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ALGEBRAIC PROPERTIES OF TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES

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Abstract. We study algebraic properties of two Toeplitz operators on the weighted Bergman space on the unit disk with harmonic symbols. In particular the product property and commutative property are discussed. Further we apply our results to solve a compactness problem of the product of two Hankel operators on the weighted Bergman space on the unit bidisk.

Keywords: Bergman space; Toeplitz operator; Hankel operator; Berezin transform

MSC 2020: 47B35

1. Introduction

Let $\mathrm{d}A_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \mathrm{d}A(z)$, $\alpha > -1$ be the normalized weighted Lebesgue area measure on the unit disk $\mathbb{D}$. The weighted Bergman space $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$ is the space of all analytic functions on $\mathbb{D}$ which are square integrable with respect to $\mathrm{d}A_\alpha$.

It is well known that the Bergman space $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$. Hence $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$ is a Hilbert space with the inner product inherited from $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$. For a function $f \in A^2(\mathbb{D}, \mathrm{d}A_\alpha)$, the norm of f is the usual $L^2(\mathbb{D}, \mathrm{d}A_\alpha)$-norm defined by

$$\|f\|_{L^2(\mathbb{D}, \mathrm{d}A_\alpha)} = \left\{ \int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}A_\alpha(z) \right\}^{1/2}.$$

Let $f \in L^\infty(\mathbb{D})$. The Toeplitz operator T_f with symbol f,

$$T_f \colon A^2(\mathbb{D}, \mathrm{d}A_\alpha) \to A^2(\mathbb{D}, \mathrm{d}A_\alpha)$$

is defined as

$$T_f u = P_\alpha(fu), \quad u \in A^2(\mathbb{D}, \mathrm{d}A_\alpha), \tag{1.1}$$

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where $P_\alpha\colon L^2(\mathbb{D}, \mathrm{d}A_\alpha) \to A^2(\mathbb{D}, \mathrm{d}A_\alpha)$ is the orthogonal projection. For a given function $f \in L^\infty(\mathbb{D})$, clearly $\|T_f\| \leqslant \|f\|_\infty$.

Brown and Halmos in [6] showed that for essentially bounded symbols f, g on the unit circle $\mathbb{T}$ the product of two Hardy space Toeplitz operators $T_fT_g = T_h$ if and only if either g is analytic or f is co-analytic, and in this case $h = fg$. Here f is analytic on $\mathbb{T}$ if all its negative Fourier coefficients vanish. Ahern and Čučković in [2] proved an analogue for the unweighted ($\alpha = 0$) Bergman space $A^2(\mathbb{D}, \mathrm{d}A)$.

After the initial work on the Hardy space on $\mathbb{T}$ and unweighted Bergman space on $\mathbb{D}$, it was natural to ask if the same result was true on the Bergman space on the higher-dimensional polydisk $\mathbb{D}^n$, $n > 1$. In particular Choe, Lee, Nam and Zheng in [7] obtained an analogue of the above theorem on $\mathbb{D}^n$, $n > 1$.

In this paper we extend the results of Ahern and Čučković (see [2]) in a different direction by introducing a weight to the measure. The α-Berezin transform plays an important role in our main proof. Therefore, we define and list some properties of the α-Berezin transform in the next section.

2. The α-Berezin transform

In general, if H is a Hilbert space of analytic functions on $\mathbb{D}$ with the property that point-evaluation functionals are bounded, then by the Riesz representation theorem the existence of the kernel function is guaranteed. Then for any bounded linear operator T on H we can define the Berezin transform of T denoted by $\widetilde{T}$ as follows.

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle, \quad z \in \mathbb{D}, \tag{2.1}$$

where k_z is the normalized kernel function in H. Now let H be the weighted Bergman space $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$. The Berezin transform of a function $u \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$ is denoted by $B_\alpha u$ and defined by

$$B_\alpha u(z) = \int_{\mathbb{D}} u(\zeta)\frac{(1-|z|^2)^{2+\alpha}}{|1-\overline{z}\zeta|^{4+2\alpha}}\,\mathrm{d}A_\alpha(\zeta). \tag{2.2}$$

where

$$k_z^\alpha(w) = \frac{(1-|z|^2)^{1+\alpha/2}}{(1-\overline{z}w)^{2+\alpha}}$$

is the normalized kernel function in $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$. It is known that the α-Berezin transform is injective on $L^1(\mathbb{D}, \mathrm{d}A_\alpha)$ and we state this result in the following lemma. Interested reader may read in [12], Chapter 6.

Lemma 1. *The α-Berezin transform is one-to-one. That is if $B_\alpha u = 0$ for $u \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$, then $u = 0$.*

The Berezin transform plays an important role in our approach of solving product properties of Toeplitz operetors. More specifically, some theorems on the range of the Berezin transform are used in our proofs. The following theorem characterizes all triples (u, f, g) such that $Bu = f\overline{g}$, where $B = B_0$ is the unweighted Berezin transform. Also we denote the set of analytic functions on $\mathbb{D}$ by $H(\mathbb{D})$ and

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \in \operatorname{Aut}(\mathbb{D}), \tag{2.3}$$

the Möbius group on $\mathbb{D}$.

Theorem 2 ([1]). *Let $f, g \in H(\mathbb{D})$ and assume they are nonconstant. If $Bu = f\overline{g}$ for some $u \in L^1(\mathbb{D})$, then there are nonconstant polynomials p and q with $\deg(pq) \leqslant 3$ and $a \in \mathbb{D}$ such that $f = p \circ \varphi_a$ and $g = q \circ \varphi_a$ and*

$$u \circ \varphi_a(z) = c_1 z^2 + c_2 \overline{z}^2 + c_3 z + c_4 \overline{z} + c_5 + d_1 \ln |z|^2 + d_2 \frac{1}{z} + d_3 \frac{1}{\overline{z}}.$$

Later Čučković and Li in [8] obtained an extension of Ahern's result by considering $Bu = f_1\overline{g}_1 + f_2\overline{g}_2$ with $g_2(z) = z^n$, $n \geqslant 1$, where f_1, f_2 and g_1 are of the same type as in the previous theorem. In 2010 Rao in [11] showed a more general version of this result as follows.

Theorem 3 ([11]). *If $Bu = \sum\limits_{k=1}^{n} f_k \overline{g}_k$, where $f_k, g_k \in H(\mathbb{D})$, then there exist finitely many points $a_k \in \mathbb{D}$, $1 \leqslant k \leqslant n$, such that*

$$u(z) = \tilde{h}(z) + \sum_{k=1}^{n} D_k \ln |z - a_k| + \frac{E_k}{(z - a_k)} + \frac{F_k}{(\overline{z} - \bar{a}_k)},$$

where D_k, E_k, F_k, $1 \leqslant k \leqslant n$ are constants, many and even all of them could vanish, and $\tilde{h} \in L^1(\mathbb{D}, \mathrm{d}A)$ and is harmonic.

Recall that the Laplacian on the complex plane is,

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \overline{z}}$$

and we denote the invariant Laplacian by

$$\widetilde{\Delta} = (1 - |z|^2)^2 \Delta.$$

It is known that the invariant Laplacian commutes with the Berezin transform.

Lemma 4 ([2]). *If $u \in C^2(\mathbb{D})$ and $u, \widetilde{\Delta} u \in L^1(\mathbb{D})$, then $\widetilde{\Delta}(Bu) = B(\widetilde{\Delta} u)$.*

The next theorem is very important for our main proof.

Theorem 5 ([3]). *If $u \in L^1(\mathbb{D})$, then for $\alpha > -1$,*

$$\widetilde{\Delta} B_\alpha u = 4(\alpha+1)(\alpha+2)(B_\alpha u - B_{\alpha+1} u).$$

The following result has been known for a long time and yet we give a short proof.

Lemma 6. *Suppose that $\alpha \in \mathbb{N}$ and $u \in C^{2\alpha}(\mathbb{D})$ and assume that*

$$u, \widetilde{\Delta} u, \widetilde{\Delta}^2 u, \ldots, \widetilde{\Delta}^\alpha u \in L^1(\mathbb{D}).$$

Then

$$B_\alpha u = B[Q_\alpha(\widetilde{\Delta})u],$$

where

$$Q_\alpha(\lambda) = \prod_{k=1}^{\alpha} \Big[1 - \frac{\lambda}{4k(k+1)}\Big]$$

is a polynomial of degree α.

Proof. Theorem 5 tells us that

$$B_{\alpha+1} u = \Big[I - \frac{\widetilde{\Delta}}{4(\alpha+1)(\alpha+2)}\Big] B_\alpha u.$$

By applying this formula n times recursively we obtain

$$B_{\alpha+n} u = \prod_{k=1}^{n} \Big[1 - \frac{\widetilde{\Delta}}{4(k+\alpha)(k+\alpha+1)}\Big] B_\alpha u,$$

where n is a positive integer. Now by letting $\alpha = 0$ we have $B_n u = Q_n(\widetilde{\Delta}) B u$, where

$$Q_n(\lambda) = \prod_{k=1}^{n} \Big[1 - \frac{\lambda}{4k(k+1)}\Big],$$

and recall that $B = B_0$ is the unweighted Berezin transform. Since α is an integer, by replacing n by α we obtain $B_\alpha u = Q_\alpha(\widetilde{\Delta}) B u$. Now it remains to show that the two operators $Q_\alpha(\widetilde{\Delta})$ and B commute. For simplicity consider $\widetilde{\Delta}^\alpha [Bu]$. Assume $u, \widetilde{\Delta} u, \widetilde{\Delta}^2 u, \ldots, \widetilde{\Delta}^\alpha u \in L^1(\mathbb{D})$. By applying Lemma 4 recursively we get

$$\widetilde{\Delta}^\alpha [Bu] = \widetilde{\Delta}^{\alpha-1} \widetilde{\Delta}[Bu] = \widetilde{\Delta}^{\alpha-1}[B(\widetilde{\Delta} u)] = \widetilde{\Delta}^{\alpha-2}[B(\widetilde{\Delta}^2 u)] \ldots = B(\widetilde{\Delta}^\alpha u),$$

and hence the result follows since Q_α is a polynomial and B is linear. $\square$

Recall that if f is a bounded complex-valued harmonic function on $\mathbb{D}$, then there exist analytic functions f_1, f_2 such that $f = f_1 + \overline{f}_2$. Here functions f_1 and f_2 are Bloch functions. Recall also that an analytic function f on $\mathbb{D}$ is said to be a *Bloch function* if $\sup\{(1-|z|^2)|f'(z)|\colon z \in \mathbb{D}\}$ is finite. We denote the space of Bloch functions on $\mathbb{D}$ by $\mathcal{B}$. We know that $\mathcal{B} \subset A^p(\mathbb{D}, \mathrm{d}A_\alpha)$ for all $1 < p < \infty$.

The following theorem tells us an important property of Bloch functions which we will use later.

Theorem 7 ([12]). *If f is analytic in $\mathbb{D}$ and $n \geqslant 2$, then $f \in \mathcal{B}$ if and only if the function $(1-|z|^2)^n f^{(n)}(z))$ is bounded in $\mathbb{D}$.*

Remark 8. Using the above theorem we can show that for any functions $f, g \in \mathcal{B}$, $\widetilde{\Delta}^k f(z)\overline{g(z)}$ is bounded in $\mathbb{D}$ for all $k \in \mathbb{N}$.

The next proposition gives us a relationship between Toeplitz operators and the α-Berezin transform and the proof directly follows by [2], where they showed it for $\alpha = 0$.

Proposition 9. *Suppose that $f = f_1 + \overline{f}_2$ and $g = g_1 + \overline{g}_2$ are bounded harmonic functions and f_1, f_2, g_1, g_2 are analytic functions on $\mathbb{D}$. Let $h \in L^\infty(\mathbb{D})$. Then the following are equivalent on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$, $\alpha > -1$.*

(1) $T_f T_g = T_h$.

(2) *For all $z \in \mathbb{D}$,*

$$f_1(z)g_1(z) + \overline{f}_2(z)\overline{g}_2(z) + f_1(z)\overline{g}_2(z)B_\alpha(h - \overline{f}_2 g_1)(z).$$

(3) *For all $z, w \in \mathbb{D}$,*

$$f_1(z)g_1(z) + \overline{f}_2(\overline{w})\overline{g}_2(\overline{w}) + f_1(z)\overline{g}_2(\overline{w}) = \int_{\mathbb{D}} \frac{h - \overline{f}_2(\zeta)g_1(\zeta)}{(1-\overline{\zeta}z)^{2+\alpha}(1-\zeta w)^{2+\alpha}}\,\mathrm{d}A_\alpha(\zeta).$$

3. Main results

We now state our main theorem for the weighted Bergman space Toeplitz operators on $\mathbb{D}$.

Theorem 10. *Let $\alpha \in \mathbb{N}$ and $f, g \in L^\infty(\mathbb{D})$. Assume that f, g are harmonic in $\mathbb{D}$. Assume $h \in L^\infty(\mathbb{D}) \cap C^{2\alpha}(\mathbb{D})$ and $\widetilde{\Delta}h, \ldots, \widetilde{\Delta}^\alpha h \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$. Then $T_f T_g = T_h$ on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$ if and only if g is analytic or f is co-analytic. In either case $h = fg$.*

P r o o f. The sufficiency is clear. We will prove the necessity. Assume that $T_fT_g = T_h$. Proposition 9 tells us that

$$B_\alpha(h - \overline{f}_2 g_1) = f_1\overline{g}_2 + f_1 g_1 + \overline{f}_2\overline{g}_2, \quad B_\alpha(h - \overline{f}_2 g_1) - f_1 g_1 - \overline{f}_2 c\overline{g}_2 = f_1\overline{g}_2.$$

Since $f_1 g_1$ and $\overline{f}_2\overline{g}_2$ are harmonic and the Berezin transform fixes harmonic functions, we have

$$B_\alpha(h - \overline{f}_2 g_1 - f_1 g_1 - \overline{f}_2\overline{g}_2) = f_1\overline{g}_2.$$

Hence

$$B_\alpha u = f_1\overline{g}_2,$$

where $u = h - \overline{f}_2 g_1 - f_1 g_1 - \overline{f}_2\overline{g}_2$. Now by Lemma 6 we have

$$B[Q_\alpha(\widetilde{\Delta})u] = f_1\overline{g}_2.$$

Note that $\widetilde{\Delta}^k u = \widetilde{\Delta}^k h - \widetilde{\Delta}^k(\overline{f}_2 g_1)$. By the hypothesis of the theorem, $\widetilde{\Delta}^k h \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$ and by Remark 8, $\widetilde{\Delta}^k(\overline{f}_2 g_1)$ is bounded in $\mathbb{D}$ for $k = 1, \ldots, \alpha$. Hence we have $Q_\alpha(\widetilde{\Delta})u \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$. Now by using Theorem 3 we conclude that

$$Q_\alpha(\widetilde{\Delta})u = c_1 z^2 + c_2\overline{z}^2 + c_3 z + c_4\overline{z} + c_5 + d_1\log|z|^2 + d_2\frac{1}{z} + d_3\frac{1}{\overline{z}},$$

for some constants $c_1, \ldots, c_5, d_1, d_2$, and d_3. But since $h \in C^{2\alpha}(\mathbb{D})$ we have $Q_\alpha(\widetilde{\Delta})u \in C(\mathbb{D})$ and hence we must have $d_1 = d_2 = d_3 = 0$. But then

$$Q_\alpha(\widetilde{\Delta})u = c_1 z^2 + c_2\overline{z}^2 + c_3 z + c_4\overline{z} + c_5,$$

and therefore $Q_\alpha(\widetilde{\Delta})u$ is a harmonic function. So

$$B[Q_\alpha(\widetilde{\Delta})u] = Q_\alpha(\widetilde{\Delta})u.$$

Now we have $Q_\alpha(\widetilde{\Delta})u = f_1\overline{g}_2$ and hence $f_1\overline{g}_2$ is harmonic. This implies that either f_1 or g_2 is a constant and therefore f is co-analytic or g is analytic. □

Further we have the following corollaries to the above theorem that are analogous to the unweighted case. We leave the proof for the interested reader.

Corollary 11. *Let $f, g \in L^\infty(\mathbb{D})$, $h \in L^\infty(\mathbb{D}) \cap C^{2\alpha}(\mathbb{D})$ and assume that f, g and h are harmonic. Suppose $T_fT_g = T_h$ on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$, $\alpha \in \mathbb{N}$. Then one of the following holds.*

(i) *f and g are analytic;*
(ii) *f and g are co-analytic;*
(iii) *f is a constant;*
(iv) *g is a constant.*

Corollary 12. *Let $f, g, \in L^\infty(\mathbb{D})$ and assume that they are harmonic. Suppose $T_f T_g = 0$ on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$, $\alpha \in \mathbb{N}$. Then either $f = 0$ or $g = 0$.*

Corollary 13. *Let $f, g, \in L^\infty(\mathbb{D})$ and assume that they are harmonic. Suppose $T_f T_g = T_{fg}$ on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$, $\alpha \in \mathbb{N}$. Then either f is co-analytic or g is analytic.*

The following proposition is a generalization of Proposition 9 and we omit the proof since it is very similar to that of Proposition 9.

Proposition 14. *Suppose that for $k = 1, \ldots, n$, functions $f_k = f_{k,1} + \overline{f}_{k,2}$ and $g_k = g_{k,1} + \overline{g}_{k,2}$ are bounded harmonic functions, $f_{k,1}$, $f_{k,2}$, $g_{k,1}$, $g_{k,2}$ are analytic functions and $h \in L^\infty(\mathbb{D})$. Then the following are equivalent on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$, $\alpha > -1$.*

(1) $\sum\limits_{k=1}^{n} T_{f_k} T_{g_k} = T_h$.

(2) *For all $z \in \mathbb{D}$,*

$$\sum_{k=1}^{n} (f_{k,1}(z) g_{k,1}(z) + \overline{f}_{k,2}(z) \overline{g}_{k,2}(z) + f_{k,1}(z) \overline{g}_{k,2}(z) = B_\alpha\Bigl(h - \sum_{k=1}^{n} \overline{f}_{k,2}(z) g_{k,1}(z)\Bigr).$$

(3) *For all $z, w \in \mathbb{D}$,*

$$\sum_{k=1}^{n} (f_{k,1}(z) g_{k,1}(z) + \overline{f}_{k,2}(\overline{w}) \overline{g}_{k,2}(\overline{w}) + f_{k,1}(z) \overline{g}_{k,2}(\overline{w}))$$

$$= \int_{\mathbb{D}} \frac{h - \sum_{k=1}^{n} \overline{f}_2(\zeta) g_1(\zeta)}{(1 - \overline{\zeta} z)^{2+\alpha} (1 - \zeta w)^{2+\alpha}} \, \mathrm{d}A_\alpha(\zeta).$$

Next we state another result. Note that unlike in Theorem 10 here we need the function h to be in $C^{2\alpha+2}(\mathbb{D})$.

Theorem 15. *Suppose f_k, g_k, $k = 1, \ldots n$ are bounded harmonic functions on $\mathbb{D}$ and h is a bounded $C^{2\alpha+2}(\mathbb{D})$ function with $\widetilde{\Delta} h, \ldots, \widetilde{\Delta}^{\alpha+1} h \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$. Assume that $\sum\limits_{k=1}^{n} T_{f_k} T_{g_k} = T_h$, then $h = \sum\limits_{k=1}^{n} f_k g_k$ and $\sum\limits_{k=1}^{n} f'_{k,1} \overline{g}'_{k,2} = 0$.*

P r o o f. Assume that $\sum\limits_{k=1}^{n} T_{f_k} T_{g_k} = T_h$, then by using Proposition 14 we get

$$\sum_{k=1}^{n} (f_{k,1} g_{k,1} + \overline{f}_{k,2} \overline{g}_{k,2} + f_{k,1} \overline{g}_{k,2}) = B_\alpha\Bigl(h - \sum_{k=1}^{n} \overline{f}_{k,2} g_{k,1}\Bigr).$$

Since the Berezin transform reproduces harmonic functions we have

$$\sum_{k=1}^{n} f_{k,1}(z) \overline{g}_{k,2} = B_\alpha\Bigl(h - \sum_{k=1}^{n} \overline{f}_{k,2} g_{k,1} + f_{k,1} g_{k,1} + \overline{f}_{k,2} \overline{g}_{k,2}\Bigr).$$

Hence we have

$$B_\alpha u = \sum_{k=1}^{n} f_{k,1}\overline{g}_{k,2}, \tag{3.1}$$

where $u = h - \sum_{k=1}^{n} (\overline{f}_{k,2}g_{k,1} + f_{k,1}g_{k,1} + \overline{f}_{k,2}\overline{g}_{k,2})$, and $\alpha \in \mathbb{N}$. Then by Lemma 6 we have

$$B[Q_\alpha(\widetilde{\Delta})u] = \sum_{k=1}^{n} f_{k,1}\overline{g}_{k,2}.$$

Now by Rao (see [11]) we conclude that $Q_\alpha(\widetilde{\Delta})u$ must be of the form

$$\tilde{h}(z) + \sum_{k=1}^{n} D_k \ln|z - a_k| + \frac{E_k}{(z - a_k)} + \frac{F_k}{(\overline{z} - \bar{a}_k)}$$

for some constants D_k, E_k, and F_k, $k = 1, \dots, n$. But our function $Q_\alpha(\widetilde{\Delta})u$ is continuous. Hence we conclude that $Q_\alpha(\widetilde{\Delta})u = \tilde{h}$ and hence $Q_\alpha(\widetilde{\Delta})u$ is harmonic. So $B[Q_\alpha(\widetilde{\Delta})u] = Q_\alpha(\widetilde{\Delta})u = \sum_{k=1}^{n} f_{k,1}\overline{g}_{k,2}$ is harmonic. Therefore we have

$$\widetilde{\Delta}[B_\alpha u] = \widetilde{\Delta}\left(\sum_{k=1}^{n} f_{k,1}\overline{g}_{k,2}\right) = 0.$$

Hence $\sum_{k=1}^{n} f'_{k,1}\overline{g}'_{k,2} = 0$. On the other hand, $\widetilde{\Delta}[B_\alpha u] = B_\alpha[\widetilde{\Delta}u] = 0$. By Lemma 1 we have $\widetilde{\Delta}u = 0$. Hence u is harmonic. Therefore by (3.1) we get

$$B_\alpha u = u = \sum_{k=1}^{n} f_{k,1}g_{k,1} = h - \sum_{k=1}^{n}(\overline{f}_{k,2}g_{k,1} + f_{k,1}g_{k,1} + \overline{f}_{k,2}\overline{g}_{k,2}).$$

That is,

$$h = \sum_{k=1}^{n}(\overline{f}_{k,2}g_{k,1} + f_{k,1}g_{k,1} + \overline{f}_{k,2}\overline{g}_{k,2} + f_{k,1}g_{k,1}) = \sum_{k=1}^{n} f_k g_k.$$

□

Now we turn to commuting properties. There is an extensive literature on commuting Toeplitz operators on various Hilbert spaces. In particular Brown and Halmos in [6] obtained the necessary and sufficient conditions on the symbols f and g in order that T_f and T_g commute. Motivated by their results on the Hardy space, later in 1991 Axler and Čučković in [4] showed an analogue on the Bergman space on $\mathbb{D}$ as follows.

Theorem 16 ([4]). *Let $f, g \in L^\infty(\mathbb{D})$ and assume that they are harmonic. Then $T_f T_g = T_g T_f$ on $A^2(\mathbb{D}, \mathrm{d}A)$ if and only if one of the following is true.*

(i) *f and g are both analytic.*

(ii) *f and g are both co-analytic.*

(iii) *There exist $a, b \in \mathbb{C}$, not both zero, such that $af + bg$ is a constant.*

A general version of the above theorem can be found in [12]. Later Čučković and Rao in [9] showed that if a Toeplitz operator with a radial symbol commutes with another Toeplitz operator, then the symbol of the second Toeplitz operator must also be a radial function. Furthermore, Axler, Čučković and Rao in [5] proved the following theorem for general domains.

Theorem 17 ([5]). *Let Ω be any bounded open domain in $\mathbb{C}$. Let f be a nonconstant bounded analytic function on Ω and assume that $T_f T_g = T_g T_f$ on $A^2(\Omega, \mathrm{d}A)$. Then g is analytic.*

As a direct consequence of Theorem 15, we show the following result. Let $[T_f, T_g] = T_f T_g - T_g T_f$ denote the commutator of two Toeplitz operators T_f and T_g.

Corollary 18. *Let $\alpha \in \mathbb{N}$. Suppose that $f_k = f_{k,1} + \overline{f}_{k,2}$ and $g_k = g_{k,1} + \overline{g}_{k,2}$ are bounded harmonic functions and $f_{k,1}$, $f_{k,2}$, $g_{k,1}$, $g_{k,2}$ are analytic functions. Let $h \in L^\infty(\mathbb{D}) \cap C^{2\alpha+2}(\mathbb{D})$. Assume $\widetilde{\Delta} h, \dots, \widetilde{\Delta}^{\alpha+1} h \in L^1(\mathbb{D}, \mathrm{d}A_\alpha)$ and*

$$\sum_{k=1}^{m} [T_{f_k}, T_{g_k}] = T_h,$$

on $A^2(\mathbb{D}, \mathrm{d}A_\alpha)$. Then $h = 0$.

4. Compactness of the product of Hankel operators on the bidisk

After proving our main result in the preceding section, now we are ready to obtain a necessary condition for the product of two Hankel operators $H_\psi^* H_\varphi$ to be compact on the weighted Bergman space on the bidisk $\mathbb{D}^2$, where φ, ψ are bounded pluriharmonic functions on $\mathbb{D}^2$.

Let $\mathrm{d}V_{\alpha,\beta}(z, w) = (1 + \alpha)(1 + \beta)(1 - |z|^2)^\alpha (1 - |w|^2)^\beta \mathrm{d}A(z)\mathrm{d}A(w)$ be the normalized weighted Lebesgue volume measure on $\mathbb{D}^2$ and $\alpha, \beta > -1$. The Bergman space $A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta})$ is the space of analytic, square integrable functions with respect to $\mathrm{d}V_{\alpha,\beta}$.

For a bounded symbol φ on $\mathbb{D}^2$, the Hankel operator

$$H_\varphi \colon A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta}) \to A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta})^\perp$$

is defined by

$$H_\varphi f = (I - P)(\varphi f), \quad f \in A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta}).$$

Here $P\colon L^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta}) \to A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta})$ is the orthogonal projection.

Čučković and Şahutoğlu in [10] studied how the boundary behavior of the symbols φ, ψ interacts with the compactness of the product of Hankel operators $H_\psi^* H_\varphi$. More precisely, they obtained the following result.

Let $\zeta_j \in \partial\mathbb{D}$. We define $D_{\zeta_j} = \{(z_1, \dots, z_n)\colon z_j = \zeta \text{ and } |z_k| < 1 \text{ for all } k \neq j\}$.

Theorem 19 ([10]). *Let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$ and take the symbols $\phi, \psi \in C(\overline{\mathbb{D}}^n)$ such that ϕ and ψ are pluriharmonic on any $(n-1)$-dimensional polydisk in the boundary of $\mathbb{D}^n$. Then $H_\psi^* H_\phi$ is compact on $A^2(\mathbb{D}^n, \mathrm{d}A)$ if and only if for every $1 \leqslant j, k \leqslant n$ such that $j \neq k$ and any $(n-1)$-dimensional polydisk D_{ζ_j}, orthogonal to the z_j-axis in the boundary of $\mathbb{D}^n$, either ϕ or ψ is analytic in z_k on D_{ζ_j}.*

Motivated by the above theorem we were interested in an analogous result in the weighted Bergman space $A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta})$. We proved the following theorem.

Theorem 20. *Let $\alpha, \beta \in \mathbb{N}$ and let $\varphi, \psi \in C(\overline{\mathbb{D}}^2)$ such that $\varphi \circ f$ and $\psi \circ f$ are harmonic for all analytic functions $f\colon \mathbb{D} \to \partial\mathbb{D}^2$. Assume $H_\psi^* H_\varphi$ is compact on $A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta})$. Then either $\varphi \circ f$ or $\psi \circ f$ is analytic for all such f.*

Proof. Assume that $H_\psi^* H_\phi$ is compact on $A^2(\mathbb{D}^2, \mathrm{d}V_{\alpha,\beta})$. Let $p \in \partial\mathbb{D}$. Now we define functions φ_p, ψ_p as

$$\varphi_p(z, w) = \varphi(z, w) - \varphi(z, p) \tag{4.1}$$

and

$$\psi_p(z, w) = \psi(z, w) - \psi(z, p). \tag{4.2}$$

We denote the disks in the boundary of $\mathbb{D}^2$ as follows.

$$D_z(w) = \{(z, w)\colon w \in \mathbb{D}, |z| = 1\} \quad \text{and} \quad D_w(z) = \{(z, w)\colon z \in \mathbb{D}, |w| = 1\}.$$

To prove inequality (4.5), let us choose a sequence of complex numbers $(p_j)_j \subset \mathbb{D}$ such that $p_j \xrightarrow{j} p$ and let us fix a function $F \in A^2(\mathbb{D}, \mathrm{d}A_\alpha(z))$ such that

$$\|F\|_{L^2(\mathbb{D}, \mathrm{d}A_\alpha)} \leqslant 1. \tag{4.3}$$

Now define a sequence of functions $f_j(z,w) = F(z)k^{\beta}_{p_j}(w)$, where $k^{\beta}_{p_j}(w)$ is the normalized weighted Bergman kernel at p_j. Then we also have

$$\|f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} \leqslant 1. \tag{4.4}$$

First we will show that for a given $\varepsilon > 0$,

$$\|H_{\varphi_p}(f_j)\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} + \|H_{\psi_p}(f_j)\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} < \varepsilon \tag{4.5}$$

for large j. Let $\varepsilon > 0$ and let $S_p = \{w \in \mathbb{D} \colon |w-p| < \delta\}$, where δ is defined below and $\widetilde{D} = \mathbb{D} \setminus S_p$. Then we have

$$\begin{aligned}
\|\varphi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} &= \int_{\mathbb{D}^2} |\varphi_p(z,w)|^2 |f_j(z,w)|^2 \,\mathrm{d}V_{\alpha,\beta} \\
&= \int_{\mathbb{D}\times S_p} |\varphi_p(z,w)|^2 |f_j(z,w)|^2 \,\mathrm{d}V_{\alpha,\beta} \\
&\quad + \int_{\mathbb{D}\times\widetilde{\mathbb{D}}} |\varphi_p(z,w)|^2 |f_j(z,w)|^2 \,\mathrm{d}V_{\alpha,\beta} \\
&\leqslant \Bigl(\sup_{\mathbb{D}\times S_p} |\varphi_p(z,w)|^2\Bigr) \int_{\mathbb{D}^2} |f_j(z,w)|^2 \,\mathrm{d}V_{\alpha,\beta} \\
&\quad + \Bigl(\sup_{\overline{\mathbb{D}}^2} |\varphi_p(z,w)|^2\Bigr) \int_{\mathbb{D}\times\widetilde{\mathbb{D}}} |f_j(z,w)|^2 \,\mathrm{d}V_{\alpha,\beta}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|\varphi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} &\leqslant \Bigl(\sup_{\mathbb{D}\times S_p} |\varphi_p(z,w)|^2\Bigr) \|f_j\|^2_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} \\
&\quad + \Bigl(\sup_{\overline{\mathbb{D}}^2} |\varphi_p(z,w)|^2\Bigr) \int_{\mathbb{D}} |F(z)|^2 \,\mathrm{d}A_\alpha(z) \int_{\widetilde{\mathbb{D}}} \frac{(1-|p_j|^2)^{2+\beta}}{|1-\bar{p}_j w|^{4+2\beta}} \,\mathrm{d}A_\beta(w) \\
&\leqslant \Bigl(\sup_{\mathbb{D}\times S_p} |\varphi_p(z,w)|^2\Bigr) \|f_j\|^2_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} \\
&\quad + \Bigl(\sup_{\overline{\mathbb{D}}^2} |\varphi_p(z,w)|^2\Bigr) \|F\|^2_{L^2(\mathbb{D},\mathrm{d}A_\alpha)} \int_{\widetilde{\mathbb{D}}} \frac{(1-|p_j|^2)^{2+\beta}}{|1-\bar{p}_j w|^{4+2\beta}} \,\mathrm{d}A_\beta(w).
\end{aligned}$$

Since $\varphi_p(z,w) \in C(\overline{\mathbb{D}}^2)$ and $\varphi_p(z,p) = 0$, we can choose $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{\mathbb{D}\times S_p} |\varphi_p(z,w)|^2 < \frac{\varepsilon}{4}.$$

Now we choose $J(\delta)$ large enough so that

$$\int_{\widetilde{\mathbb{D}}} \frac{(1-|p_j|^2)^{2+\beta}}{|1-\bar{p}_j w|^{4+2\beta}} \,\mathrm{d}A_\beta(w) < \frac{\varepsilon}{4(\sup_{\overline{\mathbb{D}^2}|\varphi_p(z,w)|^2+1})}$$

for all $j > J(\delta)$. Hence using equations (4.3) and (4.4) we get $\|\varphi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} < \frac{1}{2}\varepsilon$ and similarly $\|\psi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} < \frac{1}{2}\varepsilon$. Hence

$$\|\varphi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} + \|\psi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}A_{\alpha,\beta})} < \varepsilon \quad \text{for all } j > j(\delta).$$

Therefore

$$\begin{aligned}(4.6)\qquad &\|H_{\varphi_p}(f_j)\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} + \|H_{\psi_p}(f_j)\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})}\\ &\qquad = \|(1-P)\varphi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} + \|(1-P)\psi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})}\\ &\qquad \leqslant \|1-P\|\,\|\varphi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})} + \|1-P\|\,\|\psi_p f_j\|_{L^2(\mathbb{D}^2,\mathrm{d}V_{\alpha,\beta})}\\ &\qquad < \varepsilon \quad \text{for all } j > j(\delta).\end{aligned}$$

Let us denote $\varphi_1(z) = \varphi(z,p)$ and $\psi_1(z) = \varphi(z,p)$. Then by equations (4.1) and (4.2) we have

$$\varphi_1(z) = \varphi(z,w) - \varphi_p(z,w)$$

and

$$\psi_1(z) = \psi(z,w) - \psi_p(z,w).$$

Hence we have,

$$\begin{aligned}(4.7)\qquad |\langle H_{\varphi_1}(f_j), H_{\psi_1}(f_j)\rangle| &= |\langle (H_\varphi - H_{\varphi_p})(f_j), (H_\psi - H_{\psi_p})(f_j)\rangle|\\ &\leqslant |\langle H_\psi^* H_\varphi(f_j), f_j\rangle| + \|H_\varphi f_j\|\,\|H_{\psi_p} f_j\|\\ &\quad + \|H_{\varphi_p} f_j\|\,\|H_\psi f_j\| + \|H_{\varphi_p} f_j\|\,\|H_{\psi_p} f_j\|.\end{aligned}$$

By assumption $H_\psi^* H_\varphi$ is compact and $f_j \xrightarrow{\text{weak}} 0$, so

$$(4.8)\qquad |\langle H_\psi^* H_\varphi(f_j), f_j\rangle| \to 0.$$

Since both φ and ψ are bounded, H_φ and H_ψ are also bounded. Hence using inequality (4.6) we have

$$(4.9)\qquad \|H_\varphi f_j\|\,\|H_{\psi_p} f_j\| \xrightarrow{j} 0$$

and

$$(4.10)\qquad \|H_\psi f_j\|\,\|H_{\varphi_p} f_j\| \xrightarrow{j} 0.$$

Again using equation (4.6) we have

$$\|H_{\varphi_p} f_j\| \, \|H_{\psi_p} f_j\| \xrightarrow{j} 0. \tag{4.11}$$

Now using (4.7), (4.8), (4.9), (4.10) and (4.11) we conclude that

$$|\langle H_{\varphi_1}(f_j), H_{\psi_1}(f_j)\rangle| \xrightarrow{j} 0. \tag{4.12}$$

But here

$$\begin{aligned} H_{\varphi_1}(f_j)(z,w) &= (I-P)(\varphi_1 f_j)(z,w) \\ &= \varphi_1(z)F(z)k^{\beta}_{p_j}(w) - P(\varphi_1 F k^{\beta}_{p_j})(z,w) \\ &= \varphi_1(z)F(z)k^{\beta}_{p_j}(w) \\ &\quad - \int_{\mathbb{D}^2} \varphi_1(\zeta)F(\zeta)k^{\beta}_{p_j}(\eta)\overline{K^{\alpha}_z(\zeta)K^{\beta}_w(\eta)}\,\mathrm{d}V_{\alpha,\beta}(\zeta,\eta) \\ &= \varphi_1(z)F(z)k^{\beta}_{p_j}(w) \\ &\quad - \int_{\mathbb{D}} \varphi_1(\zeta)F(\zeta)\overline{K^{\alpha}_z(\zeta)}\,\mathrm{d}A_{\alpha}(\zeta) \int_{\mathbb{D}} k^{\beta}_{p_j}(\eta)\overline{K^{\beta}_w(\eta)}\,\mathrm{d}A_{\beta}(\eta) \\ &= \varphi_1(z)F(z)k^{\beta}_{p_j}(w) - P(\varphi_1 F)(z)k^{\beta}_{p_j}(w) \\ &= (I-P)(\varphi_1 F)(z)k^{\beta}_{p_j}(w) = (H_{\varphi_1}F)(z)k^{\beta}_{p_j}(w). \end{aligned} \tag{4.13}$$

Now using the fact that $\|k^{\beta}_{p_j}\|^2_{L^2(\mathbb{D},\mathrm{d}A_\beta)} = 1$ and by equation (4.13) we get

$$\langle H_{\varphi_1}(f_j), H_{\psi_1}(f_j)\rangle = \langle H_{\varphi_1}(F), H_{\psi_1}(F)\rangle \|k^{\beta}_{p_j}\|^2_{L^2(\mathbb{D},\mathrm{d}A_\beta)} = \langle H_{\varphi_1}(F), H_{\psi_1}(F)\rangle.$$

Therefore the compactness of $H^*_\psi H_\phi$ implies that

$$\langle H_{\varphi_1}(F), H_{\psi_1}(F)\rangle = 0 \quad \text{for all } F \in A^2(\mathbb{D}, \mathrm{d}A_\alpha).$$

Hence

$$\langle H_{\varphi_1}k^{\alpha}_z, H_{\psi_1}k^{\alpha}_z\rangle = 0, \quad z \in \mathbb{D}.$$

Now let $\varphi_1 = f_1 + \overline{f}_2$ and $\psi_1 = g_1 + \overline{g}_2$, where $f_1, f_2, g_1, g_2 \in H(\mathbb{D})$. Then

$$\begin{aligned} 0 &= \langle H_{\varphi_1}k^{\alpha}_z, H_{\psi_1}k^{\alpha}_z\rangle = \langle H_{\overline{g}_2}k^{\alpha}_z, H_{\overline{f}_2}k^{\alpha}_z\rangle = \langle H^*_{\overline{f}_2}H_{\overline{g}_2}k^{\alpha}_z, k^{\alpha}_z\rangle \\ &= \langle (T_{f_2\overline{g}_2} - T_{f_2}T_{\overline{g}_2})k^{\alpha}_z, k^{\alpha}_z\rangle. \end{aligned} \tag{4.14}$$

Hence we have

$$\begin{aligned} \langle T_{f_2\overline{g}_2}k^{\alpha}_z, k^{\alpha}_z\rangle &= \langle T_{f_2}T_{\overline{g}_2}k^{\alpha}_z, k^{\alpha}_z\rangle, \\ B_\alpha(f_2\overline{g}_2)(z) &= \overline{g}_2(z)\langle T_{f_2}k^{\alpha}_z, k^{\alpha}_z\rangle = \overline{g}_2(z)f_2(z)k^{\alpha}_z(z), \\ B_\alpha(f_2\overline{g}_2)(z) &= (f_2\overline{g}_2)(z). \end{aligned} \tag{4.15}$$

So using Lemma 6, equation (4.15) can be written as

$$B[Q_\alpha(\widetilde{\Delta})(f_2\overline{g}_2)] = f_2\overline{g}_2.$$

Now as proved in Theorem 10 we conclude that f_2 or g_2 is a constant. Hence φ_1 or ψ_1 is analytic in z and hence $\varphi|_{D_w(z)}$ or $\psi|_{D_w(z)}$ is analytic in z. $\square$

5. Final remarks

Our main result, Theorem 10, is valid only when α is a positive integer. And also this theorem is proved on $\mathbb{D}$. We would like to ask the following two questions. Firstly, is Theorem 10 true for any $\alpha > -1$? Secondly, is Theorem 10 true on the polydisk, at least when α is a positive integer?

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