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APPLICATION OF VERY WEAK FORMULATION ON HOMOGENIZATION OF BOUNDARY VALUE PROBLEMS IN POROUS MEDIA

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Abstract. The goal of this paper is to present a different approach to the homogenization of the Dirichlet boundary value problem in porous medium. Unlike the standard energy method or the method of two-scale convergence, this approach is not based on the weak formulation of the problem but on the very weak formulation. To illustrate the method and its advantages we treat the stationary, incompressible Navier-Stokes system with the non-homogeneous Dirichlet boundary condition in periodic porous medium. The nonzero velocity trace on the boundary of a solid inclusion yields a non-standard addition to the source term in the Darcy law. In addition, the homogenized model is not incompressible.

Keywords: homogenization; porous medium; Navier-Stokes system; very weak formulation

MSC 2020: 35B27, 76M50, 35Q30

1. Introduction

Modeling a flow in porous media is an important subject due to its numerous applications. A standard and very efficient approach is to use a periodic model of porous medium and the method of homogenization. The porous medium is modeled as a periodic array of solid structures with fluid flowing around them. Such geometry allows to derive a model formally and to prove rigorously the convergence of the homogenization process as the period of the medium tends to zero.

The formal computation using two-scale asymptotic expansions can be found, for instance see [3] or [16].

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The rigorous proof of convergence for the homogenization procedure can be done using the celebrated Tartar approach of oscillating test-functions (also called the *energy method*) presented in the appendix of [16] (see also [14] for the case of a non-homogeneous boundary condition). Another approach involves the two-scale convergence, introduced by Nguetseng (see [15]) and popularized by Allaire (see [2]). Those methods use the weak formulation as the starting point, making non-homogeneous Dirichlet boundary conditions difficult to treat, as they do not appear explicitly in the weak formulation. In fact, the non-homogeneous trace needs to be either lifted from the boundary or suppressed by the test function with zero trace, before the methods are applied.

One can also use a method called the *unfolding*, based on the notion of the local unfolding operator, see e.g. [6].

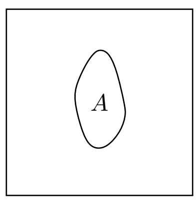
We propose here slightly different and more direct approach based on the very weak formulation of the boundary value problem. One of the advantages of such approach is that the Dirichlet boundary value is apparent in the formulation making the proofs simpler and the effects of the homogenization process immediately visible. As in the energy method, our approach does use the particular choice of the oscillating test functions but in the framework of the very weak instead of weak formulation of our boundary-value problem. To simplify the convergence of the pressure we also use the shortcut via the two-scale convergence.

2. The problem and its very weak formulation

We start by a bounded smooth domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$. Then we construct a periodic cell

$$Y^* = Y \setminus A.$$

Here, $Y =]0,1[^n$ is a unit cube and $A \subset\subset Y$ is its smooth (∂A is of class C^1) simply connected subset.



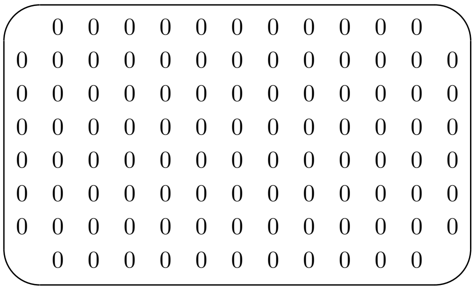
Y^*

Let

$$K_\varepsilon = \{\mathbf{k} \in \mathbf{Z}^n : \ \varepsilon(\mathbf{k} + A) \subset \Omega\}.$$

We take some small parameter $\varepsilon \ll 1$ (the average pore size) and put

$$
\begin{aligned}
Y_{\mathbf{k}}^{\varepsilon} &= \varepsilon(\mathbf{k}+Y^{*}), \quad \mathbf{k}\in\mathbf{Z}^{n}, && \text{the } \mathbf{k}\text{th pore,}\\
A_{\mathbf{k}}^{\varepsilon} &= \varepsilon(\mathbf{k}+A), \quad \mathbf{k}\in\mathbf{Z}^{n}, && \text{the } \mathbf{k}\text{th obstacle,}\\
A_{\varepsilon} &= \bigcup_{\mathbf{k}\in K_{\varepsilon}} A_{\mathbf{k}}^{\varepsilon}, && \text{the solid part,}\\
\Omega_{\varepsilon} &= \Omega\setminus A_{\varepsilon}, && \text{the fluid part.}
\end{aligned}
$$



Ω_ε

We denote by $\partial\Omega$ the exterior boundary of the domain and by $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \partial\Omega$ the boundary of the pores. We know that, due to the choice of K_ε, the perforations do not intersect with the exterior boundary, i.e.

$$\partial\Omega \cap \Gamma_\varepsilon = \emptyset.$$

Let $(\mathbf{u}_\varepsilon, p_\varepsilon)$ be the solution to the stationary Navier-Stokes problem

$$
\begin{gathered}
-\Delta\mathbf{u}_\varepsilon + \mathrm{Re}(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f} \quad \operatorname{div}\mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon,\\
\mathbf{u}_\varepsilon = \varepsilon^2\mathbf{g}_\varepsilon \quad \text{on } \partial\Omega, \quad \mathbf{u}_\varepsilon = \varepsilon^2\mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \quad \text{on } \Gamma_\varepsilon,
\end{gathered}
\tag{2.1}
$$

where the boundary velocities $\mathbf{g}_\varepsilon(\mathbf{x})$ and $\mathbf{G}(\mathbf{x},\mathbf{y})$ are assumed to be continuous vector functions, defined on whole $\bar\Omega$ and $\bar\Omega\times Y^*$, respectively. Furthermore, $\mathbf{G}\in C^1(\bar\Omega\times Y^*)$ is Y-periodic in $\mathbf{y}$ variable. The necessary condition for the existence of solution has to be imposed,

$$\int_{\partial\Omega}\mathbf{g}_\varepsilon\cdot\mathbf{n} + \int_{\Gamma_\varepsilon}\mathbf{G}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{n}_\varepsilon = 0. \tag{2.2}$$

To clarify this condition we notice that

$$\varepsilon\int_{\Gamma_\varepsilon}\mathbf{G}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{n}_\varepsilon\,\mathrm{d}S_x \to \int_\Omega\int_{\partial A}\mathbf{G}(\mathbf{x},\mathbf{y})\cdot\mathbf{n}\,\mathrm{d}S_y\,\mathrm{d}\mathbf{x}.$$

Bearing that in mind, we assume that the function has the form

$$\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{P}(\mathbf{x},\mathbf{y}) + \varepsilon \mathbf{H}(\mathbf{x},\mathbf{y}), \tag{2.3}$$

where

$$\mathbf{P}(\mathbf{x},\mathbf{y}) \cdot \mathbf{n} = 0 \quad \text{for } \mathbf{y} \in \partial A. \tag{2.4}$$

Thus, the condition (2.2) becomes

$$\int_{\partial\Omega} \mathbf{g}_\varepsilon \cdot \mathbf{n} + \varepsilon \int_{\Gamma_\varepsilon} \mathbf{H}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \cdot \mathbf{n}_\varepsilon = 0, \tag{2.5}$$

meaning that

$$\int_{\partial\Omega} \mathbf{g}_\varepsilon \cdot \mathbf{n} = -\int_\Omega \int_{\partial A} \mathbf{H}(\mathbf{x},\mathbf{y}) \cdot \mathbf{n} \, \mathrm{d}S_y \, \mathrm{d}\mathbf{x} + O(\varepsilon). \tag{2.6}$$

So, the only appropriate thing to do is to assume that

$$\mathbf{g}_\varepsilon = \mathbf{g} + \varepsilon \mathbf{j}_\varepsilon$$

with

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = -\int_\Omega \int_{\partial A} \mathbf{H}(\mathbf{x},\mathbf{y}) \cdot \mathbf{n} \, \mathrm{d}S_y \, \mathrm{d}\mathbf{x} \tag{2.7}$$

and

$$\int_{\partial\Omega} \mathbf{j}_\varepsilon \cdot \mathbf{n} = \varepsilon^{-1}\bigg(\int_\Omega \int_{\partial A} \mathbf{H}(\mathbf{x},\mathbf{y}) \cdot \mathbf{n} \, \mathrm{d}S_y \, \mathrm{d}\mathbf{x} - \varepsilon \int_{\Gamma_\varepsilon} \mathbf{H}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \cdot \mathbf{n}_\varepsilon\bigg) = O(1). \tag{2.8}$$

We summarize and in what follows we impose the following conditions on the boundary values:
(1) $\mathbf{G}(\mathbf{x},\mathbf{y}) = \mathbf{P}(\mathbf{x},\mathbf{y}) + \varepsilon \mathbf{H}(\mathbf{x},\mathbf{y})$ with $\mathbf{G},\mathbf{H} \in C^1(\bar\Omega \times Y^*)^n$ periodic in $\mathbf{y}$,
(2) $\mathbf{P}(\mathbf{x},\mathbf{y}) \cdot \mathbf{n} = 0$ for $\mathbf{y} \in \partial A$ and $\mathbf{x} \in \bar\Omega$,
(3) $\mathbf{g}_\varepsilon = \mathbf{g} + \varepsilon \mathbf{j}_\varepsilon$, where $\mathbf{g}$, $\mathbf{j}_\varepsilon \in C(\partial\Omega)^n$ satisfying (2.7) and (2.8).

The source term $\mathbf{f}$ is assumed to be an $L^2(\Omega)^n$ function.

Remark 2.1. The unusual non-homogeneous boundary condition on the solid part of the porous medium can correspond, for instance, to the flow through an array of rotating cylinders, like in [9]. The ε^2 scaling of the boundary values is chosen in order to get the source term $\mathbf{f}$ and the boundary value terms in the limit problem. If we leave out ε^2, the source term will not be present in the effective law.

The very weak formulation (see [7], [8] or [12]) for that problem reads: Find $(\mathbf{u}_\varepsilon, p_\varepsilon) \in L^2(\Omega_\varepsilon)^n \times H^{-1}(\Omega_\varepsilon)$ such that

$$
\begin{aligned}
(2.9) \qquad & \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot (-\Delta \mathbf{w} + \nabla \pi) - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \mathbf{w} \\
&= \int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{w} + \mathrm{Re} \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{w} \cdot \mathbf{u}_\varepsilon \\
&\quad + \varepsilon^2 \int_{\partial\Omega} \mathbf{g}_\varepsilon \cdot \Big(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}} + \pi \mathbf{n} \Big) + \varepsilon^2 \int_{\Gamma_\varepsilon} \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \cdot \Big(-\frac{\partial \mathbf{w}}{\partial \mathbf{n}_\varepsilon} + \pi \mathbf{n}_\varepsilon \Big)
\end{aligned}
$$

holds for any $(\mathbf{w}, \pi) \in H^2(\Omega_\varepsilon)^n \times H^1(\Omega_\varepsilon)$ is such that $\mathbf{w} = 0$ on $\partial\Omega_\varepsilon$.

That problem admits a solution even if $\mathbf{f}$ is an $H^{-1}(\Omega_\varepsilon)$ function and $\mathbf{G}$ and $\mathbf{g}_\varepsilon$ are L^2 functions on the boundary. Such solution is unique, if the Reynolds number is not too large (see [12] for details). In our case, we have a weak solution, which is then also a very weak solution, so the existence of the solution is not in question. We do not use the very weak formulation due to the lack of regularity of the data, but because it is convenient for passing to the limit, since the Dirichlet boundary value appears in the formulation.

The homogenization of the Navier-Stokes system in porous medium is a well studied problem and the reader can consult the papers, see [1], [2], [4], [5], [11], [13], and [14]. However we present here a different approach based on a very weak formulation which is the main novelty of the paper. To stress the advantages we have imposed the non-homogeneous boundary condition on the pores boundary. Due to the non-standard boundary condition, surprising additions to the usual source term appear. Furthermore, the model obtained is not incompressible.

3. A priori estimates

In addition to the previous assumptions (2.2)–(2.8), we suppose that

$$\operatorname{div}_y \mathbf{P}(\mathbf{x}, \mathbf{y}) = 0.$$

We also assume, for simplicity, that $\mathbf{P}(\mathbf{x}, \mathbf{y})$ and $\mathbf{H}(\mathbf{x}, \mathbf{y})$ vanish for $\mathbf{x}$ in the vicinity of the exterior boundary $\partial\Omega$. More precisely $\mathbf{x} \mapsto \mathbf{G}(\mathbf{x}, \mathbf{y})$ is compactly supported in Ω. Thus, for ε small enough, it equals zero in all the cells that intersect the boundary. Modifying $\mathbf{g}_\varepsilon$ to a function defined on Ω_ε is slightly more demanding as we want the modification to be equal to $\mathbf{G}(\mathbf{x}, \mathbf{x}/\varepsilon)$ on the perforations.

Proposition 3.1. *There exists a function $\overline{\mathbf{g}}_\varepsilon \in W^{1,r}(\Omega_\varepsilon)^n$ such that*

$$(3.1) \qquad \operatorname{div} \overline{\mathbf{g}}_\varepsilon = 0,$$

$$(3.2) \qquad \overline{\mathbf{g}}_\varepsilon = \mathbf{g}_\varepsilon \quad \text{on } \partial\Omega \quad \text{and} \quad \overline{\mathbf{g}}_\varepsilon = \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \quad \text{on } \Gamma_\varepsilon.$$

Furthermore, the following a priori estimates hold for $1 < r < \infty$*:*

$$\varepsilon|\nabla \overline{\mathbf{g}}_\varepsilon|_{L^r(\Omega_\varepsilon)} \leqslant C, \tag{3.3}$$

$$|\overline{\mathbf{g}}_\varepsilon|_{L^r(\Omega_\varepsilon)} \leqslant C. \tag{3.4}$$

P r o o f. We first need to modify $\mathbf{g}_\varepsilon$ to become equal to zero on the boundaries of perforations. To do so we use the Tartar restriction operator $R_\varepsilon \in \mathcal{L}(W^{1,r}(\Omega)^n, W_\varepsilon)$, where

$$W_\varepsilon = \{\varphi \in W^{1,r}(\Omega_\varepsilon)^n \colon \ \varphi = 0 \text{ on } \Gamma_\varepsilon\}.$$

There exists such an operator enjoying the properties (see Appendix in [16], where it was introduced, or [13] and [14]):

$$\varepsilon|\nabla R_\varepsilon \mathbf{w}|_{L^r(\Omega_\varepsilon)} \leqslant C(|\mathbf{w}|_{L^r(\Omega)} + \varepsilon|\nabla \mathbf{w}|_{L^r(\Omega)}), \tag{3.5}$$

$$|\operatorname{div} R_\varepsilon \mathbf{w}|_{L^r(\Omega_\varepsilon)} \leqslant C(|\operatorname{div} \mathbf{w}|_{L^r(\Omega)} + |\mathbf{w}|_{L^r(\Omega)}), \tag{3.6}$$

$$|R_\varepsilon \mathbf{w}|_{L^r(\Omega_\varepsilon)} \leqslant C(|\mathbf{w}|_{L^r(\Omega)} + \varepsilon|\nabla \mathbf{w}|_{L^r(\Omega)}). \tag{3.7}$$

Furthermore, since Γ_ε and $\partial\Omega$ do not intersect, by construction

$$R_\varepsilon \mathbf{g}_\varepsilon = \mathbf{g}_\varepsilon \quad \text{on } \partial\Omega.$$

Now the function $\mathbf{G}(\mathbf{x}, \mathbf{x}/\varepsilon) + R_\varepsilon \mathbf{g}_\varepsilon$ satisfies the required boundary condition

$$\mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + R_\varepsilon \mathbf{g}_\varepsilon = \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \quad \text{on } \Gamma_\varepsilon, \quad \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + R_\varepsilon \mathbf{g}_\varepsilon = \mathbf{g}_\varepsilon \quad \text{on } \partial\Omega,$$

but it is not divergence free. Due to (2.2), we can use the result from [10], Theorem 2.3, page 6 to prove that there exists a function $\mathbf{z}_\varepsilon \in W^{1,r}(\Omega_\varepsilon)^n$ such that

$$\begin{aligned} \operatorname{div} \mathbf{z}_\varepsilon &= \operatorname{div} \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + \operatorname{div} R_\varepsilon \mathbf{g}_\varepsilon \\ &= \varepsilon^{-1}\mathrm{div}_y \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + \mathrm{div}_x \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + \operatorname{div} R_\varepsilon \mathbf{g}_\varepsilon \\ &= (\mathrm{div}_x \mathbf{P} + \mathrm{div}_y \mathbf{H} + \varepsilon \mathrm{div}_x \mathbf{H})\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + \operatorname{div} R_\varepsilon \mathbf{g}_\varepsilon, \end{aligned} \tag{3.8}$$

$$\mathbf{z}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{3.9}$$

and

$$|\mathbf{z}_\varepsilon|_{L^r(\Omega_\varepsilon)} + \varepsilon|\nabla \mathbf{z}_\varepsilon|_{L^r(\Omega_\varepsilon)} \leqslant C\Big|(\mathrm{div}_x \mathbf{P} + \mathrm{div}_y \mathbf{H} + \varepsilon \mathrm{div}_x \mathbf{H})\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + \operatorname{div} R_\varepsilon \mathbf{g}_\varepsilon\Big|_{L^r(\Omega_\varepsilon)} \leqslant C.$$

The above result is well known (see e.g. [14] or a general discussion of the problem in [10]). Adding up the above, we have constructed the function

$$\overline{\mathbf{g}}_\varepsilon = -\mathbf{z}_\varepsilon + \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) + R_\varepsilon \mathbf{g}_\varepsilon$$

having the required properties. □

Proposition 3.2. *If* $\lim_{\varepsilon\to 0} \mathrm{Re}\,\varepsilon^{3-\delta} = 0$ *for some* $\delta > 0$, *then the following a priori estimates hold*

$$\varepsilon^{-1}|\nabla \mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C, \tag{3.10}$$

$$\varepsilon^{-2}|\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C, \tag{3.11}$$

$$|p_\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C. \tag{3.12}$$

Proof. Let δ, from the above condition on Re, be such that $0 < \delta \ll 1$. We test the equation with

$$\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon.$$

We get

$$\int_{\Omega_\varepsilon} |\nabla \mathbf{u}_\varepsilon|^2 = \varepsilon^2 \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \cdot \nabla \overline{\mathbf{g}}_\varepsilon + \int_{\Omega_\varepsilon} (\mathbf{f} - \mathrm{Re}(\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon) \cdot \{\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon\}. \tag{3.13}$$

To estimate the integrals on the right-hand side we use the Poincaré inequality

$$|\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon |\nabla(\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon)|_{L^2(\Omega_\varepsilon)},$$

the embedding $H^1 \subset L^6$

$$|\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon|_{L^6(\Omega_\varepsilon)} \leqslant C |\nabla(\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon)|_{L^2(\Omega_\varepsilon)}$$

and the interpolation for $s = 6/(3-2\delta)$ (and thus $2 < s < 6$)

$$\begin{aligned} |\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon|_{L^s(\Omega_\varepsilon)} &\leqslant |\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon|_{L^2(\Omega_\varepsilon)}^{3/s-1/2} |\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon|_{L^6(\Omega_\varepsilon)}^{3/2-3/s} \\ &\leqslant C\varepsilon^{3/s-1/2} |\nabla(\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon)|_{L^2(\Omega_\varepsilon)} \\ &= C\varepsilon^{1-\delta} |\nabla(\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon)|_{L^2(\Omega_\varepsilon)} \end{aligned} \tag{3.14}$$

with $C > 0$ independent of ε. Obviously $\delta = 3/2 - 3/s > 0$.

The first integral on the right-hand side is easily majorized by

$$C\varepsilon |\nabla \mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}.$$

Next, the use of the Poincaré inequality and (3.3) implies

$$\int_{\Omega_\varepsilon} \mathbf{f} \cdot \{\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon\} \leqslant C |\mathbf{u}_\varepsilon - \varepsilon^2 \overline{\mathbf{g}}_\varepsilon|_{L^2(\Omega_\varepsilon)} \leqslant C\varepsilon (|\nabla \mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)} + \varepsilon).$$

Now we need to estimate the inertial term

$$\mathrm{Re} \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot \varepsilon^2 \overline{\mathbf{g}}_\varepsilon.$$

We need the interpolation inequality (3.14) and the estimate (3.4) for r large enough. For $s = 6/(3-2\delta)$ (like before) and $r = 3/\delta$ (so that $1/2 + 1/s + 1/r = 1$), we have

$$\begin{aligned}\operatorname{Re}\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot\varepsilon^2\overline{\mathbf{g}}_\varepsilon &\leqslant C\operatorname{Re}\varepsilon^2|\nabla\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}|\mathbf{u}_\varepsilon|_{L^s(\Omega_\varepsilon)}|\overline{\mathbf{g}}_\varepsilon|_{L^r(\Omega_\varepsilon)}\\ &\leqslant C\operatorname{Re}\varepsilon^2|\nabla\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}|\mathbf{u}_\varepsilon|_{L^s(\Omega_\varepsilon)}\\ &\leqslant C\operatorname{Re}\varepsilon^2|\nabla\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}(|\mathbf{u}_\varepsilon-\varepsilon^2\overline{\mathbf{g}}_\varepsilon|_{L^s(\Omega_\varepsilon)}+\varepsilon^2)\\ &\leqslant C\operatorname{Re}\varepsilon^2|\nabla\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}(\varepsilon^{1-\delta}|\nabla(\mathbf{u}_\varepsilon-\varepsilon^2\overline{\mathbf{g}}_\varepsilon)|_{L^2(\Omega_\varepsilon)}+\varepsilon^2)\\ &\leqslant C\operatorname{Re}\varepsilon^{3-\delta}|\nabla\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}(|\nabla\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}+\varepsilon)\end{aligned}$$

(notice that $\delta = 3/2 - 3/s$). Finally

$$\begin{aligned}\operatorname{Re}\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot\mathbf{u}_\varepsilon &= \operatorname{Re}\varepsilon^6\int_{\partial\Omega}|\mathbf{g}_\varepsilon|^2\mathbf{g}_\varepsilon\cdot\mathbf{n}\,\mathrm{d}S_x\\ &\quad+\operatorname{Re}\varepsilon^6\int_{\Gamma_\varepsilon}\Big|\mathbf{G}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\Big|^2\mathbf{G}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{n}_\varepsilon\,\mathrm{d}S_x\\ &\leqslant C\operatorname{Re}\varepsilon^6.\end{aligned}\tag{3.15}$$

Under the condition $\lim\limits_{\varepsilon\to0}\operatorname{Re}\varepsilon^{3-\delta} = 0$ we have proved (3.10). Then the Poincaré inequality implies (3.11) and the Nečas inequality implies (3.12) (see e.g. [1], [13] or [14] for details). □

4. Convergence of the homogenization process

As usual, we extend the velocity to Ω by zero and the pressure by its mean value in each cell:

$$\widetilde{\mathbf{u}}^\varepsilon = \begin{cases}\mathbf{u}^\varepsilon & \text{in } \Omega_\varepsilon,\\ \mathbf{0} & \text{in } A_\varepsilon,\end{cases}\qquad \widetilde{p}^\varepsilon = \begin{cases}p^\varepsilon & \text{in } \Omega_\varepsilon,\\ \dfrac{1}{|Y_k^\varepsilon|}\displaystyle\int_{Y_k^\varepsilon}p^\varepsilon & \text{in } A_\varepsilon.\end{cases}\tag{4.1}$$

In the sequel we drop the "$\sim$" sign and denote the extensions simply by the same symbols $(\mathbf{u}^\varepsilon, p^\varepsilon)$. Those extensions satisfy the same a priori estimates. See [11] for details of the pressure extension. In what follows, the functions $(\mathbf{u}_\varepsilon, p_\varepsilon)$ are defined on the whole Ω and

$$\varepsilon^{-2}|\mathbf{u}_\varepsilon|_{L^2(\Omega)} \leqslant C,\quad |p_\varepsilon|_{L^2(\Omega)} \leqslant C.\tag{4.2}$$

Theorem 4.1. *Let $(\mathbf{u}_\varepsilon, p_\varepsilon)$ be the solution to the Navier-Stokes problem* (2.1) *in the porous domain Ω_ε. Then their extensions on whole Ω, given by* (4.1), *converge as*

$$\varepsilon^{-2}\mathbf{u}_\varepsilon \rightharpoonup \mathbf{v},\quad p_\varepsilon \rightharpoonup q \text{ weakly in } L^2(\Omega),$$

where $(\mathbf{v}, q)$ *is the unique solution of the Darcy problem*

$$\mathbf{v} = \mathbf{K}(\mathbf{F} - \nabla q), \quad \operatorname{div} \mathbf{v} = -\int_{\partial A} \mathbf{H}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}_y \, \mathrm{d}S_y \quad \textit{in } \Omega, \tag{4.3}$$

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} \quad \textit{on } \partial\Omega. \tag{4.4}$$

The permeability tensor $\mathbf{K}$ *is defined using the auxiliary problem* (4.5)*. The effective source term is* $\mathbf{F} = \mathbf{f} - \mathbf{K}^{-1}\mathbf{h}$ *with* $\mathbf{h} = (\mathcal{H}_1, \ldots, \mathcal{H}_n)$ *and*

$$\mathcal{H}_k(\mathbf{x}) = \int_{\partial A} \frac{\partial \mathbf{w}_k}{\partial \mathbf{n}}(\mathbf{y}) \cdot \mathbf{P}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}S_y.$$

The auxiliary function $\mathbf{w}_k$ *is the solution of the cell problem* (4.5)*.*

Proof. As our domain is complex, the main problem is the construction of the appropriate test function. Since it must have a zero trace on the boundary $\partial\Omega_\varepsilon$, it is rapidly oscillating. One simple way to construct such test function is to take $(\mathbf{w}_k(y), \pi_k(y))$, the solutions to the auxiliary problem

$$\begin{gathered} -\Delta_y \mathbf{w}_k + \nabla_y \pi_k = \mathbf{e}_k, \quad \operatorname{div}_y \mathbf{w}_k = 0 \quad \text{in } Y^*, \\ \mathbf{w}_k = 0 \quad \text{on } \partial A, \quad (\mathbf{w}_k, \pi_k) \text{ is } Y\text{-periodic}, \quad k = 1, \ldots, n. \end{gathered} \tag{4.5}$$

We pose

$$\mathbf{w}_k^\varepsilon(\mathbf{x}) = \mathbf{w}_k(\mathbf{x}/\varepsilon), \quad \pi_k^\varepsilon(\mathbf{x}) = \pi_k(\mathbf{x}/\varepsilon).$$

The oscillating functions $\mathbf{w}_k^\varepsilon$ take into account the zero trace on the grained boundary Γ_ε. To complete the test function choice, we need to take care of the exterior boundary condition. For that, we take $\varphi \in C_0^\infty(\Omega)$ and pose

$$\mathbf{w}_\varepsilon = \mathbf{w}_k^\varepsilon \varphi, \quad \pi_\varepsilon = \varepsilon^{-1} \pi_k^\varepsilon \varphi. \tag{4.6}$$

From the a priori estimates (4.2) we deduce that there exist some limits $\mathbf{v} \in H^1(\Omega)^n$ and $q \in L^2(\Omega)$ such that (at least for a subsequence)

$$\varepsilon^{-2}\mathbf{u}_\varepsilon \rightharpoonup \mathbf{v}, \quad p_\varepsilon \rightharpoonup q \quad \text{weakly in } L^2(\Omega).$$

Furthermore, there exists $Q(\mathbf{x}, \mathbf{y}) \in L^2(\Omega \times Y^*)$ such that

$$p_\varepsilon \to Q \quad \text{two-scale} \tag{4.7}$$

and $q(\mathbf{x}) = \int_{Y^*} Q(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{y}$ (see e.g. [2] for the two-scale compactness theorem). We can prove more for the pressure. Let $\mathbf{v} \in C_0^\infty(\Omega; C_{\mathrm{per}}^\infty(Y^*)^n)$ and $\mathbf{v}^\varepsilon(\mathbf{x}) = \mathbf{v}(\mathbf{x}, \mathbf{x}/\varepsilon)$, then (4.7) implies

$$\varepsilon \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \mathbf{v}^\varepsilon \to \int_\Omega \int_{Y^*} Q(\mathbf{x}, \mathbf{y}) \operatorname{div}_y \mathbf{v}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x}.$$

On the other hand

$$\begin{aligned}
\varepsilon \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \mathbf{v}^\varepsilon &= \varepsilon \int_{\Omega_\varepsilon} \nabla \mathbf{u}_\varepsilon \cdot \nabla \mathbf{v}^\varepsilon + \varepsilon \operatorname{Re} \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot \mathbf{v}^\varepsilon - \varepsilon \int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{v}^\varepsilon \\
&\leqslant \varepsilon |\nabla \mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)} (|\nabla \mathbf{v}^\varepsilon|_{L^2(\Omega_\varepsilon)} + \operatorname{Re} |\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)} |\mathbf{v}^\varepsilon|_{L^\infty(\Omega_\varepsilon)}) \\
&\quad + \varepsilon |\mathbf{v}^\varepsilon|_{L^2(\Omega_\varepsilon)} |\mathbf{f}|_{L^2(\Omega_\varepsilon)} \\
&\leqslant C(\varepsilon + \operatorname{Re} \varepsilon^3) \to 0.
\end{aligned}$$

Thus

$$\int_\Omega \int_{Y^*} Q(\mathbf{x}, \mathbf{y}) \operatorname{div}_y \mathbf{v}(\mathbf{x}, \mathbf{y}) \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{x} = 0 \Rightarrow Q(\mathbf{x}, \mathbf{y}) = q(\mathbf{x}).$$

Therefore

$$p_\varepsilon \to q(\mathbf{x}) \quad \text{two-scale.} \tag{4.8}$$

To derive the effective equations we start from the very weak formulation (2.9) with the test functions $(\mathbf{w}_\varepsilon, \pi_\varepsilon)$ given by (4.6),

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot (-\Delta \mathbf{w}_\varepsilon + \nabla \pi_\varepsilon) + \operatorname{Re} \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div} \mathbf{w}_\varepsilon \\
&\quad = \int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{w}_\varepsilon + \varepsilon^2 \int_{\partial\Omega} \mathbf{g}_\varepsilon \cdot \Big(-\frac{\partial \mathbf{w}_\varepsilon}{\partial \mathbf{n}} + \pi_\varepsilon \mathbf{n} \Big) + \varepsilon^2 \int_{\Gamma_\varepsilon} \mathbf{G}\Big(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \cdot \Big(-\frac{\partial \mathbf{w}_\varepsilon}{\partial \mathbf{n}_\varepsilon} + \pi_\varepsilon \mathbf{n}_\varepsilon \Big).
\end{aligned} \tag{4.9}$$

We pass to the limit, term-by-term, in the above very weak formulation. Obviously

$$\int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{w}_k^\varepsilon \varphi \to \int_\Omega \mathbf{f} \cdot \langle \mathbf{w}_k \rangle \varphi \tag{4.10}$$

$$\begin{aligned}
\operatorname{Re} \left| \int_{\Omega_\varepsilon} (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \cdot \mathbf{w}_k^\varepsilon \varphi \right| &\leqslant \operatorname{Re} |\nabla \mathbf{u}_\varepsilon|_{L^2(\Omega)} |\mathbf{u}_\varepsilon|_{L^2(\Omega)} |\mathbf{w}^\varepsilon|_{L^\infty(\Omega)} |\varphi|_{L^\infty(\Omega)} \\
&\leqslant C \varepsilon^3 \operatorname{Re} \to 0.
\end{aligned} \tag{4.11}$$

The two-scale convergence of the pressure (4.8) implies

$$\int_{\Omega_\varepsilon} p_\varepsilon \mathbf{w}_k^\varepsilon \cdot \nabla \varphi \to \int_\Omega q \langle \mathbf{w}_k \rangle \cdot \nabla \varphi. \tag{4.12}$$

Next we concentrate on the term

$$\begin{aligned}
\int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot (-\Delta \mathbf{w}_\varepsilon + \nabla \pi_\varepsilon) &= \varepsilon^{-2} \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot (-\Delta_y \mathbf{w}_k^\varepsilon + \nabla_y \pi_k^\varepsilon) \varphi - 2\varepsilon^{-1} \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_y \mathbf{w}_k^\varepsilon \nabla \varphi \\
&\quad - \int_{\Omega_\varepsilon} \mathbf{u}_\varepsilon \cdot \mathbf{w}_k^\varepsilon \Delta \varphi + \varepsilon^{-1} \int_{\Omega_\varepsilon} \pi_k^\varepsilon \mathbf{u}^\varepsilon \cdot \nabla \varphi.
\end{aligned}$$

The first part is the most important,

$$\varepsilon^{-2}\int_{\Omega_\varepsilon}\mathbf{u}_\varepsilon\cdot(-\Delta_y\mathbf{w}_k^\varepsilon+\nabla_y\pi_k^\varepsilon)\varphi=\varepsilon^{-2}\int_{\Omega_\varepsilon}\mathbf{e}_k\cdot\mathbf{u}_\varepsilon\varphi\to\int_\Omega v_k\varphi.$$

For the remaining terms

$$\begin{aligned}&\left|-2\varepsilon^{-1}\int_{\Omega_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla_y\mathbf{w}_k^\varepsilon\nabla\varphi-\int_{\Omega_\varepsilon}\mathbf{u}_\varepsilon\cdot\mathbf{w}_k^\varepsilon\Delta\varphi+\varepsilon^{-1}\int_{\Omega_\varepsilon}\pi_k^\varepsilon\mathbf{u}^\varepsilon\cdot\nabla\varphi\right|\\&\qquad\leqslant C\varepsilon^{-1}|\varphi|_{H^2(\Omega)}|\mathbf{u}_\varepsilon|_{L^2(\Omega_\varepsilon)}\leqslant C\varepsilon\to 0.\end{aligned}$$

Thus

$$\int_{\Omega_\varepsilon}\mathbf{u}_\varepsilon\cdot(-\Delta\mathbf{w}_\varepsilon+\nabla\pi_\varepsilon)\to\int_\Omega v_k\varphi. \tag{4.13}$$

The first of the two boundary integrals, the one on the exterior boundary, equals

$$\left|\varepsilon^2\int_{\partial\Omega}\mathbf{g}_\varepsilon\cdot\Big(-\frac{\partial\mathbf{w}_\varepsilon}{\partial\mathbf{n}}+\pi_\varepsilon\mathbf{n}\Big)\right|\leqslant C\varepsilon|\mathbf{g}_\varepsilon|_{L^2(\Gamma)}\leqslant C\varepsilon\to 0. \tag{4.14}$$

Finally, the integral on the grained boundary gives an unusual new source term. We have

$$\varepsilon^2\int_{\Gamma_\varepsilon}\mathbf{G}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\Big(-\frac{\partial\mathbf{w}_\varepsilon}{\partial\mathbf{n}_\varepsilon}+\pi_\varepsilon\mathbf{n}_\varepsilon\Big)\to 0.$$

First of all,

$$\begin{aligned}&\left|\varepsilon^3\int_{\Gamma_\varepsilon}\mathbf{H}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\Big(-\frac{\partial\mathbf{w}_\varepsilon}{\partial\mathbf{n}_\varepsilon}+\pi_\varepsilon\mathbf{n}_\varepsilon\Big)\right|\\&\quad=\left|\varepsilon^2\int_{\Gamma_\varepsilon}(-\nabla_y\mathbf{w}_k^\varepsilon\cdot\mathbf{n}_\varepsilon+\pi_k^\varepsilon\mathbf{n}_\varepsilon)\cdot\mathbf{H}(\mathbf{x},\mathbf{x}/\varepsilon)\varphi+\varepsilon^3\int_{\Gamma_\varepsilon}\mathbf{H}(\mathbf{x},\mathbf{x}/\varepsilon)\cdot\mathbf{w}_k^\varepsilon\frac{\partial\varphi}{\partial\mathbf{n}_\varepsilon}\right|\\&\quad\leqslant C\varepsilon^2|\Gamma_\varepsilon|\leqslant C\varepsilon\to 0.\end{aligned}$$

Since $\mathbf{P}\cdot\mathbf{n}_\varepsilon=0$, it remains to compute

$$\varepsilon^2\int_{\Gamma_\varepsilon}\mathbf{P}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\frac{\partial\mathbf{w}_\varepsilon}{\partial\mathbf{n}_\varepsilon}.$$

We proceed as follows:

$$\varepsilon\int_{\Gamma_\varepsilon}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}\cdot\mathbf{P}\varphi\,\mathrm{d}S_x=\varepsilon\sum_{\mathbf{m}\in K_\varepsilon}\int_{\varepsilon(\mathbf{m}+\partial A)}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}\cdot\mathbf{P}\varphi\,\mathrm{d}S_x$$

with

$$K_\varepsilon=\{\mathbf{m}\in\mathbf{Z}^n\colon\ \varepsilon(\mathbf{m}+Y^*)\subset\Omega\}.$$

Since P and φ are smooth, we have

$$|\mathbf{P}(\varepsilon\mathbf{m}+\varepsilon\mathbf{y},\mathbf{y})-\mathbf{P}(\varepsilon\mathbf{m},\mathbf{y})|\leqslant C\varepsilon|\mathbf{y}|\leqslant C\varepsilon$$

and

$$|\varphi(\varepsilon\mathbf{m}+\varepsilon\mathbf{y})-\varphi(\varepsilon\mathbf{m})|\leqslant C\varepsilon|\mathbf{y}|\leqslant C\varepsilon$$

uniformly. Thus

$$\begin{aligned}\varepsilon\int_{\varepsilon(\mathbf{m}+\partial A)}&\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}\Big(\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{P}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\varphi(\mathbf{x})\,\mathrm{d}S_x\\&=\varepsilon^n\varphi(\varepsilon\mathbf{m}+\varepsilon\mathbf{y})\int_{\partial A}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}(\mathbf{y})\cdot\mathbf{P}(\varepsilon\mathbf{m}+\varepsilon\mathbf{y},\mathbf{y})\,\mathrm{d}S_y\\&=\varepsilon^n\varphi(\varepsilon\mathbf{m})\int_{\partial A}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}(\mathbf{y})\cdot\mathbf{P}(\varepsilon\mathbf{m},\mathbf{y})\,\mathrm{d}S_y+O(\varepsilon^{n+1}),\end{aligned}$$

so that

$$\begin{aligned}\varepsilon\int_{\Gamma_\varepsilon}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}\cdot\mathbf{P}\varphi\,\mathrm{d}S_y&=\sum_{\mathbf{m}\in K_\varepsilon}\varepsilon^n\varphi(\varepsilon\mathbf{m}+\varepsilon\mathbf{y})\int_{\partial A}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}(\mathbf{y})\cdot\mathbf{P}(\varepsilon\mathbf{m}+\varepsilon\mathbf{y},\mathbf{y})\,\mathrm{d}S_y\\&=\sum_{\mathbf{m}\in K_\varepsilon}\varepsilon^n\varphi(\varepsilon\mathbf{m})\int_{\partial A}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}(\mathbf{y})\cdot\mathbf{P}(\varepsilon\mathbf{m},\mathbf{y})\,\mathrm{d}S_y+O(\varepsilon)\\&=\sum_{\mathbf{m}\in K_\varepsilon}\varepsilon^n\varphi(\varepsilon\mathbf{m})\mathcal{H}_k(\varepsilon\mathbf{m})+O(\varepsilon),\end{aligned}$$

where

$$\mathcal{H}_k(\mathbf{x})=\int_{\partial A}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}(\mathbf{y})\cdot\mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}S_y.$$

The sum

$$\sum_{\mathbf{m}\in K_\varepsilon}\varepsilon^n\varphi(\varepsilon\mathbf{m})\mathcal{H}_k(\varepsilon\mathbf{m})$$

is the Riemann integral sum (for equidistant partition) of the integral

$$\int_\Omega\varphi(\mathbf{x})\mathcal{H}_k(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_\Omega\varphi(\mathbf{x})\int_{\partial A}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}(\mathbf{y})\cdot\mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}S_y\,\mathrm{d}\mathbf{x}.$$

Thus, as $\varepsilon\to0$,

$$\varepsilon\int_{\Gamma_\varepsilon}\frac{\partial\mathbf{w}_k}{\partial\mathbf{n}}\Big(\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{P}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\varphi(\mathbf{x})\,\mathrm{d}S_x\to\int_\Omega\varphi(\mathbf{x})\mathcal{H}_k(\mathbf{x})\,\mathrm{d}\mathbf{x}.\tag{4.15}$$

In (4.10)–(4.15) we have found the limits of all the integrals in the very weak formulation. Summing up all those limits leads to

$$\int_\Omega v^k\varphi - \sum_{j=1}^n \int_\Omega q\langle w_k^j\rangle \frac{\partial\varphi}{\partial x_j} = \int_\Omega \Big(\sum_{j=1}^n \langle w_k^j\rangle f^j - \mathcal{H}_k\Big)\varphi, \tag{4.16}$$

where

$$\langle w\rangle = \int_{Y^*} w(y)\,\mathrm{d}y.$$

Denoting by

$$\mathbf{K} = [K_{ij}] \quad \text{with } K_{ij} = \langle w_i^j\rangle = \int_{Y^*} w_i^j(y)\,\mathrm{d}y = \int_{Y^*} \nabla\mathbf{w}^j\cdot\nabla\mathbf{w}^i\,\mathrm{d}y$$

the strictly positive and symmetric permeability tensor,

$$\mathbf{h} = (\mathcal{H}_1,\ldots,\mathcal{H}_n)$$

and

$$F^j = f^j - (\mathbf{K}^{-1}\mathbf{h})_j \tag{4.17}$$

we get

$$v^k = \sum_{j=1}^n K_{kj}\Big(F^j - \frac{\partial q}{\partial x_j}\Big) \quad \text{or} \quad \mathbf{v} = \mathbf{K}(\mathbf{F}-\nabla q).$$

Furthermore, for any smooth scalar function ψ

$$\int_\Gamma \mathbf{g}_\varepsilon\cdot\mathbf{n}\psi + \varepsilon\int_{\Gamma_\varepsilon} \mathbf{H}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{n}_\varepsilon\psi = \varepsilon^{-2}\int_\Omega \mathbf{u}_\varepsilon\cdot\nabla\psi \to \int_\Omega \mathbf{v}\cdot\nabla\psi.$$

Using (2.3) and the same steps as in (4.15), we get

$$\varepsilon\int_{\Gamma_\varepsilon} \mathbf{H}\Big(\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big)\cdot\mathbf{n}_\varepsilon\psi \to \int_\Omega \psi(\mathbf{x})\int_{\partial A} \mathbf{H}(\mathbf{x},\mathbf{y})\cdot\mathbf{n}_y\,\mathrm{d}S_y\,\mathrm{d}\mathbf{x},$$

implying that

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n}\psi + \int_\Omega \psi(\mathbf{x})\int_{\partial A} \mathbf{H}(\mathbf{x},\mathbf{y})\cdot\mathbf{n}_y\,\mathrm{d}S_y\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{v}\cdot\nabla\psi.$$

Thus

$$\operatorname{div}\mathbf{v} + \int_{\partial A} \mathbf{H}(\mathbf{x},\mathbf{y})\cdot\mathbf{n}_y\,\mathrm{d}S_y = 0 \quad \text{in } \Omega, \quad \mathbf{v}\cdot\mathbf{n} = \mathbf{g}\cdot\mathbf{n} \quad \text{on } \partial\Omega.$$

The problem obtained is well posed due to the condition (2.6).

Since (4.4) has a unique solution, there is only one accumulation point for $(\mathbf{u}_\varepsilon, p_\varepsilon)$, so that the whole sequence converges, not only the subsequence. $\square$

Remark 4.1. Testing the auxiliary problem (4.5) with $\mathbf{P}$, we get

$$\int_{Y^*} \nabla_y \mathbf{w}_k(\mathbf{y}) \cdot \nabla_y \mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{\partial A} \frac{\partial \mathbf{w}_k}{\partial \mathbf{n}}(\mathbf{y}) \cdot \mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}S_y + \int_{Y^*} e_k \cdot \mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$$

so that

$$\mathcal{H}_k(\mathbf{x}) = -\int_{Y^*} \nabla_y \mathbf{w}_k(\mathbf{y}) \cdot \nabla_y \mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_{Y^*} e_k \cdot \mathbf{P}(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

It is important to notice that $\mathcal{H}_k$ depends only on the boundary values of $\mathbf{P}$, which are given. It should not depend on the way we have extended $\mathbf{P}$ on the whole domain, as this extension is not unique.

Remark 4.2. It is worth noticing that the only place where we used the periodicity in the convergence proof, is the periodicity lemma, i.e., the fact that

$$\mathbf{w}_k(x/\varepsilon) \rightharpoonup \langle \mathbf{w}_k \rangle \text{ weakly in } L^2(\Omega).$$

So, assuming that the auxiliary problem

$$\begin{gathered} -\varepsilon^2 \Delta \mathbf{w}_k^\varepsilon + \varepsilon \nabla \pi_k^\varepsilon = \mathbf{e}_k, \quad \operatorname{div} \mathbf{w}_k^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ \mathbf{w}_k^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \quad k = 1, \ldots, n, \end{gathered} \tag{4.18}$$

admits a solution such that $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{M}_k$ weakly in $L^2(\Omega)$, it would be enough to prove that $\mathbf{v} = \mathbf{K}(\mathbf{F} - \nabla q)$ with $K_{ij} = \mathbf{M}_i \cdot \mathbf{e}_j$.

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