Mengling Jiang; Jianjun Liu
Finite groups with some SS-supplemented subgroups

Czechoslovak Mathematical Journal, Vol. 71 (2021), No. 3, 837–846

Persistent URL: `http://dml.cz/dmlcz/149060`

Czechoslovak Mathematical Journal, 71 (146) (2021), 837–846

FINITE GROUPS WITH SOME SS-SUPPLEMENTED SUBGROUPS

Mengling Jiang, Jianjun Liu, Chongqing

Received March 12, 2020. Published online March 24, 2021.

Abstract. A subgroup H of a finite group G is said to be SS-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is S-quasinormal in K. We analyze how certain properties of SS-supplemented subgroups influence the structure of finite groups. Our results improve and generalize several recent results.

Keywords: SS-supplemented subgroup; maximal subgroup; solvable group; minimal subgroup

MSC 2020: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite and G always denotes a finite group. Our notation and terminology are standard and the reader is referred to [4], [8]. Recall that a subgroup H of a group G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel and Deskins in 1962, see [10]. In 2012, Guo and Lu gave the definition of SS-supplemented subgroups.

Definition 1.1 ([6], Definition 2.1)**.** A subgroup H of a group G is called *SS-supplemented* in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is S-quasinormal in K. In this case, we say that K is an SS-supplement of H in G.

Theorem 1.2 ([6], Theorem 3.3)**.** *A group G is solvable if and only if every maximal subgroup M of G has a subnormal SS-supplement in G.*

This research was partially supported by the National Natural Science Foundation of China (No. 11971391), Chongqing Research Program of Basic Research and Frontier Technology (No. cstc2018jcyjAX0147), Fundamental Research Funds for the Central Universities (No. XDJK2020B052) and NSFC (No. 12071376).

DOI: 10.21136/CMJ.2021.0110-20

The research on the SS-supplemented subgroups of a given group still continues and many related results have been recently obtained, see [11], [12]. It has been proved that the SS-supplemented subgroups are suitable for describing the structure of groups. The aim of this paper is to give a generalization of the above mentioned theorems. We investigate the solvability of some normal subgroup by using certain maximal subgroups, which is a generalization of the results known. We also study the structure of groups based on the assumption that every subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ of order p or 4 (if $p = 2$) is SS-supplemented in G, where $x \in G \setminus N_G(P)$ and $G^{\mathfrak{N}_p}$ is the p-nilpotent residual of G. Some results for a group to be p-nilpotent and supersolvable are obtained and many known results are generalized.

Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking epimorphic images and such that every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*residual* of G and denoted by $G^{\mathfrak{F}}$. Throughout this paper, $\mathfrak{N}_\mathfrak{p}$ and $\mathfrak{N}$ denote the classes of p-nilpotent groups and nilpotent groups, respectively.

2. Preliminaries

In this section we present some lemmas, which are required in the proofs of our main results.

Lemma 2.1 ([6], Lemma 2.4). *Let H be an SS-supplemented subgroup of a group G. Then, the following statements hold:*

(1) *If M is a subgroup of G and $H \leqslant M$, then H is SS-supplemented in M.*
(2) *If N is a normal subgroup of G and $N \leqslant H$, then H/N is SS-supplemented in G/N.*
(3) *Let π be a set of primes. If H is a π-subgroup of G and N is a normal π'-subgroup of G, then HN/N is SS-supplemented in G/N.*

The following two lemmas are known results for S-quasinormal subgroups of a given group G.

Lemma 2.2 ([10]). *Let H be a subgroup of a group G. If H is S-quasinormal in G, then H is subnormal in G.*

Lemma 2.3 ([16], Lemma A). *If H is a p-subgroup of a group G for some prime p, then H is S-quasinormal in G if and only if $O^p(G) \leqslant N_G(H)$.*

Lemma 2.4 ([4], Lemma 14.3). *If A is a subnormal subgroup of a group G and B is a minimal normal subgroup of G, then $B \leqslant N_G(A)$.*

Lemma 2.5. *Let P be a Sylow p-subgroup of a group G and H a normal subgroup of G. If N is a normal p'-subgroup of G, then $HN \cap PN \cap P^xN = (H \cap P \cap P^{xn})N$ for some $n \in N$, where $x \in G \setminus N_G(P)$.*

P r o o f. From Sylow's theorem, we have $HN \cap PN = (HN \cap P)N = (H \cap P)N$. So $HN \cap PN \cap P^xN = (H \cap P \cap P^xN)N$. Take $P_0 = H \cap P \cap P^xN$. Then P_0 is contained in a Sylow p-subgroup of P^xN. Thus by Sylow's theorem again there exists an element n in N such that $P_0 \leqslant P^{xn}$. It follows that $P_0 = H \cap P \cap P^xN \geqslant H \cap P \cap P^{xn} \geqslant P_0$ and hence $P_0 = H \cap P \cap P^{xn}$. This implies that $HN \cap PN \cap P^xN = (H \cap P \cap P^{xn})N$. □

A 2-group is called *quaternion-free* if it has no section isomorphic to the quaternion group of order 8.

Lemma 2.6 ([5], Theorem 2.8)**.** *If a solvable group G has a Sylow 2-subgroup P which is quaternion-free, then $P \cap Z(G) \cap G^{\mathfrak{N}} = 1$.*

Lemma 2.7. *Let H be a subgroup of a group G, then $H^{\mathfrak{N}_p} \leqslant G^{\mathfrak{N}_p}$.*

P r o o f. Since $HG^{\mathfrak{N}_p}/G^{\mathfrak{N}_p} \leqslant G/G^{\mathfrak{N}_p}$ and $G/G^{\mathfrak{N}_p}$ is p-nilpotent, we have that $H/(H \cap G^{\mathfrak{N}_p})$ is p-nilpotent and so $H^{\mathfrak{N}_p} \leqslant H \cap G^{\mathfrak{N}_p}$, as desired. □

Lemma 2.8 ([1], Lemma 2)**.** *Let $\mathfrak{F}$ be a saturated formation. Assume that G is a non-$\mathfrak{F}$-group and there exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. Then*
(1) *$G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of G;*
(2) *$G^{\mathfrak{F}}$ is a p-group for some prime p;*
(3) *$G^{\mathfrak{F}}$ has exponent p if $p > 2$ and exponent is at most 4 if $p = 2$;*
(4) *$G^{\mathfrak{F}}$ is either an elementary abelian group or $(G^{\mathfrak{F}})' = Z(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.*

Lemma 2.9 ([17], Lemma 2.16)**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Let H be a normal subgroup of a group G. We define the following families of subgroups:

$$\mathfrak{M}(G) = \{M \mid M \lessdot G\},$$
$$\mathfrak{M}_{pc}(G) = \{M \mid M \in \mathfrak{M}(G),\ |G:M|_p = 1 \text{ and } |G:M| \text{ is composite}\},$$
$$\mathfrak{M}^{pcn}(G) = \{M \mid M \in \mathfrak{M}(G),\ N_G(P) \leqslant M \text{ for a Sylow } p\text{-subgroup } P \text{ of } G,\ M \text{ is nonnilpotent and } |G:M| \text{ is composite}\},$$
$$\mathfrak{M}_H(G) = \{M \mid M \in \mathfrak{M}(G) \text{ and } H \nleqslant M\}.$$

3. Main results

In this section, we firstly study the solvability of a normal subgroup H of a group G when some subgroups are assumed to be SS-supplemented subgroups of G.

Theorem 3.1. *Let H be a normal subgroup of a group G and p the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$ has a subnormal SS-supplement in G, then H is solvable.*

Proof. If $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G) = \emptyset$, then we claim that H is solvable. In fact, if $\mathfrak{M}_{pc}(G) = \emptyset$, by [13], Theorem 8, G is solvable and so is H. If $\mathfrak{M}_{pc}(G) \neq \emptyset$, then H is contained in every maximal subgroup M of G in $\mathfrak{M}_{pc}(G)$. Applying [13], Theorem 8 again, H is solvable. This proves our claim.

Now we may assume that $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G) \neq \emptyset$. Let N be a minimal normal subgroup of G, and let M/N be a maximal subgroup of $\overline{G} = G/N$ with $M/N \in \mathfrak{M}_{pc}(\overline{G}) \cap \mathfrak{M}_{\overline{H}}(\overline{G})$. Then $M \in \mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$. Furthermore, M/N has a subnormal SS-supplement in G/N by Lemma 2.1. It is clear that $(\overline{G}, \overline{H})$ satisfies the hypotheses of the theorem and so $\overline{H}$ is solvable by induction. If $N \nleqslant H$, then $H \cong \overline{H}$ is solvable, as desired. Hence, we may assume that $N \leqslant H$, and it follows that H/N is solvable. If G has two different minimal normal subgroups N_1 and N_2, then both H/N_1 and H/N_2 are solvable and so is $H/(N_1 \cap N_2)$. This implies that the group H is solvable. Hence we may assume that G has a unique minimal normal subgroup N.

Suppose that N is nonsolvable. Let q be the largest prime dividing the order of N and Q a Sylow q-subgroup of N. Then $G = N_G(Q)N$ by the Frattini argument. So there exists a maximal subgroup M of G which contains $N_G(Q)$, but $N \nleqslant M$. By hypothesis, $p \geqslant q$. If $p > q$, it is clear that $|G : M|_p = |N : M \cap N|_p$=1. If $p = q$, then $N_G(Q)$ contains a Sylow p-subgroup of G. Thus, we conclude that $|G : M|_p = 1$ in these two cases. If $|G : M| = r$ for some prime r, then, since $M_G = 1$, we have that G is isomorphic to a subgroup of the symmetric group S_r of degree r. This implies that $|G| \mid r!$, which is a contradiction as p is not a divisor of $r!$. Hence, we conclude that $M \in \mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$.

By our hypotheses, there exists a subnormal subgroup K of G such that $G = MK$ and $M \cap K$ is S-quasinormal in K. Since K is subnormal in G, Lemma 2.2 implies that $M \cap K$ is subnormal in G. We claim that $M \cap K = 1$. Otherwise, we may take a minimal subnormal subgroup L of G contained in $M \cap K$. Since $L \cap N \trianglelefteq L$, either $L \cap N = 1$ or $L \leqslant N$. If $L \cap N = 1$, then from Lemma 2.4 $NL = N \times L$ and $L \leqslant C_G(N)$=1, a contradiction. Suppose $L \leqslant N$. We have $L^G = L^{NM} = L^M \leqslant M_G = 1$, which implies $L = 1$, a contradiction. Therefore $M \cap K = 1$. By using the same arguments, we can similarly prove that all minimal subnormal subgroups of G are contained in N. Let $N = N_1 \times \ldots \times N_r$, where each N_i is isomorphic to a fixed nonabelian simple group. It follows that $N_1, \ldots, N_r$ coincide with all minimal subnormal subgroups of G. Without loss of generality, we may assume that $N_1 \leqslant K$. Then a prime p exists such that p divides $|K| = |G : M|$. By [2], Lemma 3, we can see that N is solvable, this is a contradiction. The proof is completed. □

From Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let p be the largest prime dividing the order of a group G. Then G is solvable if and only if every maximal subgroup M of G in $\mathfrak{M}_{pc}(G)$ has a subnormal SS-supplement in G.*

Proof. From Theorem 1.2, only the sufficiency requires a proof. In fact, let $G = H$ in Theorem 3.1. Then we have the corollary. □

Remark 3.3. In Theorem 3.1, the group G is not necessary solvable. For example: Let L, H be the alternating groups of degree 5 and 4, respectively, and let $G = L \times H$. Suppose that $M = L \times C_3$, where C_3 is a cyclic group of order 3 of H. Then M is a maximal subgroup of G. It is clear that $H \nleq M$ and $|G : M| = 4$. Thus $M \in \mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$ and we can also see that $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G) = \{M^g \colon g \in G\}$. Furthermore, it is easy to see that $G = MK_4$ and $M \cap K_4$ is S-quasinormal in K_4, where K_4 is the Klein four group contained in H. That is, M has a subnormal SS-supplement in G. However, G is not solvable.

Theorem 3.4. *Let H be a normal subgroup of a group G and p the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G)$ has a subnormal SS-supplement in G, then H is p-solvable.*

Proof. If $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G) = \emptyset$, then we can see that H is p-solvable by [7], Lemma 2.4. Now, we may assume that $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. If P is normal in G, then G is certainly p-solvable and so is H. So we may assume that $N_G(P) < G$.

Let N be a minimal normal subgroup of G. It is clear that G/N satisfies the hypotheses of the theorem for the normal subgroup HN/N and so HN/N is p-solvable by induction. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G.

Suppose that N is not p-solvable. Then p is a divisor of the order of N. We know that $N \cap P \in \mathrm{Syl}_p(N)$ and $P \cap N$ is not a normal subgroup of N. By the Frattini argument, we have that $G = N_G(P \cap N)N$. So there exists a maximal subgroup M of G which contains $N_G(P \cap N)$ and $M \ngeq N$. It is clear that $N_G(P) \leqslant M$. If $|G : M| = q$ is a prime, then by Sylow's theorem, we have $q = 1 + kp$ and $q \mid |N|$. This contradicts p being the largest prime which divides the order of N. Hence $|G : M|$

must be a composite number. If M is nilpotent, then the Sylow 2-subgroup M_2 of M is not identity by [14], Theorem 10.4.2. Let $M_{2'}$ be a Hall $2'$-subgroup of M. By [15], Theorem 1, $M_{2'}$ is normal in G and therefore $P \trianglelefteq G$ since P is a characteristic subgroup of $M_{2'}$. It follows that $P \cap N \trianglelefteq G$, a contradiction. Thus, $M \in \mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G)$. By the hypotheses, M has a subnormal SS-supplement subgroup K in G. By using similar arguments as in the proof of Theorem 3.1, we can get that $|K| = |G : M| \leqslant |G : N_G(P)|$ and so $p \nmid |K|$. However, K is subnormal in G, which implies that K contains N_i for some i and hence $p \mid |K|$, a contradiction. This shows that N is p-solvable and therefore H is p-solvable. The proof of the theorem is now complete. □

From Theorem 3.4, we have the following corollary.

Corollary 3.5. *Let p be the largest prime dividing the order of a group G. Then G is p-solvable if and only if every maximal subgroup M of G in $\mathfrak{M}^{pcn}(G)$ has a subnormal SS-supplement in G.*

P r o o f. Only the necessity of the condition is in doubt by Theorem 3.4. Suppose that G is p-solvable and M is a maximal subgroup of G. We argue by induction on $|G|$. Assume that $M_G \neq 1$. Set $\overline{G} = G/M_G$. By induction, we can see that $\overline{M}$ has a subnormal SS-supplement $\overline{K}$ in $\overline{G}$ and so K is a subnormal SS-supplement of M in G. Hence, we may assume that $M_G = 1$ and let N be a minimal normal subgroup of G. Then $G = MN$ and $M \cap N \leqslant M_G = 1$, which implies that N is the normal SS-supplement of M in G. □

Remark 3.6. In Theorem 3.4, the group G need not be p-solvable as the following example shows. Let $H = C_2 \times C_2 \times C_2 \times C_2$ be an elementary abelian group of order 2^4. Then there is a subgroup $M = A_5$ in the automorphism group of H, where A_5 is the alternating group of degree 5. Let $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ be the corresponding semidirect product. We can deduce that $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G) = \{M^g \colon g \in G\}$. It is clear that M has a subnormal SS-supplement H in G. That is, G satisfies the hypotheses of Theorem 3.4 for normal subgroup H. However, G is not 5-solvable.

Finally we study the p-nilpotency and supersolvability of a group G by looking at certain minimal subgroups, leading to generalizations of known results.

Theorem 3.7. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G and when $p = 2$, either cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free, then G is p-nilpotent.*

P r o o f. Suppose that the theorem is false and let G be a counterexample of minimal order. Then G is not p-nilpotent. Noticing that all its Sylow p-subgroups are conjugate in G, we see that the hypotheses of our theorem are a subgroup-closure by Lemma 2.1. Consequently, G is a minimal non-p-nilpotent group (that is, every proper subgroup of a group is p-nilpotent but is not p-nilpotent itself). Now, by a result of Itô (see [14], Theorem 10.3.3), G must be a minimal nonnilpotent group. By a result of Schmidt (see [14], Theorem 9.1.9 and Exercise 9.1.11), we know that G is of order $p^a q^b$, where q is a prime which is different from p, P is normal in G and any Sylow q-subgroup Q of G is cyclic. Moreover, $P = G^{\mathfrak{N}_p}$ and P is of exponent p when p is odd and of exponent at most 4 when $p = 2$.

Let P_1 be a minimal subgroup of P. Then by hypotheses there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K$ is S-quasinormal in K. Assume that $P_1 \cap K = 1$. Since p is the smallest prime divisor of the order of G, we get that K is normal in G. Noticing that K is a proper subgroup of G, we have that K is nilpotent. It follows that the Sylow q-subgroup of K is normal in G and therefore G is nilpotent, which is a contradiction. Hence, $P_1 \leqslant K$ and so P_1 is S-quasinormal in G. Therefore every minimal subgroup of P is S-quasinormal in G.

Let Q be a Sylow q-subgroup of G. Then P_1Q is a proper group of G and P_1Q is nilpotent by the minimality of G. It follows that $Q \subseteq C_G(P_1)$ and hence $Q \subseteq C_G(\Omega_1(P))$. If $C_G(\Omega_1(P)) < G$, then $C_G(\Omega_1(P))$ is nilpotent and so $Q \trianglelefteq G$, a contradiction. This leads to $C_G(\Omega_1(P)) = G$ and $\Omega_1(P) \leqslant Z(G)$. If $p > 2$, then from Itô's Lemma (see [9]) G is nilpotent, a contradiction. Hence $p = 2$. If P is quaternion-free, then by Lemma 2.6, we get that $\Omega_1(P) \leqslant P \cap G^{\mathfrak{N}_p} \cap Z(G) \leqslant P \cap G^{\mathfrak{N}} \cap Z(G) = 1$, a contradiction. Now assume that every cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G. Let $A = \langle a \rangle$ be a cyclic subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ with order 4. Then there exists a subgroup T of G such that $G = AT$ and $A \cap T$ is S-quasinormal in T. Noticing that $\langle a^2 \rangle \subseteq Z(G)$, we see that $\langle a^2 \rangle T$ is a subgroup of G. If $|G : T| = 4$, then $|G : \langle a^2 \rangle T| = 2$ and $\langle a^2 \rangle T$ is normal in G. This implies that the Sylow q-subgroup of $\langle a^2 \rangle T$ is normal in G and therefore G is nilpotent, this is a contradiction. If $|G : T| = 2$, then T itself is a normal subgroup and T is nilpotent. Since the normal p-complement of T is the normal p-complement of G, it follows that G is nilpotent, a contradiction. Consequently, $T = G$ and so A is S-quasinormal in G. If $A = P$, then G is nilpotent, a contradiction. Thus, $A \neq P$. Since G is a minimal nonnilpotent group and the exponent of P is at most 4, we have $P \leqslant C_G(Q)$ and therefore $G = P \times Q$, a contradiction. The proof is complete.

□

We say that a group G is a Sylow tower group of supersolvable type if $p_1 > p_2 > \ldots > p_r$ are the distinct prime divisors of the order of G, then there exists a series of normal subgroups of G,

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_r = G,$$

such that G_i/G_{i-1} is a Sylow p_i-subgroup of G/G_{i-1} for $i = 1, \ldots, r$. Given a group G, observing that $H^{\mathfrak{N}_p} \leqslant G^{\mathfrak{N}_p}$ for every subgroup H of G by Lemma 2.7 and using Lemma 2.1 and Theorem 3.7, we obtain at once the following result.

Corollary 3.8. *Let G be a group. Suppose that for every prime p dividing the order of G and for every Sylow p-subgroup P of G, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G, and when $p = 2$, either cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free. Then G is a Sylow tower group of supersolvable type.*

Theorem 3.9. *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups and N be a normal subgroup of G such that $G/N \in \mathfrak{F}$. Suppose that for every prime p dividing the order of N and for every Sylow p-subgroup P of N, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G, and when $p = 2$, every cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free. Then $G \in \mathfrak{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.1 and Corollary 3.8, we know that N is a Sylow tower group of supersolvable type. Thus if p is the largest prime dividing the order of N and P is a Sylow p-subgroup of N, then P must be normal in G and $G/P/N/P \cong G/N \in \mathfrak{F}$. It is clear that G/P satisfies the hypotheses of our theorem for its normal subgroup N/P by Lemmas 2.5 and 2.1. Then the minimality of G implies that $G/P \in \mathfrak{F}$.

Now, when G is not in $\mathfrak{F}$, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is nontrivial. Since $G/G^{\mathfrak{N}}$ is nilpotent and therefore $G/G^{\mathfrak{N}}$ belongs to $\mathfrak{F}$, necessarily $G/(P \cap G^{\mathfrak{N}})$ belongs to $\mathfrak{F}$ as well. It follows that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}}$. Furthermore, we claim that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}_p}$. Let P^* be a Sylow p-subgroup of G. As $G/G^{\mathfrak{N}_p}$ is p-nilpotent, we can see that $P^*G^{\mathfrak{N}_p} \cap O^p(G)G^{\mathfrak{N}_p} = G^{\mathfrak{N}_p}$ and so $P^* \cap O^p(G) \leqslant G^{\mathfrak{N}_p}$, which means that $P^* \cap O^p(G) = P \cap G^{\mathfrak{N}_p}$. A similar argument shows that $P^* \cap O^p(G) = P \cap G^{\mathfrak{N}}$ and this proves our claim. By [3], Theorem 3.5, there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$ and $G/M_G \notin \mathfrak{F}$. Then $G = MG^{\mathfrak{F}}$ and so $G = MF(G)$ since $G^{\mathfrak{F}}$ is a p-group, where $F(G)$ is the Fitting subgroup of G. It is now clear that M satisfies the hypotheses of our theorem for its normal subgroup $M \cap P$. Hence, the minimality of G implies that $M \in \mathfrak{F}$.

Now, by Lemma 2.8, we get that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathfrak{F}})$, G has exponent p when $p > 2$ and exponent at most 4 when $p = 2$. Let $\Phi = \Phi(G^{\mathfrak{F}})$ and A/Φ be any subgroup of $G^{\mathfrak{F}}/\Phi$ with order p, $a \in A \setminus \Phi$ and $X = \langle a \rangle$. Then $|X| = p$ or $|X| = 4$ and so X is SS-supplemented in G. Thus, there exists a subgroup K of G such that $G = XK$ and $X \cap K$ is S-quasinormal in K. Clearly, $(X\Phi/\Phi)(K/\Phi) = G/\Phi$. Assume that $X \nleq K$, then $X\Phi/\Phi \nleq K\Phi/\Phi$. Hence, the minimality of $G^{\mathfrak{F}}/\Phi$ implies that $(G^{\mathfrak{F}} \cap K)/\Phi = 1$, since $G^{\mathfrak{F}}/\Phi \cap K/\Phi \trianglelefteq G/\Phi$. By order comparison, $|G^{\mathfrak{F}}/\Phi| = p$. Assume that $X \leqslant K$, then $K = G$ and X is S-quasinormal in G. It follows that $A/\Phi = X\Phi/\Phi$ is S-quasinormal in G/Φ. By Lemma 2.3, $O^p(G/\Phi) \leqslant N_{G/\Phi}(A/\Phi)$ and so $|G/\Phi : N_{G/\Phi}(A/\Phi)| = p^a$ for some $a \in \mathbb{N}$. Thus if $\{A_1/\Phi, \ldots, A_t/\Phi\}$ is the set of all minimal subgroups of $G^{\mathfrak{F}}/\Phi$, then it follows from [8], III, 8.5 Hilfssatz, that $|G/\Phi : N_{G/\Phi}(A_i/\Phi)| = 1$ for some $i \in \{1, \ldots, t\}$. Hence, A_i/Φ is normal in G/Φ. The minimality of $G^{\mathfrak{F}}/\Phi$ also implies that $|G^{\mathfrak{F}}/\Phi| = p$.

Now $(G/\Phi)/(G^{\mathfrak{F}}/\Phi) \cong G/G^{\mathfrak{F}} \in \mathfrak{F}$ and $G^{\mathfrak{F}}/\Phi$ is a cyclic group of order p. Hence, $(G/\Phi, G^{\mathfrak{F}}/\Phi)$ satisfies the hypotheses of the theorem. If $\Phi \neq 1$, then by the minimality of G, $G/\Phi \in \mathfrak{F}$. It follows that $G \in \mathfrak{F}$, a contradiction. Thus $\Phi = 1$ and so $G^{\mathfrak{F}}$ is a cyclic group of order p. By Lemma 2.9, we can conclude that $G \in \mathfrak{F}$, a contradiction.

There remains the case, where $p = 2$ and P is quaternion-free. Let R be a Sylow r-subgroup of G with $r \neq 2$ and $G_1 = RG^{\mathfrak{F}}$. Then $G^{\mathfrak{F}}$ is a Sylow 2-subgroup of G_1. Observing that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}_p}$, we have that G_1 is 2-nilpotent by Theorem 3.7. It follows that $G^{\mathfrak{F}} \leqslant C_G(R)$ and therefore $Z(G) \cap G^{\mathfrak{F}} \neq 1$. Since $G^{\mathfrak{F}} \leqslant G^{\mathfrak{N}}$, we have $Z(G) \cap G^{\mathfrak{N}} \cap P \neq 1$, in contradiction to Lemma 2.6. This completes the proof of the theorem. □

As an immediate consequence of Theorem 3.9, we have:

Corollary 3.10. *Let G be a group. Suppose that, for every prime p dividing the order of G and for every Sylow p-subgroup P of G, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G, and when $p = 2$, every cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free. Then G is supersolvable.*

References

[1] *M. Asaad, A. Ballester-Bolinches, M. C. Pedraza Aguilera*: A note on minimal subgroups of finite groups. Commun. Algebra *24* (1996), 2771–2776. zbl MR doi

[2] *R. Baer*: Classes of finite groups and their properties. Ill. J. Math. *1* (1957), 115–187. zbl MR doi

[3] *A. Ballester-Bolinches*: h-normalizers and local definition of saturated formation of finite groups. Isr. J. Math. *67* (1989), 312–326. zbl MR doi

[4] *K. Doerk, T. Hawkes*: Finite Soluble Groups. De Gruyter Expositions in Mathematics 4. Walter de Gruyter, Berlin, 1992. zbl MR doi

[5] *L. Dornhoff*: M-groups and 2-groups. Math. Z. *100* (1967), 226–256. zbl MR doi

[6] *X. Guo, J. Lu*: On SS-supplemented subgroups of finite groups and their properties. Glasg. Math. J. *54* (2012), 481–491. zbl MR doi

[7] *X. Guo, K. P. Shum*: Cover-avoidance properties and the stucture of finite groups. J. Pure Appl. Algebra *181* (2003), 297–308. zbl MR doi

[8] *B. Huppert*: Endliche Gruppen I. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 134. Springer, Berlin, 1967. (In German.) zbl MR doi

[9] *N. Itô*: Über eine zur Frattini-Gruppe duale Bildung. Nagoya Math. J. *9* (1955), 123–127. (In German.) zbl MR doi

[10] *O. H. Kegel*: Sylow-Gruppen und Subnormalteiler endlicher Gruppen. Math. Z. *78* (1962), 205–221. (In German.) zbl MR doi

[11] *J. Lu, X. Guo, X. Li*: The influence of minimal subgroups on the stucture of finite groups. J. Algebra Appl. *12* (2013), Article ID 1250189, 8 pages. zbl MR doi

[12] *J. Lu, Y. Qiu*: On solvability of finite groups with some SS-supplemented subgroups. Czech. Math. J. *65* (2015), 427–433. zbl MR doi

[13] *N. P. Mukherjee, P. Bhattacharya*: On the intersection of a class of maximal subgroups of a finite group. Can. J. Math. *39* (1987), 603–611. zbl MR doi

[14] *D. J. S. Robinson*: A Course in the Theory of Groups. Graduate Texts in Mathematics 80. Springer, New York, 1982. zbl MR doi

[15] *J. Rose*: On finite insolvable groups with nilpotent maximal subgroups. J. Algebra *48* (1977), 182–196. zbl MR doi

[16] *P. Schmid*: Subgroups permutable with all Sylow subgroups. J. Algebra *207* (1998), 285–293. zbl MR doi

[17] *A. N. Skiba*: On weakly s-permutable subgroups of finite groups. J. Algebra *315* (2007), 192–209. zbl MR doi

Authors' address: Mengling Jiang, Jianjun Liu (corresponding author), School of Mathematics and Statistics, Southwest University, Beibei, Chongqing, 400715, P. R. China, e-mail: `991311410@qq.com`, `liujj198123@163.com`.