Kybernetika

Amin Yousefi; Mashaallah Mashinchi; Radko Mesiar

Some notes on the category of fuzzy implications on bounded lattices

Kybernetika, Vol. 57 (2021), No. 2, 332–351

Persistent URL: http://dml.cz/dmlcz/149042

KYBERNETIKA — VOLUME 57 (2021), NUMBER 2, PAGES 332–351

SOME NOTES ON THE CATEGORY OF FUZZY IMPLICATIONS ON BOUNDED LATTICES

Amin Yousefi, Mashaallah Mashinchi and Radko Mesiar

In this paper, we introduce the product, coproduct, equalizer and coequalizer notions on the category of fuzzy implications on a bounded lattice that results in the existence of the limit, pullback, colimit and pushout. Also isomorphism, monic and epic are introduced in this category. Then a subcategory of this category, called the skeleton, is studied. Where none of any two fuzzy implications are Φ-conjugate.

Keywords: fuzzy implication, t-norm, category, skeleton of category

Classification: 03B52, 03E72

1. INTRODUCTION

Fuzzy logic connectives play an important role in the theory of fuzzy sets and fuzzy logic [11, 20]. Fuzzy implications have an important role both in theory and applications [2, 3, 6]. For example in approximate reasoning, decision theory, control theory, expert systems, image processing and so on [13, 17, 21]. On the other hand, in fuzzy logic it has been very common to use lattice theory to deal with it in a more general setting [5, 9, 19]. T-norms and R-implications were defined on bounded lattices [20]. On the other hand the category theory is an abstract structure which plays an important role in mathematics composed by a collection of objects together with a collection of morphisms between them [1]. The number of t-norms especially on finite chains has been studied in many articles [9, 10, 12, 14]. We have discussed the number of t-norms and R-implications (on a finite bounded lattice L), which is clearly the same, see [19]. This number is growing exponentially and thus the study of all such implications is a difficult task. This fact has initiated our study of category of fuzzy implications. In this paper, we define equalizer, coequalizer, product and coproduct on the category $\mathfrak{FI}$ and we prove that it has limit, colimit, pullback and pushout. Then, we introduce skeleton on the category $\mathfrak{FI}$ on a bounded lattice and we show that none of two fuzzy implications in the skeleton are Φ-conjugate, where Φ denotes the family of all increasing bijections homomorphisms. In Section 4 we show that every fuzzy implication is an Ω-algebra and the category of $\mathfrak{FI}$ is a full subcategory of $\mathbf{Alg(\Omega)}$.

DOI: 10.14736/kyb-2021-2-0332

2. PRELIMINARIES

In this section, we present some main concepts that will be used in this work such as bounded lattice, triangular norms, homomorphism, fuzzy implication, category and so on. For more details about these concepts see [1–4, 7, 11, 15].

Definition 2.1. (Davey and Priestley [8]) Let L be a non-empty ordered set.

(1) If $x \vee y$ and $x \wedge y$ exist for each $x, y \in L$, then L is called a lattice.

(2) If $\bigvee S$ and $\bigwedge S$ exist for each $S \subseteq L$, then L is called a complete lattice.

Lemma 2.2. (Davey and Priestley [8]) Let L be a lattice and $x, y \in L$. Then the following are equivalent:

(i) $x \leq y$;

(ii) $x \bigvee y = y$;

(iii) $x \bigwedge y = x$.

Definition 2.3. (Davey and Priestley [8]) Let $\wedge$ and $\vee$ be two binary operations on a non-empty set L. Then, the algebraic structure $< L, \vee, \wedge >$ is a lattice if for each $x, y, z \in L$, the following properties hold:

(1) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$,

(2) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$,

(3) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$.
If there are elements 1 and 0 in L such that, for each $x \in L$, $x \wedge 1 = x$ and $x \vee 0 = x$, then $< L, \vee, \wedge, 0, 1 >$ is called a bounded lattice. Sometimes we write 1_L and 0_L.

Remark 1. (Davey and Priestley [8]) Note that Definitions 2.1 and 2.3 are equivalent.

The followings are examples of bounded lattices,

Examples 2.4. (Davey and Priestley [8])

1. $L_{\top} =< \{1\}, \vee, \wedge, 1, 1 >$, where $L = \{1\}$, $1 \vee 1 = 1$ and $1 \wedge 1 = 1$.

2. $L_{\perp} =< \{0, 1\}, \vee, \wedge, 0, 1 >$, where $L = \{0, 1\}$, $\vee$ and $\wedge$ are as max and min functions in the Boolean algebra.

3. $M_k =< \{0, a_1, a_2, \ldots, a_k, 1\}, \vee, \wedge, 0, 1 >$, where $0 < a_1, a_2, \ldots, a_k < 1$ and $a_i \vee a_j = 1$ and $a_i \wedge a_j = 0$, for each $i, j = 1, 2, \ldots k$ and $i \neq j$ as depicted diagram in Figure 1.

4. $L =< [0, 1], \vee, \wedge, 0, 1 >$, where $[0, 1]$ is unit interval in the real numbers $\mathbb{R}$, $\vee$ and $\wedge$ are as max and min functions in real numbers $\mathbb{R}$.

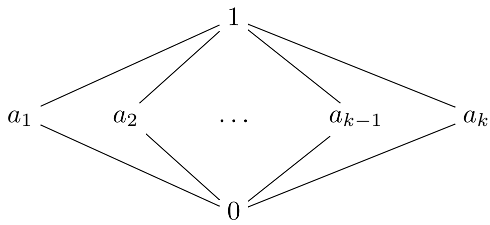


Fig. 1. The lattice M_k.

Definition 2.5. (Davey and Priestley [8]) Let L and K be lattices. A map $\phi : L \to K$ is said to be a lattice homomorphism if for each $x, y \in L$,

$$\phi(x \vee y) = \phi(x) \vee \phi(y) \quad \textit{and} \quad \phi(x \wedge y) = \phi(x) \wedge \phi(y).$$

Definition 2.6. (Davey and Priestley [8]) Let P and Q be ordered sets. A map $\varphi : P \to Q$ is said to be order-embedding if

$$x \le y \quad \text{in P} \iff \varphi(x) \le \varphi(y) \;\; \text{in Q}.$$

From now on we assume L is a bounded lattice unless otherwise stated.

Definition 2.7. (Yu et al. [20]) A binary operation T on lattice L is called a t-norm if for each $a, b, c \in L$ the following properties hold:

(1) $T(T(a, b), c) = T(a, T(b, c))$,

(2) $T(a, b) = T(b, a)$,

(3) if $b \le c$ then $T(a, b) \le T(a, c)$,

(4) $T(a, 1) = a$.

Moreover if T satisfies conditions $(1 - 3)$ and

(5) $T(a, 0) = a$, then T is called an s-norm (or t-conorm).

T_0 and S_0 are t-norm and t-conorm on a lattice L, respectively, where

$$T_0(a,b) = \begin{cases} a \wedge b & \text{if } a = 1 \text{ or } b = 1 \\ 0 & \text{otherwise} \end{cases} \qquad S_0(a,b) = \begin{cases} a \vee b & \text{if } a = 0 \text{ or } b = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Observe that for any t-norm T and t-conorm S on L it holds, $T_0 \le T \le \wedge \le \vee \le S \le S_0$.

Definition 2.8. (Baczynski et al. [3]) A function $I_L : L \times L \to L$ is called a fuzzy implication on L if for each $x, y, z \in L$ we have that

(1) if $x \le y$ then $I_L(y, z) \le I_L(x, z)$,

(2) if $y \leq z$ then $I_L(x, y) \leq I_L(x, z)$,

(3) $I_L(0, 0) = 1$,

(4) $I_L(1, 1) = 1$,

(5) $I_L(1, 0) = 0$.

The collection of all fuzzy implications on any arbitrary bounded lattice will be denoted by $\mathbb{I}$.

Definition 2.9. (Baczynski et al. [3]) A fuzzy implication I_L on bounded lattice L is said to satisfy

(i) the left neutrality property (***NP***) if

$$I_L(1, y) = y, \qquad y \in L;$$

(ii) the exchange principle (***EP***), if

$$I_L(x, I_L(y, z)) = I_L(y, I_L(x, z)), \qquad x, y, z \in L;$$

(iii) the identity principle (***IP***), if

$$I_L(x, x) = x, \qquad x \in L;$$

(iv) the ordering property (***OP***), if

$$I_L(x, y) = 1 \Longleftrightarrow x \leq y, \qquad x, y \in L;$$

(v) the left boundary condition (***LB***), if

$$I_L(0, y) = 1, \qquad y \in L;$$

(vi) the right boundary condition (***RB***), if

$$I_L(x, 1) = 1, \qquad x \in L;$$

(vii) the normality condition (***NC***)

$$I_L(0, 1) = 1.$$

Note that properties (v), (vi) and (vii) are satisfied for each fuzzy implication I_L, independently of the lattice L.

Lemma 2.10. (Yu et al. [20]) Let T be a t-norm on a complete lattice L. The function $I_T : L \times L \to L$ is a fuzzy implication which is called residual implication (in short, R-implication) generated by the t-norm T, where

$$I_T(x, y) = \bigvee \{t \in L | T(x, t) \leq y\}, \quad x, y \in L. \tag{1}$$

Definition 2.11. (Baczynski and Jayaram [4]) Let I_L and $J_{L'}$ be fuzzy implications on bounded lattices L and L', respectively, and Φ be the family of all increasing bijections (isomorphisms) from L to L'. Then, I_L and $J_{L'}$ are called Φ-conjugate if there exists $\varphi \in \Phi$ such that,

$$I_L(x, y) = \varphi^{-1}(J_{L'}(\varphi(x), \varphi(y))).$$

Then we denote it as $I_L = (J_{L'})_\varphi$.

For some infinite bounded lattices, there are many $\varphi \in \Phi$. For example in $[0, 1]$, we have $x^p \in \Phi$ for any rational positive power p. [4]

Definition 2.12. (Adámek et al. [1]) A category is a quadruple $\mathfrak{A} = (Obj(\mathfrak{A}), hom, id, \circ)$ consisting of

(1) a class $Obj(\mathfrak{A})$, whose members are called $\mathfrak{A}$-objects,

(2) for each pair (A, B) of $\mathfrak{A}$-objects, a set $hom(A, B)$, whose members are called $\mathfrak{A}$-morphisms from A to B denoted $f : A \to B$.

(3) for each $\mathfrak{A}$-object A, a morphism $id_{\mathfrak{A}} : A \to A$, called the $\mathfrak{A}$-identity on A,

(4) a composition law associating with each $\mathfrak{A}$-morphism $f : A \to B$ and each $\mathfrak{A}$-morphism $g : B \to C$ an $\mathfrak{A}$-morphism $g \circ f : A \to C$, called the composite of f and g, subject to the following conditions:

  (i) the composition is associative,

  (ii) $\mathfrak{A}$-identities act as identities with respect to composition,

  (ii) the sets $hom(A, B)$ are pairwise disjoint.

Definition 2.13. (Adámek et al. [1]) If $\mathfrak{A}$ and $\mathfrak{B}$ are categories, then a functor F from $\mathfrak{A}$ to $\mathfrak{B}$ is a function that assigns to each $\mathfrak{A}$-object A a $\mathfrak{B}$-object $F(A)$, and to each $\mathfrak{A}$-morphism $f : A \to A^{'}$ a $\mathfrak{B}$-morphism $F(f) : F(A) \to F(A^{'})$, such that F preserves both composition and $\mathfrak{A}$-identity morphisms.

Definition 2.14. (Adámek et al. [1]) An object A is called an initial object provided that for each object of B there is exactly one morphism from A to B.

Definition 2.15. (Adámek et al. [1]) An object of A is called a terminal object provided that for each object of B there is exactly one morphism from B to A.

Definition 2.16. (Adámek et al. [1]) A category $\mathfrak{A}$ is said to be a subcategory of a category $\mathfrak{B}$ provided that the following conditions are satisfied:

(1) $Ob(\mathfrak{A}) \subseteq Ob(\mathfrak{B})$,

(2) for each $A, A' \in Ob(\mathfrak{A})$, $hom_{\mathfrak{A}}(A, A') \subseteq hom_{\mathfrak{B}}(A, A')$,

(3) for each $\mathfrak{A}$-object A, the $\mathfrak{B}$-identity on A is the $\mathfrak{A}$-identity on A,

(4) the composition law in $\mathfrak{A}$ is the restriction of the composition law in $\mathfrak{B}$ to the morphisms of $\mathfrak{A}$,

(5) $\mathfrak{A}$ is called a full subcategory of $\mathfrak{B}$ if, in addition to the above conditions $(1-4)$, $hom_{\mathfrak{A}}(A, A_0) = hom_{\mathfrak{B}}(A, A_0)$, for each $A, A_0 \in Ob(\mathfrak{A})$.

Definition 2.17. (Adámek et al. [1]) A diagram in a category $\mathfrak{A}$ is a functor $D : \mathfrak{I} \to \mathfrak{A}$ with codomain $\mathfrak{A}$. The domain, $\mathfrak{I}$ is called the scheme of the diagram.

Definition 2.18. (Adámek et al. [1]) Let $D : \mathfrak{I} \to \mathfrak{A}$ be a diagram.

(1) An $\mathfrak{A}$-source $(D : A \xrightarrow{f_i} D_i)_{i \in Obj(\mathfrak{I})}$ is said to be natural for D provided that for each $\mathfrak{I}$-morphism $i \xrightarrow{d} j$, the triangle

$$\begin{array}{ccc} A & & \\ {\scriptstyle f_i}\downarrow & \searrow^{f_j} & \\ D_i & \xrightarrow[Dd]{} & D_j \end{array}$$

commutes.

(2) A limit of D is a natural source $(B \xrightarrow{b_i} D_i)_{i \in Obj(\mathfrak{I})}$ for D with the (universal) property that each natural source $(D : A \xrightarrow{f_i} D_i)_{i \in Obj(\mathfrak{I})}$ for D uniquely factors through it; i. e., for every such source there exists a unique morphism $f : A \to B$ with $f_i = b_i \circ f$ for each $i \in Obj(\mathfrak{I})$.

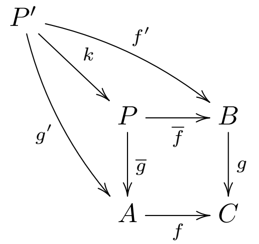
Definition 2.19. (Adámek et al. [1]) A square

$$\begin{array}{ccc} P & \xrightarrow{\overline{f}} & B \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

is called a pullback square provided that it commutes and that for any commuting square of the form

$$\begin{array}{ccc} P' & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

there exists a unique morphism $P' \xrightarrow{k} P$ for which the following diagram commutes.



Definition 2.20. (Adámek et al. [1]) A full subcategory $\mathfrak{A}$ of a category $\mathfrak{B}$ is called isomorphism-dense provided that every $\mathfrak{B}$-object is isomorphic to some $\mathfrak{A}$-object.

Definition 2.21. (Adámek et al. [1]) A skeleton of a category $\mathfrak{C}$ is a full, isomorphism-dense subcategory of $\mathfrak{C}$ in which no two distinct objects are isomorphic.

Theorem 2.22. (Adámek et al. [1])

(1) Every category has a skeleton.

(2) Any two skeletons of a category are isomorphic.

(3) Any skeleton of a category $\mathfrak{C}$ is equivalent to $\mathfrak{C}$.

Theorem 2.23. (Bedregal [5]) Let A be the collection of all t-norms on a bounded lattice and $hom(T_L, T_{L'})$, $T_L, T_{L'} \in A$, be the set of all lattice homomorphisms so that for each ρ from L to $L^{'}$, we have

$$\rho(T_L(x,y)) = T_{L'}(\rho(x), \rho(y)), \qquad \text{for all} \qquad x, y \in L.$$

Note that T_L act on L and $T_{L'}$ on L'. Then quadruple $T - NORM = (A, hom, id, \circ)$ is the category of t-norms.

Remark 2. (Yousefi and Mashinchi [18]) Let $I_L : L \times L \to L$ and $J_{L'} : L' \times L' \to L'$ be two fuzzy implications and φ be a lattice homomorphism such that the diagram in Figure 2 commutes and φ satisfies the property $\varphi(I_L(x,y)) = J_{L'}(\varphi(x), \varphi(y))$ for each $x, y \in L$. We denote this φ with extra property by $\varphi^* : I_L \to J_{L''}$. So φ^* is just φ with the extra property, $\varphi(I_L(x,y)) = J_{L'}(\varphi(x), \varphi(y))$.

Lemma 2.24. (Yousefi and Mashinchi [18]) Let $\mathbb{I}$ be the collection of all fuzzy implications on bounded lattices. Let $\varphi^* : I_L \to J_{L'}$ and $\psi^* : J_{L'} \to K_{L''}$ be two morphisms defined in Remark 2, where $\psi^* \circ \varphi^* = (\psi \circ \varphi)^*$ and 1^* is identity lattice homomorphism, where I_L, $J_{L'}$ and $K_{L''}$ are fuzzy implications on bounded lattices L, L' and L'' respectively. Then $\mathfrak{FI} = (\mathbb{I}, hom(I_L, J_{L'}), 1^*, \circ)$ is a category.
We call $\mathfrak{FI}$ the category of fuzzy implications.

Remark 3. (Yousefi and Mashinchi [18]) Note that on a singleton lattice $L_\top$, there is a unique binary operation which is simultaneously t-norm, t-conorm and a fuzzy implication. Therefore $I_\top : L_\top \times L_\top \to L_\top$ that defined by $I_\top(1,1) = 1$ is a fuzzy implication on $L_\top$.

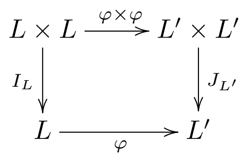


Fig. 2. Arrow in category $\mathfrak{FI}$.

Theorem 2.25. (Yousefi and Mashinchi [18]) Let I_L be a fuzzy implication on a bounded lattice L and $\varphi_\top : L \to L_\top$ be defined by $\varphi_\top(x) = 1$ for each $x \in L$, Then $\varphi_\top^*$ is the unique morphism from I_L to $I_\top$. Thus, $I_\top$ is a terminal object of category $\mathfrak{FI}$.

Lemma 2.26. (Yousefi and Mashinchi [18]) Let $I_\perp : L_\perp \times L_\perp \to L_\perp$ be defined by $I_\perp(0,0) = I_\perp(1,1) = I_\perp(0,1) = 1$ and $I_\perp(1,0) = 0$. Then $I_\perp$ is a fuzzy implication (implication in classical logic) on the bounded lattice $L_\perp$.

Theorem 2.27. (Yousefi and Mashinchi [18]) Let I_L be a fuzzy implication on a bounded lattice L and $\varphi_\perp : L_\perp \to L$ be defined by $\varphi_\perp(0) = 0$ and $\varphi_\perp(1) = 1$. Then $\varphi_\perp^*$ is the unique morphism from $I_\perp$ to I_L. Thus $I_\perp$ is an initial object of category $\mathfrak{FI}$.

Lemma 2.28. (Yousefi and Mashinchi [18]) Let $\mathbb{I}_T$ be the collection of all R-implications on any complete lattice and φ^* be a t-norm morphism in the category T-NORM and φ be order-embedding on the lattice, then $\mathfrak{FRI} = (\mathbb{I}_T, hom(I_T, I_{T'}))$ is a subcategory of $\mathfrak{FI}$.

Theorem 2.29. (Yousefi and Mashinchi [18]) Let F be a function from $\mathfrak{FRI}$ to T-NORM such that $F(I_T) = T$ and $F(\varphi) = \varphi$. Then F is a functor.

Theorem 2.30. Let F be functor in the Theorem 2.29 and G be a function from T-NORM to $\mathfrak{FRI}$ such that $G(T) = I_T$ and $G(\varphi) = \varphi$. Then G is a functor and $F \circ G = G \circ F = id$ and so F is isomorphism.

3. SOME RESULTS ON THE CATEGORY OF FUZZY IMPLICATIONS($\mathfrak{FI}$)

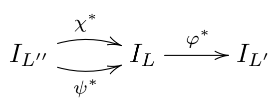


Fig. 3. φ^* is a monic in $\mathfrak{FI}$.

Theorem 3.1. Let $\varphi : L \to L'$ be a lattice monomorphism. Then $\varphi^* : I_L \to I_{L'}$ is a monic in $\mathfrak{F}\mathfrak{I}$.

Proof. If ψ^* and χ^* are two morphisms from $I_{L''}$ to I_L such that $\varphi^* \circ \psi^* = \varphi^* \circ \chi^*$ as depicted diagram in Figure 3 , then $\varphi \circ \psi(I_{L''}(x, y)) = \varphi \circ \chi(I_{L''}(x, y))$. So $\varphi(\psi(I_{L''}(x, y))) = \varphi(\chi(I_{L''}(x, y)))$. Since φ is a lattice monomorphism, we have $\psi(I_{L''}(x, y)) = \chi(I_{L''}(x, y))$. Hence $\psi^* = \chi^*$ and φ^* is monic. □

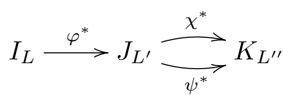


Fig. 4. φ^* is an epic in $\mathfrak{F}\mathfrak{I}$.

Theorem 3.2. If $\varphi : L \to L'$ is an epimorphism as a lattice homomorphism, then $\varphi^* : I_L \to J_{L'}$ is epic as an arrow in $\mathfrak{F}\mathfrak{I}$.

Proof. Let φ be an epimorphism from bounded lattice L to bounded lattice L' and $\psi^* \circ \varphi^* = \chi^* \circ \varphi^*$ for some parallel arrow, ψ^* and $\chi*$ from $J_{L'}$ to $K_{L''}$ as depicted diagram in Figure 4. Now let x' and y' be in L', then there exist x and y in L such that $\varphi(x) = x'$ and $\varphi(y) = y'$, and

$$\begin{aligned}
\psi^*(J_{L'}(x', y')) &= \psi^*(J_{L'}(\varphi(x), \varphi(y))) \\
&= \psi^*(\varphi^*(I_L(x, y))) \\
&= \psi^* \circ \varphi^*(I_L(x, y)) \\
&= \chi^* \circ \varphi^*(I_L(x, y)) \\
&= \chi^*(\varphi^*(I_L(x, y))) \\
&= \chi^*(J_{L'}(\varphi(x), \varphi(y))) \\
&= \chi^*(J_{L'}(x', y')).
\end{aligned}$$

Therefore $\psi^* = \chi^*$ and φ^* is epic. □

Theorem 3.3. Let I_L and $J_{L'}$ be fuzzy implications and there exist an epic $\varphi^*(I_L) = J_{L'}$, then

(1) If I_L satisfies **LB**, then $J_{L'}$ satisfy **LB**.

(2) If I_L satisfies **RB**, then $J_{L'}$ satisfy **RB**.

(3) If I_L satisfies **NP**, then $J_{L'}$ satisfy **NP**.

(4) If I_L satisfies **IP**, then $J_{L'}$ satisfy **IP**.

(5) If I_L satisfies **EP**, then $J_{L'}$ satisfy **EP**.

(6) If I_L satisfies **NC**, then $J_{L'}$ satisfy **NC**.

Proof. Case 5 is proved, other cases can be proved similarly. Let $x, y, z \in L'$ and φ^* be epic, then there exist $a, b, c \in L$ such that $\varphi(a) = b$, $\varphi(b) = y$ and $\varphi(c) = c$, so that

$$\begin{aligned} J_{L'}(x, J_{L'}(y, z)) &= J_{L'}(\varphi(a), J_{L'}(\varphi(b), \varphi(c)) \\ &= J_{L'}(\varphi(a), \varphi(I_L(b, c))) \\ &= \varphi(I_L(a, I_L(b, c))) \\ &= \varphi(I_L(b, I_L(a, c))) \\ &= J_{L'}(\varphi(b), \varphi(I_L(a, c))) \\ &= J_{L'}(\varphi(b), J_{L'}(\varphi(a), \varphi(c)) \\ &= J_{L'}(y, J_{L'}(x, z)). \end{aligned}$$

□

Theorem 3.4. Let I_L and $J_{L'}$ be fuzzy implications in $\mathbb{I}$ and define

$$I_L \times J_{L'}((x_1, x_2), (y_1, y_2)) := (I_L(x_1, y_1), J_{L'}(x_2, y_2)),\ \forall (x_1, x_2)\ \&\ (y_1, y_2) \in L \times L'.$$

Then $I_L \times J_{L'}$ is a product on the category $\mathfrak{FI}$.

Proof. Let I_L and $J_{L'}$ be fuzzy implications on lattices L and L', respectively. Then we prove that $I_L \times J_{L'}$ is fuzzy implication on the lattice $L \times L'$. Note that we have

$$\begin{aligned} I_L \times J_{L'}((0,0),(0,0)) &= (I_L(0,0), J_{L'}(0,0)) = (1,1) \\ I_L \times J_{L'}((1,1),(1,1)) &= (I_L(1,1), J_{L'}(1,1)) = (1,1) \\ I_L \times J_{L'}((0,0),(1,1)) &= (I_L(0,1), J_{L'}(0,1)) = (1,1) \\ I_L \times J_{L'}((1,1),(0,0)) &= (I_L(1,0), J_{L'}(1,0)) = (0,0). \end{aligned}$$

Now if $(x_1, x_2) \leq (y_1, y_2)$, then $x_1 \leq y_1$ and $x_2 \leq y_2$. So $I_L(y_1, z_1) \leq I_L(x_1, z_1)$ and $J_{L'}(y_2, z_2) \leq J_{L'}(x_2, z_2)$. Hence $(I_L(y_1, z_1), J_{L'}(y_2, z_2)) \leq (I_L(x_1, z_1), J_{L'}(x_2, z_2))$. Thus $I_L \times J_{L'}((y_1, y_2), (z_1, z_2)) \leq I_L \times J_{L'}((x_1, x_2), (z_1, z_2))$. Similarly, if $(x_1, x_2) \leq (y_1, y_2)$, then $I_L \times J_{L'}((z_1, z_2), (x_1, x_2)) \leq I_L \times J_{L'}((z_1, z_2), (y_1, y_2))$. Therefore $I_L \times J_{L'}$ is fuzzy implication. Now let π_1 and π_2 be two projections from $L \times L'$ to L and L', respectively. Then we prove π_1^* and π_2^* are two projections from $I_L \times J_{L'}$ to I_L and J'_L, respectively. Note that

$$\begin{aligned} \pi_1(I_L \times J_{L'}((x_1, x_2), (y_1, y_2))) &= \pi_1((I_L(x_1, y_1), J_{L'}(x_2, y_2))) \\ &= I_L(x_1, y_1) = I_L(\pi_1(x_1, x_2), \pi_1(y_1, y_2)). \end{aligned}$$

Thus π_1^* is a projection. Similarly, π_2^* is a projection. Finally, we prove that $I_L \times J_{L'}$ satisfies the universal property. Let $K_{L''}$ be a fuzzy implication on the bounded lattice L'' and let ρ_1 and ρ_2 be fuzzy implications morphism from $K_{L''}$ to I_L and $J_{L'}$, respectively.

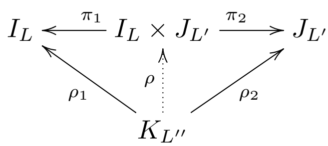


Fig. 5. $I_L \times J_{L'}$ is a product on the category $\mathfrak{FI}$.

Then, we show there exists a morphism ρ from $K_{L''}$ to $I_L \times J_{L'}$ such that diagram in Figure 5 commutes.

Let $\rho(x) = (\rho_1(x), \rho_2(x))$, then

$$\begin{aligned}\rho(K_{L''}(x,y)) &= (\rho_1(K_{L''}(x,y)), \rho_2(K_{L''}(x,y)))\\ &= (I_L(\rho_1(x), \rho_1(y)), J_{L'}(\rho_2(x), \rho_2(y)))\\ &= I_L \times J_{L'}((\rho_1(x), \rho_2(x)), (\rho_1(y), \rho_2(y)))\\ &= I_L \times J_{L'}(\rho(x), \rho(y)).\end{aligned}$$

So, ρ is a morphism. Now suppose that ρ' is another morphism from $K_{L''}$ to $I_L \times J_{L'}$ such that the diagram in Figure 5, commutes. Then, $\pi_1(\rho'(K_{L''}(x,y))) = \rho_1(K_{L''}(x,y))$ and $\pi_2(\rho'(K_{L''}(x,y))) = \rho_2(K_{L''}(x,y))$. Thus $\rho' = \rho$ and hence ρ is unique. □

Theorem 3.5. Let I_L and $J_{L'}$ be two fuzzy implication with the same property in $\mathbb{I}$, then $I_L \times J_{L'}$ preserves properties of I_L and $J_{L'}$. That is

(1) If I_L and $J_{L'}$ satisfy **LB**, then $I_L \times J_{L'}$ satisfy **LB**.

(2) If I_L and $J_{L'}$ satisfy **RB**, then $I_L \times J_{L'}$ satisfy **RB**.

(3) If I_L and $J_{L'}$ satisfy **NP**, then $I_L \times J_{L'}$ satisfy **NP**.

(4) If I_L and $J_{L'}$ satisfy **EP**, then $I_L \times J_{L'}$ satisfy **EP**.

(5) If I_L and $J_{L'}$ satisfy **IP**, then $I_L \times J_{L'}$ satisfy **IP**.

(6) If I_L and $J_{L'}$ satisfy **OP**, then $I_L \times J_{L'}$ satisfy **OP**.

(7) If I_L and $J_{L'}$ satisfy **NC**, then $I_L \times J_{L'}$ satisfy **NC**.

Proof. Cases 4 and 6 are proved, other cases can be proved similarly.

(4):

$$\begin{aligned}I_L \times J_{L'}((m,n), I_L \times J_{L'}((p,q),(r,s))) &= I_L \times J_{L'}((m,n),(I_L(p,r), J_{L'}(q,s)))\\ &= (I_L(m, I_L(p,r)), J_{L'}(n, J_{L'}(q,s)))\\ &= (I_L(p, I_L(m,r)), J_{L'}(q, J_{L'}(n,s)))\\ &= I_L \times J_{L'}((p,q),(I_L(m,r), J_{L'}(n,s)))\\ &= I_L \times J_{L'}((p,q), I_L \times J_{L'}((m,n),(r,s)))\end{aligned}$$

(6):

$$
\begin{aligned}
I_L \times J_{L'}((m,n),(p,q)) = (1,1) &\Leftrightarrow (I_L(m,p), J_{L'}(p,q)) = (1,1)\\
&\Leftrightarrow I_L(m,p) = 1 \quad \& \quad J_{L'}(p,q) = 1\\
&\Leftrightarrow m < p \quad \& \quad n < q\\
&\Leftrightarrow (m,n) < (p,q)
\end{aligned}
$$

□

Theorem 3.6. If I_L and $J_{L'}$ are two objects in category $\mathfrak{F}\mathfrak{I}$.Then $I_L \oplus J_{L'}$ is coproduct with coproduct injections, $\varphi^* : I_L \to I_L \oplus J_{L'}$ and $\psi^* : J_{L'} \to I_L \oplus J_{L'}$, where $I_L \oplus J_{L'}((m,n),(p,q)) = (I_L(m,p), J_{L'}(n,q))$ for all $m,p \in L$ and $n,q \in L'$.

Proof. Let $\varphi : L \to L \times L'$ and $\psi : L' \to L \times L'$ be two lattice homomorphism such that $\varphi(x) = (x, 0_{L'})$ and $\psi(y) = (0_L, y)$.Then $\varphi^* : I_L \to I_L \oplus J_{L'}$ and $\psi^* : J_{L'} \to I_L \oplus J_{L'}$ are two arrows in category $\mathfrak{F}\mathfrak{I}$. Now let there exist a fuzzy implication $K_{L''}$ on bounded lattice L'' and $f^* : I_L \to K_{L''}$ and $g^* : J_{L'} \to K_{L''}$ be two arrows. Then we define arrow $[f^*, g^*](I_L, J_{L'}) = f^*(I_L) \wedge g^*(J_{L'})$. Therefore,

$$
\begin{aligned}
\varphi^*(I_L(x,y)) &= I_L \oplus J_{L'}(\varphi(x), \varphi(y))\\
&= I_L \oplus J_{L'}((x, 0_{L'}), (y, 0_{L'}))\\
&= (I_L(x,y), J_{L'}(0_{L'}, 0_{L'}))\\
&= (I_L(x,y), 1_{L'})\\
&= f(I_L(x,y)) \wedge g(1_{L'})\\
&= f(I_L(x,y)) \wedge 1_{L''}\\
&= f(I_L(x,y)).
\end{aligned}
$$

and we also have

$$
\begin{aligned}
\psi^*(J_{L'}(x,y)) &= I_L \oplus J_{L'}(\psi(x), \psi(y))\\
&= I_L \oplus J_{L'}((0_L, x), (0_L, y))\\
&= (I_L(0_L, 0_L), J_{L'}(x,y))\\
&= (1_L, J_{L'}(x,y))\\
&= f(1_L) \wedge g(J_{L'}(x,y))\\
&= 1_{L''} \wedge g(J_{L'}(x,y))\\
&= g(J_{L'}(x,y)).
\end{aligned}
$$

Hence diagram in Figure 6 commutes, and $I_L \oplus J_{L'}$ is coproduct with coproduct injections, $\varphi^* : I_L \to I_L \oplus J_{L'}$ and $\psi^* : J_{L'} \to I_L \oplus J_{L'}$. □

Theorem 3.7. Let φ^* and ψ^* be two parallel morphisms from I_L to $J_{L'}$ in $\mathbb{I}$ and $P \times P = \{(x,y) \in L \times L | \varphi(I_L(x,y)) = \psi(I_L(x,y))\}$ and $E_P = I_L|_{P \times P}$. Then the inclusion morphism $i^* : E_P \to I_L$ is an equalizer of φ^* and ψ^* as depicted diagram in Figure 7.

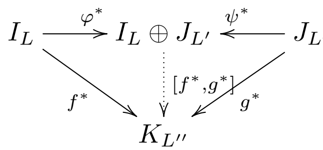


Fig. 6. $I_L \oplus J_{L'}$ is coproduct on $\mathfrak{F}\mathfrak{I}$.

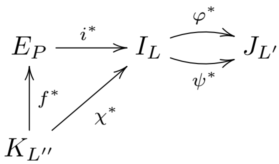


Fig. 7. $i^* : E_P \to I_L$ is an equalizer of φ^* and ψ^*.

Proof. For each $(x, y) \in P \times P$, we have:

$$\begin{aligned} \varphi \circ i(E_P(x, y)) &= \varphi(i(E_P(x, y))) \\ &= \varphi(I_L(i(x), i(y))) \\ &= \varphi(I_L(x, y)) \\ &= \psi(I_L(x, y)) \\ &= \psi(I_L(i(x), i(y))) \\ &= \psi(i(E_P(x, y))) \\ &= \psi \circ i(E_P(x, y)). \end{aligned}$$

Thus $\varphi^* \circ i^* = \psi^* \circ i^*$. Now let $K_{L''}$ be an object in category $\mathfrak{F}\mathfrak{I}$ and χ^* be a morphism from $K_{L''}$ to I_L such that $\varphi^* \circ \chi^* = \psi^* \circ \chi^*$. Then χ^* is a morphism from $K_{L''}$ to E_P, $i^* \circ \chi^* = \chi^*$ and χ^* is unique, because if there exists a morphism f^* from $K_{L''}$ to E_P such that $i^* \circ f^* = \chi^*$, we have

$$\begin{aligned} \chi(K_{L''}(x, y)) = i \circ f(K_{L''}(x, y)) &= i(E_P(f(x), f(y))) \\ &= I_L(i(f(x)), i(f(y))) \\ &= I_L(f(x), f(y)) \\ &= f(K_{L''}(x, y)). \end{aligned}$$

Thus $i^* : E_P \to I_L$ is an equalizer of φ^* and ψ^*. □

Theorem 3.8. Let φ^* and ψ^* be two parallel arrows from I_L to $J_{L'}$ in category $\mathfrak{FI}$ and let $\sim$ be the smallest equivalence relation on L' such that $\varphi(I_L(x,y)) \sim \psi(I_L(x,y))$ for all $x,y \in L$. Then $\chi^* : J_{L'} \to J_{L'/\sim}$ is a coequalizer, where $\chi^*(J_{L'}(x,y)) = [J_{L'}(x,y)]$.

Proof. First we should prove that $\chi^* \circ \varphi^* = \chi^* \circ \psi^*$. Note that $\chi^*(J_{L'}(x,y)) = [J_{L'}(x,y)] = J_{L'/\sim}([x],[y])$. Where $[x]$ is equivalence class of x.

$$\begin{aligned}
\chi^* \circ \varphi^*(I_L(x,y)) &= \chi(J_{L'}(\varphi(x),\varphi(y))) \\
&= [J_{L'}(\varphi(x),\varphi(y))] \\
&= J_{L'/\sim}([\varphi(x)],[\varphi(y)]) \\
&= J_{L'/\sim}([\psi(x)],[\psi(y)]) \\
&= [J_{L'}(\psi(x),\psi(y))] \\
&= \chi(J_{L'}(\psi(x),\psi(y))) \\
&= \chi^* \circ \psi^*(I_L(x,y))
\end{aligned}$$

Now if there exist a fuzzy implication $K_{L''}$ on bounded lattice L'' and an arrow $f^* : J_{L'} \to K_{L''}$ such that $f^* \circ \varphi^* = f^* \circ \psi^*$, then we defined $\overline{f} : J_{L'/\sim} \to K_{L''}$ such that $\overline{f}([x]) = f(x)$. clearly $\overline{f}$ is an arrow and diagram in Figure 8 commutes. So $\chi^* : J_{L'} \to J_{L'/\sim}$ is coequalizer. □

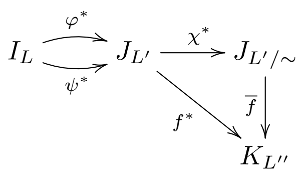


Fig. 8. $\chi^* : J_{L'} \to J_{L'/\sim}$ is a coequalizer.

Remark 4. From Theorems 3.4 and 3.7 the category $\mathfrak{FI}$ has limit and pullback. From Theorems 3.8 and 3.6 the category $\mathfrak{FI}$ has colimit and pushout.

Lemma 3.9. Let I_L and $J_{L'}$ are Φ-conjugates in $\mathbb{I}$, then there exists a morphism from I_L to $J_{L'}$.

Proof. Let I_L and $J_{L'}$ are Φ-conjugates, then there exists $\varphi \in \Phi$ such that $I_L(x,y) = \varphi^{-1}(J_{L'}(\varphi(x),\varphi(y)))$. So $\varphi(I_L(x,y)) = J_{L'}(\varphi(x),\varphi(y))$. Thus there exists a morphism from I_L to $J_{L'}$. □

In the following example, we show that not all morphisms are Φ-conjugate.

Example 3.10. Let $L = \{0, a, b, c, 1\}$ and $L' = \{0, a', 1\}$ be two lattices such that, $0 < a, b < c < 1$ and $0 < a' < 1$. Then φ is a morphism from L to L', where

$$\varphi(0) = \varphi(a) = 0, \quad \varphi(b) = \varphi(c) = a', \quad \varphi(1) = 1.$$

Let I_L and $I_{L'}$ be fuzzy implications defined as in Table 1.

I_L	0	a	b	c	1
0	1	1	1	1	1
a	1	1	1	1	1
b	b	b	1	1	1
c	b	b	1	1	1
1	0	0	b	b	1

$I_{L'}$	0	a'	1
0	1	1	1
a'	a'	1	1
1	0	a'	1

Tab. 1. Fuzzy implications I_L and $I_{L'}$.

Then φ is a lattice homomorphism from L to L' with extra property $\varphi(I_L(x, y)) = I_{L'}(\varphi(x), \varphi(y))$ for each $x, y \in L$. So φ^* is a morphism from I_L to $I_{L'}$ by Lemma 2.24. But φ is not bijective hence $\varphi \notin \Phi$, therefore I_L is not Φ-conjugate of $I_{L'}$.

Definition 3.11. (Baczynski and Jayaram [4]) Let I be a fuzzy implication and $I_\varphi = I$ for each $\varphi \in \Phi$. Then I is self-conjugate or invariant.

Remark 5. There are many fuzzy implications which are invariant in the lattice $[0, 1]$. For example, Godel, Rescher and Weber fuzzy implications are invariant. See [4]. In particular, any $\{0, 1\}$ -valued fuzzy implication is invariant under any $\varphi \in \Phi$, i.e., it is self-conjugate.

Theorem 3.12. The morphism $\varphi^* : I_L \to J_{L'}$ is isomorphism if the lattice homomorphism $\varphi : L \to L'$ is lattice isomorphism.

Proof. Let φ be an isomorphism. Then there exists the homomorphism $\psi : L' \to L$ such that $\varphi \circ \psi = 1_{L'}$ and $\psi \circ \varphi = 1_L$ so,

$$\varphi^* \circ \psi^* = (\varphi \circ \psi)^* = (1_{L'})^* = 1_{J_{L'}}$$

and

$$\psi^* \circ \varphi^* = (\psi \circ \varphi)^* = (1_L)^* = 1_{I_L}.$$

Thus φ^* is an isomorphism. □

Remark 6. The morphism $\varphi^* : I_L \to J_{L'}$ is isomorphism if and only if I_L is Φ-conjugate of $J_{L'}$ and $I_L = (J_{L'})_\varphi$.

Proof. Let φ^* be isomorphism. Then φ is increasing bijection function and $\varphi(I_L(x, y)) = J_{L'}(\varphi(x), \varphi(y))$. Hence $\varphi^{-1} \circ \varphi(I_L(x, y)) = \varphi^{-1}(J_{L'}(\varphi(x), \varphi(y)))$.
So $I_L(x, y) = \varphi^{-1}(J_{L'}(\varphi(x), \varphi(y)))$ and I_L is Φ-conjugate of $J_{L'}$. The converse is easy. □

Lemma 3.13. Let $\sim$ be the relation on objects of category $\mathfrak{FI}$ defined by,

$$I_L \sim J_{L'} \Leftrightarrow \varphi^*(I_L) = J_{L'} \text{ where } \varphi^* \text{ is an isomorphism.}$$

Then $\sim$ is an equivalence relation.

Proof. Let φ be identity morphism, then $\varphi^*(I_L) = I_L$, so $I_L \sim I_L$. Now suppose $I_L \sim J_{L'}$, hence there exist an isomorphism φ^* such that $\varphi^*(I_L) = J_{L'}$. So $(\varphi^*)^{-1}(J_{L'}) = I_L$ and $(\varphi^*)^{-1}$ is isomorphism, therefore, $J_{L'} \sim I_L$. If $I_L \sim J_{L'}$ and $J_{L'} \sim K_{L''}$, then there exist two isomorphism φ^* and ψ^* such that $\varphi^*(I_L) = J_{L'}$ and $\psi^*(J_{L'}) = K_{L''}$. So $\psi^* \circ \varphi^*$ is isomorphism and $\psi^* \circ \varphi^*(I_L) = K_{L''}$, therefore $I_L \sim K_{L''}$. Thus $\sim$ is an equivalence relation. □

Since $\sim$ is equivalence relation, so $\|I_L\| = \{J_{L'} \in \mathbb{I} | J_{L'} \sim I_L\}$ is the equivalence class of I. So, we have the following lemma.

Lemma 3.14. $FIL/\sim = \{\|I_L\| | I_L \in \mathbb{I}\}$ is a partition of $\mathbb{I}$.

Theorem 3.15. Let $SKL = \{I_L \in \mathbb{I} | \; \|I_L\| \neq \|J_{L'}\|, \quad \forall I_L, J_{L'} \in SKL\}$.
Then $\mathfrak{SKL} = (SKL, hom, id, \circ)$ is a full subcategory of $\mathfrak{FI}$.

Remark 7. Note that SKL is not empty.

(1) All of the invariant fuzzy implications such as Godel, Rescher and Weber and some others are in SKL. So it is not empty.

(2) Let $\mathbb{N}$ be set of natural numbers and let $L = \mathbb{N} \cup \{\top\}$ and $L' = \{\sqrt{x} | x \in \mathbb{N}\} \cup \{\top\}$ be bounded lattices with the top element $\top$, then $\varphi : L \to L'$ is a lattice homomorphism in Φ, where
$\varphi(x) = \begin{cases} \top & \text{if } x = \top \\ \sqrt{x} & \text{if } x < \top \end{cases}$ and the inverse of φ is $\varphi^{-1}(x) = \begin{cases} \top & \text{if } x = \top \\ x^2 & \text{if } x < \top \end{cases}$
Then

$$I_L(x, y) = \begin{cases} \top & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

is a fuzzy implication on L and $J_{L'} = \varphi^{-1}(I_L(\varphi(x), \varphi(y)))$ is a fuzzy implication on L'. So I_L and $J_{L'}$ are Φ- conjugate, therefore if I_L is in SKL, then $J_{L'}$ is not in SKL. In other words, both I_L and $J_{L'}$ cannot be in SKL. See Remark 8.

Remark 8. Any two different members of SKL are not isomorphic. So any two elements in SKL are not Φ-cojugate.

Remark 9. Let $L = [0, 1]$ and $I_L(x, y) = min(1, 1 - x + y)$ be Lukasiewicz fuzzy implication. Then $I_L^\lambda(x, y) = [min(1, 1 - x^\lambda + y^\lambda)]^{1/\lambda}$ for each $\lambda > 0$ and $\lambda \neq 1$ and I_L^λ are φ_λ-conjugate of I_L and hence it is a family of fuzzy implications in $\mathbb{I}$, which are not in SKL. So SKL is proper subset of $\mathbb{I}$.

P r o o f . Let $\varphi_\lambda(x) = x^{1/\lambda}$ be an increasing bijection function from $[0,1]$ to $[0,1]$. Then $\varphi_\lambda^{-1}(I_L(\varphi_\lambda(x), \varphi_\lambda(y))) = I_L^\lambda(x,y)$, so I_L^λ for each $\lambda > 0$ are Φ- conjugate of I_L, so $\|I_L\| = \|I_L^\lambda\|$ and I_L^λ is not in SKL for each $\lambda > 0$ and $\lambda \neq 1$. □

Theorem 3.16. $\mathfrak{SKL}$ as defined in Theorem 3.15 is the unique skeleton of $\mathfrak{FI}$.

P r o o f . By Theorem 3.15, we have that $\mathfrak{SKL}$ is a full subcategory of category $\mathfrak{FI}$. Let I_L and $J_{L'}$ be in SKL and suppose there exists an isomorphism from I_L to $J_{L'}$. Then by Theorem 3.13, $\|I_L\| = \|J_{L'}\|$. So $I_L = J_{L'}$, by Theorem 3.15. Let $I_L \in \mathfrak{FI}$. Then by Theorem 3.15, there exist $J_{L'} \in SKL$ such that $J_{L'} \in \|I_L\|$. So, by Theorem 3.13, I_L is an isomorphism to $J_{L'}$. If we introduce another skeleton $\mathfrak{S}$ on the category $\mathfrak{FI}$. Then by Theorem 2.22 we have that $\mathfrak{S}$ is isomorphism to $\mathfrak{SKL}$. So $\mathfrak{SKL}$ is unique up to isomorphism. □

Remark 10. Let $\mathbb{I}_f$ be the collection of all fuzzy implications on finite bounded lattices. Then $\mathfrak{FI}_f$ is the category of all fuzzy implications on finite bounded lattices and SKL_f is the skeleton of $\mathfrak{FI}_f$. In this category, skeleton is not proper. In other words, SKL_f is equal to $\mathbb{I}_f$, the class of all fuzzy implications on finite bounded lattices. This means that any two different fuzzy implication on finite bounded lattices are not Φ-conjugate. Therefore to study similar properties of any two fuzzy implications, we need new tools rather than conjugacy.

Category theory and universal algebra are closely related. Both of these theories can be a way to study algebraic structures. In the following section, we prove that any fuzzy implication can be interpreted as an Ω-algebra and the category $\mathfrak{FI}$ is a full subcategory of the category $\mathbf{Alg}(\Omega, E)$, where $\mathbf{Alg}(\Omega, E)$ stands for category of universal algebra [1] (see Lemma 4.3 and Lemma 4.4).

4. RELATIONS BETWEEN UNIVERSAL ALGEBRAS CATEGORY AND FUZZY IMPLICATIONS CATEGORY

Definition 4.1. (Pierce et al. [16]) Let Ω be a set of operator symbols, equipped with a mapping ar from elements of Ω to natural numbers; for each $\omega \in \Omega$, $ar(\omega)$ is the arity of ω. An Ω-algebra A is a set $|A|$ (the carrier of A)equipped with a system of functions a_ω, $\omega \in \Omega$, such that for each operator ω of arity $ar(\omega)$, a function $a_\omega : |A|^{ar(\omega)} \to |A|$, called the interpretation of ω, maps $ar(\omega)$-tuples of elements of the carrier back into the carrier. An Ω-homomorphism from an Ω-algebra A to an Ω-algebra B is a function $h : |A| \to |B|$ such that for each operator $\omega \in \Omega$ and tuple $X_1, X_2 ..., X_{ar(\omega)}$ of elements of $|A|$, the following equation holds: $h(a_\omega(x_1, X_2, ..., X_{ar(\omega)}) = b_\omega(h(X_1), h(X_2), ..., h(x_{ar(\omega)}))$. The category $\mathbf{Alg}(\Omega)$ has Ω-algebras as objects and Ω-homomorphisms as arrows. This construction can be refined by adding to the signature Ω a set E of equations between expressions built from elements of Ω and a set $\{x, y, z, ...\}$ of variable symbols. Then the Ω-algebras A for which the equations in E are satisfied under all assignments of elements of $|A|$ to the variable symbols form the objects of a category $\mathbf{Alg}(\Omega, E)$.

For example, if

- $\Omega = \{., e\}$
- $ar(.) = 2$
- $ar(e) = 0$
- $E = \{x.(y.z) = (x.y).z,\ e.x = x,\ x.e = x\}$,

then $\mathbf{Alg}(\Omega, E)$ is another name for category of monoids.

Example 4.2. If,

- $|A| = L \times L \times L$, such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
- $\Omega = \{*, (0,0,1), (1,1,1), (1,0,0)\}$,
- $ar(*) = 2,\ ar((0,0,1)) = ar((1,1,1)) = ar((1,0,0)) = 0$,
- $E = \{(x,y,z) * (x',y',z') = (x \wedge x', y \vee y', z'')\}$ such that z'' is from set of $\{l \in L | z \vee z' < l\}$,

then A is an Ω-algebra and $\mathbf{Alg}(\Omega, E)$ is a category.

Lemma 4.3. Let I_L be a fuzzy implication on the bounded lattice L and $I_L(x,y) = z$ implies $(x,y,z) \in L \times L \times L$, then $I_L = \{(x,y,z) \in L \times L \times L | z = I_L(x,y)\}$ is Ω-algebra defined in Example 4.2.

Proof. Let I_L be a fuzzy implication on L, then $I_L(0,0) = 1$, $I_L(1,1) = 1$ and $I_L(1,0) = 0$. Therefore $(0,0,1)$, $(1,1,1)$ and $(1,0,0)$ are constant in algebra. Let $x_1 \leq x_2$ then $(x_1, y, z_1) * (x_2, y, z_2) = (x_1 \wedge x_2, y \vee y, z')$. On the other hand $z_1 > z_2$ so $z_1 > z_1 \vee z_2$. Therefore $z' = I_L(x_1 \wedge x_2, y \vee y) = I_L(x_1, y) = z_1$ and $*$ is well defined and I_L is the Ω-algebra. □

Lemma 4.4. Category $\mathfrak{FI}$ is a full subcategory of $\mathbf{Alg}(\Omega, E)$.

Proof. Let $I_L \in Ob(\mathfrak{FI})$, then by Lemma 4.3, I_L is an Ω-algebra. So $Ob(\mathfrak{FI}) \subseteq Ob(\mathbf{Alg}(\Omega, E))$. Now suppose I_L and $J_{L'}$ be in $Ob(\mathfrak{FI})$ and φ^* be a morphism in $hom_{\mathfrak{FI}}(I_L, J_{L'})$. Then $\varphi(I_L(x,y)) = J_{L'}(\varphi(x), \varphi(y))$, so
$\varphi(x,y,z) = (\varphi(x), \varphi(y), \varphi(z))$. Therefore φ is an arrow in $\mathbf{Alg}(\Omega, E)$, because φ is a lattice homomorphism. Hence $hom_{\mathfrak{FI}}(I_L, J_{L'}) \subseteq hom_{\mathbf{Alg}(\Omega,E)}(I_L, J_{L'})$. Clearly, identity and composition are preserving and
$hom_{\mathfrak{FI}}(I_L, J_{L'}) = hom_{\mathbf{Alg}(\Omega,E)}(I_L, J_{L'})$ for all I_L and $J_{L'}$ in $ob(\mathfrak{FI})$. So $\mathfrak{FI}$ is a full subcategory of the category $\mathbf{Alg}(\Omega, E)$). □

Remark 11. Many familiar mathematical objects (e.g., semigroups, monoids, groups, abelian groups, rings, lattices, boolean algebras, vector spaces, etc.) can be defined by means of operations and equations. Moreover, the corresponding categories can be obtained as full subcategories of categories of the form $\mathbf{Alg}(\Omega)$, consisting of those objects that satisfy suitable equations. On the other hand, all of these structures alone are important and useful for studying as they have special properties. So $\mathfrak{FI}$ is not isomorphism or equivalent to $\mathbf{Alg}(\Omega)$, however $\mathfrak{FI}$ is an algebraic structure as a category.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we have defined monic, epic and isomorphism on the category of fuzzy implications ($\mathfrak{FI}$). We have defined product, coproduct, equalizer and coequalizer on $\mathfrak{FI}$ in such a way it has pullback and limit, but also pushout and colimit. We have introduced the skeleton of $\mathfrak{FI}$ and we have shown that the members of the skeleton are not Φ-conjugate and we shown that $\mathfrak{FI}$ is a full subcategory of $\mathbf{Alg}(\Omega, E)$, therefore $\mathfrak{FI}$ is an algebraic structure category.
For the future work, one can introduce the category of S-implications and Q-implications and then study the relations between them. Also seeking for applications of these categories in real-world problems is worth to follow.

6. ACKNOWLEDGMENT

The third author was supported by the grant APVV-18-0052.

(Received December 17, 2019)

REFERENCES

[1] J. Adámek, H. Herrlich, and G. E. Strecker: Abstract and Concrete Categories: The Joy of Cats. Wiley, 1990.

[2] M. Baczynski: On the applications of fuzzy implication functions. In: Soft Computing Applications, Springer, Berlin Heidelberg 2013, pp. 9–10. DOI:10.1007/978-3-642-33941-7_4

[3] M. Baczynski, G. Beliakov, H. B. Sola, and A. Pradera: Advances in Fuzzy Implication Functions. Springer, Berlin Heidelberg 2013.

[4] M. Baczynski and B. Jayaram: Fuzzy Implications. Springer, Berlin Heidelberg 2008.

[5] B. C. Bedregal: Bounded lattice t-norms as an interval category. In: International Workshop on Logic, Language, Information, and Computation, Springer 2007, pp. 26–37.

[6] R. Bělohlávek: Granulation and granularity via conceptual structures: A perspective from the point of view of fuzzy concept lattices. In: Data mining, rough sets and granular computing, Springer, Berlin Heidelberg 2002, pp. 265–289. DOI:10.1007/978-3-7908-1791-1_13

[7] G. Birkhoff: Lattice Theory. American Mathematical Society, Rhode Island 1940.

[8] B. A. Davey and H. A. Priestley: Introduction to Lattices and Order. Cambridge University Press 2002.

[9] B. De Baets and R. Mesiar: Triangular norms on product lattices Fuzzy Sets and Systems *104* (1999),1, 61–75. DOI:10.1016/s0165-0114(98)00259-0

[10] B. De Baets and R. Mesiar: Discrete Triangular Norms. In:Topological and Algebraic Structures in Fuzzy Sets, Springer 2003, pp. 389–400. DOI:10.1007/978-94-017-0231-7_16

[11] P. Hájek: Metamathematics of Fuzzy Logic. Volume 4, Springer Science and Business Media, 2013.

[12] E P. Klement, R. Mesiar, and E. Pap: Triangular Norms Volume 8, Springer Science and Business Media, 2013.

[13] C. C. Lee: Fuzzy logic in control systems: fuzzy logic controller. IEEE Trans. Systems Man Cybernet. *20* (1990), 2, 404–418. DOI:10.1109/21.52551

[14] G. Mayor, J. Suñer, and J. Torrens: Operations on Finite Settings: from Triangular Norms to Copulas. In: Copulas and Dependence Models with Applications, Springer 2017, pp. 157–170. DOI:10.1007/978-3-319-64221-5_10

[15] H. T. Nguyen and E. A. Walker: A first Course in Fuzzy Logic. Chapman and Hall/CRC Press 2006.

[16] B. C. Pierce, M. R. Garey, and A. Meyer: Basic Category Theory for Computer Scientists. MIT Press 1991.

[17] M. Togai and H. Watanabe: Expert system on a chip: An engine for real-time approximate reasoning. In: Proc. ACM SIGART international symposium on Methodologies for intelligent systems, ACM 1986, pp. 147–154.

[18] A. Yousefi and M. Mashinchi: Categories of fuzzy implications and R-implications on bounded lattices. In: 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS), IEEE 2018, pp. 40–42. DOI:10.1109/cfis.2018.8336622

[19] A. Yousefi and M. Mashinchi: Counting T-norms and R-implications on Bounded Lattices. In: 9th National Conference on Mathematics of Payame Noor University, On CD 2019, pp. 726–731.

[20] Y. Yu, J. N. Mordeson, and S. C. Cheng: Elements of L-algebra. Lecture Notes in Fuzzy Mathematics and Computer Science, Creighton University, Omaha 1994.

[21] L. A. Zadeh: A computational approach to fuzzy quantifiers in natural languages. Computers Math. Appl. *9*, Elsevier (1983), 149–184. DOI:10.1016/0898-1221(83)90013-5

Amin Yousefi, Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Shahid Bahonar University of Kerman, Kerman. Iran.
e-mail: aminyoosofi@gmail.com

Mashaallah Mashinchi, Department of Statistics, Faculty of Mathematics and Computer Science, Shahid Bahonar University of Kerman, Kerman. Iran.
e-mail: mashinchi@uk.ac.ir

Radko Mesiar, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia, and Institute of Information Theory and Automation, The Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: mesiar@math.sk