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AN EXAMPLE OF A REFLEXIVE LORENTZ GAMMA SPACE
WITH TRIVIAL BOYD AND ZIPPIN INDICES

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Abstract. We show that for every $p \in (1, \infty)$ there exists a weight w such that the Lorentz Gamma space $\Gamma_{p,w}$ is reflexive, its lower Boyd and Zippin indices are equal to zero and its upper Boyd and Zippin indices are equal to one. As a consequence, the Hardy-Littlewood maximal operator is unbounded on the constructed reflexive space $\Gamma_{p,w}$ and on its associate space $\Gamma'_{p,w}$.

Keywords: Lorentz Gamma space; reflexivity; Boyd indices; Zippin indices

MSC 2020: 46E30, 42B25

1. INTRODUCTION

Let m denote the standard Lebesgue measure on \mathbb{R}^n . The Hardy-Littlewood maximal operator M is one of the most important operators in harmonic analysis. For a function $f \in L^1_{\text{loc}}(\mathbb{R}^n, m)$ it is defined by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{m(Q)} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes. We are interested in the following question: Given a class \mathfrak{S} of Banach function spaces (see [1], [9]), is it true that the Hardy-Littlewood maximal operator M is bounded on each reflexive space $X(\mathbb{R}^n, m) \in \mathfrak{S}$ or on its associate space $X'(\mathbb{R}^n, m)$? It is well known that the answer is “yes” for the class of all Lebesgue spaces. We show

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that the answer is “no” for the class of Lorentz Gamma spaces $\Gamma_{p,w}(\mathbb{R}^n, m)$ (being a proper subclass of the class of all rearrangement-invariant Banach function spaces).

A measurable function $w: (0, \infty) \rightarrow [0, \infty)$ is called a *weight function*. Following [6], page 289, we say that a weight function belongs to the class \mathcal{D}_p , $1 \leq p < \infty$, whenever for all $x \in (0, \infty)$,

$$0 < W(x) := \int_0^x w(t) dt < \infty, \quad W_p(x) := x^p \int_x^\infty t^{-p} w(t) dt < \infty.$$

Let (\mathcal{R}, μ) be a totally σ -finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. The Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is the set of all μ -measurable functions $f: \mathcal{R} \rightarrow \mathbb{C}$ such that

$$\|f\|_{\Gamma_{p,w}} := \left(\int_0^\infty (f^{**}(t))^p w(t) dt \right)^{1/p} < \infty,$$

where

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(x) dx$$

and f^* is the decreasing rearrangement of f (see e.g. [1], Chapter 2, Section 1 or [7], Chapter II, §2 for its definition and properties). The Lorentz Gamma spaces were introduced by Sawyer (see [10]), but they can be traced to earlier works of Calderón, Hunt, O’Neil and others. These spaces have been studied by many authors (see e.g. [3], [4], [5], [6] and [9], Chapter 10 and also the references in the above monograph). The Lorentz Gamma spaces are examples of rearrangement-invariant (or symmetric) Banach function spaces (see [1], Chapter 2 and [7], Chapter II for the theory of rearrangement-invariant Banach function spaces). We should also note that one can study quasi-Banach Lorentz Gamma spaces $\Gamma_{p,w}$ if one admits that $0 < p < \infty$, see e.g. [6].

It is well known that Boyd and Zippin indices play important role in many questions related to interpolation properties of rearrangement-invariant spaces and boundedness behavior of classical operators of harmonic analysis (like the Hardy-Littlewood maximal operator or the Hilbert transform) on rearrangement-invariant spaces. The Boyd indices

$$0 \leq \alpha(X) \leq \beta(X) \leq 1$$

and Zippin indices

$$0 \leq p(X) \leq q(X) \leq 1$$

of a rearrangement-invariant space $X(\mathcal{R}, \mu)$ were introduced in [2] and [11], respectively. We refer to [1], Chapter 3, [7], Chapter 2, and [8] for their definitions, properties, and applications in interpolation theory and harmonic analysis. Boyd indices of Lorentz Gamma spaces $\Gamma_{p,w}$ were studied in [4], [6].

In this paper we construct an example of a reflexive Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$, whose Boyd and Zippin indices behave “badly”, (one says that the indices are trivial in this case) by rather elementary techniques:

$$\alpha(\Gamma_{p,w}) = p(\Gamma_{p,w}) = 0, \quad \beta(\Gamma_{p,w}) = q(\Gamma_{p,w}) = 1.$$

It may well be that examples of reflexive rearrangement-invariant spaces $X(\mathcal{R}, \mu)$ with both Boyd indices being trivial are known to experts or can be obtained from known results, but we were unable to find a published example of such space. Moreover, we believe that an example of a reflexive Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ with trivial Boyd and Zippin indices is of interest because it leads to the negative answer to the question posed above not only for the whole class of rearrangement-invariant Banach function spaces, but to its narrower subclass of Lorentz Gamma spaces.

For a precise definition of what we mean by a “rearrangement-invariant Banach function space” and by its “associate space”, see [1], Chapter 1, Definitions 1.1 and 2.1; Chapter 2, Definition 4.1 and Section 2 below. Definitions of Boyd and Zippin indices are also given in Section 2.

Theorem 1.1 (Main result). *Let (\mathcal{R}, μ) be a totally σ -finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. If $p \in (1, \infty)$ and*

$$(1.1) \quad w(x) := \begin{cases} \frac{1}{x(1 - \log x)^2}, & 0 < x < 1, \\ \frac{x^{p-1}}{(1 + \log x)^2}, & x \geq 1, \end{cases}$$

then the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is a reflexive rearrangement-invariant Banach function space and its Boyd and Zippin indices are trivial, that is,

$$\alpha(\Gamma_{p,w}) = p(\Gamma_{p,w}) = 0, \quad \beta(\Gamma_{p,w}) = q(\Gamma_{p,w}) = 1.$$

Combining Theorem 1.1 with the Lorentz-Shimogaki theorem (see e.g. [1], Chapter 3, Theorem 5.17), we arrive at the following corollary.

Corollary 1.2. *Let $p \in (1, \infty)$ and a weight w be given by (1.1). Then the Lorentz Gamma space $\Gamma_{p,w}(\mathbb{R}^n, m)$ is a reflexive rearrangement-invariant Banach function space and the Hardy-Littlewood maximal operator M is unbounded on the space $\Gamma_{p,w}(\mathbb{R}^n, m)$ and on its associate space $\Gamma'_{p,w}(\mathbb{R}^n, m)$.*

The paper is organized as follows. In Section 2, we recall the notions of a rearrangement-invariant Banach function space $X(\mathcal{R}, \mu)$ and its associate space $X'(\mathcal{R}, \mu)$, and give the definitions of its Boyd indices $\alpha(X)$, $\beta(X)$ and Zippin indices $p(X)$, $q(X)$. In Section 3, we show that the Lorentz Gamma spaces $\Gamma_{p,q}(\mathcal{R}, \mu)$ fall into the class of rearrangement-invariant Banach function spaces if $1 \leq p < \infty$ and $w \in \mathcal{D}_p$. Further, we formulate necessary and sufficient conditions for the reflexivity of $\Gamma_{p,w}(\mathcal{R}, \mu)$. In Section 4, we prove Theorem 1.1 and Corollary 1.2.

2. REARRANGEMENT-INVARIANT BANACH FUNCTION SPACES AND THEIR INDICES

2.1. Banach function spaces. Let (\mathcal{R}, μ) be a totally σ -finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. The set of all μ -measurable complex-valued functions on \mathcal{R} is denoted by $\mathfrak{M}(\mathcal{R}, \mu)$. Let $\mathfrak{M}^+(\mathcal{R}, \mu)$ be the subset of functions in $\mathfrak{M}(\mathcal{R}, \mu)$ whose values lie in $[0, \infty]$. The characteristic function of a μ -measurable set $E \subset \mathcal{R}$ is denoted by χ_E . Following [1], Chapter 1, Definition 1.1, a mapping

$$\varrho: \mathfrak{M}^+(\mathcal{R}, \mu) \rightarrow [0, \infty]$$

is called a *Banach function norm* if, for all functions f, g, f_n ($n \in \mathbb{N}$) in $\mathfrak{M}^+(\mathcal{R}, \mu)$, for all constants $a \geq 0$, and for all μ -measurable subsets E of \mathcal{R} , the following axioms hold:

- (A1) $\varrho(f) = 0 \Leftrightarrow f = 0$ a.e., $\varrho(af) = a\varrho(f)$, $\varrho(f + g) \leq \varrho(f) + \varrho(g)$,
- (A2) $0 \leq g \leq f$ a.e. $\Rightarrow \varrho(g) \leq \varrho(f)$ (the lattice property),
- (A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \varrho(f_n) \uparrow \varrho(f)$ (the Fatou property),
- (A4) $\mu(E) < \infty \Rightarrow \varrho(\chi_E) < \infty$,
- (A5) $\mu(E) < \infty \Rightarrow \int_E f(x) d\mu(x) \leq C_E \varrho(f)$

with $C_E \in (0, \infty)$, which may depend on E and ϱ but is independent of f . When functions differing only on a set of μ -measure zero are identified, the set $X(\mathcal{R}, \mu)$ of all functions $f \in \mathfrak{M}(\mathcal{R}, \mu)$ for which $\varrho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathcal{R}, \mu)$, the norm of f is defined by

$$\|f\|_X := \varrho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathcal{R}, \mu)$ becomes a Banach space (see [1], Chapter 1, Theorems 1.4 and 1.6). If ϱ is a Banach function norm, its associate norm ϱ' is defined on $\mathfrak{M}^+(\mathcal{R}, \mu)$ by

$$\varrho'(g) := \sup \left\{ \int_{\mathcal{R}} f(x)g(x) d\mu(x) : f \in \mathfrak{M}^+(\mathcal{R}, \mu), \varrho(f) \leq 1 \right\}, \quad g \in \mathfrak{M}^+(\mathcal{R}, \mu).$$

It is a Banach function norm itself, see [1], Chapter 1, Theorem 2.2. The Banach function space $X'(\mathcal{R}, \mu)$ determined by the Banach function norm ϱ' is called the *associate space* (Köthe dual) of $X(\mathcal{R}, \mu)$. The associate space $X'(\mathcal{R}, \mu)$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathcal{R}, \mu)]^*$.

2.2. Rearrangement-invariant Banach function spaces. Let $\mathfrak{M}_0(\mathcal{R}, \mu)$ and $\mathfrak{M}_0^+(\mathcal{R}, \mu)$ be the classes of a.e. finite functions in $\mathfrak{M}(\mathcal{R}, \mu)$ and $\mathfrak{M}^+(\mathcal{R}, \mu)$, respectively. The distribution function μ_f of $f \in \mathfrak{M}_0(\mathcal{R}, \mu)$ is given by

$$\mu_f(\lambda) := \mu\{x \in \mathcal{R} : |f(x)| > \lambda\}, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathfrak{M}_0(\mathcal{R}, \mu)$ are said to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$. The decreasing rearrangement of $f \in \mathfrak{M}_0(\mathcal{R}, \mu)$ is the function defined by

$$f^*(t) := \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad t \geq 0.$$

We use here the standard convention that $\inf \emptyset = \infty$.

A Banach function norm $\varrho: \mathfrak{M}^+(\mathcal{R}, \mu) \rightarrow [0, \infty]$ is called *rearrangement-invariant* if for every pair of equimeasurable functions $f, g \in \mathfrak{M}_0^+(\mathcal{R}, \mu)$, the equality $\varrho(f) = \varrho(g)$ holds. In that case, the Banach function space $X(\mathcal{R}, \mu)$ generated by ϱ is said to be a rearrangement-invariant Banach function space (or simply a rearrangement-invariant space). Lebesgue spaces $L^p(\mathcal{R}, \mu)$, $1 \leq p \leq \infty$, Orlicz spaces $L^\Phi(\mathcal{R}, \mu)$, and Lorentz spaces $L^{p,q}(\mathcal{R}, \mu)$ are classical examples of rearrangement-invariant Banach function spaces (see e.g. [1] and the references therein). By [1], Chapter 2, Proposition 4.2, if a Banach function space $X(\mathcal{R}, \mu)$ is rearrangement-invariant, then its associate space $X'(\mathcal{R}, \mu)$ is also rearrangement-invariant.

2.3. Boyd and Zippin indices. A measurable function $\varrho: (0, \infty) \rightarrow (0, \infty)$ is said to be submultiplicative if

$$\varrho(x_1 x_2) \leq \varrho(x_1) \varrho(x_2) \quad \forall x_1, x_2 \in (0, \infty).$$

The behavior of a measurable submultiplicative function ϱ in neighborhoods of zero and infinity is described by the quantities

$$(2.1) \quad \begin{aligned} \alpha(\varrho) &:= \sup_{x \in (0,1)} \frac{\log \varrho(x)}{\log x} = \lim_{x \rightarrow 0} \frac{\log \varrho(x)}{\log x}, \\ \beta(\varrho) &:= \inf_{x \in (1,\infty)} \frac{\log \varrho(x)}{\log x} = \lim_{x \rightarrow \infty} \frac{\log \varrho(x)}{\log x}, \end{aligned}$$

where

$$(2.2) \quad -\infty < \alpha(\varrho) \leq \beta(\varrho) < \infty,$$

see [7], Chapter 2, Theorem 1.3. The numbers $\alpha(\varrho)$ and $\beta(\varrho)$ are called the *lower* and *upper indices* of the measurable submultiplicative function ϱ .

Let (\mathcal{R}, μ) be a totally σ -finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. Suppose $X(\mathcal{R}, \mu)$ is a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm ϱ . In this case, the Luxemburg representation theorem (see [1], Chapter 2, Theorem 4.10) provides a unique rearrangement-invariant Banach function norm $\bar{\varrho}$ over the half-line $\mathbb{R}_+ = (0, \infty)$ equipped with the Lebesgue measure m , defined by

$$\bar{\varrho}(h) := \sup \left\{ \int_0^\infty g^*(t)h^*(t) dt : \varrho'(g) \leq 1 \right\},$$

and such that $\varrho(f) = \bar{\varrho}(f^*)$ for all $f \in \mathfrak{M}_0^+(\mathcal{R}, \mu)$. The rearrangement-invariant Banach function space generated by $\bar{\varrho}$ is denoted by $\bar{X}(\mathbb{R}_+, m)$.

For each $t > 0$ let E_t denote the dilation operator defined on $\mathfrak{M}(\mathbb{R}_+, m)$ by

$$(E_t f)(s) = f(st), \quad 0 < s < \infty.$$

With $X(\mathcal{R}, \mu)$ and $\bar{X}(\mathbb{R}_+, m)$ as above, let $h(t, X)$ denote the norm of $E_{1/t}$ as an operator on $\bar{X}(\mathbb{R}_+, m)$. By [1], Chapter 3, Proposition 5.11, for each $t > 0$, the operator E_t is bounded on $\bar{X}(\mathbb{R}_+, m)$ and the function $h(\cdot, X)$ is increasing (and hence, measurable) and submultiplicative on $(0, \infty)$. The indices of $h(\cdot, X)$ are called the *Boyd indices* of the rearrangement-invariant Banach function space $X(\mathcal{R}, \mu)$ and are denoted by

$$\alpha(X) := \alpha(h(\cdot, X)), \quad \beta(X) := \beta(h(\cdot, X)).$$

Following [1], Chapter 2, Definition 5.1, for each finite value t let $E \subset \mathcal{R}$ be such that $\mu(E) = t$ and let

$$\varphi_X(t) := \|\chi_E\|_X.$$

The function φ_X so defined is called the fundamental function of the rearrangement-invariant Banach function space $X(\mathcal{R}, \mu)$. Following [11], page 271 (see also [8], page 28), for a given rearrangement-invariant Banach function space $X(\mathcal{R}, \mu)$ with the fundamental function φ_X , let us consider the function

$$M(t, X) := \sup_{0 < x < \infty} \frac{\varphi_X(tx)}{\varphi_X(x)}, \quad t \in (0, \infty).$$

It is easy to check that this function is increasing (and hence, measurable) and submultiplicative on $(0, \infty)$. The indices of $M(\cdot, X)$ are called the *Zippin* (or fundamental) *indices* of the rearrangement-invariant Banach function space $X(\mathcal{R}, \mu)$ and are denoted by

$$p(X) := \alpha(M(\cdot, X)), \quad q(X) := \beta(M(\cdot, X)).$$

It is easy to see that $M(t, X) \leq h(t, X)$ for $t \in (0, \infty)$. Combining this inequality with [1], Chapter 3, Proposition 5.13, we conclude that

$$(2.3) \quad M(t, X) \leq h(t, X) \leq \max\{1, t\}, \quad t \in (0, \infty).$$

It follows from (2.1)–(2.3) that

$$0 \leq \alpha(X) \leq p(X) \leq q(X) \leq \beta(X) \leq 1.$$

The lower Boyd (or Zippin) index is said to be trivial if $\alpha(X) = 0$ (or $p(X) = 0$). Analogously, the upper Boyd (or Zippin) index is said to be trivial if $\beta(X) = 1$ (or $q(X) = 1$).

Note that for the Lebesgue spaces $L^p(\mathcal{R}, \mu)$, $1 \leq p \leq \infty$, all these indices are equal to $1/p$. Hence the Lebesgue space $L^p(\mathcal{R}, \mu)$ is reflexive if and only if its Boyd and Zippin indices are nontrivial. In contrast, Theorem 1.1 asserts that an analogous result fails for more general Lorentz Gamma spaces $\Gamma_{p,w}(\mathcal{R}, \mu)$.

3. LORENTZ GAMMA SPACES

3.1. Lorentz Gamma spaces are rearrangement-invariant Banach function spaces. The following lemma contains well known information on Lorentz Gamma spaces (see e.g. [4], Section 2 and [6], Section 0). We give its proof here for completeness.

Lemma 3.1. *Let (\mathcal{R}, μ) be a totally σ -finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. If $1 \leq p < \infty$ and $w \in \mathcal{D}_p$, then the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is a rearrangement-invariant Banach function space. The fundamental function of $\Gamma_{p,w}(\mathcal{R}, \mu)$ is given by*

$$(3.1) \quad \varphi_{\Gamma_{p,w}}(t) = (W(t) + W_p(t))^{1/p}, \quad t \in (0, \infty).$$

Proof. Axioms (A1)–(A3) in the definition of a Banach function space are satisfied in view of [1], Chapter 2, Proposition 3.2 and Theorem 3.4.

If $\mu(E) = 0$, then $\chi_E^{**}(x) = 0$ for $x \in (0, \infty)$ and $\|\chi_E\|_{\Gamma_{p,w}} = 0$. Let $E \subset \mathcal{R}$ be a set of measure $t > 0$. Then $\chi_E^* = \chi_{[0,t]}$ and $\chi_E^{**}(x) = \max\{1, t/x\}$. Hence

$$\varphi_{\Gamma_{p,w}}(t) = \|\chi_E\|_{\Gamma_{p,w}} = \left(\int_0^t w(x) dx + \int_t^\infty (t/x)^p w(x) dx \right)^{1/p} = (W(t) + W_p(t))^{1/p}.$$

If $w \in \mathcal{D}_p$, then the right-hand side of the above inequality is finite. Therefore $\mu(E) < \infty \Rightarrow \|\chi_E\|_{\Gamma_{p,w}} < \infty$. Thus, Axiom (A4) is satisfied and the fundamental function of the space $\Gamma_{p,w}$ is given by (3.1).

Since $w \in \mathcal{D}_p$, we have $0 < W(t) < \infty$ for all $t > 0$. Let $E \subset \mathcal{R}$ be a set of positive measure t and $f \in \Gamma_{p,w}(\mathcal{R}, \mu)$. By the Hardy-Littlewood inequality (see [1], Chapter 2, inequality (3.1) and Theorem 2.2),

$$(3.2) \quad \int_E |f(y)| \, d\mu(y) \leq \int_0^t f^*(x) \, dx = t f^{**}(t).$$

Taking into account that f^{**} is decreasing (see [1], Chapter 2, Proposition 3.2), we have

$$(3.3) \quad \begin{aligned} t f^{**}(t) &= \frac{t}{(W(t))^{1/p}} f^{**}(t) \left(\int_0^t w(x) \, dx \right)^{1/p} \\ &\leq \frac{t}{(W(t))^{1/p}} \left(\int_0^t (f^{**}(x))^p w(x) \, dx \right)^{1/p} \leq \frac{t}{(W(t))^{1/p}} \|f\|_{\Gamma_{p,w}}. \end{aligned}$$

It follows from (3.2)–(3.3) that

$$\mu(E) < \infty \Rightarrow \int_E |f(y)| \, d\mu(y) \leq C_E \|f\|_{\Gamma_{p,w}},$$

where $C_E = \mu(E)/(W(\mu(E)))^{1/p}$. Thus, Axiom (A5) is satisfied. \square

3.2. Reflexivity of Lorentz Gamma spaces. Now we recall necessary and sufficient conditions for the reflexivity of Lorentz Gamma spaces.

Lemma 3.2. *Let (\mathcal{R}, μ) be a totally σ -finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. Suppose that $w \in \mathcal{D}_p$ is a weight such that*

$$(3.4) \quad \int_0^x w(t) t^{-p} \, dt = \infty \quad \forall x \in (0, \infty).$$

Then the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is reflexive if and only if

$$W(\infty) = \int_0^\infty w(t) \, dt = \infty \quad \text{and} \quad V(\infty) = \int_0^\infty v(t) \, dt = \infty,$$

where $1/p + 1/p' = 1$ and

$$v(t) = \frac{t^{p'-1} W(t) W_p(t)}{(W(t) + W_p(t))^{p'+1}}, \quad t \in (0, \infty).$$

The above lemma is proved in [3], Lemma 6.4 in the case of $(\mathcal{R}, \mu) = (\mathbb{R}_+, m)$. Its proof is a combination of three ingredients. Two of these ingredients, namely, see [1], Chapter 1.1, Corollary 4.4 and [4], Theorem A, are proved for the measure spaces (\mathcal{R}, μ) under consideration. It remains to observe that, although the last ingredient (see [6], Proposition 1.1 (1)) is proved for (\mathbb{R}_+, m) , an inspection of its proof shows that it is also valid for (\mathcal{R}, μ) .

4. PROOF OF THE MAIN RESULTS

4.1. Proof of Theorem 1.1. Taking into account (1.1), a straightforward calculation gives that

$$(4.1) \quad W(x) = \frac{1}{1 - \log x} \quad \text{for } x \in (0, 1],$$

$$(4.2) \quad W_p(x) = \frac{x^p}{1 + \log x} \quad \text{for } x \in [1, \infty).$$

Thus $w \in \mathcal{D}_p$. By Lemma 3.1, the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is a rearrangement-invariant Banach function space.

Let us show that the space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is reflexive. Using L'Hôpital's rule, one can easily derive from (1.1) that

$$(4.3) \quad \lim_{x \rightarrow \infty} \frac{W(x)}{x^p(1 + \log x)^{-2}} = \frac{1}{p},$$

$$(4.4) \quad \lim_{x \rightarrow 0} \frac{W_p(x)}{(1 - \log x)^{-2}} = \lim_{x \rightarrow 0} \frac{\int_x^\infty t^{-p} w(t) dt}{x^{-p}(1 - \log x)^{-2}} = \frac{1}{p}.$$

It follows from (4.2) and (4.3) that

$$(4.5) \quad W(\infty) = \lim_{x \rightarrow \infty} W(x) = \infty$$

and

$$(4.6) \quad \begin{aligned} v(x) &= \frac{x^{p'-1} W(x) W_p(x)}{(W(x) + W_p(x))^{p'+1}} \\ &= x^{p'-1+2p-p(p'+1)} \frac{1 + o(1)}{p(1 + \log x)^3} \left(\frac{1 + o(1)}{p(1 + \log x)^2} + \frac{1}{(1 + \log x)} \right)^{-(p'+1)} \\ &= \frac{1}{px} (1 + \log x)^{p'-2} (1 + o(1)) \left(\frac{1 + o(1)}{p(1 + \log x)} + 1 \right)^{-(p'+1)} \\ &= \frac{1}{px} (1 + \log x)^{p'-2} (1 + o(1)) \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Since $p' > 1$, i.e., $2 - p' < 1$, it follows from (4.6) that

$$(4.7) \quad V(\infty) = \int_0^\infty v(x) dx \geq \text{const} \int_1^\infty \frac{1}{x(1 + \log x)^{2-p'}} dx = \infty.$$

It is easy to see that (3.4) holds for w defined by (1.1). Then it follows from (4.5) and (4.7) that the conditions of Lemma 3.2 are satisfied and hence the space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is reflexive.

Let us calculate the Boyd and Zippin indices of the space $\Gamma_{p,w}(\mathcal{R}, \mu)$. By Lemma 3.1, the fundamental function of $\Gamma_{p,w}(\mathcal{R}, \mu)$ is given by the formula

$$(4.8) \quad \varphi_{\Gamma_{p,w}}(x) = (W(x) + W_p(x))^{1/p}, \quad x \in (0, \infty).$$

If $t \in [1, \infty)$, then it follows from (4.2) and (4.3) that

$$(4.9) \quad \sup_{0 < x < \infty} \frac{W(tx) + W_p(tx)}{W(x) + W_p(x)} \geq \lim_{x \rightarrow \infty} \frac{W(tx) + W_p(tx)}{W(x) + W_p(x)} \\ = t^p \lim_{x \rightarrow \infty} \left(\frac{1 + \log x}{1 + \log(tx)} \right)^2 \lim_{x \rightarrow \infty} \frac{1 + o(1) + p(1 + \log(tx))}{1 + o(1) + p(1 + \log x)} = t^p.$$

If $t \in (0, 1)$, then it follows from (4.1) and (4.4) that

$$(4.10) \quad \sup_{0 < x < \infty} \frac{W(tx) + W_p(tx)}{W(x) + W_p(x)} \geq \lim_{x \rightarrow 0} \frac{W(tx) + W_p(tx)}{W(x) + W_p(x)} \\ = \lim_{x \rightarrow 0} \left(\frac{1 - \log x}{1 - \log(tx)} \right)^2 \lim_{x \rightarrow 0} \frac{p(1 - \log(tx)) + 1 + o(1)}{p(1 - \log x) + 1 + o(1)} = 1.$$

Combining (4.8)–(4.10), we conclude that for $t \in (0, \infty)$,

$$(4.11) \quad M(t, \Gamma_{p,w}) = \sup_{0 < x < \infty} \frac{\varphi_{\Gamma_{p,w}}(tx)}{\varphi_{\Gamma_{p,w}}(x)} = \sup_{0 < x < \infty} \left(\frac{W(tx) + W_p(tx)}{W(x) + W_p(x)} \right)^{1/p} \\ \geq \max\{1, t\}.$$

Then it follows from (2.3) and (4.11) that

$$M(t, \Gamma_{p,w}) = h(t, \Gamma_{p,w}) = \max\{1, t\}, \quad t \in (0, \infty).$$

Applying (2.1) to the submultiplicative functions $M(\cdot, \Gamma_{p,w})$ and $h(\cdot, \Gamma_{p,w})$, we get

$$\alpha(\Gamma_{p,w}) = p(\Gamma_{p,w}) = \lim_{t \rightarrow 0} \frac{\log \max\{1, t\}}{\log t} = 0, \\ \beta(\Gamma_{p,w}) = q(\Gamma_{p,w}) = \lim_{t \rightarrow \infty} \frac{\log \max\{1, t\}}{\log t} = 1,$$

which completes the proof. □

4.2. Proof of Corollary 1.2. In view of Theorem 1.1, the Lorentz Gamma space $\Gamma_{p,w}(\mathbb{R}^n, m)$ is a reflexive rearrangement-invariant Banach function space such that its Boyd indices are trivial, that is, $\alpha(\Gamma_{p,w}) = 0$ and $\beta(\Gamma_{p,w}) = 1$. Since $\beta(\Gamma_{p,w}) = 1$, the Hardy-Littlewood maximal operator M is unbounded on the space $\Gamma_{p,w}(\mathbb{R}^n, m)$ in view of the Lorentz-Shimogaki theorem (see e.g. [1], Chapter 3, Theorem 5.17). By [1], Chapter 3, Proposition 5.13, $\beta(\Gamma'_{p,w}) = 1 - \alpha(\Gamma_{p,w}) = 1$. Now, applying [1], Chapter 3, Theorem 5.17 to the associate space $\Gamma'_{p,w}(\mathbb{R}^n, m)$, we conclude that the operator M is unbounded on the space $\Gamma'_{p,w}(\mathbb{R}^n, m)$. \square

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