}(y)\,\mathrm{d}y\right| &= \left|\int_{|y|\leqslant\varepsilon} \left(\int_0^y \frac{\mathrm{e}^{s}}{2}(y - s)^2\,\mathrm{d}s\right) k_{\mathrm{VG}}(y)\,\mathrm{d}y\right| \\ &\leqslant \frac{1}{6} \int_{|y|\leqslant\varepsilon} |y^3| \mathrm{e}^{|y|} k_{\mathrm{VG}}(y)\,\mathrm{d}y \\ &\leqslant \frac{1}{9\nu} \max(1, \mathrm{e}^{\varepsilon - \lambda_{\mathrm{n}}\varepsilon})\varepsilon^3. \end{aligned} \tag{2.43}$$

Comparing (2.42)–(2.43) and (2.24)–(2.25) one concludes that the approximations of ω and $w(x + y, \hat{t})$ produce similar errors proportional to ε only provided that

$$\omega + \beta(\varepsilon) \approx \omega(\varepsilon) - \frac{\sigma^2(\varepsilon)}{2}. \tag{2.44}$$

Consequently, taking all the above into account, it is possible to represent the modified option pricing problem as the initial-boundary value one for an unknown function $u(x, \hat{t})\colon \Omega \times (0, T) \to \mathbb{R}_0^+$ governed by

$$\frac{\partial u}{\partial \hat{t}} + \mathcal{D}^{\varepsilon}(u) = \mathcal{I}^{\varepsilon}(u) + \mathcal{R}^{\varepsilon} \quad \text{in } \Omega \times (0, T), \tag{2.45}$$

where $\mathcal{I}^\varepsilon = \mathcal{I}_\varepsilon^- + \mathcal{I}_\varepsilon^+$ is given by (2.35)–(2.36), and

$$\mathcal{D}^\varepsilon(u) = -\frac{\sigma^2(\varepsilon)}{2}\frac{\partial^2 u}{\partial x^2} - \Big(r - q + \omega(\varepsilon) - \frac{\sigma^2(\varepsilon)}{2}\Big)\frac{\partial u}{\partial x} + (r + \lambda(\varepsilon))u, \tag{2.46}$$

and $u(x, \hat{t})$ satisfies the (restricted) initial condition

$$u(x, 0) = w_0(x) \quad \text{in } \Omega \tag{2.47}$$

and a couple of appropriate boundary conditions

$$u(-x_{\max}, \hat{t}) = u_{\rm L}(\hat{t}) = \begin{cases} 0 & \text{for a call,} \\ {\rm e}^{-r\hat{t}} - {\rm e}^{-x_{\max} - q\hat{t}} & \text{for a put,} \end{cases} \tag{2.48}$$

$$u(x_{\max}, \hat{t}) = u_{\rm U}(\hat{t}) = \begin{cases} {\rm e}^{x_{\max} - q\hat{t}} - {\rm e}^{-r\hat{t}} & \text{for a call,} \\ 0 & \text{for a put.} \end{cases} \tag{2.49}$$

Although the Dirichlet boundary conditions (2.48)–(2.49) reflect the asymptotic behaviour of the options (2.32)–(2.33), they are inaccurate in general, see [18]. However, from the point view of financial engineering, this fact is marginal provided that the zone of financial interest, i.e., the domain $\Omega^* \subset \Omega$, in which option values are desirable to know, is sufficiently distant from the far-field boundary $\partial\Omega$.

In what follows, we discuss estimates for localization and truncation errors related to approximation of the original option pricing contract by the artificial one paying a rebate corresponding to the discounted and shifted payoff.

Theorem 2.1. *Let w be the (viscosity) solution of (2.21)–(2.22) and let w_Ω^ε be the (classical) solution of (2.45)–(2.49), both related to the European put option contract. Then,*

$$|w(x, \hat{t}) - w_\Omega^\varepsilon(x, \hat{t})| \leqslant C(\varepsilon {\rm e}^{\max(\lambda_{\rm p}, \lambda_{\rm n})\varepsilon} + {\rm e}^{-\alpha(x_{\max} - |x|)}) \quad \textit{for } x \in \Omega, \tag{2.50}$$

where $0 < \alpha < \min(\lambda_{\rm p}, \lambda_{\rm n})$ and the constant C does not depend on $x_{\max}$ and ε.

Proof. The proof is based on the application of the probabilistic approach from [3] and [7]. Accordingly, we split the error in two parts and treat each part separately, i.e.,

$$|w(x, \hat{t}) - w_\Omega^\varepsilon(x, \hat{t})| \leqslant |w(x, \hat{t}) - w^\varepsilon(x, \hat{t})| + |w^\varepsilon(x, \hat{t}) - w_\Omega^\varepsilon(x, \hat{t})|, \tag{2.51}$$

where the function $w^\varepsilon(x, \hat{t})$ satisfies the Cauchy problem

$$\frac{\partial w^\varepsilon}{\partial \hat{t}} + \mathcal{D}_\varepsilon(w^\varepsilon) = I_3(w^\varepsilon) \ \text{ in } \mathbb{R} \times (0, T), \tag{2.52}$$

$$w^\varepsilon(x, 0) = w_0(x) \quad \text{in } \mathbb{R}. \tag{2.53}$$

Since w_0 is Lipschitz continuous (the proof is left to the reader), by a direct application of [8], Theorem 5.1, arising from [7], Proposition 6.2, we have

$$|w(x,\hat{t}) - w^{\varepsilon}(x,\hat{t})| \leqslant C_1 \frac{\int_{|y|\leqslant\varepsilon} |y|^3 k_{\mathrm{VG}}(y)\,\mathrm{d}y}{\sigma^2(\varepsilon)} \leqslant C_2 \varepsilon \mathrm{e}^{\max(\lambda_{\mathrm{p}},\lambda_{\mathrm{n}})\varepsilon} \tag{2.54}$$

with C_1, C_2 independent of ε. The second inequality in (2.54) results from the estimates

$$\int_{|y|\leqslant\varepsilon} |y|^3 k_{\mathrm{VG}}(y)\,\mathrm{d}y \leqslant \frac{1}{\nu}\int_{|y|\leqslant\varepsilon} y^2\,\mathrm{d}y = \frac{2}{3\nu}\varepsilon^3, \tag{2.55}$$

$$\int_{|y|\leqslant\varepsilon} y^2 k_{\mathrm{VG}}(y)\,\mathrm{d}y \geqslant \frac{1}{\nu}\min(\mathrm{e}^{-\lambda_{\mathrm{p}}\varepsilon}, \mathrm{e}^{-\lambda_{\mathrm{n}}\varepsilon})\int_{|y|\leqslant\varepsilon} |y|\,\mathrm{d}y = \frac{1}{\nu}\varepsilon^2 \mathrm{e}^{-\max(\lambda_{\mathrm{p}},\lambda_{\mathrm{p}})\varepsilon}. \tag{2.56}$$

In the following, we equivalently express the problem (2.45)–(2.49) with the knock-out condition (2.31) as

$$\frac{\partial w^{\varepsilon}_{\Omega}}{\partial \hat{t}} + \mathcal{D}_{\varepsilon}(w^{\varepsilon}_{\Omega}) = I_3(w^{\varepsilon}_{\Omega}) \quad \text{in } \Omega \times (0,T), \tag{2.57}$$

$$w^{\varepsilon}_{\Omega}(x,0) = w_0(x) \quad \text{in } \Omega, \tag{2.58}$$

$$w^{\varepsilon}_{\Omega}(x,\hat{t}) = \mathrm{e}^{-r\hat{t}} w_0(x + (r-q)\hat{t}) \quad \text{in } \mathbb{R}\setminus\Omega. \tag{2.59}$$

More precisely, the solution of (2.45)–(2.49) represents a solution of the localized problem (2.57)–(2.59) restricted to Ω. Since the payoff of a put option is bounded, i.e., $\operatorname*{ess\,sup}_{x\in\mathbb{R}} w_0(x) = 1$, and the function k_{VG} satisfies

$$\int_{|y|>1} \mathrm{e}^{\alpha|y|} k_{\mathrm{VG}}(y)\,\mathrm{d}y < \infty \quad \text{for } 0 < \alpha < \min(\lambda_{\mathrm{p}}, \lambda_{\mathrm{n}}), \tag{2.60}$$

the assumptions of [8], Proposition 4.1, are fulfilled for the process $L^{\varepsilon}(t)$ and thus we obtain the estimate

$$|w^{\varepsilon}(x,\hat{t}) - w^{\varepsilon}_{\Omega}(x,\hat{t})| \leqslant C_3 \mathrm{e}^{-\alpha(x_{\max}-|x|)} \quad \text{for } x \in \Omega \tag{2.61}$$

with C_3 independent of $x_{\max}$. Finally, putting (2.54) and (2.61) together, we obtain the desired estimate (2.50). $\square$

R e m a r k 2.2. The solution w of (2.21)–(2.22) is considered to be a viscosity solution, the existence and uniqueness of which is guaranteed by the Lipschitz continuity and polynomial growth at infinity of w_0, for more details see [9]. Furthermore, the solution w^{ε}_{Ω} of (2.45)–(2.49) is the classical solution, the existence and uniqueness of which is a direct consequence of the Feynman-Kac representation of w^{ε}_{Ω} in terms of the process (2.40), see [7]. Therefore, the point-wise evaluation of the error in (2.50) makes sense, because the classical solution w^{ε}_{Ω} is also a viscosity solution under given regularity; for further reading we refer to [9].

R e m a r k 2.3. The estimate (2.50) illustrates that the localization error has an exponential decay proportional to the size of $|\Omega|$ and the truncation error decreases almost linearly with respect to the size of ε. As noted above these estimates cannot be directly applied to call options, and to obtain similar results (for calls) one can use the put-call parity to transform the option pricing problem to put options, i.e.,

$$w_{\rm call}(x,\hat{t}) - w_{\rm put}(x,\hat{t}) = {\rm e}^{x-q\hat{t}} - {\rm e}^{-r\hat{t}}. \tag{2.62}$$

Finally, we derive the variational formulation to (2.45)–(2.49). In the first instance, we recall the well-known Lebesgue space $L^2(\Omega)$ with the norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$ induced by the inner product $(u,v) = \int_\Omega uv\,{\rm d}x$, and the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega)\colon v(-x_{\max}) = v(x_{\max}) = 0\}$. The detailed definition of Lebesgue, Sobolev and Bochner spaces can be found in [25].

Next, we follow standard steps. Considering a test function v from the space $\mathcal{C}_0^\infty(\Omega)$, that is densely embedded in $H_0^1(\Omega)$, we multiply (2.45) by v and employ integration by parts for diffusion terms. As a result, we define the bilinear form

$$\begin{aligned}\mathcal{D}^\varepsilon(u,v) = \frac{\sigma^2(\varepsilon)}{2}\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,{\rm d}x + \Big(\frac{\sigma^2(\varepsilon)}{2} - \omega(\varepsilon) - r + q\Big)\int_\Omega \frac{\partial u}{\partial x} v\,{\rm d}x \\ + (r + \lambda(\varepsilon))\int_\Omega uv\,{\rm d}x\end{aligned} \tag{2.63}$$

and introduce the following concept of a weak solution similarly as in [20].

Definition 2.1. The variational formulation of (2.45)–(2.49) reads: Find $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ such that the following conditions are satisfied:

$$u - u_{\rm D} \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \tag{2.64}$$

where $u_{\rm D}(\hat{t})$ fulfills

$$u_{\rm D}(\hat{t})|_{x=-x_{\max}} = u_{\rm L}(\hat{t}) \quad \text{and} \quad u_{\rm D}(\hat{t})|_{x=x_{\max}} = u_{\rm U}(\hat{t})$$

a.e. $\hat{t} \in (0,T)$,

$$\Big(\frac{\partial u}{\partial \hat{t}}(\hat{t}),v\Big) + \mathcal{D}^\varepsilon(u(\hat{t}),v) = (\mathcal{I}^\varepsilon(u(\hat{t})),v) + (\mathcal{R}^\varepsilon(\hat{t}),v) \qquad \forall\, v \in H_0^1(\Omega) \text{ a.e. } \hat{t} \in (0,T), \tag{2.65}$$

$$(u(0),v) = (w_0,v) \quad \forall\, v \in H_0^1(\Omega). \tag{2.66}$$

R e m a r k 2.4. Since we avoid the singularity of the original integral operator and at the same time obtain the parabolicity of the modified price equation, to prove the existence and uniqueness of the weak solution of (2.64)–(2.66), we may apply results from [30], specifically Theorem 2.3 and Theorem 3.4, introduced in that paper.

2.4. Extension to American options. In contrast to the European style option, an American style option can be exercised before the expiry of the contract, deriving thus a more complex problem. In this case, we have to encompass an additional constraint to the problem (2.21)–(2.22) that $w(x,\hat{t}) \geqslant w_0(x)$ at any time $\hat{t} \in (0,T)$. In other words, the value of an American option cannot fall below its payoff function.

This American feature leads to a moving-boundary problem [6], where apart from solving the pricing equation, it is also necessary to determine two regions separated by a free boundary $\mathcal{E}$ driven by the optimal exercise price. Thus, in the exercise region $\Omega_{\mathrm{E}} \subset \mathbb{R}$, it is optimal to exercise the option early and we solve the problem

$$\left.\begin{aligned} &\frac{\partial w}{\partial \hat{t}} + \mathcal{D}(w) - \mathcal{I}(w) > 0, \\ &w(x,\hat{t}) = w_0(x) \end{aligned}\right\} \quad \text{in } \Omega_{\mathrm{E}} \times (0,T). \tag{2.67}$$

While in the continuation region, it is not optimal to exercise early and we solve the problem

$$\left.\begin{aligned} &\frac{\partial w}{\partial \hat{t}} + \mathcal{D}(w) - \mathcal{I}(w) = 0, \\ &w(x,\hat{t}) > w_0(x) \end{aligned}\right\} \quad \text{in } (\mathbb{R} \setminus \Omega_{\mathrm{E}}) \times (0,T). \tag{2.68}$$

Moreover, to guarantee the well-posedness of (2.67)–(2.68), the continuity of the price function w and its derivative $\partial w/\partial x$ is required on the free boundary $\mathcal{E}$, see [33].

There are several approaches how to handle the early exercise feature, among the widely used ones let us cite the linear complementarity problem with penalty techniques [34] or operator splitting methods [21]. In this paper we follow the penalty approach and reformulate both the problems (2.67) and (2.68) into one equation valid everywhere in both regions, i.e.,

$$\frac{\partial w}{\partial \hat{t}} + \mathcal{D}(w) - \mathcal{I}(w) - g = 0 \quad \text{in } \mathbb{R} \times (0,T), \tag{2.69}$$

where the penalty term $g = g(x,\hat{t})$ is defined so as to ensure the American constraint and it satisfies the conditions

$$g(x,\hat{t}) = \begin{cases} \text{zero} & \text{if } w(x,\hat{t}) > w_0(x), \\ \text{positive} & \text{if } w(x,\hat{t}) = w_0(x). \end{cases} \tag{2.70}$$

Analogously to the case of European options, we transfer (2.69) to the localized and modified option pricing problem

$$\frac{\partial w}{\partial \hat{t}} + \mathcal{D}^{\varepsilon}(w) = \mathcal{I}^{\varepsilon}(w) + \mathcal{R}^{\varepsilon} + g \quad \text{in } \Omega \times (0,T), \tag{2.71}$$

subjected to the same initial condition (2.47) and boundary conditions set in accordance with the early exercise feature. More precisely, if $-x_{\max}$ or $x_{\max}$ belongs to Ω_{E}, then the value of w at that point is given by the initial condition (2.47), else the values correspond to the boundary conditions for European counterparts (2.48)–(2.49). In line with this, terms (2.38) and (2.39) are slightly modified by avoiding the parameters r and q.

Finally, note that the enforcement of the American constraint can be viewed as an additional nonlinear source term g in the pricing PIDE. Obviously, this penalty approach can be unified for both European and American exercise features, if we set $g(x, \hat{t}) = 0$ in the whole domain for the case of a European exercise right. There are several ways to define such a penalty term g that forces the solution of (2.71) to be equal to the payoff in the exercise region Ω_{E}, one of them arising from [34] is discussed in Section 3.2.

3. Discretization

The more rigorous approach using PIDE forms the basis of advanced option pricing models. As a result, there is a need to successfully solve these complex governing equations containing a combination of differential and integral operators. Unsurprisingly, analytical option pricing formulae are available only for simple option contracts or under very strong limitations on the market conditions. Therefore, in general, the pricing equation has to be carefully solved numerically. This is also the case of the VG model presented, where we simultaneously treat the convection dominated character of the differential part and the non-smooth kernel of the integral part.

Taking these properties into account, the DG method combined with a modified numerical quadrature approach for convolution integrals represents a promising numerical tool for such option pricing models that improves the valuation process. Within the DG approach the numerical solution is composed by piecewise polynomial functions on the finite element mesh without any requirements on the inter-element continuity of the solution; for a detailed overview see [11]. More specifically, the discontinuous treatment enables to apply upwind stabilization among elements as well as to handle the American early exercise constraint in a more natural way. As a result, the numerical scheme obtained is sufficiently robust with respect to option styles and market conditions.

We follow the standard discretization steps in this section. At first, a partition of Ω is introduced together with the appropriate function spaces. Then, the derivation of DG formulation for the option pricing problem is presented employing a method of lines for space semidiscretization and including a numerical quadrature approach to the integral operator. Finally, a suitable time discretization is applied that results to a numerical scheme consisting of a sequence of systems of linear algebraic equations.

3.1. Partitions and function spaces. We start with a discretization of the computational domain Ω and recall the standard notation from [11]. Let $\mathcal{T}_h$, $h > 0$, denote a partition of the closure $\overline{\Omega} = [-x_{\max}, x_{\max}]$ of the domain Ω into N closed subintervals (called elements) $J_k = [x_{k-1}, x_k]$ with a uniform length h (to simplify a discretization of the integral term, see Section 3.2). Obviously, $\mathcal{T}_h = \{J_k, 1 \leqslant k \leqslant N\}$.

Over the fixed partition $\mathcal{T}_h$ we define the finite dimensional space of discontinuous piecewise polynomial functions

$$S_h^p(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega)\colon\ v|_{J_k} \in P_p(J_k)\ \forall\, J_k \in \mathcal{T}_h\}, \tag{3.1}$$

where $P_p(J_k)$ denotes the space of all polynomials of the order less than or equal to p defined on J_k. We recall that our aim is to construct an approximate solution of (2.45) or its American counterpart (2.71) lying in the space S_h^p. The DG methods can handle different polynomial degrees over elements in general, but here we consider only p-uniform approximations for simplicity.

Since the functions $v \in S_h^p$ are discontinuous across partition nodes in general, we introduce the operator jump $[\cdot]$ and mean value $\langle\cdot\rangle$ as

$$[v(x_k)] = v(x_k^-) - v(x_k^+), \quad \langle v(x_k)\rangle = \frac{1}{2}(v(x_k^-) + v(x_k^+)), \quad x_k \in \Omega, \tag{3.2}$$

where $v(x_k^+) = \lim\limits_{\Delta\to 0+} v(x_k + \Delta)$ and $v(x_k^-) = \lim\limits_{\Delta\to 0+} v(x_k - \Delta)$. For endpoints $\{x_0, x_N\}$, we simply put $[v(x_0)] = -v(x_0^+)$, $\langle v(x_0)\rangle = v(x_0^+)$, $[v(x_N)] = v(x_N^-)$ and $\langle v(x_N)\rangle = v(x_N^-)$.

3.2. DG formulation for option pricing problem. In order to derive the spatial semidiscrete problem in the sense of the DG approach, we assume that $u(\hat{t})$ as a solution of (2.64)–(2.66) is a sufficiently regular function lying in the suitable space, see broken Sobolev spaces [11].

Following the standard techniques as in [20], we proceed in the same way as in the variational formulation, with the only difference that we apply the integration by parts to convection terms as well, cf. (2.63), and include discontinuities across partition nodes with the aid of operators (3.2). As a result, it is necessary to deal with the definition of the convective flux

$$\Big(\frac{\sigma^2(\varepsilon)}{2} - \omega(\varepsilon) - r + q\Big)u$$

at the partition nodes $x_k \in \overline{\Omega}$. Therefore, to ensure the propagation of the information through these nodes in the proper direction, the concept of upwinding (see [11]) within the following numerical flux H is employed, i.e.,

(3.3)
$$H(u(x_k^-), u(x_k^+)) = \begin{cases} \Big(\dfrac{\sigma^2(\varepsilon)}{2} - \omega(\varepsilon) - r + q\Big) u(x_k^-) & \text{if } \dfrac{\sigma^2(\varepsilon)}{2} - \omega(\varepsilon) \geqslant r - q, \\ \Big(\dfrac{\sigma^2(\varepsilon)}{2} - \omega(\varepsilon) - r + q\Big) u(x_k^+) & \text{if } \dfrac{\sigma^2(\varepsilon)}{2} - \omega(\varepsilon) < r - q, \end{cases}$$

where the choice of $u(x_0^-)$ and $u(x_N^+)$ for the boundary points $-x_{\max}$ and $x_{\max}$ has to satisfy the prescribed Dirichlet boundary conditions (2.48) and (2.49), respectively.

Further, to introduce the semidiscrete variant of (2.63), we recall three slightly modified bilinear forms from [20] defined on $S_h^p \times S_h^p$, that is,

$$(3.4)\qquad a_h^\varepsilon(u, v) = \frac{\sigma^2(\varepsilon)}{2} \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \,\mathrm{d}x - \frac{\sigma^2(\varepsilon)}{2} \sum_{k=0}^{N} \Big\langle \frac{\partial u}{\partial x}(x_k) \Big\rangle [v(x_k)] + \frac{\sigma^2(\varepsilon)}{2} \sum_{k=0}^{N} \Big\langle \frac{\partial v}{\partial x}(x_k) \Big\rangle [u(x_k)],$$

$$(3.5)\qquad b_h^\varepsilon(u, v) = \Big(r - q + \omega(\varepsilon) - \frac{\sigma^2(\varepsilon)}{2}\Big) \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} u \frac{\partial v}{\partial x} \,\mathrm{d}x + \sum_{k=0}^{N} H(u(x_k^-), u(x_k^+))[v(x_k)],$$

$$(3.6)\qquad J_h^\varepsilon(u, v) = \frac{\sigma^2(\varepsilon)}{2h} \sum_{k=0}^{N} [u(x_k)][v(x_k)].$$

Then, we define the compound form

$$(3.7)\qquad \mathcal{D}_h^\varepsilon(u, v) = a_h^\varepsilon(u, v) + b_h^\varepsilon(u, v) + J_h^\varepsilon(u, v) + (r + \lambda(\varepsilon))(u, v) \quad \forall\, u, v \in S_h^p,$$

that represents the form (2.63) in the sense of the non-symmetric interior penalty Galerkin method; for the detailed explanation see [11].

As in [20], we introduce the linear form ℓ_h^ε to enforce the fulfillment of the boundary conditions and to encompass the remainders from the large jumps, specifically

$$(3.8)\quad \ell_h(v)(\hat{t}) = (\mathcal{R}^\varepsilon(\hat{t}), v) + \frac{\sigma^2(\varepsilon)}{2} \Big(-\frac{\partial v}{\partial x}(x_0^+) u_{\mathrm{L}}(\hat{t}) + \frac{\partial v}{\partial x}(x_N^-) u_{\mathrm{U}}(\hat{t}) + \frac{u_{\mathrm{L}}(\hat{t}) v(x_0^+)}{h} + \frac{u_{\mathrm{U}}(\hat{t}) v(x_N^-)}{h} \Big).$$

In contrast to the BS framework, the semidiscrete variant of the integral operator from (2.45) has to be also included. We formally obtain $L^2(\Omega)$-inner product of a test function v with a convolution integral, which can be written under (2.35)–(2.36) as

$$\mathcal{I}^{\varepsilon}(u)(x,\hat{t}) = \begin{cases} \int_{\Omega} u(z,\hat{t}) k^{\varepsilon}_{\mathrm{VG}}(z-x)\,\mathrm{d}z, & x \in (-x_{\max}+\varepsilon, x_{\max}-\varepsilon), \\ 0, & x \in \Omega \setminus (-x_{\max}+\varepsilon, x_{\max}-\varepsilon), \end{cases} \tag{3.9}$$

where $k^{\varepsilon}_{\mathrm{VG}}(y) = k_{\mathrm{VG}}(y)(1-\mathbb{1}_{|y|<\varepsilon})$. In particular, it is much more essential how the convolution integral (3.9) is numerically treated. A commonly used direct approximations of this term are based on the standard quadrature methods, see, e.g., [26]. Apart from the wavelet transform [30], another, relatively modern technique represents the integral terms as solutions of proper PDEs and leads to the local pseudo-differential formulation; for more details see [22].

In the following paragraphs we recall and modify the numerical quadrature approach primarily designed for finite activity processes, i.e., for convolution integrals with smooth kernels. Therefore, to avoid discontinuity of the function $k^{\varepsilon}_{\mathrm{VG}}$ we follow the simplest way and introduce a new function

$$\hat{k}^{\varepsilon}_{\mathrm{VG}}(y) = \begin{cases} k^{\varepsilon}_{\mathrm{VG}}(y), & |y| \geqslant \varepsilon, \\ \dfrac{k^{\varepsilon}_{\mathrm{VG}}(\varepsilon) - k^{\varepsilon}_{\mathrm{VG}}(-\varepsilon)}{2\varepsilon} y + \dfrac{k^{\varepsilon}_{\mathrm{VG}}(-\varepsilon) + k^{\varepsilon}_{\mathrm{VG}}(\varepsilon)}{2}, & |y| < \varepsilon, \end{cases} \tag{3.10}$$

that creates a continuous function from $k^{\varepsilon}_{\mathrm{VG}}$ by connecting the function values $k^{\varepsilon}_{\mathrm{VG}}(-\varepsilon)$ and $k^{\varepsilon}_{\mathrm{VG}}(\varepsilon)$ using a linear function defined on the interval $[-\varepsilon, \varepsilon]$. Then, we can write

$$\begin{aligned} \int_{\Omega} u(z,\hat{t}) k^{\varepsilon}_{\mathrm{VG}}(z-x)\,\mathrm{d}z &= \int_{\Omega} u(z,\hat{t}) \hat{k}^{\varepsilon}_{\mathrm{VG}}(z-x)\,\mathrm{d}z \\ &\quad - \int_{x-\varepsilon}^{x+\varepsilon} u(z,\hat{t}) \Big(\frac{k_{\mathrm{VG}}(\varepsilon) - k_{\mathrm{VG}}(-\varepsilon)}{2\varepsilon}(z-x) + \frac{k_{\mathrm{VG}}(-\varepsilon) + k_{\mathrm{VG}}(\varepsilon)}{2} \Big)\,\mathrm{d}z. \end{aligned} \tag{3.11}$$

Now, we employ two different quadrature rules with the same orders of accuracy. For simplicity, we assume that quadrature nodes coincide with the partition nodes of $\mathcal{T}_h$. In line with [26], the first term in (3.11) is evaluated by the composite trapezoidal quadrature rule. For the second term in (3.11), the midpoint rule is employed. More precisely,

$$\begin{aligned} &\int_{\Omega} u(z,\hat{t}) \hat{k}^{\varepsilon}_{\mathrm{VG}}(z-x)\,\mathrm{d}z \\ &\qquad \approx \frac{h}{2} \sum_{k=0}^{N-1} (u(x_k^+,\hat{t}) \hat{k}^{\varepsilon}_{\mathrm{VG}}(x_k - x) + u(x_{k+1}^-,\hat{t}) \hat{k}^{\varepsilon}_{\mathrm{VG}}(x_{k+1} - x)) - \eta(\varepsilon) u(x,\hat{t}), \end{aligned} \tag{3.12}$$

where $\eta(\varepsilon) = (k_{\mathrm{VG}}(-\varepsilon) + k_{\mathrm{VG}}(\varepsilon))\varepsilon$. In order to guarantee that both quadrature rules in (3.12) produce proportionally the same errors, one has to assume $\varepsilon = h/2$ and odd N. On the basis of the above considerations, one can conclude that

$$\mathcal{I}^{\varepsilon}(u)(x,\hat{t}) \approx \mathcal{I}^{\varepsilon}_h(u)(x,\hat{t}) - \eta(\varepsilon, x)u(x,\hat{t}),$$

where $\eta(\varepsilon, x) = \eta(\varepsilon)\mathbb{1}_{|x|<x_{\max}-\varepsilon}$ and

(3.13)
$$\mathcal{I}^{\varepsilon}_h(u)(x,\hat{t}) = \begin{cases} h\displaystyle\sum_{k=0}^{N} \vartheta_k \langle u(x_k,\hat{t})\rangle \hat{k}^{\varepsilon}_{\mathrm{VG}}(x_k - x), & x \in (-x_{\max}+\varepsilon, x_{\max}-\varepsilon), \\ 0, & x \in \Omega \setminus (-x_{\max}+\varepsilon, x_{\max}-\varepsilon) \end{cases}$$

with the weights $\vartheta_0 = \vartheta_N = \frac{1}{2}$ and $\vartheta_1 = \ldots = \vartheta_{N-1} = 1$. Unfortunately, the simplicity of the above numerical quadrature approach is partly compensated by higher computational demandingness, cf. the multinomial method in [5].

Furthermore, to handle the American early exercise feature, we were inspired by [34] and introduce the DG formulation of the penalty term g in the form

$$\begin{aligned} (3.14)\qquad (g(\hat{t}), v) &= c_p \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \mathbb{1}_{\mathrm{exe}}(\hat{t})(w_0 - u(\hat{t}))v\,\mathrm{d}x \\ &= \underbrace{c_p \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \mathbb{1}_{\mathrm{exe}}(\hat{t}) w_0 v\,\mathrm{d}x}_{g_h(v)} - \underbrace{c_p \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \mathbb{1}_{\mathrm{exe}}(\hat{t}) u(\hat{t}) v\,\mathrm{d}x}_{\mathcal{G}_h(u,v)}, \end{aligned}$$

which can be split into the linear form g_h and the bilinear form $\mathcal{G}_h$. The function $\mathbb{1}_{\mathrm{exe}}(\hat{t})$ in (3.14) is defined as an indicator function of the optimal early exercise region Ω_{E} at the time instant $\hat{t}$ and $c_p > 0$ is a suitably defined large number that represents a weight with which the early exercise of the option is enforced. Its value is specified later in Section 3.3 and Section 4. For European options, $\Omega_{\mathrm{E}} = \emptyset$ and the term (3.14) vanishes.

Finally, we define the semidiscrete problem to (2.45)–(2.49) or its American counterpart (2.71) with the relevant boundary conditions and construct the solution $u_h(\hat{t}) \in S^p_h$ with continuous time running, i.e., the so-called method of lines. The DG semidiscrete formulation of the option pricing problem reads: Find $u_h \in H^1(0,T;S^p_h)$ such that the following conditions are satisfied:

$$\begin{aligned} (3.15)\quad \Big(\frac{\partial u_h(\hat{t})}{\partial \hat{t}}, v_h\Big) &+ \mathcal{D}^{\varepsilon}_h(u_h(\hat{t}), v_h) + \mathcal{G}_h(u_h(\hat{t}), v_h) \\ &= (\mathcal{I}^{\varepsilon}_h(u_h(\hat{t})), v_h) - (\eta(\varepsilon,x)u_h(\hat{t}), v_h) + \ell_h(v_h)(\hat{t}) + g_h(v_h) \\ &\qquad \forall\, v_h \in S^p_h,\ \forall\, \hat{t} \in (0,T), \end{aligned}$$

$$(3.16)\quad (u_h(0), v_h) = (w_0, v_h) \quad \forall\, v_h \in S^p_h.$$

In other words, the problem (3.15) represents a system of ordinary differential equations for the unknown function $u_h(\hat{t})$ subjected to the initial condition (3.16).

Remark 3.1. Since $\partial w_0/\partial x$ is discontinuous at $x = 0$, to preserve the regularity of $u_h(0)$, it is sufficient to capture the support of w_0 exactly. Therefore, one can require that $x = 0$ is a partition node of $\mathcal{T}_h$, see [20]. If odd N is assumed as in (3.12), the central element $[-h/2, h/2]$ is bisected to two equal parts. Then, the definitions of (3.6) and (3.13) are slightly changed. This modification is left to the reader.

3.3. Fully discrete option pricing problem. In what follows, we discretize (3.15)–(3.16) with respect to the temporal variable $\hat{t}$. In order to speed up numerical simulations, the proposed numerical scheme should have no restrictive condition on the length of the time step, preserve the sparsity of a system of linear algebraic equations (resulting from this fully discrete problem) and be easy to implement. Therefore, the simple way is to realize the discretization in time by a semi-implicit Euler scheme, where the non-local integral term $(\mathcal{I}_h^\varepsilon(\cdot), \cdot)$ is evaluated only explicitly.

Let $0 = \hat{t}_0 < \hat{t}_1 < \ldots < \hat{t}_M = T$ be a partition of the interval $[0, T]$ with the constant time step $\tau = T/M$ (for simplicity) and denote by $u_h^m \in S_h^p$ the approximation of the solution $u_h(\hat{t})$ at the time level $\hat{t}_m \in [0, T]$. Then, the fully discrete formulation of the option pricing problem results in a sequence of problems of the form: Find $u_h^{m+1} \in S_h^p$, $m = 0, \ldots, M-1$, such that

$$\begin{aligned}(3.17)\quad &(u_h^{m+1}, v_h) + \tau \mathcal{D}_h^\varepsilon(u_h^{m+1}, v_h) + \tau(\eta(\varepsilon, x)u_h^{m+1}, v_h) + \tau\mathcal{G}_h(u_h^{m+1}, v_h)\\ &= (u_h^m, v_h) + \tau\ell_h(v_h)(\hat{t}_{m+1}) + \tau(\mathcal{I}_h^\varepsilon(u_h^m), v_h) + \tau g_h(v_h) \quad \forall\, v_h \in S_h^p\end{aligned}$$

with the starting data, defined by projections $(u_h^0, v_h) = (w_0, v_h)$ for all $v_h \in S_h^p$. This simple treatment is balanced by the fact that the resulting accuracy of the numerical scheme is only of the first order in time. As an improvement of (3.17) we refer to the scheme of the second order accuracy in [20].

For practical purpose, we use in the evaluation of forms $\mathcal{G}_h$ and g_h an element-wise approximation of the early exercise region Ω_{E}, defined by the approximate solution from the previous time level as in [19], i.e.,

$$(3.18)\qquad \mathbb{1}_{\mathrm{exe}}(\hat{t}_m)|_{J_k} \approx \begin{cases} 1 & \text{if } u_h^{m-1}\Big(\dfrac{x_{k-1}+x_k}{2}\Big) < w_0\Big(\dfrac{x_{k-1}+x_k}{2}\Big),\\[2ex] 0 & \text{if } u_h^{m-1}\Big(\dfrac{x_{k-1}+x_k}{2}\Big) \geqslant w_0\Big(\dfrac{x_{k-1}+x_k}{2}\Big)\end{cases}$$

for $\hat{t}_m \in (0, T)$ and $J_k \in \mathcal{T}_h$. Moreover, the value of c_p should be proportional to $1/\tau$ in order to avoid the influence of time stepping on the enforcement of the early exercise feature, especially for small time steps, see (3.17).

Remark 3.2. The existence and uniqueness of the discrete solution in (3.17) are standardly guaranteed under the boundedness and ellipticity of the form $(\cdot,\cdot) + \tau\mathcal{D}_h^\varepsilon(\cdot,\cdot) + \tau(\eta(\varepsilon,x)\cdot,\cdot) + \tau\mathcal{G}_h(\cdot,\cdot)$ on the left-hand side. Since $\mathcal{G}_h$ can be observed as the weighted variant of the L^2-inner product, we can follow the same steps as in [20].

The properties of the scheme (3.17) depend on the orders of polynomial approximation essentially. Intuitively, in the continuation region $\Omega \setminus \Omega_{\mathrm{E}}$, the rate of convergence is driven by properties of the NIPG variant of the DG method and quadrature rules applied. On the other hand, in the early exercise region Ω_{E}, as $c_p \gg 0$, the scheme (3.17) tends to the equation $\mathcal{G}_h(u_h^{m+1}, v_h){=}g_h(v_h)$ that in fact represents the L^2-projection of the payoff function w_0 onto the space S_h^p restricted to Ω_{E}. Here the rate of convergence is driven by the approximating properties of the space S_h^p.

As one can easily recognize, the problems (3.17) result into a sequence of solving linear algebraic equations. Let $\mathcal{B} = \{\varphi_j\}_{j=1}^{N(p+1)}$ denote the basis of the space S_h^p. Then the discrete solution u_h^m is identified with the real vector of its basis coefficients $U_m = \{\xi_j^m\}_{j=1}^{N(p+1)} \in \mathbb{R}^{N(p+1)}$ with respect to the basis $\mathcal{B}$ and (3.17) reads

$$(\mathbf{M} + \tau\mathbf{D} + \tau\widehat{\mathbf{M}} + \tau\mathbf{G})U_{m+1} = \mathbf{M}U_m + \tau(F_{m+1} + G_m), \quad m = 0, 1, \ldots, \tag{3.19}$$

with the initial vector U_0 related to u_h^0.

The system matrix in (3.19) is a composition of the mass matrix $\mathbf{M} = \{m_{ij}\}_{i,j=1}^{N(p+1)}$, its slight modification $\widehat{\mathbf{M}} = \{\widehat{m}_{ij}\}_{i,j=1}^{N(p+1)}$, the matrix $\mathbf{D} = \{d_{ij}\}_{i,j=1}^{N(p+1)}$ and the penalty matrix $\mathbf{G} = \{g_{ij}\}_{i,j=1}^{N(p+1)}$, defined element-wisely as $m_{ij} = (\varphi_j, \varphi_i)$, $\widehat{m}_{ij} = (\eta(\varepsilon,x)\varphi_j, \varphi_i)$, $d_{ij} = \mathcal{D}_h^\varepsilon(\varphi_j, \varphi_i)$ and $g_{ij} = \mathcal{G}_h(\varphi_j, \varphi_i)$, respectively. Taking into account the properties of the basis $\mathcal{B}$ together with the definition of the form $\mathcal{D}_h^\varepsilon$, one can conclude that the system matrices are non-symmetric and sparse with a block structure.

The right-hand side of (3.19) contains two vectors evaluated at different time levels $\hat{t}_m$ and $\hat{t}_{m+1}$, more precisely $G_m = \{(\mathcal{I}_h^\varepsilon(u_h^m), \varphi_j) + g_h(\varphi_j)\}_{j=1}^{N(p+1)}$ that includes terms arising from quadrature rules and enforces the American constraint, and $F_{m+1} = \{\ell_h(\varphi_j)(\hat{t}_{m+1})\}_{j=1}^{N(p+1)}$, that encompasses remainders from the large jumps together with the fulfillment of boundary conditions.

4. Numerical experiments

In this section, we illustrate through two numerical experiments of option pricing in the VG model with European as well as American exercise features the usage and capabilities of the numerical scheme presented.

The whole implementation is done in the solver Freefem++ (see [16]). As far as solving linear systems (3.19) of given properties is concerned, the GMRES method is used without preconditioner, since the number of unknowns is relatively small in the experimental study considered. Moreover, in order to handle numerically the American early exercise feature, we choose $c_p = 10^3/\tau$ in (3.14) according to [34]. The value c_p set seems to be sufficient and taking larger values does not improve the results significantly.

From the practical point of view we are most interested in the option values in the zone of financial interest Ω^*, which is usually associated with the underlying prices in the range $[0.75\mathcal{K}, 1.25\mathcal{K}]$, thus we set $\Omega^* = [\ln(0.75), \ln(1.25)] \subset \Omega$ in all forthcoming experiments. Accordingly, we restrict the computational domain to $\Omega = (-3, 3)$. The value $x_{\max} = 3$ is experimentally set to eliminate the localization component of error (2.50) under the given set $(\nu, \lambda_{\mathrm{n}}, \lambda_{\mathrm{p}})$ with respect to the discretization error (related to Ω^*) on the finest space-time grid (i.e., for sufficiently small parameters h, τ and ε). For the detailed procedure we refer to [20].

Regarding the experimental error analysis, an important indicator for financial practitioners is the worst-case pricing scenario, which is evaluated as the discrete l^∞-norm

$$e^M_{h,\infty}(P_p) = \max_{x_i \in \Omega^*} |\langle u^M_h(x_i)\rangle - u_{\mathrm{ref}}(x_i, T)|, \tag{4.1}$$

where x_i are all nodes lying in the zone of financial interest (associated with degrees of freedom for the finest mesh), u^M_h is the P_p approximate solution (constructed on the grid with the mesh size h and evaluated at $\hat{t} = \hat{t}_M = T$) and u_{ref} is a given reference solution. Since analytical pricing formulae are not available in general, especially for American options, one can consider u_{ref} (in a certain sense) as the approximate solution computed on the sufficiently fine time-space grid. In that case, we write also $\langle u^M_{\mathrm{ref}}(x_i)\rangle$ in (4.1). On the other hand, since the discrete solutions are discontinuous across the partition nodes, we also evaluate errors in the L^2-norm to illustrate convergence properties of the numerical scheme in a more credible way, i.e.,

$$e^M_{h,2}(P_p) = \|u^M_h - u_{\mathrm{ref}}(T)\|_{L^2(\Omega^*)}. \tag{4.2}$$

At this point, it should be emphasized that the aim of the forthcoming numerical experiments is to substantiate the design of the numerical scheme as the whole. We are aware that the presented results are preliminary and more thorough numerical analysis is needed, especially in the issue of discretization errors. But it would be beyond the scope of this article and it is left for a future research.

4.1. European call option. Within the first set of numerical experiments we evaluate the European call options under the model parameters

$$T = 0.25,\quad \mathcal{K} = 100,\quad r = 0.1,\quad q = 0.0,\quad \nu = 0.3,\quad \lambda_{\rm n} = 13.653,\quad \lambda_{\rm p} = 33.153, \tag{4.3}$$

which are artificially set to be the representatives of parameter values of practical significance.

We start with a point-wise comparison of the proposed numerical scheme to MC simulations performed according to [14] with 10^7 samples. This approach is the most common from the practical point of view, when options are numerically priced. We compute the piecewise linear ($p = 1$) and quadratic ($p = 2$) numerical solutions on a sequence of grids with the consecutively expanding number of elements N. The time step is fixed sufficiently small as $\tau = T/400$ to suppress the influence of the time discretization on the results. A particular case of numerical solution is depicted in Figure 2 (left).

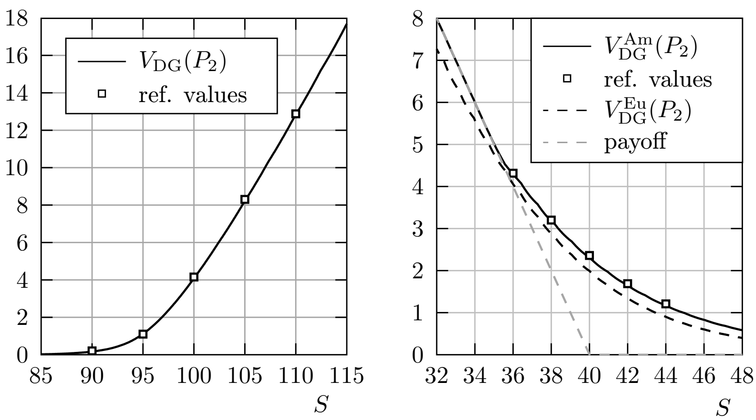

Figure 2. The piecewise quadratic option prices evaluated at $\hat{t} = T$ on a mesh with $N = 512$: European call (left) and American put (right).

The computed pricing functions are evaluated at underlying prices $S_{\rm ref} \in \{90, 95, 100, 105, 110\}$ at the time state $\hat{t} = T$ and recorded in Table 1 with the MC values as a proxy of the exact option prices. The results obtained reveal that the presented DG approach with quadratic approximations seems to be successfully applicable to the VG option pricing problem. More precisely, the numerical option prices match tightly the reference ones as the space-mesh is finer and the polynomial order increases. However, this behaviour does not have to be strictly monotonous in the case of the point-wise evaluation within the DG approach, since the point-wise behaviour is influenced to certain extent by a position of the reference point with respect to the partition nodes.

DG(P_p)	N	$S = 90$	$S = 95$	$S = 100$	$S = 105$	$S = 110$
$p = 1$	32	0.98424	2.11433	3.38527	7.95426	12.7910
	64	0.14464	1.69416	3.75683	8.02284	12.3793
	128	0.19617	1.23641	4.00082	8.25562	13.0759
	256	0.18812	1.22217	4.23043	8.59100	13.4296
	512	0.17101	1.17748	4.22910	8.56719	13.3662
	1024	0.16652	1.13503	4.14308	8.42880	13.1702
$p = 2$	32	0.54205	1.79065	3.66037	7.65324	12.1325
	64	0.22890	1.28376	3.81029	7.81889	12.3171
	128	0.18101	1.15525	3.97338	8.03154	12.5958
	256	0.16971	1.20231	4.09739	8.21763	12.8232
	512	0.16305	1.10190	4.14886	8.29462	12.8958
	1024	0.16122	1.09845	4.15733	8.31574	12.9097
MC simulations		0.161	1.103	4.162	8.305	12.89

Table 1. European case: Comparison of the approximate option values with the reference values at five reference nodes for P_1 and P_2 approximations on a sequence of meshes.

N	$\frac{\text{convection}}{\text{diffusion}}$	$e^M_{h,2}(P_2)$	ratio	$e^M_{h,\infty}(P_2)$	ratio
32	34	$1.5433\mathrm{e}-01$	—	$8.9854\mathrm{e}-03$	—
64	104	$8.8434\mathrm{e}-02$	1.745	$7.1431\mathrm{e}-03$	1.258
128	341	$1.0754\mathrm{e}-01$	0.822	$9.5735\mathrm{e}-03$	0.746
256	1199	$6.9258\mathrm{e}-02$	1.553	$6.5092\mathrm{e}-03$	1.471
512	4446	$2.7230\mathrm{e}-02$	2.544	$3.3293\mathrm{e}-03$	1.955
1024	17063	$7.9732\mathrm{e}-03$	3.415	$1.4984\mathrm{e}-03$	2.222

Table 2. European case: Errors (in the L^2- and l^∞-norm) and the corresponding convergence ratios for P_2 approximations on a sequence of meshes.

Secondly, we investigate the numerical scheme with respect to the errors (4.1) and (4.2). Since the parameters ε and h are bonded together, the equation (2.45) becomes convection dominated thanks to the refinement. Therefore, we would like to illustrate also the robustness of the scheme with respect to the character of the differential operator (2.46), i.e., the ratio $\frac{\text{convection}}{\text{diffusion}}$. According to the point-wise observations we consider P_2 approximations and the same time step. The results obtained are listed in Table 2, which is divided into two panels associated with the particular error and the corresponding convergence ratio (related to two consecutive grids). Actually, the evaluated errors are the scaled ones in the original underlying prices and option values by the factor $\mathcal{K}$, calculated over the interval $[0.75\mathcal{K}, 1.25\mathcal{K}]$. The observed ratios indicate how fast the numerical solutions converge to the reference solution u_{ref}^M given by the piecewise quadratic approximation computed on the grid with 4096 elements with $\tau = T/400$. In case of the L^2-norm, the decrease seems to be closer to the quadratic rate, while the l^∞-norm shows rather a linear character. As stated before, these observations are preliminary and they should be confronted (within the future research) with theoretical results to provide a clear statement.

4.2. American put option. The second numerical benchmark is performed on the reference data from [5], where the American put option prices under the VG process are evaluated using a multinomial method. In the following setting for the VG model, we take the parameters

$$T = 1.0,\ \mathcal{K} = 40,\ r = 0.06,\ q = 0.0,\ \nu = 0.2,\ \lambda_{\mathrm{n}} = 13.50781,\ \lambda_{\mathrm{p}} = 18.50781, \tag{4.4}$$

which correspond to the values $\theta = -0.1$ and $\sigma = 0.2$ in the original VG process.

N	$S = 36$	$S = 38$	$S = 40$	$S = 42$	$S = 44$
32	4.00080	2.00481	0.02206	0.20102	0.19041
64	4.01528	2.38182	1.71172	1.12280	0.75179
128	4.28127	2.77698	2.01871	1.49855	1.09643
256	4.29901	3.18462	2.34633	1.70440	1.23528
512	4.31640	3.20918	2.37141	1.70809	1.23301
1024	4.30214	3.20549	2.37175	1.69989	1.22739
Ref. val. [5]	4.3173	3.2034	2.3767	1.6947	1.2267

Table 3. American case: Comparison of the approximate option values with the reference values at five reference nodes for P_2 approximations on a sequence of meshes.

We proceed similarly as in the previous experiment. Within the first part we analyze the point-wise behaviour of P_2 numerical solutions computed on a sequence of uniformly refined grids with the fixed time step $\tau = T/800$. In Table 3 we compare approximate option prices (associated to reference the underlying prices $S_{\mathrm{ref}} \in \{36, 38, 40, 42, 44\}$ and time $\hat{t} = T$) with values obtained by a multinomial method from [5]. In this respect, the results obtained are quite comparable with the reference ones, as in the case of the European style option.

The second part aims to determine the behaviour of both the errors considered with respect to the discretization parameter h. The reference solution (related to the zone of financial interest $[30, 50]$ in the original underlying prices) is again given by the approximate one computed for the discretization parameters $p = 2$, $N = 4096$ and $\tau = T/800$. Table 4 has the same format as Table 2 in the preceding experiment. Except for minor differences, the errors exhibit the behaviour similar as for the case of the European style option, thus we come to the same conclusions, including the robustness of the scheme with respect to the convection-diffusion character of the pricing equation.

N	$\frac{\text{convection}}{\text{diffusion}}$	$e^M_{h,2}(P_2)$	ratio	$e^M_{h,\infty}(P_2)$	ratio
32	11	7.0415e − 01	—	5.8778e − 02	—
64	36	6.7800e − 01	1.039	5.8514e − 02	1.005
128	125	2.5000e − 01	2.712	1.6273e − 02	3.560
256	454	1.6509e − 01	1.514	1.5465e − 02	1.052
512	1723	6.0034e − 02	2.750	8.4185e − 03	1.837
1024	6696	1.7627e − 02	3.406	2.8998e − 03	2.903

Table 4. American case: Errors (in the L^2- and l^∞-norm) and the corresponding convergence ratios for P_2 approximations on a sequence of meshes.

Finally, to conclude this section, we illustrate the American constraint in Figure 2 (right) that shows a general relationship between the American and European option prices with respect to their payoff. It is apparent that the American option prices do not fall below the values of the payoff. Moreover, one can easily observe other typical findings, namely that the American options cost more than their European counterparts.

5. Conclusion

Pricing of options is very challenging and no less important part of financial engineering and the numerical techniques take a crucial part in this field, especially if no closed-form pricing formulae exist for the particular market conditions. In this contribution we have presented the methodological concept based on the DG method that forms and improves the numerical valuation of options under the VG process, including different options styles (European vs. American). The presented experiments provide a brief insight into the performance of the method and although the results obtained are comparable with the reference values of selected benchmarks, more thorough numerical (error) analysis is needed and various improvements of the numerical scheme are welcome within the further investigation. Furthermore, the natural extension of this scheme is its application to option pricing under the Carr-Geman-Madan-Yor (CGMY) process.

References

[1] *M. Abramowitz, I. A. Stegun* (eds.): Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. U. S. Department of Commerce, Washington, 1964. zbl MR

[2] *A. Almendral, C. W. Oosterlee*: On American options under the variance gamma process. Appl. Math. Finance *14* (2007), 131–152. zbl MR doi

[3] *A. Bensoussan, J. L. Lions*: Contrôle impulsionnel et inéquations quasi variationnelles. Gauthier-Villars, Paris, 1982. (In French.) zbl MR

[4] *F. Black, M. Scholes*: The pricing of options and corporate liabilities. J. Polit. Econ. *81* (1973), 637–654. zbl MR doi

[5] *N. Cantarutti, J. Guerra*: Multinomial method for option pricing under variance gamma. Int. J. Comput. Math. *96* (2019), 1087–1106. MR doi

[6] *P. Carr, R. Jarrow, R. Myneni*: Alternative characterizations of American put options. Math. Finance *2* (1992), 87–106. zbl doi

[7] *R. Cont, P. Tankov*: Financial Modelling with Jump Processes. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, 2004. zbl MR doi

[8] *R. Cont, E. Voltchkova*: A finite difference scheme for option pricing in jump diffusion and exponential Lévy models. SIAM J. Numer. Anal. *43* (2005), 1596–1626. zbl MR doi

[9] *R. Cont, E. Voltchkova*: Integro-differential equations for option prices in exponential Lévy models. Finance Stoch. *9* (2005), 299–325. zbl MR doi

[10] *F. Delbaen, W. Schachermayer*: A general version of the fundamental theorem of asset pricing. Math. Ann. *300* (1994), 463–520. zbl MR doi

[11] *V. Dolejší, M. Feistauer*: Discontinuous Galerkin Method: Analysis and Applications to Compressible Flow. Springer Series in Computational Mathematics 48. Springer, Cham, 2015. zbl MR doi

[12] *E. Eberlein*: Jump-type Lévy processes. Handbook of Financial Time Series. Springer, Berlin, 2009, pp. 439–455. zbl doi

[13] *L. Feng, V. Linetsky*: Pricing options in jump-diffusion models: An extrapolation approach. Oper. Res. *56* (2008), 304–325. zbl MR doi

[14] *M. C. Fu*: Variance-gamma and Monte Carlo. Advances in Mathematical Finance. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston, 2007, pp. 21–35. zbl MR doi

[15] *E. G. Haug*: The Complete Guide to Option Pricing Formulas. McGraw-Hill, New York, 2007.

[16] *F. Hecht*: New development in freefem++. J. Numer. Math. *20* (2012), 251–265. zbl MR doi

[17] *A. Hirsa, D. B. Madan*: Pricing American options under variance gamma. J. Comput. Finance *7* (2004), 63–80. doi

[18] *J. Hozman, T. Tichý*: On the impact of various formulations of the boundary condition within numerical option valuation by DG method. Filomat *30* (2016), 4253–4263. zbl MR doi

[19] *J. Hozman, T. Tichý*: DG framework for pricing European options under one-factor stochastic volatility models. J. Comput. Appl. Math. *344* (2018), 585–600. zbl MR doi

[20] *J. Hozman, T. Tichý, M. Vlasák*: DG method for pricing European options under Merton jump-diffusion model. Appl. Math., Praha *64* (2019), 501–530. zbl MR doi

[21] *S. Ikonen, J. Toivanen*: Operator splitting methods for American option pricing. Appl. Math. Lett. *17* (2004), 809–814. zbl MR doi

[22] *A. Itkin*: Pricing Derivatives Under Lévy Models: Modern Finite-Difference and Pseudo-Differential Operators Approach. Pseudo-Differential Operators. Theory and Applications 12. Birkhäuser, Basel, 2017. zbl MR doi

[23] *J. Jacod, A. N. Shiryaev*: Limit Theorems for Stochastic Processes. Grundlehren der Mathematischen Wissenschaften 288. Springer, Berlin, 1987. zbl MR doi

[24] *S. Kozpınar, M. Uzunca, B. Karasözen*: Pricing European and American options under Heston model using discontinuous Galerkin finite elements. Math. Comput. Simul. *177* (2020), 568–587. zbl MR doi

[25] *A. Kufner, O. John, S. Fučík*: Function Spaces. Monographs and Textbooks on Mechanics of Solids and Fluids. Mechanics: Analysis 3. Noordhoff International Publishing, Leyden, 1977. zbl MR

[26] *Y. Kwon, Y. Lee*: A second-order finite difference method for option pricing under jump-diffusion models. SIAM J. Numer. Anal. *49* (2011), 2598–2617. zbl MR doi

[27] *D. B. Madan, P. P. Carr, E. C. Chang*: The variance gamma process and option pricing. Eur. Finance Rev. *2* (1998), 79–105. zbl doi

[28] *D. B. Madan, F. Milne*: Option pricing with V. G. martingale components. Math. Finance *1* (1991), 39–55. zbl doi

[29] *D. B. Madan, E. Seneta*: The variance gamma (V.G.) model for share market returns. J. Business *63* (1990), 511–524. doi

[30] *A.-M. Matache, T. von Petersdorff, C. Schwab*: Fast deterministic pricing of options on Lévy driven assets. M2AN, Math. Model. Numer. Anal. *38* (2004), 37–71. zbl MR doi

[31] *R. C. Merton*: Theory of rational option pricing. Bell J. Econ. Manage. Sci. *4* (1973), 141–183. zbl MR doi

[32] *D. P. Nicholls, A. Sward*: A discontinuous Galerkin method for pricing American options under the constant elasticity of variance model. Commun. Comput. Phys. *17* (2015), 761–778. zbl MR doi

[33] *H. Y. Wong, J. Zhao*: An artificial boundary method for American option pricing under the CEV model. SIAM J. Numer. Anal. *46* (2008), 2183–2209. zbl MR doi

[34] *R. Zvan, P. A. Forsyth, K. R. Vetzal*: Penalty methods for American options with stochastic volatility. J. Comput. Appl. Math. *91* (1998), 199–218. zbl MR doi

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