Czechoslovak Mathematical Journal

Erfan Manouchehri; Ali Soleyman Jahan

The linear syzygy graph of a monomial ideal and linear resolutions

Czechoslovak Mathematical Journal, Vol. 71 (2021), No. 3, 785–802

Persistent URL: `http://dml.cz/dmlcz/149056`

Czechoslovak Mathematical Journal, 71 (146) (2021), 785–802

THE LINEAR SYZYGY GRAPH OF A MONOMIAL IDEAL AND LINEAR RESOLUTIONS

Erfan Manouchehri, Marivan, Ali Soleyman Jahan, Sanadaj

Received March 4, 2020. Published online November 18, 2020.

Abstract. For each squarefree monomial ideal $I \subset S = k[x_1, \ldots, x_n]$, we associate a simple finite graph G_I by using the first linear syzygies of I. The nodes of G_I are the generators of I, and two vertices u_i and u_j are adjacent if there exist variables x, y such that $xu_i = yu_j$. In the cases, where G_I is a cycle or a tree, we show that I has a linear resolution if and only if I has linear quotients and if and only if I is variable-decomposable. In addition, with the same assumption on G_I, we characterize all squarefree monomial ideals with a linear resolution. Using our results, we characterize all Cohen-Macaulay codimension 2 monomial ideals with a linear resolution. As another application of our results, we also characterize all Cohen-Macaulay simplicial complexes in the case, where $G_\Delta \cong G_{I_{\Delta^\vee}}$ is a cycle or a tree.

Keywords: monomial ideal; linear resolution, linear quotient; variable-decomposability; Cohen-Macaulay simplicial complex

MSC 2020: 13D02, 13F55, 13F20

1. Introduction

Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$. Then there is a minimal graded free S-resolution for I of the form $0 \mapsto F_p \mapsto \ldots \mapsto F_1 \mapsto F_0 \mapsto I \mapsto 0$, where $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ and $S(-j)$ denotes the free S-module obtained by shifting the degrees of S by j. The numbers $\beta_{ij} = \beta_{ij}(I)$ are called the *graded Betti numbers* of I. Recall that I has a d-linear resolution over k if $\beta_{ij}(I) = 0$ for all $j \neq i + d$. Let $\varphi\colon F_0 \mapsto I$ be the map which sends the basis element e_is of F_0 to the generators u_i of I. Recall that I has linear relations if the kernel of φ is generated by linear forms. Note that if the ideal I has a linear resolution or has linear relations, then all of its generators have the same degree. In general, it is not easy to find ideals with linear resolution. Note that the free S-resolution of a monomial ideal and its linearity depends in general on the characteristic of the base field.

DOI: 10.21136/CMJ.2020.0099-20

We denote by $G(I)$ the unique minimal monomial set of generators of the monomial ideal I. We say that I has linear quotients if there exists an order $\sigma = u_1, \ldots, u_m$ of $G(I)$ such that the colon ideal $(u_1, \ldots, u_{i-1})\colon (u_i)$ is generated by a subset of the variables, for $i = 2, \ldots, m$. Ideals with linear quotients were introduced by Herzog and Takayama, see [12]. Note that having linear quotients is a purely combinatorial property of an ideal I and hence does not depend on the characteristic of the base field. Suppose that I is a graded ideal generated in degree d. It is known that if I has linear quotients, then I has a d-linear resolution, see [10], Proposition 8.2.1.

The concept of variable-decomposable monomial ideal was first introduced by Rahmati and Yassemi (see [14]) as a concept dual to vertex-decomposable simplicial complexes. We denote by $\Delta^\vee$ the Alexander dual of Δ. In the case, where I is the Stanley-Reisner ideal of $\Delta^\vee$, they proved that I is variable-decomposable if and only if Δ is vertex-decomposable. Also they proved that if a monomial ideal I is variable-decomposable, then it has linear quotients. Hence, for monomial ideal generated in one degree we have the following implications:

$$I \text{ is variable-decomposable } \Rightarrow I \text{ has linear quotients } \Rightarrow I \text{ has a linear resolution.}$$

However, there are ideals with linear resolution but without linear quotients (see [5]) and ideals with linear quotients which are not variable-decomposable, see [14], Example 2.24.

The problem of characterizing ideals with 2-linear resolution is completely solved by Fröberg in [9] (see also [13]). Any ideal of S which is generated by squarefree monomials of degree 2 can be assumed as edge ideal of a finite simple graph. Fröberg proved that the edge ideal of a finite simple graph G has a linear resolution if and only if the complementary graph $\overline{G}$ of G is chordal. Trying to generalize the result of Fröberg for monomial ideals generated in degree d, $d \geqslant 3$, is an interesting problem which several mathematicians including Emtander (see [7]) and Woodroofe (see [16]) have worked on.

It is known that monomial ideals with 2-linear resolution have linear quotients, see [11]. Let $I = I_{\Delta^\vee}$ be a squarefree monomial ideal generated in degree d which has a linear resolution. By a result of Eagon-Reiner (see [6]), one has that Δ is a Cohen-Macaulay complex of dimension $n - d$. In [2] the authors proved that if Δ is a Cohen-Macaulay simplicial complex of codimension 2, then Δ is vertex-decomposable. Hence, by [14], Theorem 2.10, $I_{\Delta^\vee}$ is a variable-decomposable monomial ideal generated in degree 2. Therefore for squarefree monomial ideals generated in degree 2, we have:

$$\begin{aligned} I \text{ has a linear resolution } &\Leftrightarrow I \text{ has linear quotients} \\ &\Leftrightarrow I \text{ is variable-decomposable ideal.} \end{aligned}$$

So it is natural to look for some other classes of monomial ideals with the same property. In this paper, we investigate some classes of monomial ideals with this property via combinatorial properties of simple graph G_I, which we associated to a squarefree monomial ideal I generated in degree $d \geqslant 2$. We show that in the cases, where G_I is a cycle or a tree, these three statements are equivalent.

The paper proceeds as follows. In Section 2, we associated a simple graph G_I to a squarefree monomial ideal I generated in degree $d \geqslant 2$. Let C_m denote the m-cycle on vertex set $V = \{1, \ldots, n\}$. In Theorem 2.2, we show that if $G_I \cong C_m$, $m \geqslant 4$, then I has a linear resolution if and only if it has linear quotients and this is equivalent to saying that I is a variable-decomposable. With the same assumption on G_I, we characterize all monomial ideals with a linear resolution.

In Section 3, we consider the monomial ideal I, where G_I is a tree. We prove that if I has linear relations, then G_I is a tree if and only if $\operatorname{projdim}(I) = 1$ (see Theorem 3.2). In Theorem 3.3 we show that if G_I is a tree, then the following are equivalent:

(a) I has a linear resolution.
(b) I has linear relations.
(c) $G_I^{(u,v)}$ is a connected graph for all u and v in $G(I)$.
(d) If $u = u_1, u_2, \ldots, u_s = v$ is the unique path between u and v in G_I, then $F(u_j) \subset F(u_i) \cup F(u_k)$ for all $1 \leqslant i \leqslant j \leqslant k \leqslant s$.
(e) L has a linear resolution for all $L \subseteq I$, where $G(L) \subset G(I)$ and G_L is a path.

In addition, it is shown that I has a linear resolution if and only if it has linear quotients if and only if it is variable-decomposable, provided that G_I is a tree (see Theorem 3.4).

In Section 4, as an application of our results in Corollary 4.1, we characterize all Cohen-Macaulay monomial ideals of codimension 2 with a linear resolution. Let $t \geqslant 2$ and $I_t(C_n)$ ($I_t(L_n)$) be the path ideal of length t for n-cycle C_n (n-path L_n). We show that $I_t(C_n)$ ($I_t(L_n)$) has a linear resolution if and only if $t = n - 2$ or $t = n - 1$ ($t \geqslant \frac{1}{2}n$), see Corollaries 4.2 and 4.3.

Finally, we consider the simplicial complex $\Delta = \langle F_1, \ldots, F_m \rangle$, where F_is are the facets of Δ. It is shown that Δ is connected in codimension one if and only if $G_{I_{\Delta^\vee}}$ is a connected graph, see Lemma 5.1. In Corollary 5.1, we show that $I_{\Delta^\vee}$ has linear relations if and only if $\Delta^{(F,G)}$ is connected in codimension one for all facets F and G of Δ. Also, we introduce a simple graph G_Δ on vertex set $\{F_1, \ldots, F_m\}$ which is isomorphic to $G_{I_{\Delta^\vee}}$. As an other application of our results, we show that if G_Δ is a cycle or a tree, then the following are equivalent:

(a) Δ is Cohen-Macaulay.
(b) Δ is pure shellable.
(c) Δ is pure vertex-decomposable.

In addition, with the same assumption on G_Δ, all Cohen-Macaulay simplicial complexes are characterized.

2. Monomial ideals whose G_I is a cycle

First, we recall some definitions and known facts which will be useful later.

Proposition 2.1 ([10], Proposition 8.2.1)**.** *Suppose $I \subseteq S$ is a monomial ideal generated in degree d. If I has linear quotients, then I has a d-linear resolution.*

Let $u = x_1^{a_1} \dots x_n^{a_n}$ be a monomial in S. Set $F(u) := \{i\colon\ a_i > 0\} = \{i\colon\ x_i \mid u\}$. For another monomial v we set $[u, v] = 1$ if $x_i^{a_i} \nmid v$ for all $i \in F(u)$. Otherwise, we set $[u, v] \neq 1$. For a monomial ideal $I \subseteq S$, set $I_u = (u_i \in G(I)\colon\ [u, u_i] = 1)$ and $I^u = (u_j \in G(I)\colon\ [u, u_j] \neq 1)$.

Definition 2.1 ([14])**.** Let I be a monomial ideal with $G(I) = \{u_1, \dots, u_m\}$. A monomial $u = x_1^{a_1} \dots x_n^{a_n}$ is called *shedding* if $I_u \neq 0$ and for each $u_i \in G(I_u)$ and $l \in F(u)$ there exists $u_j \in G(I^u)$ such that $u_j : u_i = x_l$. A monomial ideal I is *r-decomposable* if $m = 1$ or else has a shedding monomial u with $|F(u)| \leqslant r+1$ such that the ideals I_u and I^u are r-decomposable. A monomial ideal is decomposable if it is r-decomposable for some $r \geqslant 0$. A 0-decomposable ideal is called *variable-decomposable.*

Example 2.1. Let $I = (x_1x_2x_3, x_1^2x_2, x_2x_3^2)$. It is easy to see that x_1 is a shedding monomial for I, $I^{x_1} = (x_1x_2x_3, x_1^2x_2)$ and $I_{x_1} = (x_2x_3^2)$. It is clear that x_1^2 is a shedding monomial for I^{x_1} and hence, I is a decomposable ideal.

In [14] the authors proved the following result:

Theorem 2.1. *Let I be a monomial ideal. Then I is decomposable if and only if it has linear quotients.*

Let G be a finite simple graph on vertex set $[n] = \{1, \dots, n\}$ with edge set $E(G)$. A path of length t is a sequence $i_1, i_2, \dots, i_t, i_{t+1}$ of $t+1$ distinct vertices, where $\{i_j, i_{j+1}\}$ is an edge for $1 \leqslant j \leqslant t$. A cycle is a path that begins and ends at the same vertex. A connected graph G is called a *tree* if it has no cycle. A vertex i is called a *leaf* if there exists a vertex j, called *branch* of i, such that $\{i, j\}$ is an edge in G and for each vertex $k \neq j$, $\{i, k\}$ is not an edge in G. An induced subgraph of G on $B \subset [n]$ is a graph with vertex set B together with any edges whose endpoints are both in B.

We associate to an ideal I a simple graph G_I whose vertices are labeled by the elements of $G(I)$. Two vertices u_i and u_j are adjacent if there exist variables x, y such that $xu_i = yu_j$. We called it the *first syzygies graph* of I. This graph was first introduced by Bigdeli, Herzog and Zaare-Nahandi, see [3].

Remark 2.1. If I is a squarefree monomial ideal, then two types of 3-cycle u_{i_1}, u_{i_2}, u_{i_3}, u_{i_1} may appear in G_I.

(i) If $F(u_{i_1}) = A\cup\{j,k\}$, $F(u_{i_2}) = A\cup\{i,k\}$ and $F(u_{i_3}) = A\cup\{i,j\}$, then we have $x_ie_{i_1} - x_ke_{i_3} = (x_ie_{i_1} - x_je_{i_2}) + (x_je_{i_2} - x_ke_{i_3})$ and, hence, one of the linear forms can be written as a linear combination of two other linear forms.

(ii) If $F(u_{i_1}) = A\cup\{i\}$, $F(u_{i_2}) = A\cup\{j\}$ and $F(u_{i_3}) = A\cup\{k\}$, then the three linear forms are independent.

The number of the minimal generating set of $\ker(\varphi)$ in degree $d+1$ is $\beta_{1(d+1)}$ and $\beta_{1(d+1)} \leqslant |E(G_I)|$. It is clear that equality holds if G_I has no C_3 of type (i). In this paper, we assume that G_I has no 3 cycle of type (i). Our aim is to study minimal free resolution of I via some combinatorial properties of G_I. Set $x_F := \prod\limits_{i\in F} x_i$ for each $F\subset [n] = \{1,\ldots,n\}$.

Remark 2.2. Let I be a squarefree monomial ideal. If $u_i = x_{F_i}$ and $u_j = x_{F_j}$ are two elements in $G(I)$ such that $w_iu_i = w_ju_j$, then there exists a monomial $w\in S$ such that $w_i = wx_{F_j\setminus F_i}$ and $w_j = wx_{F_i\setminus F_j}$.

Lemma 2.1. *If I is a squarefree monomial ideal and $u_{i_1}, u_{i_2},\ldots,u_{i_{t-1}}, u_{i_t}$ is a path in G_I, then one can obtain minimal monomials (with respect to divisibility) w_i and w_j from the given path such that $w_ie_{i_1} - w_je_{i_t}$ belong to* $\ker(\varphi)$ *and* $\deg w_i = \deg w_j \leqslant t-1$.

Proof. It is clear that $t-1$ linear forms $(x_{k_1}e_{i_1} - x_{k'_2}e_{i_2}), (x_{k_2}e_{i_2} - x_{k'_3}e_{i_3}),\ldots, (x_{k_t-1}e_{i_t-1} - x_{k'_t}e_{i_t})$ belong to $\ker(\varphi)$. Hence $(x_{k_2}x_{k_1}e_{i_1} - x_{k'_2}x_{k'_3}e_{i_3})\in\ker\varphi$. Again, we have $(x_{k_3}x_{k_2}x_{k_1}e_{i_1} - x_{k'_4}x_{k'_3}x_{k'_2}e_{i_4})\in\ker\varphi$. Continuing these procedures, we obtain w_i and w_j with the required property. □

The following example shows that the inequality $\deg w_i = \deg w_j \leqslant t-1$ can be pretty strict.

Example 2.2. Consider the monomial ideal $I = (u,v,w,z)\subset k[x_1,\ldots,x_5]$, where $u = x_1x_2x_3$, $w = x_1x_2x_4$, $z = x_1x_4x_5$ and $v = x_3x_4x_5$. It is easy to see that u,w,z,v is a path of length 3 between u and v, but $(x_4x_5e_u - x_1x_2e_v)\in\ker\varphi$.

Lemma 2.2. *Let I be a squarefree monomial ideal which has linear relations. Then G_I is a connected graph.*

Proof. For any $u_i, u_j\in G(I)$, there exist minimal monomials w_i and w_j such that $w_iu_i = w_ju_j$ and hence $w_ie_i - w_je_j\in\ker(\varphi)$. Since $\ker(\varphi)$ is generated by linear forms, one has:

$$w_ie_i - w_je_j = f_{i_1}(x_{k_1}e_i - x_{k'_2}e_{i_2}) + \ldots + f_{i_t}(x_{k_t}e_{i_t} - x_{k'_{t+1}}e_j),$$

where $f_{ij}\in S$ for $j = 0,\ldots,t$. Therefore $u_i, u_{i_2},\ldots,u_{i_t}, u_j$ is a path in G_I. □

The following example shows that the converse of Lemma 2.2 is not true in general.

Example 2.3. Consider the monomial ideal $I = (u, v, w, z, q) \subset k[x_1, \ldots, x_6]$, where $u = x_1x_2x_3$, $v = x_1x_2x_4$, $w = x_1x_4x_5$, $z = x_4x_5x_6$ and $q = x_3x_5x_6$. It is easy to see that G_I is a connected graph. However, I does not have linear relations. Computation with CoCoA (see [1]) shows that I has the minimal free S-resolution

$$0 \mapsto S(-6) \mapsto S(-4)^4 + S(-5) \mapsto S(-3)^5 \mapsto I \mapsto 0.$$

Remark 2.3. Let I be a squarefree monomial ideal and $u_{i_1}, u_{i_2}, \ldots, u_{i_t}$ be a path in G_I. Then by Lemma 2.1, there are monomials w, w' and f_{i_j} in S such that $we_{i_1} - w'e_{i_t} \in \ker(\varphi)$ and

$$we_{i_1} - w'e_{i_t} = f_{i_1}(x_{k_1}e_{i_1} - x_{k'_2}e_{i_2}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k'_t}e_{i_t}).$$

(i) If $r \in F(u_{i_t})$ and $r \notin F(u_{i_1})$, then x_r is the coefficient of some e_{i_j} in the equation which is given above.

(ii) If w and w' are minimal monomials (with respect to dividing) and $F(u_{i_j}) \subseteq F(u_{i_1}) \cup F(u_{i_t})$ for each j, $1 \leqslant j \leqslant t$, then $w = x_{F(u_{i_t})\setminus F(u_{i_1})}$ and $w' = x_{F(u_{i_1})\setminus F(u_{i_t})}$. Let $x_l \mid w$, by part (i), x_l is the coefficient of some e_{i_r} which appears in the above equation. Hence, there exist u_{i_j} such that $l \in F(u_{i_j})$. Since $F(u_{i_j}) \subseteq F(u_{i_1}) \cup F(u_{i_t})$ and $l \notin F(u_{i_1})$, one has $l \in F(u_{i_t})$. So $x_l \nmid w'$. Similarly, for arbitrary x_r with $x_r \mid w'$, one has $x_r \nmid w$. Hence, we conclude that $w = x_{F(u_{it})\setminus F(u_{i_1})}$ and $w' = x_{F(u_{i_1})\setminus F(u_{i_t})}$.

Remark 2.4. Let w_{i_1} and w_{i_t} be two monomials in S such that $w_{i_1}e_{i_1} - w_{i_t}e_{i_t} \in \ker(\varphi)$ and

$$w_{i_1}e_{i_1} - w_{i_t}e_{i_t} = f_{i_1}(x_{k_1}e_{i_1} - x_{k'_2}e_{i_2}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k'_t}e_{i_t}).$$

If $x_i \nmid u_{i_1}$ and there exist u_{i_r} $(2 \leqslant r \leqslant t)$ such that $x_i \mid u_{i_r}$, then $x_i \mid w_{i_1}$. We may assume that r is the smallest number with the property that $x_i \mid u_{i_r}$. We know $f_{i_{r-2}}(x_{k_{r-2}}e_{i_{r-2}} - x_{k'_{r-1}}e_{i_{r-1}}) + f_{i_{r-1}}(x_i e_{i_{r-1}} - x_{k'_r}e_{i_r})$ is a part of the above equation. Since in the above equation $e_{i_{r-1}}$ must be eliminated, we have $f_{i_{r-1}}x_i = f_{i_{r-2}}x_{k'_{r-1}}$. Hence, $x_i \mid f_{i_{r-2}}$. Also, $e_{i_{r-2}}$ must be eliminated and hence one has $f_{i_{r-2}}x_{k_{r-2}} = f_{i_{r-3}}x_{k'_{r-2}}$. Therefore $x_i \mid f_{i_{r-3}}$. Continuing these procedures yields $x_i \mid f_{i_1}$, i.e., $x_i \mid w_{i_1}$. Similarly, if $x_i \nmid u_{i_t}$ and there exist u_{i_r} $(1 \leqslant r \leqslant t-1)$ such that $x_i \mid u_{i_r}$, then $x_i \mid w_{i_t}$.

For all $u, v \in G(I)$ let $G_I^{(u,v)}$ be the induced subgraph of G_I on vertex set $V(G_I^{(u,v)}) = \{w \in G(I)\colon F(w) \subseteq F(u) \cup F(v)\}$. The following fact was proved by Bigdeli, Herzog and Zaare-Nahandi, see [3]. Here we present a different proof.

Proposition 2.2. *Let I be a squarefree monomial ideal which is generated in degree d. Then I has linear relations if and only if $G_I^{(u,v)}$ is connected for all $u, v \in G(I)$.*

Proof. Assume that I has linear relations and $u, v \in G(I)$. We know that $x_{F(v)\setminus F(u)}e_u - x_{F(u)\setminus F(v)}e_v \in \ker(\varphi)$. Since $\ker(\varphi)$ is generated by linear forms, one has:

$$x_{F(v)\setminus F(u)}e_u - x_{F(u)\setminus F(v)}e_v = f_{i_1}(x_{k_1}e_{i_1} - x_{k'_2}e_{i_2}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k'_t}e_t).$$

Hence, $u = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ is a path in G_I. It is enough to show that $F(u_{i_j}) \subseteq F(u_{i_1}) \cup F(u_{i_t})$ for all i_j, $1 < j < t$. Assume to the contrary that there exists k, $1 < k < t$, such that $F(u_{i_k}) \nsubseteq F(u_{i_1}) \cup F(u_{i_t})$. Let $l \in F(u_{i_k})$ and $l \notin F(u_{i_1}) \cup F(u_{i_t})$. By Remark 2.4, $x_l \mid x_{F(v)\setminus F(u)}$ and $x_l \mid x_{F(u)\setminus F(v)}$, which is a contradiction.

Conversely, $\ker(\varphi)$ is generated by $x_{F_v\setminus F_u}e_u - x_{F_u\setminus F_v}e_v$, where $u, v \in G(I)$. By our assumption, $G_I^{(u,v)}$ is a connected graph for all $u, v \in G(I)$. Therefore there exists a path $u = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = v$ between u and v in $G^{(u,v)}$. By Remark 2.3, one has

$$x_{F(v)\setminus F(u)}e_{i_1} - x_{F(u)\setminus F(v)}e_{i_t} = f_{i_1}(x_{k_1}e_{i_1} - x_{k'_2}e_{i_2}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}}e_{i_{t-1}} - x_{k'_t}e_t).$$

Hence, $x_{F(v)\setminus F(u)}e_{i_1} - x_{F(v)\setminus F(u)}e_{i_t}$ is a combination of linear forms. □

Lemma 2.3. *Let I be a squarefree monomial ideal. Then one can assign to each cycle of G_I an element in $\ker(\psi)$, where $\psi\colon F_1 \mapsto F_0$ sends the basis element g_is of F_1 to elements of the minimal generating set of $\ker(\varphi)$.*

Proof. Let $u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t}, u_{i_1}$ be a cycle in G_I. Then we have two paths u_{i_1}, u_{i_2} and $u_{i_2}, \ldots, u_{i_t}, u_{i_1}$. Since $\{u_{i_1}, u_{i_2}\} \in E(G_I)$, there exist variables x and y such that $xe_{i_1} - ye_{i_2} \in \ker(\varphi)$. Since $xe_{i_1} - ye_{i_2}$ is an element in the minimal set of generators of $\ker(\varphi)$, there exists a basis element g of F_1 such that $\psi(g) = xe_{i_1} - ye_{i_2}$.

By Lemma 2.1, there exist monomials w_1 and w_2 in S such that $w_1e_{i_1} - w_2e_{i_2} = f_{i_2}(x_{k_2}e_{i_2} - x_{k'_3}e_{i_3}) + \ldots + f_{i_t}(x_{k_t}e_{i_t} - x_{k'_{t+1}}e_{i_1}) = \psi\Big(\sum\limits_{j=2}^{t} f_{i_j}g_{i_j}\Big)$. Remark 2.2 implies that $w_1 = hx_{F(u_{i_2})\setminus F(u_{i_1})} = hx$ and $w_2 = hx_{F(u_{i_1})\setminus F(u_{i_2})} = hy$. Therefore, we have

$$h(xe_{i_1} - ye_{i_2}) = w_1e_{i_1} - w_2e_{i_2}.$$

This implies that $h\psi(g) = \psi\Big(\sum\limits_{j=2}^{t} f_{i_j}g_{i_j}\Big)$ and hence $\Big(hg - \sum\limits_{j=2}^{t} f_{i_j}g_{i_j}\Big) \in \ker\psi$. Since $g \neq g_{i_j}$ for all $1 \leqslant j \leqslant r$, one has $\Big(hg - \sum\limits_{j=2}^{t} f_{i_j}g_{i_j}\Big) \neq 0$. □

Lemma 2.4. *Let w be an element of a minimal set of generators of* $\ker(\psi)$. *If $w = \sum h_i g_i$, where g_i is a basis element of F_1 and $0 \neq h_i \in S$ for each i, then h_i is a monomial.*

Proof. Without loss of generality, we may assume that $\psi(g_1) = t'_1 e_1 - t_2 e_2$. Let $u \in \operatorname{supp}(h_1)$ be a monomial. Since ut_2e_2 must be eliminated, there exists a basis element g_j of F_1 such that $\psi(g_j) = (t'_2 e_2 - t_3 e_l)$. Without loss of generality, we may assume $j = 2$ and $l = 3$. Hence, $t_2 u/t'_2 = u' \in \operatorname{supp}(h_2)$. Again, since $u' t_3 e_3$ must be eliminated, without loss of generality, we may assume there exists a basis element g_3 of F_1 such that $\psi(g_3) = (t'_3 e_3 - t_4 e_4)$. Therefore $t_3 u'/t'_3 = u'' \in \operatorname{supp}(h_3)$. Continuing this procedure yields $\psi(g_l) = (t'_l e_l - t_1 e_1)$ and $t_l u^{l-2}/t'_l = u^{l-1} \in \operatorname{supp}(h_l)$. Hence, we obtain a cycle in G_I in this way. Now if there exists another monomial $v \in \operatorname{supp}(h_1)$ with $u \neq v$, then by a similar argument one can find a new cycle in G_I. Hence, Lemma 2.3 implies that w is a combination of some other elements of $\ker(\psi)$, a contradiction. So h_i is a monomial. $\square$

Lemma 2.5. *Let I be a squarefree monomial ideal which has linear relations. Then corresponding to every element in a minimal set of generators of* $\ker(\psi)$ *there is a cycle in G_I.*

Proof. Let $\sum\limits_{i=1}^{n} h_i g_i$ be an element in a minimal set of generators of $\ker(\psi)$. Then $\psi\Big(\sum\limits_{i=1}^{n} h_i g_i\Big) = \sum\limits_{i=1}^{n} h_i \psi(g_i) = 0$, therefore $-h_1\psi(g_1) = \sum\limits_{i=2}^{n} h_i \psi(g_i)$. Assume that $\psi(g_1) = x_{i_1} e_{i_1} - x_{i_2} e_{i_2}$. So u_{i_1}, u_{i_2} is a path in G_I.

The left-hand side of the above equation is of the form $w_{i_1} e_{i_1} - w_{i_2} e_{i_2}$. By the proof of Lemma 2.2, the right-hand side of the above equation is of the form

$$f_{i_2}(x_{k_2} e_{i_2} - x_{k'_3} e_{i_3}) + f_{i_3}(x_{k_3} e_{i_3} - x_{k'_4} e_{i_4}) + \ldots + f_{i_t}(x_{k_t} e_{i_t} - x_{k'_{t+1}} e_{i_1}),$$

where $e_{i_t} \neq e_{i_2}$. If $e_{i_t} = e_{i_2}$, then $x_{k'_{t+1}} = x_{i_1}$ and $x_{k_t} = x_{i_2}$. Hence, g_1 appears on the right-hand side of the equation, a contradiction. Thus, $u_{i_2}, u_{i_3}, \ldots, u_{i_t}, u_{i_1}$ is a path which is different from the path u_{i_1}, u_{i_2}. $\square$

From now, we assume that I is a squarefree monomial ideal generated in one degree, n is the smallest integer such that $I \subset k[x_1, \ldots, x_n]$ and $I \neq uJ$, where u is a monomial and J a monomial ideal.

Theorem 2.2. *Let $I \subset k[x_1, \ldots, x_n]$ be a squarefree monomial ideal such that $G_I \cong C_m$, $m \geqslant 4$. Then the following conditions are equivalent:*

(a) *I has a linear resolution.*

(b) $m = n$ *and after a suitable relabeling of variables, one has the generators of* I *of the forms* $u_i = \prod\limits_{j=i+1}^{n-2+i} x_j$ *for* $1 \leqslant i \leqslant n$, *where* $x_{n+k} = x_k$.

(c) I *is variable-decomposable ideal.*

(d) I *has linear quotients.*

P r o o f. (a) $\Rightarrow$ (b) Assume that I has a linear resolution. Since G_I is a cycle, by Lemmas 2.3 and 2.5, $\ker(\psi) = (w)$. Let $w = \sum\limits_{i=1}^{m} h_i g_i$. Without loss of generality, we may assume that $G_I = u_1, u_2, \ldots, u_m, u_1$. Then

$$\begin{aligned}\psi(w) &= \sum_{i=1}^{m} h_i \psi(g_i)\\ &= h_1(x_{t_1}e_1 - x_{t'_2}e_2) + h_2(x_{t_2}e_2 - x_{t'_3}e_3) + \ldots + h_m(x_{t_m}e_m - x_{t'_1}e_1) = 0.\end{aligned}$$

Therefore $h_1 x_{t_1} e_1 = h_m x_{t'_1} e_1$. Since I has d-linear resolution and $\deg(e_i) = d$, we conclude that $\deg(h_i) = 1$ for $i = 1, \ldots, m$. Consequently, $h_1 = x_{t'_1}$ and $h_m = x_{t_1}$. By a similar argument, $h_j = x_{t'_j}$ and $h_j = x_{t_{j+1}}$. Hence, $x_{t_{j+1}} = x_{t'_j}$ for all $1 \leqslant j \leqslant m-1$. So $\ker(\varphi)$ is minimally generated by the following linear forms:

$$(x_{t_1}e_1 - x_{t_3}e_2), (x_{t_2}e_2 - x_{t_4}e_3), \ldots, (x_{t_m}e_m - x_{t_2}e_1).$$

For an arbitrary variable x_i in S there exits u_i and u_j in $G(I)$ such that $x_i \mid u_i$ and $x_i \nmid u_j$. Hence, by Remark 2.3 $x_i \in \{x_{t_1}, x_{t_2}, \ldots, x_{t_m}\}$. It is clear that the variables $x_{t_1}, x_{t_2}, \ldots, x_{t_m}$ are distinct and hence $n = m$.

Set $x_{t_{-1}} = x_{t_{m-1}}$, $x_{t_{m+1}} = x_{t_1}$, $e_0 = e_m$ and $e_{m+1} = e_1$. For $1 \leqslant i \leqslant m-1$ we have $\varphi(x_{t_{i-2}}e_{i-1} - x_{t_i}e_i) = 0$ and hence, $x_{t_i} \mid u_{i-1}$ and $x_{t_i} \nmid u_i$. Also, from $\varphi(x_{t_i}e_{i+1} - x_{t_{i+2}}e_{i+2}) = 0$ we have $x_{t_i} \nmid u_{i+1}$ and $x_{t_i} \mid u_{i+2}$. By Remark 2.3 $x_{t_i} \mid u_j$ for $j \neq i, i+1$.

(b) $\Rightarrow$ (c) It is easy to see that $u = x_1$ is a shedding variable for I, $I_{x_1} = \langle u_1, u_2\rangle$ and $I^{x_1} = \langle u_3, \ldots, u_n\rangle$. Also, it is clear that I_{x_1} is variable decomposable and x_2 is a shedding variable for I^{x_1}. Now we have $(I^{x_1})^{x_2} = \langle u_4, \ldots, u_n\rangle$ and $(I^{x_1})_{x_2} = \langle u_3\rangle$. Continuing these procedures yields that I^{x_1} is variable-decomposable. Hence, I is a variable-decomposable ideal.

(c) $\Rightarrow$ (d) follows from Theorem 2.1.

(d) $\Rightarrow$ (a) follows from Proposition 2.1. $\square$

As an immediate consequence of Theorem 2.2 we have the following corollaries:

Corollary 2.1. *Let* I *be a squarefree monomial ideal generated in degree* d *and* $G_I \cong C_m$. *If* $d + 2 < n$ *or* $m \neq n$, *then* I *can not have a* d-*linear resolution.*

Corollary 2.2. *Let $I \subset S$ be a squarefree monomial ideal generated in degree 2 and assume that $G_I \cong C_m$, $m \geqslant 4$. Then I has a linear resolution if and only if $m = 4$.*

Example 2.4. Consider the monomial ideal $I = (xy, zy, zq, qx) \subset k[x, y, z, q]$. It is clear that G_I is 4-cycle, $d = 2$, $n = 4$ and $d + 2 = n$. Computation with CoCoA (see [1]) shows that I has the minimal free 2-linear resolution

$$0 \mapsto S(-4) \mapsto S(-3)^4 \mapsto S(-2)^4 \mapsto I \mapsto 0.$$

Example 2.5. Let $I = (xyz, yzq, zqw, qwe, wex, xye) \subset k[x, y, z, q, e, w]$. Then $G_I \cong C_6$. Therefore I does not have a 3-linear resolution since $d = 3$, $n = 6$ and $d + 2 < n$. Computation with CoCoA (see [1]) shows that I has the minimal free S-resolution:

$$0 \mapsto S(-6) \mapsto S(-4)^6 \mapsto S(-3)^6 \mapsto I \mapsto 0.$$

Remark 2.5. Let I be a squarefree monomial ideal. If $G_I \cong C_3$, then I has linear quotients. Hence I has a linear resolution.

Let I be a squarefree monomial ideal generated in degree 2. We may assume that $I = I(G)$ is the edge ideal of a graph G. Hence, by Fröberg's result, $I(G)$ has a linear resolution if and only if $\overline{G}$ is a chordal graph. If $G \cong C_m$, then $\overline{G}$ is chordal if and only if $m = 3$ or $m = 4$. In this situation $G \cong C_m$ if and only if $G_I \cong C_m$. Hence, in this case our result coincides to Fröberg's result.

3. Linear resolution of monomial ideals whose G_I is a tree

Let I be a squarefree monomial ideal such that G_I is a tree. In this section we study linear resolutions of such monomial ideals. We know that each path is a tree, therefore first we consider the following:

Theorem 3.1. *Let $I = (u_1, \ldots, u_m)$ be a squarefree monomial ideal generated in degree d. If $G_I = u_1, u_2, \ldots, u_m$ is a path, then the following conditions are equivalent:*

(a) *I has a linear resolution.*

(b) *for any $1 \leqslant j \leqslant k \leqslant i \leqslant m$*

$$F(u_k) \subseteq F(u_i) \cup F(u_j).$$

(c) *I is variable-decomposable ideal.*

(d) *I has linear quotients.*

P r o o f. (a) $\Rightarrow$ (b) Suppose on the contrary that there exist $1 \leqslant j < k < i \leqslant m$ and $l \in F(u_k)$ such that $l \notin F(u_i) \cup F(u_j)$. Since I has a linear resolution, we have $x_{F(u_i)\setminus F(u_j)} e_j - x_{F(u_j)\setminus F(u_i)} e_i = f_i(x_{k_1} e_i - x_{k'_2} e_{i+1}) + f_{i+1}(x_{k_2} e_{i+1} - x_{k'_3} e_{i+2}) + \ldots + f_{j-1}(x_{k_{j-1}} e_{j-1} - x_{k'_t} e_j)$. By Remark 2.4, $x_l \mid x_{F(u_j)\setminus F(u_i)}$ and $x_l \mid x_{F(u_i)\setminus F(u_j)}$, which is a contradiction.

(b) $\Rightarrow$ (c) Let $F(u_2) \setminus F(u_1) = \{l\}$. From the facts that $F(u_2) \subseteq F(u_1) \cup F(u_i)$, $l \in F(u_2)$ and $u_2 : u_1 = x_l$, we conclude that $l \in F(u_i)$ for all $2 \leqslant i \leqslant m$, $I_{x_l} = \langle u_1 \rangle$ and x_1 is a shedding monomial. By induction on m, I^{x_l} is variable-decomposable, since I^{x_l} is a path of length $m-1$.

(c) $\Rightarrow$ (d) follows from Theorem 2.1.

(d) $\Rightarrow$ (a) follows from Proposition 2.1. $\square$

Theorem 3.2. *If I is a squarefree monomial ideal which has linear relations, then G_I is a tree if and only if* projdim$(I) = 1$.

P r o o f. If G_I is a tree, then G_I has no cycle. Therefore by Lemma 2.5, $\ker(\psi) = 0$. Hence, the linear resolution of I is of the form

$$0 \mapsto F_1 \mapsto F_0 \mapsto I \mapsto 0, \quad \text{and} \quad \operatorname{projdim}(I) = 1.$$

Conversely, assume that projdim$(I) = 1$. Then $\ker(\psi) = 0$ and by Lemma 2.3, G_I has no cycle. Since I has linear relations, by Lemma 2.2, G_I is a connected graph. Therefore G_I is a tree. $\square$

Proposition 3.1. *Let I be a squarefree monomial ideal with* projdim$(I) = 1$. *Then I has a linear resolution if and only if G_I is a connected graph.*

P r o o f. Assume that G_I is a connected graph. Since projdim$(I) = 1$, Lemma 2.3 implies that G_I has no cycle and hence it is a tree. So it is enough to show that I has linear relations. For $u_i, u_j \in G(I)$ there exists a unique path between u_i and u_j in $G(I)$. Assume that $we_i - w'e_j = f_{i_1}(x_{k_1} e_{i_1} - x_{k'_2} e_{i_2}) + \ldots + f_{i_{t-1}}(x_{k_{t-1}} e_{i_{t-1}} - x_{k'_t} e_{i_t})$ is an element of $\ker(\varphi)$ which is obtained from this path. If $we_i - w'e_j = x_{F(u_j)\setminus F(u_i)} e_i - x_{F(u_i)\setminus F(u_j)} e_j$, we are done. So assume that the equality does not hold. Then $x_{F(u_j)\setminus F(u_i)} e_i - x_{F(u_i)\setminus F(u_j)} e_j$ belongs to the minimal set of generators of $\ker(\varphi)$. Hence, there exists $g \in F_1$ such that $\psi(g) = x_{F(u_j)\setminus F(u_i)} e_i - x_{F(u_i)\setminus F(u_j)} e_j$. Remark 2.2 implies that there exists a monomial $h \in S$ such that $h\psi(g) = we_i - w'e_j = \sum\limits_{j=1}^{t-1} f_{i_j} \psi(g_{i_j})$. Therefore $\psi\Big(hg - \sum\limits_{j=1}^{t-1} f_{i_j} g_{i_j}\Big) = 0$ and $hg - \sum\limits_{j=1}^{t-1} f_{i_j} \psi(g_{i_j}) \neq 0$, a contradiction.

The converse follows from Lemma 2.2. $\square$

Proposition 3.2. *Let $I = (u_1, \ldots, u_m)$ be a squarefree monomial ideal generated in degree d which has linear quotients. Assume that G_I is a tree and v is a monomial in degree d which is a leaf in $G_{(I,v)}$. Then the following conditions are equivalent:*
(a) *(I, v) has a linear resolution.*
(b) *Let u_i be the branch of v and $F(u_i) \setminus F(v) = \{l\}$. Then $l \in \bigcap_{t=1}^{m} F(u_t)$.*
(c) *(I, v) has linear quotients.*

Proof. (a) $\Rightarrow$ (b) Suppose on the contrary that there exists a $1 \leqslant j \leqslant m$ such that $l \notin F(u_j)$. Let $v, u_i = u_{i_1}, u_{i_2}, \ldots, u_{i_{t-1}}, u_{i_t} = u_j$ be the unique path between v and u_j. Without loss of generality, we may assume that $l \in F(u_{i_r})$ for all r, $1 \leqslant r \leqslant t-1$. Since (I, v) has a linear resolution, we have $x_{F(u_j)\setminus F(v)}e_v - x_{F(v)\setminus F(u_j)}e_j = f_0(x_{i_0}e_v - x'_{i_1}e_{i_1}) + f_1(x_{i_1}e_{i_1} - x'_{i_2}e_{i_2}) + \ldots + f_{t-1}(x_{i_{t-1}}e_{i_{t-1}} - x'_{i_t}e_t)$. By Remark 2.4, one has $x_l \mid x_{F(u_j)\setminus F(v)}$ and $x_l \mid x_{F(v)\setminus F(u_j)}$, a contradiction.

(b) $\Rightarrow$ (c) Assume that I has linear quotients with respect to an ordering v_1, $v_2, \ldots, v_m$ of $G(I)$. Since by our assumption $\{l\} = F(u_i) \setminus F(v)$ and $l \in F(u_j)$ for any $1 \leqslant j \leqslant m$, we conclude that the order $v_1, v_2, \ldots, v_m, v$ is an admissible order for (I, v).

(c) $\Rightarrow$ (a) follows from Proposition 2.1. □

Proposition 3.3. *Let $I = (u_1, \ldots, u_m)$ be a squarefree monomial ideal generated in degree d. If G_I is a tree, then I has a linear resolution if and only if L has a linear resolution for all $L \subseteq I$, where $G(L) \subset G(I)$ and G_L is a path.*

Proof. Assume that I has a linear resolution. Since G_I is a tree, we have $\operatorname{projdim}(I) = 1$. So if $L \subset I$ with $G(L) \subset G(I)$ and G_L is a path, then L has linear relations and $\operatorname{projdim}(L) = 1$. Therefore L has a linear resolution.

For the converse, by our assumption there exists a monomial ideal $J_0 \subset I$ such that $G(J_0) = \{u_{i_1}, \ldots, u_{i_t}\} \subset G(I)$, G_{J_0} is a path and J_0 has linear resolution. Therefore J_0 has linear quotients. Take $v \in V(G_I) \setminus V(G_{J_0})$ such that v and u_{i_j} are adjacent in G_I for some $1 \leqslant j \leqslant t$. Set $F(u_{i_j}) \setminus F(v) = \{l\}$. Since J_0 has linear quotients, there exists a path between u_{i_r} and u_{i_j} for all $1 \leqslant r \leqslant t$. Therefore we have path $u_{i_r}, \ldots, u_{i_j}, v$ in G_I. By our hypothesis $L = \langle u_{i_r}, \ldots, u_{i_j}, v\rangle$ has a linear resolution and Proposition 3.1 implies that $F(u_{i_j}) \subseteq F(v) \cup F(u_{i_r})$. Therefore $\{l\} \in F(u_{i_r})$ and Proposition 3.2 implies that $J_1 = \langle J_0, v\rangle$ has linear quotients. Now replace J_0 by J_1 and do the same procedure until we obtain I. □

Theorem 3.3. *Let I be a squarefree monomial ideal which is generated in degree d. If G_I is a tree, then the following conditions are equivalent:*
(a) *I has a linear resolution.*
(b) *I has linear relations.*

(c) $G_I^{(u,v)}$ *is a connected graph for all* u *and* v *in* $G(I)$.

(d) *If* $u = u_1, u_2, \ldots, u_s = v$ *is the unique path between* u *and* v *in* G_I. *Then* $F(u_j) \subset F(u_i) \cup F(u_k)$ *for all* $1 \leqslant i \leqslant j \leqslant k \leqslant s$.

(e) L *has a linear resolution for all* $L \subseteq I$, *where* $G(L) \subset G(I)$ *and* G_L *is a path.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Leftrightarrow$ (c) follows from Proposition 2.2.

(c) $\Rightarrow$ (d) for all $1 \leqslant i \leqslant j \leqslant k \leqslant s$, $G_I^{(u_i,u_k)}$ is connected and u_j is a vertex of this graph. Therefore $F(u_j) \subset F(u_i) \cup F(u_k)$.

(d) $\Rightarrow$ (e) follows from Proposition 3.1.

(e) $\Rightarrow$ (a) follows by Proposition 3.3. $\square$

Theorem 3.4. *Let* I *be a squarefree monomial ideal generated in degree* d. *If* G_I *is a tree, then the following are equivalent:*

(a) I *has a linear resolution.*

(b) I *is variable-decomposable ideal.*

(c) I *has linear quotients.*

Proof. (a) $\Rightarrow$ (b) G_I is a tree and I has a linear resolution, hence $\operatorname{projdim}(I) = 1$. Without loss of generality we may assume that u_1 is a leaf in G_I and u_2 is its branch. Set $F(u_2) \setminus F(u_1) = \{l\}$. Proposition 2.2 implies that $G_I^{(u_1,u_i)}$ is a connected graph for all u_i. If $l \notin F(u_i)$ for some $i > 2$, then $F(u_2) \nsubseteq F(u_1) \cup F(u_i)$ and $u_2 \notin V(G_I^{(u_1,u_i)})$. Therefore $G_I^{(u_1,u_i)}$ is not connected, a contradiction. Hence $I_{x_l} = \{u_1\}$ and $G(I^{x_l}) = G(I) \setminus \{u_1\}$. It is easy to see that x_l is a shedding variable. Since $G_{I^{x_l}}$ is a tree and has linear relations, by induction on $|G(I)|$, we conclude that I^{x_l} is variable-decomposable. Therefore I is a variable-decomposable ideal.

(b) $\Rightarrow$ (c) follows from Theorem 2.1.

(c) $\Rightarrow$ (a) follows by Proposition 2.1. $\square$

4. Linear resolution of some classes of monomial ideals

In this section, as applications of our obtained results, we determine the linearity of resolution for some classes of monomial ideals.

Let I be a squarefree Cohen-Macaulay monomial ideal of codimension 2 which is generated in one degree. Since $\operatorname{projdim}(I) = 1$, as a corollary of Proposition 3.1 and Theorem 3.4, we have:

Corollary 4.1. *Let* I *be a squarefree Cohen-Macaulay monomial ideal of codimension 2. Then* I *has a linear resolution if and only if* G_I *is a connected graph. Indeed, in this case* G_I *is a tree and the following conditions are equivalent:*

(i) *I has a linear resolution.*
(ii) *I has linear quotients.*
(iii) *I is variable decomposable.*

The following example shows that there are Cohen-Macaulay monomial ideals of codimension 2 with and without a linear resolution.

Example 4.1.
(i) Let $I = (xy, yz, zt) \subset K[x, y, z, t]$. Computation with CoCoA (see [1]) shows that I is Cohen-Macaulay of codimension 2 and has the minimal free 2-linear resolution.
(ii) Let $I = (xy, zt) \subset K[x, y, z, t]$. Again, using CoCoA (see [1]) shows that I is Cohen-Macaulay of codimension 2 which does not have a linear resolution.

Remark 4.1. Let I be a squarefree monomial ideal generated in degree d. If G_I is a complete graph, then the following statements hold.
(a) I has a linear resolution.
(b) I is variable-decomposable ideal.
(c) I has linear quotients.

In [4] Conca and De Negri introduced a path ideal of a graph. The path ideal of G of length t is the monomial ideal $I_t(G) = \left\langle \prod_{j=1}^{t} x_{i_j} \right\rangle$, where $i_1, \ldots, i_t$ is a path in G. In [8], Proposition 4.1, it is shown that $S/I_2(C_n)$ is vertex decomposable (shellable, Cohen-Macaulay) if and only if $n = 3$ or 5. In [15], the authors showed that if $2 < t \leqslant n$, then $S/I_t(C_n)$ is sequentially Cohen-Macaulay if and only if $t = n$, $t = n - 1$ or $t = \frac{1}{2}(n-1)$. In [2] it is shown that $S/I_t(C_n)$ is Cohen-Macaulay if and only if it is shellable and if and only if $I_t(C_n)$ is vertex decomposable.

It is easy to see that if $t < n - 1$, then $G_{I_t(C_n)} \cong C_n$. Hence, by Theorem 2.2, $I_t(C_n)$ has a linear resolution if and only if $t = n - 2$. For $t = n - 1$, since $G_{I_t(G)}$ is a complete graph, $I_t(G)$ has a linear resolution. Also, in these cases, having a linear resolution is equivalent to having linear quotients and it is equivalent to variable decomposability of $I_t(C_n)$. Hence we have:

Corollary 4.2. *$I_t(C_n)$ has a linear resolution if and only if $t = n-2$ or $t = n-1$. In addition, the following conditions are equivalent:*
(a) *$I_t(C_n)$ has a linear resolution.*
(b) *$I_t(C_n)$ is variable-decomposable ideal.*
(c) *$I_t(C_n)$ has linear quotients.*

Corollary 4.3. *Let L_n be a path on vertex set $\{1, \ldots, n\}$ and $I_t(L_n)$ be the path ideal of L_n. Then $I_t(L_n)$ has a linear resolution if and only if $t \geqslant \frac{1}{2}n$. Also $I_t(C_n)$*

has linear resolution if and only if it has linear quotients and this is equivalent to saying that $I_t(C_n)$ is variable decomposable.

Proof. Let $L_n = 1, \ldots, n$ be a path. It is easy to see that $G_{I_t(L_n)} \cong L_{n-t+1}$ and $I_t(L_n) = \Bigl(\prod_{i=1}^{t} x_i, \ldots, \prod_{i=t+1}^{2t} x_i, \ldots, \prod_{i=n-t+1}^{n} x_i\Bigr)$. If $n - t + 1 > t + 1$, then $F(u_2) \nsubseteq F(u_1) \cup F(u_n)$. Hence Theorem 3.1 implies that $I_t(G)$ does not have a linear resolution. If $n - t + 1 \leqslant t + 1$, i.e., $t \geqslant \frac{1}{2}n$, then it is clear that for any $1 \leqslant j \leqslant k \leqslant i \leqslant m$ one has:

$$F(u_k) \subseteq F(u_i) \cup F(u_j).$$

Therefore, by Theorem 3.1, $I_t(G)$ has a linear resolution and the equivalent conditions hold. $\square$

5. Cohen-Macaulay simplicial complex

A simplicial complex Δ over a set of vertices $[n] = \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that $\{i\} \in \Delta$ for all i and if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a face and the dimension of a face F is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The maximal faces of Δ under inclusion are called facets and the set of all facets is denoted by $\mathcal{F}(\Delta)$. The dimension of the simplicial complex Δ is the maximal dimension of its facets. A subcomplex of Δ is a simplicial complex whose facets are also facets of Δ. We say that a simplicial complex Δ is connected if for each F and G of $\mathcal{F}(\Delta)$ there exists a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that $F_i \cap F_{i+1} \neq \emptyset$ for $i = 0, \ldots, q - 1$.

Let Δ be a simplicial complex on $[n]$ with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$. The Stanley-Reisner ideal of Δ is a squarefree monomial ideal $I_\Delta = (x_{i_1} \ldots x_{i_p} \mid \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta)$. The Alexander dual of Δ is the simplicial complex $\Delta^\vee = (\{x_1, \ldots, x_n\} \setminus F \mid F \notin \Delta)$. For each $F \subset [n]$ we set $\overline{F_i} = [n] \setminus F_i$ and $P_F = (x_j \colon j \in F)$. It is well known that $I_\Delta = \bigcap_{i=1}^{m} P_{\overline{F_i}}$ and $I_{\Delta^\vee} = (x_{\overline{F_i}} \colon i = 1, \ldots, m)$, see [10].

The simplicial complex Δ is called *pure* if its facets have the same dimension. It is easy to see that Δ is pure if and only if $I_{\Delta^\vee}$ is generated in one degree. The k-algebra $k[\Delta] = S/I_\Delta$ is called the Stanley-Reisner ring of Δ. We say that Δ is Cohen-Macaulay over k if $k[\Delta]$ is Cohen-Macaulay. It is known that Δ is Cohen-Macaulay over k if and only if $I_{\Delta^\vee}$ has a linear resolution, see [6].

The simplicial complex Δ is called *shellable* if its facets $F_1, F_2, \ldots, F_m$ can be ordered so that for all $2 \leqslant i \leqslant m$, the subcomplex $(F_1, \ldots, F_{i-1}) \cap (F_i)$ is pure of dimension $\dim(F_i) - 1$.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{F \cup G\colon\ F \in \Delta_1,\ G \in \Delta_2\}$. For a face F in Δ, the link, deletion and star of F in Δ are, respectively, denoted by $\mathrm{link}_\Delta F$, $\Delta \setminus F$ and $\star_\Delta F$ and defined by $\mathrm{link}_\Delta F = \{G \in \Delta\colon\ F \cap G = \emptyset,\ F \cup G \in \Delta\}$, $\Delta \setminus F = \{G \in \Delta\colon\ F \nsubseteq G\}$ and $\star_\Delta F = (F) * \mathrm{link}_\Delta F$.

A face F in Δ is called a *shedding face* if every face G of $\star_\Delta F$ satisfies the following exchange property: for every $i \in F$ there is $j \in [n] \setminus G$ such that $(G \cup \{j\}) \setminus \{i\}$ is a face of Δ. A simplicial complex Δ is recursively defined to be k-decomposable if either Δ is a simplex or else has a shedding face F with $\dim(F) \leqslant k$ such that both $\Delta \setminus F$ and $\mathrm{link}_\Delta F$ are k-decomposable. 0-decomposable simplicial complexes are called *vertex decomposable*.

It is clear that $x_{\overline{F_i}}$ and $x_{\overline{F_j}}$ are adjacent in $G_{I_{\Delta^\vee}}$ if and only if F_i and F_j are connected in codimension one, i.e., $|F_i \cap F_j| = |F_i| - 1$. A simplicial complex Δ is called connected in codimension one or strongly connected if for any two facets F and G of Δ there exists a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that F_i and F_{i+1} are connected in codimension one for each $i = 1, \ldots, q-1$. Hence, we have the following:

Lemma 5.1. *A simplicial complex Δ is connected in codimension one if and only if $G_{I_{\Delta^\vee}}$ is a connected graph.*

For facets F and G of Δ, we introduce a subcomplex $\Delta^{(F,G)} = (L \in \mathcal{F}(\Delta)\colon\ F \cap G \subset L)$. It is easy to see that $\Delta^{(F,G)}$ is connected in codimension one if and only if $G_{I_{\Delta^\vee}}^{(x_{\overline{F}}, x_{\overline{G}})}$ is a connected graph. Hence, by Proposition 2.2 we have:

Corollary 5.1. *Let Δ be a pure simplicial complex on vertex set $[n]$. Then $I_{\Delta^\vee}$ has linear relations if and only if $\Delta^{(F,G)}$ is connected in codimension one for all facets F and G of Δ.*

Suppose that Δ is a pure simplicial complex of dimension d, i.e., $|F_i| = d+1$ for all i. We associate to Δ a simple graph G_Δ whose vertices are labeled by the facets of Δ. Two vertices F_i and F_j are adjacent if F_i and F_j are connected in codimension one, i.e., $|F_i \cap F_{i+1}| = d$. Then it is easy to see that $|\ \overline{F_i} \cap \overline{F_j}\ | = n - d - 2$. Therefore $x_{\overline{F_i}}$ and $x_{\overline{F_j}}$ are adjacent in $G_{I_{\Delta^\vee}}$ and hence $G_\Delta \cong G_{I_{\Delta^\vee}}$.

Now assume that $G_\Delta \cong G_{I_{\Delta^\vee}}$ is a path. Proposition 3.1 implies that $I_{\Delta^\vee}$ has a linear resolution if and only if for any $1 \leqslant j \leqslant k \leqslant i \leqslant m$, $\overline{F_k} \subseteq \overline{F_i} \cup \overline{F_j}$. Hence, by Eagon-Reiner (see [6]), in this case Δ is Cohen-Macaulay. Therefore we have:

Corollary 5.2. *Let $\Delta = (F_1, \ldots, F_m)$ be a pure simplicial complex. If $G_\Delta = F_1, F_2, \ldots, F_m$ is a path, then Δ is Cohen-Macaulay if and only if $F_i \cap F_j \subseteq F_k$*

for any $1 \leqslant j \leqslant k \leqslant i \leqslant m$*. Moreover, in this case the following conditions are equivalent:*

(a) $\Delta^{(F,G)}$ *is connected in codimension one for all facets* F *and* G *in* Δ*.*
(b) Δ *is Cohen-Macaulay.*
(c) Δ *is shellabe.*
(d) Δ *is vertex decomposable simplicial complex.*

Also as a consequence of Theorem 2.2 we have:

Corollary 5.3. *Let* $\Delta = (F_1, \dots, F_m)$ *be a pure simplicial complex on vertex set* $[n]$*. If* $G_\Delta \cong C_m$*, then* Δ *is Cohen-Macaulay if and only if* $m = n$ *and with a suitable relabeling of vertexes, we have* $F_i = \{i+1, i+2, \dots, i+n-2\}$*, where* $n + i = i$*. Moreover, in this case* Δ *is shellabe and vertex decomposable simplicial complex.*

As another corollary of Theorems 3.3 and 3.4 we have:

Corollary 5.4. *Let* $\Delta = (F_1, \dots, F_m)$ *be a pure simplicial complex. If* G_Δ *is a tree, then the following conditions are equivalent:*

(a) $\Delta^{(F,G)}$ *is connected in codimension one for all facets* F *and* G *in* Δ*.*
(b) *If* $F = F_1, F_2, \dots, F_s = G$ *is the unique path in* G_Δ *from* F *to* G*, then* $F_i \cap F_k \subset F_j$ *for all* $1 \leqslant i \leqslant j \leqslant k \leqslant s$*.*
(c) Δ *is Cohen-Macaulay.*
(d) Δ *is shellable.*
(e) Δ *is vertex decomposable.*

References

[1] *J. Abbott, A. M. Bigatti, G. Lagorio*: CoCoA-5: A system for doing Computations in Commutative Algebra. Available at `http://cocoa.dima.unige.it/`. sw
[2] *S. M. Ajdani, A. S. Jahan*: Vertex decomposability of 2-CM and Gorenstein simplicial complexes of codimension 3. Bull. Malays. Math. Sci. Soc. (2) *39* (2016), 609–617. zbl MR doi
[3] *M. Bigdeli, J. Herzog, R. Zaare-Nahandi*: On the index of powers of edge ideals. Commun. Algebra *46* (2018), 1080–1095. zbl MR doi
[4] *A. Conca, E. De Negri*: M-sequences, graph ideals, and ladder ideals of linear type. J. Algebra *211* (1999), 599–624. zbl MR doi
[5] *A. Conca, J. Herzog*: Castelnuovo-Mumford regularity of product of ideals. Collect. Math. *54* (2003), 137–152. zbl MR
[6] *J. A. Eagon, V. Reiner*: Resolutions of Stanley-Reisner rings and Alexander duality. J. Pure Appl. Algebra *130* (1998), 265–275. zbl MR doi
[7] *E. Emtander*: A class of hypergraphs that generalizes chordal graphs. Math. Scand. *106* (2010), 50–66. zbl MR doi
[8] *C. A. Francisco, A. Van Tuyl*: Sequentially Cohen-Macaulay edge ideals. Proc. Am. Math. Soc. *135* (2007), 2327–2337. zbl MR doi

[9] *R. Fröberg*: On Stanley-Reisner rings. Topics in Algebra. Part 2. Commutative Rings and Algebraic Groups. Banach Center Publications 26. Polish Academy of Sciences, Institute of Mathematics, Warszaw, 1990, pp. 57–70. zbl MR

[10] *J. Herzog, T. Hibi*: Monomial Ideals. Graduate Texts in Mathematics 260. Springer, London, 2011. zbl MR doi

[11] *J. Herzog, T. Hibi, X. Zheng*: Monomial ideals whose powers have a linear resolution. Math. Scand. *95* (2004), 23–32. zbl MR doi

[12] *J. Herzog, Y. Takayama*: Resolutions by mapping cones. Homology Homotopy Appl. *4* (2002), 277–294. zbl MR doi

[13] *M. Morales*: Simplicial ideals, 2-linear ideals and arithmetical rank. J. Algebra *324* (2010), 3431–3456. zbl MR doi

[14] *R. Rahmati-Asghar, S. Yassemi*: k-decomposable monomial ideals. Algebra Colloq. *22* (2015), 745–756. zbl MR doi

[15] *S. Saeedi Madani, D. Kiani, N. Terai*: Sequentially Cohen-Macaulay path ideals of cycles. Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. *54* (2011), 353–363. zbl MR

[16] *R. Woodroofe*: Chordal and sequentially Cohen-Macaulay clutters. Electron. J. Comb. *18* (2011), Article ID P208, 20 pages. zbl MR

Authors' address: Erfan Manouchehri (corresponding author), Ali Soleyman Jahan, Department of Mathematics, University of Kurdistan, P.O.Box 66177-15175, Sanadaj, Iran, e-mail: erfanm6790@yahoo.com, solymanjahan@gmail.com.