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NOTES ON THE AVERAGE NUMBER OF SYLOW SUBGROUPS OF FINITE GROUPS

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Abstract. We show that if the average number of (nonnormal) Sylow subgroups of a finite group is less than $\frac{29}{4}$ then G is solvable or $G/F(G) \cong A_5$. This generalizes an earlier result by the third author.

Keywords: Fitting subgroup; Sylow subgroup; composition factor

MSC 2020: 20D20

1. Introduction

All groups considered in this paper are finite. Given a group G, we define the average class size of G to be $\operatorname{acs}(G) = |G|/k(G)$, where $k(G)$ is the number of conjugacy classes of G. Using this notation, Theorem 11 of [2] (which also follows from the earlier results of Lescot, as mentioned in the addendum), asserts that if $\operatorname{acs}(G) < \frac{40}{3}$ then either G is solvable or $G \cong A_5 \times T$. In particular, since $\operatorname{acs}(A_5) = 12$ this implies that if $\operatorname{acs}(G) < 12$ then G is solvable.

An analog of the final part of this result for Sylow numbers was considered in [5]. Let us introduce some notation from [5] to be used in this note. Given a prime p, $\nu_p(G)$ stands for the number of Sylow p-subgroups of G. Let $\mathcal{S} =$

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$\{p \text{ prime}\colon \nu_p(G) > 1\}$ and put $\operatorname{asn}(G) = \Bigl(\sum\limits_{p\in\mathcal{S}} \nu_p(G)\Bigr)/|\mathcal{S}|$. Theorem A of [5] asserts that if $\operatorname{asn}(G) < 7$ then G is solvable. Our goal here is to extend this result to include also the analog of the first part of the result of Lescot-Guralnick-Robinson. Our main result is the following.

Theorem 1.1. *Let G be a finite nonsolvable group. Assume that $\operatorname{asn}(G) < \frac{29}{4}$. Then $G/F(G) \cong A_5$. Furthermore, if $Z(G) = 1$ then $G \cong A_5$.*

In this case, we cannot get a factorization as a direct product of A_5 and a nilpotent group because $\operatorname{asn}(\mathrm{SL}(2,5)) = \operatorname{asn}(A_7) = 7$. We expect that it should be possible to find $C > \frac{29}{4}$ such that if G is nonsolvable and $\operatorname{asn}(G) < C$ then we still get that $G/F(G) \cong A_5$. Our main aim in the proof, rather than trying to find the best possible value of C, was to keep the proof as elementary as possible. In fact, as in [5], Burnside's p^aq^b-theorem is the most advanced result that we are using.

We also prove the following result, which can be compared with Theorem B of [4].

Theorem 1.2. *Let G be a finite group. Assume that $\operatorname{asn}(G) < \frac{7}{2}$. Then G is supersolvable. Furthermore, if $Z(G) = 1$, then $G \cong S_3$.*

2. Proofs

We start with two elementary lemmas.

Lemma 2.1. *If N is a normal subgroup of a finite group G, then $\nu_p(N)\nu_p(G/N)$ divides $\nu_p(G)$. In particular, if $S_1, \dots, S_t$ are the composition factors of G including repetitions, then $\nu_p(S_1) \cdot \ldots \cdot \nu_p(S_t) \mid \nu_p(G)$.*

Proof. See [3] for the first part. The second part is an immediate consequence. □

Lemma 2.2. *Let S be a simple subgroup of A_7. Then $S \cong A_5, A_6, A_7$ or $\mathrm{PSL}(2,7)$. Furthermore, if A_6, A_7 or $\mathrm{PSL}(2,7)$ is a composition factor of a finite group G, then $\operatorname{asn}(G) \geqslant \frac{89}{6}$.*

Proof. The first part is a group theory exercise, or it can be checked with GAP, see [1]. For the second part, assume first that $S = A_6$. As another exercise or using GAP, one can check that $\nu_2(A_6) = 45$, $\nu_3(A_6) = 10$ and $\nu_5(A_6) = 36$. The average of these three integers is $\frac{91}{3}$. Using Lemma 2.1, we get $\nu_2(G) \geqslant 45$, $\nu_3(G) \geqslant 10$ and $\nu_5(G) \geqslant 36$. If we want to find a group G with $\operatorname{asn}(G)$ as low as possible among the groups that satisfy these conditions, it is an arithmetic exercise (using Sylow's

Theorem) to check that we cannot do better than having $\nu_7(G)=8$, $\nu_{11}(G)=12$, $\nu_{13}(G)=14$, $\nu_{17}(G)=18$ and $\nu_{19}(G)=20$. In this case, $\operatorname{asn}(G)\geqslant\frac{164}{8}=\frac{41}{2}$.

If $S=A_7$ we can argue as in the previous case to see that $\operatorname{asn}(G)>\frac{41}{2}$. (In fact, it is much bigger.)

If $S=\mathrm{PSL}(2,7)$, then $\nu_2(S)=21$, $\nu_3(S)=28$ and $\nu_7(S)=8$. Arguing as in the case when $G=A_6$, we see that $\operatorname{asn}(G)\geqslant\frac{89}{6}$. □

Now we are ready to prove Theorem 1.1.

P r o o f of Theorem 1.1. Let S be a nonabelian composition factor of G. Assume first that $\nu_p(S)\geqslant 8$ for every prime divisor p of $|S|$. Assume that 5 divides $|S|$. Then $\nu_5(G)\geqslant\nu_5(S)\geqslant 11$ using Lemma 2.1 and Sylow's Theorem. If 2 divides $|S|$, then $\nu_2(G)\geqslant\nu_2(S)\geqslant 9$. Even if $\nu_3(G)=4$ we get that the average of $\nu_2(G),\nu_3(G)$ and $\nu_5(G)$ is at least 8, so $\operatorname{asn}(G)\geqslant 8$, a contradiction. Therefore, 2 does not divide $|S|$. If 3 divides $|S|$, then $\nu_3(G)\geqslant\nu_3(S)\geqslant 10$. As before, we get that the average of $\nu_2(G)$, $\nu_3(G)$ and $\nu_5(G)$ is at least 8, so $\operatorname{asn}(G)\geqslant 8$, another contradiction. But then all Sylow numbers that are bigger than 1 are at least 8, so $\operatorname{asn}(G)\geqslant 8$. It follows that 5 does not divide $|S|$.

If 6 divides $|S|$, then $\nu_2(G)\geqslant\nu_2(S)\geqslant 9$ and $\nu_3(G)\geqslant\nu_3(S)\geqslant 10$ (using Sylow's Theorem and the fact that $\nu_p(S)\geqslant 8$ for every prime divisor p of $|S|$). As before, one can see that this implies that $\operatorname{asn}(G)\geqslant 8$. This contradiction and Burnside's Theorem imply that there exist two different primes $u,v\geqslant 7$ such that $uv\mid|S|$. Arguing as in the last paragraph of the proof of Theorem 2.2 of [4], we get that $\operatorname{asn}(G)\geqslant\frac{39}{5}>\frac{29}{4}$. This is the final contradiction.

Therefore $\nu_p(S)<8$ for some prime divisor p of $|S|$. Then S has a proper subgroup of index $\leqslant 7$ and we deduce that S is isomorphic to a simple subgroup of S_7. By Lemma 2.2, we deduce that $S=A_5$. Therefore, $\nu_2(S)=5$, $\nu_3(S)=10$ and $\nu_5(S)=6$. By Lemma 2.1, $\nu_p(S)\mid\nu_p(G)$ for every prime p. Since $\operatorname{asn}(G)<\frac{29}{4}$, it follows that $\nu_2(G)=5$, $\nu_3(G)=10$ and $\nu_5(G)=6$.

Assume that there exists a prime $q\geqslant 7$ that divides $|G|$. If $\nu_q(G)>1$ then $\nu_q(G)\geqslant 8$ and since the average of 5, 10, 6 and 8 is $\frac{29}{4}$, $\operatorname{asn}(G)\geqslant\frac{29}{4}$. It follows that $\nu_q(G)=1$ for every $q\geqslant 7$.

Let N be the largest normal solvable subgroup of G and let M/N be a chief factor of G. We know that M/N is a direct product of copies of A_5. Using Lemma 2.1, we see that $M/N=A_5$. Let $C/N=C_{G/N}(M/N)$. Notice that $C/N\times M/N\trianglelefteq G/N$. Using Lemma 2.1 again we see that C/N is solvable, so $C=N$. It follows that G/N is isomorphic to a subgroup of $\operatorname{Aut}(A_5)=S_5$. If $G/N=S_5$ then $\nu_2(G/N)=15$ and $\operatorname{asn}(G)>\frac{29}{4}$, a contradiction. Hence, $G=M$.

By Lemma 2.1, $\nu_p(N)=1$ for every prime p, so N is nilpotent. The first part of the statement follows. Now, we assume that $Z(G)=1$ and we want to prove that

$N = 1$. By way of contradiction, assume that $N > 1$. Let $R \in \mathrm{Syl}_r(N)$ for some prime $r \mid |N|$ and let P be a Sylow subgroup of G for some prime $p \in \{2,3,5\} - \{r\}$. Since $R \trianglelefteq G$, P normalizes R and $RP \leqslant G$. On the other hand, $R \leqslant N \leqslant N_G(P)$ (otherwise $\nu_p(G) > \nu_p(G/N)$ and we saw in the third paragraph of the proof that this is not the case), so $[P, R] = 1$. Therefore, R centralizes all the Sylow p-subgroups of G for all primes $p \neq r$. Since G/N is generated by its Sylow p-subgroups for any $p \in \{2,3,5\}$ it follows that $Z(R) \leqslant Z(G)$. This contradicts the hypothesis $Z(G) = 1$. It follows that $G = A_5$. □

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. Notice that in order to have $\mathrm{asn}(G) < \frac{7}{2}$, we must have $\nu_2(G) = 3$, $\nu_p(G) = 1$ for every prime $p \geqslant 3$. Therefore, G has a normal nilpotent Hall $2'$-subgroup N and $G = PN$, where $P \in \mathrm{Syl}_2(G)$. Since $|G : N_G(P)| = 3$, we have that $|N : C_N(P)| = |N : N \cap N_G(P)| = 3$. Since $C_N(P)$ has its prime index in the nilpotent subgroup N, we deduce that $C_N(P) \trianglelefteq N$. Clearly, P normalizes $C_N(P)$, so $C_N(P) \trianglelefteq G$. Observe that any chief series of N that contains $C_N(P)$ consists of normal subgroups of G. Extending this chief series to a chief series of G, we see that G is supersolvable.

Assume now that $Z(G) = 1$. If $C_N(P) > 1$, we can take a minimal normal subgroup M of N contained in $C_N(P)$. This subgroup is central in G and this is a contradiction. We conclude that $C_N(P) = 1$ so $|N| = 3$. Since $Z(G) = 1$, we deduce that $|P| = 2$ and $G = S_3$. □

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