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MATCHINGS IN COMPLETE BIPARTITE GRAPHS AND THE r-LAH NUMBERS

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Abstract. We give a graph theoretic interpretation of r-Lah numbers, namely, we show that the r-Lah number $\left\lfloor {n \atop k} \right\rfloor_r$ counting the number of r-partitions of an $(n+r)$-element set into $k+r$ ordered blocks is just equal to the number of matchings consisting of $n-k$ edges in the complete bipartite graph with partite sets of cardinality n and $n+2r-1$ ($0 \leqslant k \leqslant n$, $r \geqslant 1$). We present five independent proofs including a direct, bijective one. Finally, we close our work with a similar result for r-Stirling numbers of the second kind.

Keywords: r-Lah number; number of matchings; complete bipartite graph; r-Stirling number of the second kind

MSC 2020: 05C70, 05C31, 05A19, 11B73

1. The r-Lah numbers

In the middle 1950s, Lah wrote two papers (see [11], [12]), one in actuarial mathematics and another in mathematical statistics, in which he defined the numbers named after him as the coefficients in the equation connecting rising and falling factorials. Namely, $x^{\bar{n}} = \sum\limits_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor x^{\underline{k}}$ ($n \geqslant 0$), where $x^{\bar{n}} = \prod\limits_{i=0}^{n-1}(x+i)$ and $x^{\underline{k}} = \prod\limits_{i=0}^{k-1}(x-i)$. Combinatorially, the Lah number $\left\lfloor {n \atop k} \right\rfloor$ counts the number of partitions of an n-element set into k ordered blocks ($0 \leqslant k \leqslant n$), which means that the order of the elements in each block matters.

Nyul and Rácz in [18] considered the r-generalization of Lah numbers and proved many identities and properties. We note that these numbers are close relatives of

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r-Stirling numbers of the second kind, which will be discussed in a later section. To define these numbers, we need the following notion: A partition of a set with at least r elements is called an r-*partition* if r distinguished elements belong to distinct blocks. Then, the r-Lah number $\left\lfloor {n \atop k} \right\rfloor_r$ gives the number of r-partitions of an $(n+r)$-element set into $k+r$ ordered blocks ($0 \leqslant k \leqslant n$, $r \geqslant 0$). Obviously, $\left\lfloor {n \atop k} \right\rfloor_0 = \left\lfloor {n \atop k} \right\rfloor$ and $\left\lfloor {n \atop k} \right\rfloor_1 = \left\lfloor {n+1 \atop k+1} \right\rfloor$.

Belbachir, Belkhir in [1], Belbachir, Bousbaa in [2], and Shattuck in [22] derived some additional identities for r-Lah numbers, while Mihoubi and Rahmani in [17] encountered them as the values of special substitutions into the so-called partial r-Bell polynomials.

Here, we stop for a moment to prove a second-order recurrence for r-Lah numbers in which the coefficients are independent of the lower parameter. This proposition is closely related to Theorem 3.5 in [19], but it seems to be missing from the current list of r-Lah number identities.

Proposition 1.1. *If* $n \geqslant 2$, $1 \leqslant k \leqslant n-1$ *and* $r \geqslant 0$, *then*

$$\left\lfloor {n+1 \atop k} \right\rfloor_r = \left\lfloor {n \atop k-1} \right\rfloor_r + (2n+2r)\left\lfloor {n \atop k} \right\rfloor_r - n(n+2r-1)\left\lfloor {n-1 \atop k} \right\rfloor_r.$$

P r o o f. In order to enumerate the r-partitions of an $(n+r+1)$-element set into $k+r$ ordered blocks, we choose a non-distinguished element. There exist $\left\lfloor {n \atop k-1} \right\rfloor_r$ such partitions if this element forms a singleton. Otherwise, after r-partitioning the other $n+r$ elements into $k+r$ ordered blocks, we can put the chosen element in $2n+2r$ places, before or after any element. This would give $(2n+2r)\left\lfloor {n \atop k} \right\rfloor_r$ possibilities, but we have to subtract the number of the cases counted twice, namely, when the chosen element is inserted between two other elements.

These duplicated cases can be of two types depending on the previous element to our chosen element. On the one hand, if the previous element is one of the n non-distinguished elements, then this ordered pair can be positioned before one of the remaining $n+r-1$ elements in any of their r-partitions into $k+r$ ordered blocks, hence the number of duplicated cases of the first type is $n(n+r-1)\left\lfloor {n-1 \atop k} \right\rfloor_r$. On the other hand, if the previous element is distinguished, then the element following the chosen one has to be among the n non-distinguished elements, and the ordered pair of the chosen and this latter element certainly goes after any of the r distinguished elements, thus the number of duplicated possibilities of the second type is similarly $nr\left\lfloor {n-1 \atop k} \right\rfloor_r$. □

In the last few years, further generalizations and several variants of r-Lah numbers have been studied. Shattuck in [23] and [25] examined q-analogues of r-Lah numbers and restricted r-Lah numbers, while Schlosser and Yoo in [21] treated elliptic r-Lah numbers. We mention another generalization of r-Lah numbers by Shattuck, see [24]. The curiosity of them is that they involve not only r-Lah numbers, but also r-Stirling numbers of both kinds.

Cheon and Jung in [5] defined r-Whitney-Lah numbers. They were extensively studied by Gyimesi and Nyul (see [10]) through a newly given combinatorial interpretation. See also [6], [16], [20] as further references.

Finally, we recall the definition of r-Lah polynomials $L_{n,r}(x) = \sum\limits_{k=0}^{n} \left\lfloor{n \atop k}\right\rfloor_r x^k$, which were systematically studied by Nyul and Rácz, see [19]. Their counterparts using the above mentioned r-Whitney-Lah numbers as coefficients, the r-Dowling-Lah polynomials, were investigated by Gyimesi, see [8].

In this paper, we provide a graph theoretic interpretation of r-Lah numbers. We offer five proofs, from which the last one is the most important, since it is a direct proof based on a bijective correspondence between r-partitions of a finite set into ordered blocks and matchings in a complete bipartite graph. It turns out that this interpretation of r-Lah numbers works even if r is a half-integer. Finally, we show a similar result for r-Stirling numbers of the second kind.

2. Main result

In order to state and prove our main theorem, we have to introduce the following notation.

Whenever we say that a complete bipartite graph $K_{m,n}$ has partite sets A and B, then we always assume that $|A| = m$ and $|B| = n$ in this order.

Simply counting matchings in complete bipartite graphs is not a difficult task, see, e.g., [7], but our purpose is to connect this question directly to r-Lah numbers.

For $n, r \geqslant 1$ and $0 \leqslant k \leqslant n$, denote by $l_r(n,k)$ the number of $(n-k)$-element matchings in the complete bipartite graph $K_{n,n+r-1}$. By convention, we allow the empty matching as the only 0-element matching. In the degenerate case $n = 0$, this graph becomes an edgeless graph, therefore $l_r(0,0) = 1$.

We immediately obtain the following special values by simple arguments:

- ▷ $l_r(n,0) = r^{\bar{n}}$,
- ▷ $l_r(n,1) = (r+1)^{\bar{n}} - r^{\bar{n}}$ $(n \geqslant 1)$,
- ▷ $l_r(n,n-1) = n(n+r-1)$ $(n \geqslant 1)$,
- ▷ $l_r(n,n) = 1$.

Using these numbers as coefficients, for $n \geqslant 0$ and $r \geqslant 1$, we can define the polynomial

$$\mathcal{L}_{n,r}(x) = \sum_{k=0}^{n} l_r(n,k)x^k,$$

which is the reciprocal polynomial of the matching generating polynomial of the complete bipartite graph $K_{n,n+r-1}$.

Now, we are in position to formulate our main result, which states that r-Lah numbers count matchings in those complete bipartite graphs where the difference of the cardinalities of the partite sets is $2r-1$.

Theorem 2.1. *If* $0 \leqslant k \leqslant n$ *and* $r \geqslant 1$, *then*

$$l_{2r}(n,k) = \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r, \quad \mathcal{L}_{n,2r}(x) = L_{n,r}(x).$$

Remark 2.1. The theorem holds even for $r = 0$ if we assume $1 \leqslant k \leqslant n$, and the proofs also work, sometimes with slight modifications. However, the condition $r \geqslant 1$ is mathematically non-restrictive, since $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_0 = \left\lfloor \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rfloor_1$ and the underlying bipartite graphs are isomorphic. Therefore, we keep it in the rest of the paper.

3. Proofs

In this section, we present five independent proofs of Theorem 2.1. Four of them use the properties of the numbers $l_r(n,k)$ and the polynomials $\mathcal{L}_{n,r}(x)$, namely, two different types of recurrences, a polynomial identity between shifted rising and falling factorials, and an explicit formula, respectively. But one can ask for a direct proof of such a simple-looking formula. So finally, the last (not the shortest, but maybe the nicest) one is a bijective proof, which we think brings new insight and understanding to this interesting connection.

3.1. Proof 1.

Proposition 3.1. *If* $n \geqslant 2$, $1 \leqslant k \leqslant n-1$ *and* $r \geqslant 1$, *then*

$$l_r(n+1,k) = l_r(n,k-1) + (2n+r)l_r(n,k) - n(n+r-1)l_r(n-1,k),$$

and if $n, r \geqslant 1$, *then*

$$\mathcal{L}_{n+1,r}(x) = (x+2n+r)\mathcal{L}_{n,r}(x) - n(n+r-1)\mathcal{L}_{n-1,r}(x).$$

P r o o f. We enumerate the $(n-k+1)$-element matchings in the complete bipartite graph $K_{n+1,n+r}$ with partite sets A and B. Let $v \in A$ and $w \in B$ be two vertices.

If both v and w are unmatched, then we actually have a matching in $K_{n,n+r-1}$ obtained by deleting v and w, hence their number is $l_r(n,k-1)$.

If v is matched, then its pair can be any of the $n+r$ vertices from B, and we have to choose an $(n-k)$-element matching in $K_{n,n+r-1}$ after deletion of v and its pair, thus we have $(n+r)l_r(n,k)$ such matchings. Similarly, we have $(n+1)l_r(n,k)$ matchings when w is matched. But, of course, we counted the cases where both v and w are matched twice, thus the number of these cases has to be subtracted.

Obviously, there exist $l_r(n,k)$ matchings if v and w are the pairs of each other. Otherwise, the pairs of v and w can be chosen in $n+r-1$ and n ways, respectively, and we need to find additionally an $(n-k-1)$-element matching in $K_{n-1,n+r-2}$ after deletion of these four vertices. Hence, the number of these latter possibilities is $n(n+r-1)l_r(n-1,k)$. $\square$

Theorem 2.1 follows by comparing Proposition 3.1 with Proposition 1.1 (see also [19], Theorem 3.5), together with the special values $l_{2r}(n,0) = (2r)^{\bar{n}} = \left\lfloor {n \atop 0} \right\rfloor_r$, $l_{2r}(n,n-1) = n(n+2r-1) = \left\lfloor {n \atop n-1} \right\rfloor_r$ and $l_{2r}(n,n) = 1 = \left\lfloor {n \atop n} \right\rfloor_r$.

3.2. Proof 2.

Proposition 3.2. *If* $1 \leqslant k \leqslant n$ *and* $r \geqslant 1$*, then*

$$l_r(n+1,k) = l_r(n,k-1) + (n+k+r)l_r(n,k),$$
$$\mathcal{L}_{n+1,r}(x) = x\mathcal{L}'_{n,r}(x) + (x+n+r)\mathcal{L}_{n,r}(x).$$

P r o o f. We enumerate again the $(n-k+1)$-element matchings in the complete bipartite graph $K_{n+1,n+r}$ with partite sets A and B. Let $v \in A$ and $w \in B$ be two vertices.

As we saw in the previous proof, we have $l_r(n,k-1)$ matchings if both v and w are unmatched, while $(n+r)l_r(n,k)$ matchings if v is matched. Finally, if v is unmatched, but w is matched, then first we pick an $(n-k)$-element matching in $K_{n,n+r-1}$ obtained by deleting v and w, thereafter the pair of w can be chosen from A in k ways, which gives $kl_r(n,k)$ additional possibilities. $\square$

Theorem 2.1 follows by comparing Proposition 3.2 with Theorem 3.1 in [18], together with the special values $l_{2r}(n,0) = (2r)^{\bar{n}} = \left\lfloor {n \atop 0} \right\rfloor_r$ and $l_{2r}(n,n) = 1 = \left\lfloor {n \atop n} \right\rfloor_r$.

3.3. Proof 3.

Proposition 3.3. *If* $n \geqslant 0$ *and* $r \geqslant 1$, *then*

$$(x+r)^{\bar{n}} = \sum_{k=0}^{n} l_r(n,k) x^{\underline{k}}.$$

P r o o f. We may assume that $n \geqslant 1$. Consider the complete bipartite graph $K_{n,n+r-1}$ with partite sets A and B, and extend B with m new vertices ($m \geqslant n$). The number of the matchings in the extended graph $K_{n,m+n+r-1}$ which cover A is simply $(m+n+r-1)^{\underline{n}} = (m+r)^{\bar{n}}$.

Now, we count these matchings in another way. Denote by k the number of vertices in A whose pair is one of the new vertices ($k = 0, \ldots, n$). We begin with an $(n-k)$-element matching in the original graph, and then we choose the pairs of the remaining k vertices in A from the m new vertices. This gives $l_r(n,k)m^{\underline{k}}$ possibilities for a fixed k.

Therefore, $(m+r)^{\bar{n}} = \sum\limits_{k=0}^{n} l_r(n,k)m^{\underline{k}}$ holds for all $m \geqslant n$, which completes the proof. □

Theorem 2.1 follows by comparing Proposition 3.3 with Theorem 3.2 in [18].

3.4. Proof 4.

Proposition 3.4. *If* $0 \leqslant k \leqslant n$ *and* $r \geqslant 1$, *then*

$$l_r(n,k) = \frac{n!}{k!}\binom{n+r-1}{k+r-1}.$$

P r o o f. We may assume that $n \geqslant 1$. Consider the complete bipartite graph $K_{n,n+r-1}$ with partite sets A and B. To find the number of $(n-k)$-element matchings in this graph, first we choose the $n-k$ matched vertices in B, then find their pairs from A, whence the result is

$$\binom{n+r-1}{n-k} n^{\underline{n-k}} = \frac{n!}{k!}\binom{n+r-1}{k+r-1}.$$

□

Theorem 2.1 follows by comparing Proposition 3.4 with Theorem 3.7 in [18].

3.5. Proof 5. Last but not least, we present a fifth proof which is based on a direct, bijective assignment.

Let $X = \{x_1, \ldots, x_n, y_1, \ldots, y_r\}$ be an $(n + r)$-element set with distinguished elements $y_1, \ldots, y_r$. Further, consider the complete bipartite graph $K_{n,n+2r-1}$ with partite sets $A = \{x_1, \ldots, x_n\}$ and $B = \{\overleftarrow{y_1}, \overrightarrow{y_1}, \ldots, \overleftarrow{y_r}, \overrightarrow{y_r}, 1, \ldots, n-1\}$. Then, we provide a bijection between the r-partitions of X into $k + r$ ordered blocks and the $(n - k)$-element matchings in the complete bipartite graph $K_{n,n+2r-1}$, as follows.

Pick an r-partition of X into $k + r$ ordered blocks. First we write the ordered blocks containing the distinguished elements $y_1, \ldots, y_r$ in this order, thereafter we arrange the other ordered blocks in increasing order of the index of their first elements.

If some x_i is the leading element of an ordered block containing no distinguished element, then x_i, as a vertex from A, is unmatched. If x_i is the first element of the ordered block containing y_j, then x_i is matched with $\overleftarrow{y_j}$, while if x_i is the element just after y_j in its ordered block, then x_i is matched with $\overrightarrow{y_j}$. If there are no elements before/after y_j in its ordered block, then $\overleftarrow{y_j}/\overrightarrow{y_j}$ is unmatched. For any other x_i, we count the non-distinguished elements which stand before x_i (in its own ordered block or in a preceding ordered block) and match x_i with this number between 1 and $n-1$ as a vertex from B.

Obviously, the result of this construction is a matching in the graph $K_{n,n+2r-1}$. Since the number of unmatched vertices from A is k, the number of ordered blocks containing no distinguished element, we have that the cardinality of the obtained matching is $n - k$.

In order to show that the above assignment is indeed bijective, we describe how to reconstruct the r-partition from a matching. Pick an $(n-k)$-element matching in the graph $K_{n,n+2r-1}$. First, we place $y_1, \ldots, y_r$ into distinct ordered blocks in this order. If $\overleftarrow{y_i}/\overrightarrow{y_i}$ is matched, which means that there is at least one non-distinguished element before/after y_i in its ordered block, then we put its pair before/after y_i into this ordered block, which will be the leading element/the element just after y_i. For an unmatched vertex from A, we open a new ordered block and place it there, where it will be the leading element. These new ordered blocks stand after the above ordered blocks, and we arrange them in increasing order of the index of the unmatched vertices. Finally, we consider the vertices $1, \ldots, n-1 \in B$ in this order. If some j is matched, then we put its pair directly after the previously placed jth non-distinguished element from the left.

This procedure gives an r-partition of X, because $y_1, \ldots, y_r$ trivially belong to distinct ordered blocks. On the other hand, there are k unmatched vertices in A, hence we opened k new ordered blocks beyond these blocks, and the number of ordered blocks is $k + r$. $\square$

Remark 3.1. To make this proof clearer, we illustrate the above assignment with an example. Let $n = 11$, $k = 2$ and $r = 3$. Then, the 3-partition

$$\{(x_8, x_5, y_1), (x_6, y_2, x_2, x_{11}, x_9), (y_3, x_4), (x_7), (x_{10}, x_1, x_3)\}$$

of the 14-element set $\{x_1, \ldots, x_{11}, y_1, y_2, y_3\}$ into 5 ordered blocks corresponds to the 9-element matching

$$\{\{x_8, \overleftarrow{y_1}\}, \{x_6, \overleftarrow{y_2}\}, \{x_2, \overrightarrow{y_2}\}, \{x_4, \overrightarrow{y_3}\}, \{x_5, 1\}, \{x_{11}, 4\}, \{x_9, 5\}, \{x_1, 9\}, \{x_3, 10\}\}$$

of the complete bipartite graph $K_{11,16}$ with partite sets $A = \{x_1, \ldots, x_{11}\}$ and $B = \{\overleftarrow{y_1}, \overrightarrow{y_1}, \overleftarrow{y_2}, \overrightarrow{y_2}, \overleftarrow{y_3}, \overrightarrow{y_3}, 1, \ldots, 10\}$.

4. Some other properties

In [18], we gave two different expressions of r-Lah numbers in terms of $(r-s)$-Lah numbers, but we provided a combinatorial proof only for Theorem 3.4 in [18], whereas we proved Theorem 3.5 in [18] by algebraic manipulations. Although it is possible to prove the latter result combinatorially using r-partitions into ordered blocks (see Appendix of the present paper), it is more straightforward to do so by the above graph theoretic interpretation of r-Lah numbers.

Proposition 4.1. *If $0 \leqslant k \leqslant n$ and $0 \leqslant s < r$, then*

$$l_r(n,k) = \sum_{j=k}^{n} l_{r-s}(n,j) \binom{j}{k} s^{\underline{j-k}}.$$

P r o o f. We may assume that $n \geqslant 1$. We enumerate the $(n-k)$-element matchings in the complete bipartite graph $K_{n,n+r-1}$ with partite sets A and B, where C is an s-element subset of B.

Denote by $j-k$ the number of matched vertices in A whose pair is in C ($j = k, \ldots, n$). To obtain such a matching, we begin with an $(n-j)$-element matching in $K_{n,n+r-s-1}$ with partite sets A and $B \setminus C$, then choose the other $j-k$ matched vertices from the j still unmatched vertices of A, and finally their pairs from C. Altogether, we have $l_{r-s}(n,j)\binom{j}{j-k}s^{\underline{j-k}}$ possibilities for a fixed j. □

We continue with four further recurrences, which obviously do not have counterparts for r-Lah numbers.

Proposition 4.2.

(1) *If* $1 \leqslant k \leqslant n$ *and* $r \geqslant 1$, *then*

$$l_r(n+1,k) = l_{r+1}(n,k-1) + (n+r)l_r(n,k),$$
$$\mathcal{L}_{n+1,r}(x) = x\mathcal{L}_{n,r+1}(x) + (n+r)\mathcal{L}_{n,r}(x).$$

(2) *If* $1 \leqslant k \leqslant n$ *and* $r \geqslant 1$, *then*

$$l_r(n+1,k) = l_{r+1}(n,k-1) + (k+r)l_{r+1}(n,k),$$
$$\mathcal{L}_{n+1,r}(x) = (x+r)\mathcal{L}_{n,r+1}(x) + x\mathcal{L}'_{n,r+1}(x).$$

(3) *If* $0 \leqslant k \leqslant n$ *and* $r \geqslant 2$, *then*

$$l_r(n+1,k) = l_{r-1}(n+1,k) + (n+1)l_r(n,k),$$
$$\mathcal{L}_{n+1,r}(x) = \mathcal{L}_{n+1,r-1}(x) + (n+1)\mathcal{L}_{n,r}(x).$$

(4) *If* $0 \leqslant k \leqslant n$ *and* $r \geqslant 2$, *then*

$$l_r(n+1,k) = l_{r-1}(n+1,k) + (k+1)l_{r-1}(n+1,k+1),$$
$$\mathcal{L}_{n+1,r}(x) = \mathcal{L}_{n+1,r-1}(x) + \mathcal{L}'_{n+1,r-1}(x).$$

Proof. We sketch the proof for the last identity by enumerating the $(n-k+1)$-element matchings in the complete bipartite graph $K_{n+1,n+r}$ with partite sets A and B. Let $w \in B$ be a vertex.

If w is unmatched, then we actually have a matching in $K_{n+1,n+r-1}$ after deletion of w. While, if w is matched, then first we choose an $(n-k)$-element matching in $K_{n+1,n+r-1}$, and there remain $k+1$ possibilities for the pair of w. $\square$

Remark 4.1. We note that Proposition 3.2 follows from Proposition 4.2 by combining the first and fourth identities.

We close this section with the observation that our main result immediately implies Theorem 3.8 in [19], whose proof was based on the counterpart of Proposition 3.2 and some mathematical analysis.

Proposition 4.3. *If* $n, r \geqslant 1$, *then all roots of* $\mathcal{L}_{n,r}(x)$ *are negative real numbers.*

Proof. It is well-known that all roots of the matching generating polynomial of a graph are negative real numbers (see, e.g., [14]), consequently the same holds for its reciprocal polynomial, in particular for $\mathcal{L}_{n,r}(x)$. $\square$

5. The r-Stirling numbers of the second kind

If the blocks in the definition of r-Lah numbers are unordered, then we obtain the r-Stirling numbers of the second kind defined independently by Carlitz (see [4]), Broder (see [3]) and Merris (see [15]). More precisely, $\left\{{n \atop k}\right\}_r$ counts the number of r-partitions of a set with $n+r$ elements into $k+r$ unordered nonempty subsets. We note that Gyimesi and Nyul (see [9]) gave an interpretation of r-Stirling numbers of the second kind in connection with the so-called combinatorial subspaces, which notion is of particular importance in Ramsey theory.

It is known that classical Stirling numbers of the second kind count the number of matchings in a special bipartite graph with partite sets of equal cardinalities, see, e.g., Problem 4.31 in [13], where the solution makes use of Hamiltonian paths in transitive tournaments. (We note that this is also mentioned in [14], but incorrectly.)

Here, we extend this result and provide a graph theoretic interpretation of r-Stirling numbers of the second kind, which is similar to Theorem 2.1, but the bijective proof can be given more easily.

Define the bipartite graph $G_{n,n+r-1}$ as the spanning subgraph of $K_{n,n+r-1}$ with partite sets $A = \{v_{r+1}, \ldots, v_{n+r}\}$ and $B = \{w_1, \ldots, w_{n+r-1}\}$, in which the vertices v_i and w_j are adjacent if and only if $i > j$ ($i = r+1, \ldots, n+r$, $j = 1, \ldots, n+r-1$).

Proposition 5.1. *If $0 \leqslant k \leqslant n$ and $r \geqslant 1$, then the number of $(n-k)$-element matchings in the graph $G_{n,n+r-1}$ is $\left\{{n \atop k}\right\}_r$.*

Proof. We construct a bijective mapping between the r-partitions of the $(n+r)$-element set $X = \{x_1, \ldots, x_{n+r}\}$ with distinguished elements $x_1, \ldots, x_r$ into $k+r$ unordered blocks and the $(n-k)$-element matchings in the graph $G_{n,n+r-1}$.

Pick an r-partition of X into $k+r$ blocks, where we write the elements in increasing order of their indices in each block. If x_i stands just after x_j in a block ($i > j$), then we choose the edge $\{v_i, w_j\}$, where $r+1 \leqslant i \leqslant n+r$ since two distinguished elements cannot be in the same block.

The number of unmatched vertices in A is k, because it is the number of blocks containing no distinguished element, and v_i is unmatched if and only if the non-distinguished element x_i is listed first in its block. This means that the chosen edges constitute a matching whose cardinality is $n-k$.

We can reconstruct the r-partition from an $(n-k)$-element matching, as follows. We go through the elements of B in increasing order of their indices. If the vertex w_j is matched with v_i and x_j is not listed yet, then open a new block beginning with x_j and x_i. If w_j is matched with v_i and x_j is already listed, then write x_i after x_j in its block. If w_j is unmatched and x_j is already listed, then its block is simply closed. If w_j is unmatched, but x_j has not appeared yet, then x_j forms a singleton. Finally, whenever x_{n+r} is written down, then that block is closed, otherwise, if it does not happen, then it forms a singleton.

By definition of the graph $G_{n,n+r-1}$, the distinguished elements $x_1, \dots, x_r$ belong to distinct blocks, hence we arrive at an r-partition of X. The number of blocks is $k+r$, the number of unmatched vertices in B plus 1 (because of x_{n+r}). □

Remark 5.1. Again, we illustrate the bijection in the proof with an example. Let $n=10$, $k=2$ and $r=3$. Then, the 3-partition

$$\{\{x_1,x_8,x_{11}\},\{x_2,x_5,x_9,x_{12}\},\{x_3,x_7\},\{x_4,x_6,x_{13}\},\{x_{10}\}\}$$

of the 13-element set $\{x_1,\dots,x_{13}\}$ into 5 unordered blocks corresponds to the 8-element matching

$$\{\{v_8,w_1\},\{v_{11},w_8\},\{v_5,w_2\},\{v_9,w_5\},\{v_{12},w_9\},\{v_7,w_3\},\{v_6,w_4\},\{v_{13},w_6\}\}$$

of the bipartite graph $G_{10,12}$ with partite sets $A=\{v_4,\dots,v_{13}\}$ and $B=\{w_1,\dots,w_{12}\}$.

6. Appendix

As we mentioned before Proposition 4.1, the proof of Theorem 3.5 in [18] did not use any combinatorial argument in that paper. We eliminate this lack based on the nice idea suggested by the referee.

For $0 \leqslant k \leqslant n$ and $0 \leqslant s \leqslant r$, we enumerate the $\left\lfloor {n \atop k} \right\rfloor_r$ r-partitions of an $(n+r)$-element set into $k+r$ ordered blocks. If we consider such a partition and delete the first s distinguished elements, then each of their ordered blocks splits into two ordered subsets. After omitting the possibly empty ordered sets, denote by j the number of ordered blocks which contain no distinguished element ($j=k,\dots,\min\{n,k+2s\}$).

For a fixed j, the n non-distinguished elements together with the last $r-s$ distinguished ones can be $(r-s)$-partitioned into $j+r-s$ ordered blocks in $\left\lfloor {n \atop j} \right\rfloor_{r-s}$ ways, and, in addition, we put the first s distinguished elements into separate ordered blocks. At this point, the number of ordered blocks is $j+r$, which is more than it should be by $j-k$. The k untouched ordered blocks containing no distinguished element can be chosen in $\binom{j}{k}$ ways, while the other $j-k$ ordered blocks containing no distinguished element should be inserted before or after the first s distinguished elements in their ordered blocks, but into different positions of the $2s$ possible ones.

Therefore, we conclude that

$$\left\lfloor {n \atop k} \right\rfloor_r = \sum_{j=k}^{\min\{n,k+2s\}} \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}} = \sum_{j=k}^{n} \left\lfloor {n \atop j} \right\rfloor_{r-s} \binom{j}{k} (2s)^{\underline{j-k}},$$

because $(2s)^{\underline{j-k}} = 0$ if $j \geqslant k+2s+1$.

Remark 6.1. Under the same assumptions, a similar approach leads us to the analogous identity

$$\left\{ {n \atop k} \right\}_r = \sum_{j=k}^{n} \left\{ {n \atop j} \right\}_{r-s} \binom{j}{k} s^{\underline{j-k}}$$

for r-Stirling numbers of the second kind. But the proof is trickier in case of the formula

$$\left[{n \atop k} \right]_r = \sum_{j=k}^{n} \left[{n \atop j} \right]_{r-s} \binom{j}{k} s^{j-k}$$

for r-Stirling numbers of the first kind, because the idea should be combined with the standard cycle representation of permutations (when the smallest number is listed first in each cycle, and the cycles are sorted in decreasing order of their first elements). We leave the details of this argument to the reader.

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