UNIFORMLY STARLIKE FUNCTIONS AND UNIFORMLY CONVEX FUNCTIONS RELATED TO THE PASCAL DISTRIBUTION

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Received March 9, 2020. Published online November 30, 2020.
Communicated by Grigore Sălăgean

Abstract. In this article, we aim to find sufficient conditions for a convolution of analytic univalent functions and the Pascal distribution series to belong to the families of uniformly starlike functions and uniformly convex functions in the open unit disk $\mathbb{U}$. We also state corollaries of our main results.

Keywords: uniformly starlike function; uniformly convex function; starlike function; convex function; Pascal distribution; convolution

MSC 2020: 30C45, 33C10, 33C20

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C}\colon\ |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. Moreover, for functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

DOI: 10.21136/MB.2020.0044-20

the convolution of f and g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n \, b_n \, z^n, \quad z \in \mathbb{U}.$$

It is well known that special functions play an important role in Geometric Function Theory, particularly in the proof given by de Branges (see [5]) for the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in recent years. There is a widespread literature dealing with geometric properties of various types of special functions, especially for the generalized Gaussian hypergeometric functions (see [4], [7], [10], [16], [17] and references cited therein).

A variable χ is said to have *Pascal distribution* if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$$(1-q)^m, \quad \frac{qm(1-q)^m}{1!}, \quad \frac{q^2 m(m+1)(1-q)^m}{2!}, \quad \frac{q^3 m(m+1)(m+2)(1-q)^m}{3!}, \ldots,$$

respectively, where q and m are called parameters, and thus

$$P(\chi = k) = \binom{k+m-1}{m-1} q^k (1-q)^m, \quad k = 0, 1, 2, 3, \ldots$$

Recently, El-Deeb (see [6]) and Çakmak et al. (see [3]) introduced a power series whose coefficients are probabilities of the Pascal distribution

$$\Phi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m z^n, \quad z \in \mathbb{U} \tag{1.2}$$

where $m \geqslant 1$; $0 \leqslant q \leqslant 1$ and we note that, by ratio test, the radius of convergence of the above series is infinity.

We consider the linear operator

$$\mathcal{I}_q^m \colon \mathcal{A} \to \mathcal{A}$$

defined by the convolution

$$\mathcal{I}_q^m f(z) = \Phi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m a_n z^n, \quad z \in \mathbb{U}.$$

For our convenience throughout this paper, we use the following formulas:

$$\begin{aligned}
(1.3)\qquad &\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n=\frac{1}{(1-q)^m},\\
&\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n=\frac{1}{(1-q)^{m-1}},\\
&\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{1}{(1-q)^{m+1}},\\
&\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n=\frac{1}{(1-q)^{m+2}},\quad 0\leqslant q<1.
\end{aligned}$$

Here, in our present research, we consider the following subclasses which were studied earlier by Rosy et al. (see [15]) and Subramanian et al. (see [18]).

Definition 1.1 (see [15]). For $\beta>0$, a function $f\in\mathcal{A}$ of the form (1.1) is said *to be in the subclass* $\mathcal{USD}(\beta)$ of the normalized univalent function class of $\mathcal{S}$ if it satisfies the following inequality:

$$\Re(f'(z))\geqslant\beta|zf''(z)|,\quad z\in\mathbb{U}.$$

Definition 1.2 (see [18]). For $\beta>0$, a function $f\in\mathcal{A}$ of the form (1.1) is said *to be in the subclass* $\mathcal{USN}(\beta)$ if it satisfies the following inequality:

$$\Re\Big(\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\Big)>\beta,\quad (z,\zeta)\in\mathbb{U}\times\mathbb{U}.$$

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see [4], [7], [16], [17]) and by recent investigations on Poisson distribution series (see [1], [2], [8], [9], [12], [13], [11], [14]), in the present paper we determine sufficient conditions for the function $\Psi^m_{q,\mu}(z)$ given by

$$\begin{aligned}
(1.4)\qquad \Psi^m_{q,\mu}(z)&=(1-\mu)\Phi^m_q(z)+\mu z(\Phi^m_q(z))'\\
&=z+\sum_{n=2}^{\infty}(1+n\mu-\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n,\quad 0\leqslant\mu\leqslant 1
\end{aligned}$$

to belong to the above-defined classes $\mathcal{USD}(\beta)$ and $\mathcal{USN}(\beta)$.

2. Main results and their consequences

To prove the main results in our present investigation, we shall need the following lemmas.

Lemma 2.1 (see [15]). *A function f of the form* (1.1) *is in the class* $\mathcal{USD}(\beta)$ *if*

$$\sum_{n=2}^{\infty}(n(1-\beta)+n^2\beta)|a_n| \leqslant 1. \tag{2.1}$$

Lemma 2.2 (see [18]). *A function f of the form* (1.1) *is in the class* $\mathcal{USN}(\beta)$ *if*

$$\sum_{n=2}^{\infty}(n(3-\beta)-2)|a_n| \leqslant 1-\beta. \tag{2.2}$$

Our first main result is asserted by Theorem 2.3 below.

Theorem 2.3. *Let $m \geqslant 1$ and $0 \leqslant q < 1$. Then $\Psi_{q,\mu}^{m}(z) \in \mathcal{USD}(\beta)$ if*

$$\begin{aligned}\mu\beta\frac{q^3m(m+1)(m+2)}{(1-q)^{m+3}} &+ (\mu+\beta+4\mu\beta)\frac{q^2m(m+1)}{(1-q)^{m+2}} \\ &+ (1+2\mu+2\beta+2\mu\beta)\frac{qm}{(1-q)^{m+1}} \leqslant 1.\end{aligned} \tag{2.3}$$

Proof. Let

$$\Phi_q^m(z) = z + \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n, \quad 0 \leqslant \mu \leqslant 1.$$

Taking $z=1$, we have

$$\Phi_q^m(1) - 1 = (1-q)^m \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}. \tag{2.4}$$

By simple calculation we have the following:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^{n}-1, \tag{2.5}$$

$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}=qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^{n}, \tag{2.6}$$

$$\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}=q^{2}m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^{n}, \tag{2.7}$$

and

$$\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\binom{n+m-2}{m-1}q^{n-1} = q^{3}m(m+1)(m+2)\sum_{n=0}^{\infty}\binom{n+m+2}{m+2}q^{n}. \tag{2.8}$$

Since $\Psi_{q,\mu}^{m}(z)\in\mathcal{USD}(\beta)$, by Lemma 2.1 it suffices to show that

$$\sum_{n=2}^{\infty}(1+n\mu-\mu)(n(1-\beta)+n^{2}\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}\leqslant 1. \tag{2.9}$$

We now let

$$S(n,\lambda,\beta,\alpha)=\sum_{n=2}^{\infty}(1+n\mu-\mu)(n(1-\beta)+n^{2}\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m},$$

so that

$$\begin{aligned} S(n,\lambda,\beta,\alpha)=\mu\beta\sum_{n=2}^{\infty}n^{3}\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m} \\ +(\mu+\beta-2\mu\beta)\sum_{n=2}^{\infty}n^{2}\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m} \\ +(1-\mu-\beta+\mu\beta)\sum_{n=2}^{\infty}n\binom{n+m-2}{m-1}q^{n-1}(1-q)^{m}. \end{aligned}$$

Writing

$$\begin{aligned} n^{3}&=(n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1,\\ n^{2}&=(n-1)(n-2)+3(n-1)+1 \end{aligned}$$

and

$$n=(n-1)+1$$

we get

$$
\begin{aligned}
S(n,\lambda,\beta,\alpha) &= \mu\beta(1-q)^m \\
&\quad \times \sum_{n=2}^{\infty}((n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1)\binom{n+m-2}{m-1}q^{n-1} \\
&\quad + (\mu+\beta-2\mu\beta)(1-q)^m \sum_{n=2}^{\infty}((n-1)(n-2)+3(n-1)+1)\binom{n+m-2}{m-1}q^{n-1} \\
&\quad + (1-\mu-\beta+\mu\beta)(1-q)^m \sum_{n=2}^{\infty}((n-1)+1)\binom{n+m-2}{m-1}q^{n-1} \\
&= \mu\beta(1-q)^m \sum_{n=2}^{\infty}\Bigg((n-1)(n-2)(n-3)\binom{n+m-2}{m-1}q^{n-1} \\
&\quad + 6(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1} \\
&\quad + 7(n-1)\binom{n+m-2}{m-1}q^{n-1} + \binom{n+m-2}{m-1}q^{n-1}\Bigg) \\
&\quad + (\mu+\beta-2\mu\beta)(1-q)^m \sum_{n=2}^{\infty}\Bigg((n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1} \\
&\quad + 3(n-1)\binom{n+m-2}{m-1}q^{n-1} + \binom{n+m-2}{m-1}q^{n-1}\Bigg) \\
&\quad + (1-\mu-\beta+\mu\beta)(1-q)^m \sum_{n=2}^{\infty}\Bigg((n-1)\binom{n+m-2}{m-1}q^{n-1} + \binom{n+m-2}{m-1}q^{n-1}\Bigg).
\end{aligned}
$$

From (2.5), (2.6), (2.7) and (2.8), we get

$$
\begin{aligned}
S(n,\lambda,\beta,\alpha) &= \mu\beta(1-q)^m \\
&\quad \times \Bigg(q^3 m(m+1)(m+2)\sum_{n=0}^{\infty}\binom{n+m+2}{m+2}q^n + 6q^2 m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\
&\quad + 7qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n + \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\Bigg) \\
&\quad + (\mu+\beta-2\mu\beta)(1-q)^m\Bigg(q^2 m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\
&\quad + 3qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n + \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\Bigg) \\
&\quad + (1-\mu-\beta+\mu\beta)(1-q)^m\Bigg(qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n + \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\Bigg).
\end{aligned}
$$

Now by using (1.3), we have

$$
\begin{aligned}
S(n,\lambda,\beta,\alpha) &= \mu\beta(1-q)^m \\
&\quad\times\Big(\frac{q^3m(m+1)(m+2)}{(1-q)^{m+3}}+\frac{6q^2m(m+1)}{(1-q)^{m+2}}+\frac{7qm}{(1-q)^{m+1}}+\frac{1}{(1-q)^m}-1\Big)\\
&\quad+(\mu+\beta-2\mu\beta)(1-q)^m\Big(\frac{q^2m(m+1)}{(1-q)^{m+2}}+\frac{3qm}{(1-q)^{m+1}}+\frac{1}{(1-q)^m}-1\Big)\\
&\quad+(1-\mu-\beta+\mu\beta)(1-q)^m\Big(\frac{qm}{(1-q)^{m+1}}+\frac{1}{(1-q)^m}-1\Big).
\end{aligned}
$$

By simple computation,

$$
\begin{aligned}
S(n,\lambda,\beta,\alpha) = \mu\beta\frac{q^3m(m+1)(m+2)}{(1-q)^3} &+ (\mu+\beta+4\mu\beta)\frac{q^2m(m+1)}{(1-q)^2}\\
&+(1+2\mu+2\beta+2\mu\beta)\frac{qm}{1-q}+1-(1-q)^m.
\end{aligned}
$$

But this last expression is bounded above by 1 if (2.3) holds. Thus the proof of Theorem 2.3 is completed. □

Theorem 2.4. *Let $m \geqslant 1$ and $0 \leqslant q < 1$. Then $\Phi_q^m(z) \in \mathcal{USD}(\beta)$ if*

$$
\beta\frac{q^2m(m+1)}{(1-q)^{m+2}}+(2\beta+1)\frac{qm}{(1-q)^{m+1}} \leqslant 1. \tag{2.10}
$$

P r o o f. According to Lemma 2.1, it suffices to show that

$$
\sum_{n=2}^{\infty}(n(1-\beta)+n^2\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leqslant 1.
$$

We note that

$$
\Psi_{q,0}^m(z) = \Phi_q^m(z).
$$

Hence, by taking $\mu = 0$ in (2.9), we get the above inequality. Therefore, by setting $\mu = 0$ in Theorem 2.3, we get the desired result given in (2.10). □

Corollary 2.5. *Let $m \geqslant 1$ and $0 \leqslant q < 1$. Then $\Phi_q^m(z) \in \mathcal{USD}(0)$ if and only if*

$$
\frac{qm}{(1-q)^{m+1}} \leqslant 1. \tag{2.11}
$$

Theorem 2.6. *Let $m \geqslant 1$ and $0 \leqslant q < 1$. Then $\Psi^m_{q,\mu}(z) \in \mathcal{USN}(\beta)$ if*

$$(3-\beta)\mu\frac{q^2m(m+1)}{(1-q)^{m+2}} + (3-\beta+4\mu-2\beta\mu)\frac{qm}{(1-q)^{m+1}} \leqslant 1-\beta. \tag{2.12}$$

Proof. By virtue of Lemma 2.2, it suffices to show that

$$\sum_{n=2}^{\infty}(1+n\mu-\mu)((3-\beta)n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leqslant 1-\beta.$$

We now let

$$S(n,\lambda,\beta,\alpha) = \sum_{n=2}^{\infty}(1+n\mu-\mu)((3-\beta)n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m,$$

that is,

$$\begin{aligned}S(n,\lambda,\beta,\alpha) &= \mu(3-\beta)\sum_{n=2}^{\infty}n^2\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ &\quad + (3-\beta-3\mu-2\mu+\beta\mu)\sum_{n=2}^{\infty}n\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ &\quad - 2(1-\mu)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m,\end{aligned}$$

which, upon writing $n^2 = (n-1)(n-2)+3(n-1)+1$ and $n=(n-1)+1$, yields

$$\begin{aligned}S(n,\lambda,\beta,\alpha) &= \mu(3-\beta)\sum_{n=2}^{\infty}(n-2)(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ &\quad + (3-\beta+4\mu-2\beta\mu)\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\ &\quad + (1-\beta)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.\end{aligned}$$

From (2.5), (2.6) and (2.7), we get

$$\begin{aligned}S(n,\lambda,\beta,\alpha) &= \mu(3-\beta)(1-q)^m q^2 m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\ &\quad + (3-\beta+4\mu-2\beta\mu)(1-q)^m qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n \\ &\quad + (1-\beta)(1-q)^m\biggl(\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\biggr).\end{aligned}$$

Again by using (1.3) we get

$$S(n,\lambda,\beta,\alpha)=\mu(3-\beta)\frac{q^2m(m+1)}{(1-q)^2}+(3-\beta+4\mu-2\beta\mu)\frac{qm}{1-q}+(1-\beta)(1-(1-q)^m).$$

But this last expression is bounded above by $1-\beta$ if the condition (2.12) holds. Thus the proof of Theorem 2.6 is completed. □

By taking $\mu=0$ in Theorem 2.6, we can easily deduce the following corollary.

Corollary 2.7. *Let $m\geqslant 1$ and $0\leqslant q<1$. Then $\Phi_q^m(z)\in\mathcal{USN}(\beta)$ if*

$$\frac{(3-\beta)qm}{(1-q)^{m+1}}\leqslant 1-\beta. \tag{2.13}$$

Furthermore, $\Phi_q^m(z)\in\mathcal{USN}(0)$ if

$$\frac{3qm}{(1-q)^{m+1}}\leqslant 1.$$

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