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S-DEPTH ON ZD-MODULES AND LOCAL COHOMOLOGY

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Abstract. Let R be a Noetherian ring, and I and J be two ideals of R. Let S be a Serre subcategory of the category of R-modules satisfying the condition C_I and M be a ZD-module. As a generalization of the S-depth(I,M) and depth(I,J,M), the S-depth of (I,J) on M is defined as $S\text{-depth}(I,J,M) = \inf\{S\text{-depth}(\mathfrak{a},M)\colon\ \mathfrak{a} \in \widetilde{\mathrm{W}}(I,J)\}$, and some properties of this concept are investigated. The relations between S-depth(I,J,M) and $H^i_{I,J}(M)$ are studied, and it is proved that $S\text{-depth}(I,J,M) = \inf\{i\colon\ H^i_{I,J}(M) \notin S\}$, where S is a Serre subcategory closed under taking injective hulls. Some conditions are provided that local cohomology modules with respect to a pair of ideals coincide with ordinary local cohomology modules under these conditions. Let $\mathrm{Supp}_R H^i_{I,J}(M)$ be a finite subset of $\mathrm{Max}(R)$ for all $i < t$, where M is an arbitrary R-module and t is an integer. It is shown that there are distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k \in \mathrm{W}(I,J)$ such that $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}_1}(M) \oplus H^i_{\mathfrak{m}_2}(M) \oplus \ldots \oplus H^i_{\mathfrak{m}_k}(M)$ for all $i < t$.

Keywords: depth; local cohomology; Serre subcategory; ZD-module

MSC 2020: 13C15, 13C60, 13D45

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with nonzero identity, I and J are two ideals of R, M is an R-module and t is an integer. For notation and terminology not given in this paper, the reader is referred to [5], [6], and [13] if necessary.

The local cohomology theory has been a useful and significant tool in commutative algebra and algebraic geometry. There are some extensions of this theory. Bijan-Zadeh in [4] introduced the local cohomology modules with respect to a system of ideals. As a special case of these generalized modules, Takahashi, Yoshino, and Yoshizawa in [13] defined the local cohomology modules with respect to a pair of ideals. To be more precise, let $\Gamma_{I,J}(M) = \{x \in M\colon\ \exists\, t \in \mathbb{N},\ I^t x \subseteq Jx\}$. It is easy to see that $\Gamma_{I,J}(M)$ is a submodule of M and $\Gamma_{I,J}(-)$ is a covariant R-linear functor from the category of R-modules to itself. For an integer i, the local cohomology functor $H^i_{I,J}(-)$ with respect to (I,J) is defined to be the ith right derived functor of $\Gamma_{I,J}(-)$. Also $H^i_{I,J}(M)$ is called the *ith local cohomology module of M with respect to (I,J)*. If $J=0$, then $H^i_{I,J}(-)$ coincides with the ordinary local cohomology functor $H^i_I(-)$.

Let $\widetilde{\mathrm{W}}(I,J)=\{\mathfrak{a}\leqslant R\colon I^t\subseteq J+\mathfrak{a}$ for some positive integer $t\}$. One can see that $x\in\Gamma_{I,J}(M)$ if and only if $\mathrm{Ann}_R(x)\in\widetilde{\mathrm{W}}(I,J)$. Let $\mathrm{W}(I,J)=\{\mathfrak{p}\in\mathrm{Spec}(R)\colon I^t\subseteq J+\mathfrak{p}$ for some positive integer $t\}$. It is shown in [13], Corollary 1.8, that $x\in\Gamma_{I,J}(M)$ if and only if $\mathrm{Supp}_R Rx\subseteq \mathrm{W}(I,J)$.

The notion of ZD-modules was introduced by Evans, see [9]. An R-module M is called a *ZD-module* (*zero-divisor module*) if, for any submodule N of M, the set of zero-divisors of M/N is a finite union of the associated prime ideals of M/N. According to [8], Example 2.2, the class of ZD-modules contains finitely generated, Laskerian, weakly Laskerian, linearly compact, Matlis reflexive, and minimax modules. Also, it contains modules whose quotients have finite Goldie dimension and modules with finite support, in particular, Artinian modules.

Let S be a Serre subcategory of the category of R-modules. As a generalization of the regular sequences, Aghapournahr and Melkersson in [2] introduced the S-sequences. Let S satisfy the condition C_I and M be finitely generated such that $M/IM\notin S$. They showed that all maximal S-sequences on M in I have the same length. This common length, denoted by S-depth(I,M), was called the S-depth of I on M. In [10], we generalized this concept to the ZD-modules. Let S satisfy the condition C_I, M be a ZD-module, and let I contain a maximal S-sequence on M. We proved, in [10], Theorem 2.1, that all maximal S-sequences on M in I have the same length. Also, it was shown that if $M/IM\notin S$, then I contains maximal S-sequences on M, see [10], Proposition 2.1.

The notion of depth of a pair of ideals (I,J) on a finitely generated R-module M was introduced in [1] as $\mathrm{depth}(I,J,M)=\inf\{\mathrm{depth}(\mathfrak{a},M)\colon \mathfrak{a}\in\widetilde{\mathrm{W}}(I,J)\}$. Let S be a Serre subcategory of the category of R-modules satisfying the condition C_I and M be a ZD-module. In Section 2, we define the S-depth of a pair of ideals (I,J) on M as $S\text{-depth}(I,J,M)=\inf\{S\text{-depth}(\mathfrak{a},M)\colon \mathfrak{a}\in\widetilde{\mathrm{W}}(I,J)\}$. It is easy to see that S-depth(I,J,M) is a generalization of S-depth(I,M) and depth(I,J,M). We also investigate some properties of the S-depth(I,J,M).

In Section 3, the relations between the local cohomology modules of M with respect to (I,J) and S-depth(I,J,M) are studied. Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. As one of the main results of this paper, we prove that $S\text{-depth}(I,J,M)=\inf\{i\colon H^i_{I,J}(M)\notin S\}$, see Theorem 3.7.

In Section 4, we get some isomorphisms on local cohomology modules with respect to a pair of ideals and ordinary local cohomology modules. Let $\operatorname{Supp}_R H^i_{I,J}(M)$ be a finite subset of $\operatorname{Max}(R)$ for all $i < t$, where M is an arbitrary R-module. It is proved in Proposition 4.2, that there are distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k \in \mathrm{W}(I, J)$ such that $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}_1}(M) \oplus H^i_{\mathfrak{m}_2}(M) \oplus \ldots \oplus H^i_{\mathfrak{m}_k}(M)$ for all $i < t$.

2. S-depth of a pair of ideals on ZD-modules

Recall that R is a Noetherian ring, I and J are two ideals of R and M is an R-module.

Definition 2.1. A full subcategory of the category of R-modules is said to be a *Serre subcategory*, if it is closed under taking submodules, quotients, and extensions. A Serre subcategory S is said to *satisfy the condition* C_I, if for any I-torsion R-module M, $0 :_M I \in S$ implies that $M \in S$.

It is shown that if a Serre subcategory is closed under taking injective hulls, then it satisfies the condition C_I, see [2], Lemma 2.2. By [2], Example 2.4, the class of zero modules, Artinian modules, I-cofinite Artinian modules, modules with finite support, and the class of R-modules N with $\dim_R N \leqslant k$, where k is a non-negative integer, are Serre subcategories closed under taking injective hulls and hence satisfy the condition C_I. Also, by [2], Example 2.5 the class of I-cofinite Artinian modules is a Serre subcategory of the category of R-modules satisfying the condition C_I, but is not closed under taking injective hulls. In the rest of the paper, S denotes a Serre subcategory of the category of R-modules.

Aghapournahr and Melkersson in [2] introduced the notion of S-sequences on M as follows.

Definition 2.2. An element a of R is called *S-regular on M*, if $0 :_M a \in S$. A sequence $a_1, \ldots, a_t$ is an *S-sequence on M*, if a_i is S-regular on $M/(a_1, \ldots, a_{i-1})M$ for $i = 1, \ldots, t$.

When S is the class of zero modules, Artinian R-modules, modules with finite support, and the class of R-modules N with $\dim_R N \leqslant k$, where k is a non-negative integer, then S-sequences on M are poor M-sequences, filter-regular sequences, generalized regular sequences, and M-sequences in dimension greater than k, respectively, see [2], Example 2.8.

Let $Z_R(M)$ denote the set of zero-divisors of M. As a generalization of finitely generated modules, Evans in [9] introduced ZD-modules as follows.

Definition 2.3. An R-module M is said to be a zero-divisor module (ZD-module) if for any submodule N of M, the set $Z_R(M/N)$ is a finite union of prime ideals in $\mathrm{Ass}_R M/N$.

According to [8], Example 2.2, the class of ZD-modules contains finitely generated, Laskerian, weakly Laskerian, linearly compact, and Matlis reflexive modules. Also, it contains modules whose quotients have finite Goldie dimension and modules with finite support, in particular, Artinian modules. Therefore, the class of ZD-modules is much larger than that of finitely generated modules.

Let S satisfy the condition C_I and M be a ZD-module. We proved in [10], Theorem 2.1, that if I contains a maximal S-sequence on M, then all maximal S-sequences on M in I have the same length. This common length is called the S-depth of I on M and denoted by S-depth(I, M). Now, we generalize this notion to a pair of ideals.

Definition 2.4. Let S satisfy the condition C_I and M be a ZD-module. The S-depth of (I, J) on M, denoted by S-depth(I, J, M), is defined as

$$S\text{-}\operatorname{depth}(I, J, M) = \inf\{S\text{-}\operatorname{depth}(\mathfrak{a}, M) \colon \mathfrak{a} \in \widetilde{\mathrm{W}}(I, J)\}.$$

Let S satisfy the condition C_I and M be a ZD-module. If S is the class of zero modules, then S-depth(I, J, M) coincides with depth(I, J, M) which was defined for finite modules in [1] as $\operatorname{depth}(I, J, M) = \inf\{\operatorname{depth}(\mathfrak{a}, M) \colon \mathfrak{a} \in \widetilde{\mathrm{W}}(I, J)\}$. We show that S-depth$(I, J, M) = S$-depth(I, M) whenever $J = 0$. For this, it is clear from the definition that S-depth$(I, J, M) \leqslant S$-depth(I, M). To prove the converse inequality, suppose that $J = 0$ and $\mathfrak{a} \in \widetilde{\mathrm{W}}(I, J)$. There is a positive integer t such that $I^t \subseteq \mathfrak{a}$. Now in view of [10], Proposition 3.1, we have S-depth$(I, M) = S$-depth$(I^t, M) \leqslant S$-depth$(\mathfrak{a}, M)$, and therefore S-depth$(I, M) \leqslant S$-depth(I, J, M).

The following lemma is the key for the next results.

Lemma 2.5. *Let I' and J' be two ideals of R. Then:*

(i) *If $I \subseteq I'$, then $\widetilde{\mathrm{W}}(I, J) \supseteq \widetilde{\mathrm{W}}(I', J)$.*
(ii) *If $J \subseteq J'$, then $\widetilde{\mathrm{W}}(I, J) \subseteq \widetilde{\mathrm{W}}(I, J')$.*
(iii) $\widetilde{\mathrm{W}}(I, J) = \widetilde{\mathrm{W}}(\sqrt{I}, J) = \widetilde{\mathrm{W}}(I, \sqrt{J})$.
(iv) $\widetilde{\mathrm{W}}(II', J) = \widetilde{\mathrm{W}}(I \cap I', J)$.
(v) $\widetilde{\mathrm{W}}(I, JJ') = \widetilde{\mathrm{W}}(I, J \cap J')$.
(vi) *If $\mathfrak{a} \in \widetilde{\mathrm{W}}(I, J)$, then $\widetilde{\mathrm{W}}(\mathfrak{a}, J) \subseteq \widetilde{\mathrm{W}}(I, J)$.*

P r o o f. The proof is easy, and is left to the reader. $\square$

Proposition 2.6. *Let S satisfy the condition C_I. Let M be a ZD-module, and I' and J' be two ideals of R. Then:*

(i) *If $I \subseteq I'$, then $S\text{-depth}(I, J, M) \leqslant S\text{-depth}(I', J, M)$.*

(ii) *If $J \subseteq J'$, then $S\text{-depth}(I, J, M) \geqslant S\text{-depth}(I, J', M)$.*

(iii) $S\text{-depth}(I, J, M) = S\text{-depth}(\sqrt{I}, J, M) = S\text{-depth}(I, \sqrt{J}, M)$.

(iv) $S\text{-depth}(II', J, M) = S\text{-depth}(I \cap I', J, M)$.

(v) $S\text{-depth}(I, JJ', M) = S\text{-depth}(I, J \cap J', M)$.

(vi) *If $\mathfrak{a} \in \widetilde{\mathrm{W}}(I, J)$, then*

$$S\text{-}\operatorname{depth}(I, J, M) \leqslant S\text{-}\operatorname{depth}(\mathfrak{a}, J, M) \leqslant S\text{-}\operatorname{depth}(\mathfrak{a}, M).$$

Proof. The claim follows by Lemma 2.5. □

Proposition 2.7. *Let S satisfy the condition C_I and let $0 \to U \to M \to N \to 0$ be an exact sequence of ZD-modules. Then:*

(i) $S\text{-depth}(I, J, M) \geqslant \min\{S\text{-depth}(I, J, U),\ S\text{-depth}(I, J, N)\}$.

(ii) $S\text{-depth}(I, J, U) \geqslant \min\{S\text{-depth}(I, J, M),\ S\text{-depth}(I, J, N) + 1\}$.

(iii) $S\text{-depth}(I, J, N) \geqslant \min\{S\text{-depth}(I, J, U) - 1,\ S\text{-depth}(I, J, M)\}$.

Proof. We just prove (i) and the other parts are proved similarly. It follows by Proposition 2.6 (vi) and [10], Proposition 3.2 that

$$S\text{-}\operatorname{depth}(\mathfrak{a}, M) \geqslant \min\{S\text{-}\operatorname{depth}(I, J, U),\ S\text{-}\operatorname{depth}(I, J, N)\}$$

for all $\mathfrak{a} \in \widetilde{\mathrm{W}}(I, J)$. Now the claim follows easily. □

3. S-depth of (I, J) and local cohomology with respect to (I, J)

In this section, we study the relations on $S\text{-depth}(I, J, M)$ and local cohomology modules of M with respect to (I, J).

Throughout this section, we assume that

$$0 \to M \to E^0 \xrightarrow{d^0} E^1 \to \ldots \to E^i \xrightarrow{d^i} E^{i+1} \to \ldots$$

is a minimal injective resolution of M, where $E^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} (E_R(R/\mathfrak{p}))^{\mu^i(\mathfrak{p}, M)}$ is a decomposition of E^i as the direct sum of indecomposable injective R-modules and $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$.

The following lemma plays a key role in the next results, and is a generalization of [10], Corollary 3.2.

Lemma 3.1. *Let S be a Serre subcategory closed under taking injective hulls. The following conditions are equivalent:*
(i) $H^i_{I,J}(M) \in S$ *for all* $i < t$.
(ii) $\Gamma_{I,J}(E^i) \in S$ *for all* $i < t$.

Proof. It is easy to see that $\operatorname{Ker}\Gamma_{I,J}(d^i) = \operatorname{Ker} d^i \cap \Gamma_{I,J}(E^i)$ for all i. Also, it follows by [3], Lemma 4.4, that $\Gamma_{I,J}(E^i)$ is injective. Therefore $\Gamma_{I,J}(E^i)$ is the injective hull of $\operatorname{Ker}\Gamma_{I,J}(d^i)$, and the claim follows by [10], Lemma 3.2. $\square$

The next result is a characterization of Artinian local cohomology modules with respect to a pair of ideals.

Corollary 3.2. *The following statements are equivalent:*
(i) $H^i_{I,J}(M)$ *is Artinian for all* $i < t$.
(ii) $\operatorname{Supp}_R\Gamma_{I,J}(E^i)$ *is a finite subset of* $\operatorname{Max}(R)$, *and* $\mu^i(\mathfrak{m}, M)$ *is finite for all* $\mathfrak{m} \in \operatorname{Supp}_R\Gamma_{I,J}(E^i)$ *and all* $i < t$.

Proof. The claim follows easily by Lemma 3.1. $\square$

The following result is a generalization of [7], Theorem 2.4.

Proposition 3.3. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then*

$$\inf\{i\colon\ H^i_{I,J}(M) \notin S\} = \inf\{\operatorname{depth} M_{\mathfrak{p}}\colon\ \mathfrak{p} \in \mathrm{W}(I,J) \text{ and } R/\mathfrak{p} \notin S\}.$$

Proof. Put $t = \inf\{\operatorname{depth} M_{\mathfrak{p}}\colon\ \mathfrak{p} \in \mathrm{W}(I,J)$ and $R/\mathfrak{p} \notin S\}$. It follows by [10], Corollary 3.1, that if $\mathfrak{p} \in \mathrm{W}(I,J)$ and $R/\mathfrak{p} \notin S$, then $\mu^i(\mathfrak{p}, M) = 0$ for all $i < t$. Since, by [13], Proposition 1.11, we have $\Gamma_{I,J}(E^i) \cong \bigoplus\limits_{\mathfrak{p}\in \mathrm{W}(I,J)} (E_R(R/\mathfrak{p}))^{\mu^i(\mathfrak{p},M)}$, therefore $\Gamma_{I,J}(E^i) \in S$ for all $i < t$. Now it follows by Lemma 3.1 that $H^i_{I,J}(M) \in S$ for all $i < t$ and hence $t \leqslant \inf\{i\colon\ H^i_{I,J}(M) \notin S\}$. To prove the converse inequality, it is enough to show that $H^t_{I,J}(M) \notin S$. By assumption, there is $\mathfrak{q} \in \mathrm{W}(I,J)$ with $R/\mathfrak{q} \notin S$ such that $t = \operatorname{depth} M_{\mathfrak{q}}$. It follows by [10], Corollary 3.1, that $\mu^t(\mathfrak{q}, M) \neq 0$. Therefore $\Gamma_{I,J}(E^t) \notin S$ and hence $H^t_{I,J}(M) \notin S$ by Lemma 3.1. $\square$

Corollary 3.4. *Let M be a ZD-module. Then*

$$\inf\{i\colon\ H^i_{I,J}(M) \text{ is not Artinian}\} = \inf\{i\colon\ \operatorname{Supp}_R H^i_{I,J}(M) \nsubseteq \operatorname{Max} R\}.$$

P r o o f. It follows by Proposition 3.3 that

$$\begin{aligned}\inf\{i\colon\ H^i_{I,J}(M) \text{ is not Artinian}\} &= \inf\{\operatorname{depth} M_{\mathfrak p}\colon\ \mathfrak p \in \mathrm W(I,J) - \operatorname{Max}(R)\}\\ &= \inf\{i\colon\ \operatorname{Supp}_R H^i_{I,J}(M) \nsubseteq \operatorname{Max} R\}.\end{aligned}$$

□

Corollary 3.5. *Let M be a ZD-module. If $\operatorname{Supp}_R \Gamma_{I,J}(E^i) \subseteq \operatorname{Max}(R)$ for all $i < t$, then $H^i_{I,J}(M)$ is Artinian for all $i < t$.*

P r o o f. The claim follows by Lemma 3.1 and Corollary 3.4. □

Now, we get a formula on the relation between S-depth(I,J,M) and local cohomology modules of M with respect to (I,J). In this direction, the following result immediately follows.

Lemma 3.6. *Let S satisfy the condition C_I and let M be a ZD-module. Then* S-depth$(I,J,M) \geqslant \inf\{i\colon\ H^i_{I,J}(M) \notin S\}$.

P r o o f. The claim follows by [11], Corollary 2.7 and [10], Corollary 3.1. □

The next theorem is one of the main results of this paper, and is a generalization of [10], Lemma 3.1.

Theorem 3.7. *Let S be a Serre subcategory closed under taking injective hulls and M be a ZD-module. Then* S-depth$(I,J,M) = \inf\{i\colon\ H^i_{I,J}(M) \notin S\}$.

P r o o f. Put $t = S$-depth(I,J,M). Then S-depth$(\mathfrak a, M) \geqslant t$ for all $\mathfrak a \in \widetilde{\mathrm W}(I,J)$. It follows by [10], Lemma 3.1, that $H^i_{\mathfrak a}(M) \in S$ for all $\mathfrak a \in \widetilde{\mathrm W}(I,J)$ and all $i < t$. Thus $\Gamma_{\mathfrak a}(E^i) \in S$ for all $\mathfrak a \in \widetilde{\mathrm W}(I,J)$ and all $i < t$ by [10], Corollary 3.2. Since $\Gamma_{\mathfrak a}(E^i) \cong \bigoplus_{\mathfrak p \in \mathrm V(\mathfrak a)} (E_R(R/\mathfrak p))^{\mu^i(\mathfrak p, M)}$, thus if $\mathfrak p \in \mathrm W(I,J)$ and $R/\mathfrak p \notin S$, then $\mu^i(\mathfrak p, M) = 0$ for all $i < t$. Now it follows by [10], Corollary 3.1, that $\operatorname{depth} M_{\mathfrak p} \geqslant t$ for all $\mathfrak p \in \mathrm W(I,J)$ with $R/\mathfrak p \notin S$. Hence $\inf\{\operatorname{depth} M_{\mathfrak p}\colon\ \mathfrak p \in \mathrm W(I,J)$ and $R/\mathfrak p \notin S\} \geqslant t$, and therefore $\inf\{i\colon\ H^i_{I,J}(M) \notin S\} \geqslant t$ by Proposition 3.3. The converse inequality follows by Lemma 3.6. □

Corollary 3.8. *Let S be a Serre subcategory closed under taking injective hull. Let M be a ZD-module and J' an ideal of R such that $J' \subseteq J$. Then*

$$S\text{-}\operatorname{depth}(I+J', J, M) = S\text{-}\operatorname{depth}(I,J,M).$$

In particular, S-depth$(I+J,J,M) = S$-depth(I,J,M).

P r o o f. The claim follows by Theorem 3.7 and [13], Proposition 1.4, equation (6).

□

The following result is a generalization of [1], Proposition 2.3.

Proposition 3.9. *Let S be a Serre subcategory closed under taking injective hulls, and M be a ZD-module. Then*

$$\begin{aligned} S\text{-}\operatorname{depth}(I,J,M) &= \inf\{\operatorname{depth} M_{\mathfrak p}\colon\ \mathfrak p\in \mathrm{W}(I,J) \text{ and } R/\mathfrak p\notin S\}\\ &= \inf\{S\text{-}\operatorname{depth}(\mathfrak p,M)\colon\ \mathfrak p\in \mathrm{W}(I,J) \text{ and } R/\mathfrak p\notin S\}. \end{aligned}$$

P r o o f. The first equality follows by Proposition 3.3 and Theorem 3.7. For the second, let $\mathfrak p\in \mathrm{W}(I,J)$ with $R/\mathfrak p\notin S$. It follows by [10], Theorem 3.1, that

$$S\text{-}\operatorname{depth}(\mathfrak p,M)=\inf\{\operatorname{depth} M_{\mathfrak q}\colon\ \mathfrak q\in \mathrm{V}(\mathfrak p) \text{ and } R/\mathfrak q\notin S\}\leqslant \operatorname{depth} M_{\mathfrak p}.$$

Therefore

$$\begin{aligned} &\inf\{S\text{-}\operatorname{depth}(\mathfrak p,M)\colon\ \mathfrak p\in \mathrm{W}(I,J) \text{ and } R/\mathfrak p\notin S\}\\ &\qquad\qquad\leqslant \inf\{\operatorname{depth} M_{\mathfrak p}\colon\ \mathfrak p\in \mathrm{W}(I,J) \text{ and } R/\mathfrak p\notin S\}. \end{aligned}$$

To prove the converse inequality, we have by Proposition 2.6 (vi) that

$$S\text{-}\operatorname{depth}(I,J,M)\leqslant S\text{-}\operatorname{depth}(\mathfrak p,M)$$

for all $\mathfrak p\in \mathrm{W}(I,J)$. Therefore

$$S\text{-}\operatorname{depth}(I,J,M)\leqslant \inf\{S\text{-}\operatorname{depth}(\mathfrak p,M)\colon\ \mathfrak p\in \mathrm{W}(I,J) \text{ and } R/\mathfrak p\notin S\}.$$

Now the claim follows from the first equality. □

4. Some isomorphisms on local cohomology modules

In this section, we get some identities between local cohomology modules with respect to a pair of ideals and ordinary local cohomology modules. The following theorem is one of the main results of this paper.

Theorem 4.1. *Let $\operatorname{Supp}_R H^i_{I,J}(M)$ be finite for all $i<t$. Then there is an ideal $\mathfrak a\in \widetilde{\mathrm{W}}(I,J)$ such that $H^i_{I,J}(M)\cong H^i_{\mathfrak a}(M)$ for all $i<t$.*

P r o o f. Let $0 \to M \to E^0 \to \ldots \to E^i \to \ldots$ be a minimal injective resolution of M, where $E^i \cong \bigoplus_{\mathfrak{p}\in\operatorname{Spec}(R)} (E_R(R/\mathfrak{p}))^{\mu^i(\mathfrak{p},M)}$. It follows by [13], Proposition 1.11, that $\Gamma_{I,J}(E^i) \cong \bigoplus_{\mathfrak{p}\in\mathrm{W}(I,J)} (E_R(R/\mathfrak{p}))^{\mu^i(\mathfrak{p},M)}$ and so $\operatorname{Supp}_R\Gamma_{I,J}(E^i) = \{\mathfrak{p} \in \mathrm{W}(I,J)\colon \mu^i(\mathfrak{p},M) \neq 0\}$. It follows by the hypothesis and Lemma 3.1 that $\operatorname{Supp}_R\Gamma_{I,J}(E^i)$ is finite for all $i < t$. Put $\bigcup_{i<t} \operatorname{Supp}_R\Gamma_{I,J}(E^i) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k\}$ and $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_k$. Then $\mathfrak{a} \in \widetilde{\mathrm{W}}(I,J)$ and $\operatorname{Supp}_R\Gamma_{I,J}(E^i) = \{\mathfrak{p} \in \mathrm{V}(\mathfrak{a})\colon \mu^i(\mathfrak{p},M) \neq 0\}$ for all $i < t$. Hence

$$\Gamma_{I,J}(E^i) \cong \bigoplus_{\mathfrak{p}\in\mathrm{V}(\mathfrak{a})} (E_R(R/\mathfrak{p}))^{\mu^i(\mathfrak{p},M)} \cong \Gamma_{\mathfrak{a}}(E^i)$$

for all $i < t$. Therefore, $H^i_{I,J}(M) \cong H^i_{\mathfrak{a}}(M)$ for all $i < t$. □

Proposition 4.2. *Let* $\operatorname{Supp}_R H^i_{I,J}(M)$ *be a finite subset of* $\operatorname{Max}(R)$ *for all* $i < t$. *Then there are distinct maximal ideals* $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k \in \mathrm{W}(I,J)$ *such that*

$$H^i_{I,J}(M) \cong H^i_{\mathfrak{m}_1}(M) \oplus H^i_{\mathfrak{m}_2}(M) \oplus \ldots \oplus H^i_{\mathfrak{m}_k}(M)$$

for all $i < t$.

P r o o f. In view of Theorem 4.1 and its proof, one can see that there are distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k \in \mathrm{W}(I,J)$ such that $\bigcup_{i<t} \operatorname{Supp}_R\Gamma_{I,J}(E^i) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k\}$ and $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}_1\cap\mathfrak{m}_2\cap\ldots\cap\mathfrak{m}_k}(M)$ for all $i < t$. Now, by the Mayer-Vietoris sequence (see [5], Subsection 3.2.3) $H^i_{\mathfrak{m}_1\cap\mathfrak{m}_2\cap\ldots\cap\mathfrak{m}_k}(M) \cong H^i_{\mathfrak{m}_1}(M) \oplus H^i_{\mathfrak{m}_2}(M) \oplus \ldots \oplus H^i_{\mathfrak{m}_k}(M)$, and the claim follows. □

The following result is a generalization of [12], Corollary 2.10.

Corollary 4.3. *Let* $(R,\mathfrak{m})$ *be a local ring. If* $\operatorname{Supp}_R H^i_{I,J}(M) \subseteq \{\mathfrak{m}\}$ *for all* $i < t$, *then* $H^i_{I,J}(M) \cong H^i_{\mathfrak{m}}(M)$ *for all* $i < t$.

P r o o f. The claim immediately follows by Proposition 4.2. □

Corollary 4.4. *If* $(R,\mathfrak{m})$ *is a local ring, then*

$$\inf\{i\colon H^i_{I,J}(M) \ncong H^i_{\mathfrak{m}}(M)\} = \inf\{i\colon \operatorname{Supp}_R H^i_{I,J}(M) \nsubseteq \{\mathfrak{m}\}\}.$$

P r o o f. The claim follows by Corollary 4.3. □

The following result generalizes [7], Proposition 2.5.

Corollary 4.5. *Let $(R, \mathfrak{m})$ be a local ring and M be a ZD-module. Then*

$$\inf\{i\colon\ H^i_{I,J}(M) \ncong H^i_{\mathfrak{m}}(M)\} = \inf\{i\colon\ H^i_{I,J}(M) \text{ is not Artinian}\}.$$

Proof. The claim follows by Corollaries 3.4 and 4.4. $\square$

Corollary 4.6. *Let $\mathrm{Supp}_R H^i_{I,J}(M)$ be a finite subset of $\mathrm{Max}(R)$ for all $i < t$. Then $H^i_{\mathfrak{a}}(M)$ is a submodule of $H^i_{I,J}(M)$ for all $\mathfrak{a} \in \widetilde{\mathrm{W}}(I,J)$, and all $i < t$.*

Proof. It follows by [11], Corollary 2.6, that $\mathrm{Supp}_R H^i_{\mathfrak{a}}(M)$ is a finite subset of $\mathrm{Max}(R)$ for all $\mathfrak{a} \in \widetilde{\mathrm{W}}(I,J)$ and all $i < t$. Since $\mathrm{Supp}_R \Gamma_{\mathfrak{a}}(E^i) \subseteq \mathrm{Supp}_R \Gamma_{I,J}(E^i)$ for all $\mathfrak{a} \in \widetilde{\mathrm{W}}(I,J)$ and all $i < t$, the claim follows by Proposition 4.2 and its proof. $\square$

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