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Applications of Mathematics, Vol. 66 (2021), No. 4, 583–597

Persistent URL: http://dml.cz/dmlcz/148973

66 (2021) APPLICATIONS OF MATHEMATICS No. 4, 583–597

GLOBAL EXISTENCE AND L_p DECAY ESTIMATE OF SOLUTION FOR CAHN-HILLIARD EQUATION WITH INERTIAL TERM

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Received July 22, 2019. Published online May 20, 2021.

Abstract. The Cauchy problem of the Cahn-Hilliard equation with inertial term in multi space dimension is considered. Based on detailed analysis of Green's function, using fixed-point theorem, we get the global existence in time of classical solution with large initial data. Furthermore, we get L_p decay rate of the solution.

Keywords: Cahn-Hilliard equation with inertial term; large initial data; classical solution; L_p decay

MSC 2020: 35M11

1. Introduction

This paper is concerned with global existence and its decay estimate of classical solution to the Cauchy problem for the Cahn-Hilliard equation with inertial term:

$$\begin{cases} \eta u_{tt} + u_t + \Delta^2 u - \Delta f(u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = u_0(x), u_t = u_1(x) & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases} \tag{1.1}$$

Here $\eta > 0$ is a given constant, Δ is the usual Laplace operator. In the nonlinear term $\Delta f(u)$, $f(u)$ has the form $|u|^{\theta+1}$ or $|u|^{\theta-q+1} \cdot u^q$, where θ, q are positive integers and $\theta - q + 1 \geqslant 0$, $\theta \geqslant 1$.

In the case $\eta = 0$, (1.1) reduces to

$$u_t + \Delta^2 u - \Delta f(u) = 0. \tag{1.2}$$

This work was supported by the National Natural Science Foundation of China (11571092).

DOI: 10.21136/AM.2021.0180-19

Relation (1.2) is the well-known Cahn-Hilliard equation which describes phase separation of a binary mixture and u denotes the relative concentration of one phase. Many mathematicians devoted their energy to the equation and the main properties of the solution such as qualitative behavior; asymptotics are now well understood (see e.g. [1], [2], [4], [5], [11], [19], [20]). In order to model non-equilibrium decompositions caused by deep supercooling in certain glasses, Galenko et al. in [6], [7], [8], [9] proposed to add inertial term ηu_{tt} to (1.2). The modified equation (1.1) shows a good agreement with experiments performed on glasses (see [10], [12]).

Equation (1.1) is a hyperbolic equation with relaxation while (1.2) is a parabolic one, so they have different mathematical features. Equation (1.1) presents some mathematical difficulties because the solutions does not get smoother in finite time anymore. Thus, previous work for (1.1) mainly focused on the so-called energy bounded solution and quasi-strong solution, see [13], [14], [15]. As observed in [14], (1.1) bears some similarities with the semilinear damped wave equation. However, in contrast to that case, it is not easy to obtain temporal global existence of classical solution only by usual energy method because of weak dissipation of (1.1). As [17] pointed out, multiplying (1.1) by u and u_t respectively, one can get

$$(1.3)\quad \frac{\mathrm{d}}{\mathrm{d}t}\int u^2+\eta u_t^2+(\triangle u)^2\,\mathrm{d}x+\int \eta u_t^2+(\triangle u)^2\,\mathrm{d}x=\int \eta u_t\triangle f(u)+u\triangle f(u)\,\mathrm{d}x.$$

Since there are no estimates of $\|u(\cdot,t)\|_{L_2}$ or $\|Du(\cdot,t)\|_{L_2}$ on the left-hand side of (1.3), the term $\int u\triangle f(u)\,\mathrm{d}x$ cannot be controlled by the standard energy method. Thus, Wang and Wu [17] introduced a long wave short wave method. With the method, they obtained the global existence and L_2 decay rate of classical solution of (1.1), however, only for the case of $n\geqslant 3$ with small initial data.

This paper proposes a new method to solve the problem for all n and with large initial data. This method uses frequency decomposition and estimates the long wave and short wave separately. During the estimation of long and short waves, explicit analysis of Green's function is applied instead of energy integration. The advantage of this method is that the differential equation can be turned into an integral one, thus much more accurate calculation can be conducted. Finally, the global existence in time of classical solutions and its L_p decay rates of (1.1) with large initial data in all space dimension are obtained.

Notations in this paper are introduced below: $L_p, W^{m,p}$ denote the usual Lebesgue and Sobolev spaces on $\mathbb{R}^n$ and $H^m=W^{m,2}$, with norms $\|\cdot\|_{L_p}$, $\|\cdot\|_{W^{m,p}}$, $\|\cdot\|_{H^m}$, respectively, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is multi-index, and $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_n$. Fourier transform to the variable x of function $f(x,t)$ is $\hat{f}(\xi,t)$, that is $\hat{f}(\xi,t)=\int f(x,t)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$, where i is imaginary unit. Thus, the inverse Fourier transform to the variable ξ of $\hat{f}(\xi,t)$ is defined as

$$f(x,t) = F^{-1}(\hat{f})(x,t) = (2\pi)^{-n/2}\int \hat{f}(\xi,t)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi.$$

Symbol $*$ denotes convolution of space variable in this paper. The constant C is assumed to be positive.

The main result of this paper is the following theorem:

Theorem 1.1. *Suppose* $s \geqslant 1 + [\frac{n}{2}]$, $u_0 \in H^{s+2} \cap L_1$, $u_1 \in H^s \cap L_1$. *The Cauchy problem* (1.1) *admits a unique, global classical solution* $u(x,t)$ *satisfying:*

$$u \in L_\infty(0,\infty; H^s(\mathbb{R}^n)), \quad \|\partial_x^\alpha u\|_{L_2} \leqslant C(1+t)^{-n/8-|\alpha|/4} \quad \text{for } |\alpha| \leqslant s.$$

If $1 \leqslant n \leqslant 3$, $n\theta + n - 4 \geqslant 0$, *we can go further and get*

$$\|\partial_x^\beta u\|_{L_p} \leqslant C(1+t)^{-|\beta|/4-n(1-1/p)/4} \quad \text{for } |\beta| \leqslant s-2, \quad 2 \leqslant p \leqslant \infty.$$

2. Explicit analysis of Green's function

Green's function G of (1.1) is defined as

$$\begin{cases} (\eta\partial_{tt} + \partial_t + \Delta^2)G = 0, & x \in \mathbb{R}^n,\ t > 0, \\ G|_{t=0} = 0, \\ G_t|_{t=0} = \delta(x). \end{cases} \tag{2.1}$$

By Fourier transformation to variable x, we deduce

$$\begin{cases} (\eta\partial_{tt} + \partial_t + |\xi|^4)\widehat{G}(\xi,t) = 0, & \xi \in \mathbb{R}^n,\ t > 0, \\ \widehat{G}|_{t=0} = 0, \\ \widehat{G}_t|_{t=0} = 1. \end{cases} \tag{2.2}$$

The problem (2.2) is an ODE, its solution is

$$\widehat{G}(\xi,t) = \frac{1}{\lambda_+ - \lambda_-}(\mathrm{e}^{\lambda_+ t} - \mathrm{e}^{\lambda_- t}) = \mathrm{e}^{\lambda_+ t}\cdot\frac{1-\mathrm{e}^{(\lambda_- - \lambda_+)t}}{\lambda_+ - \lambda_-}, \tag{2.3}$$

where

$$\lambda_+ = \frac{1}{2\eta}\Big(-1 + \sqrt{1-4\eta|\xi|^4}\Big), \quad \lambda_- = \frac{1}{2\eta}\Big(-1 - \sqrt{1-4\eta|\xi|^4}\Big). \tag{2.4}$$

By Duhamel's principle, solution of (1.1) can be represented as

$$u(x,t) = G(x,t) * (u_0 + u_1) + \eta \partial_t G * u_0 + \int_0^t G(\cdot, t-\tau) * \Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau. \tag{2.5}$$

Let

$$x(\xi) = \begin{cases} 1, & |\xi| \leqslant \dfrac{2}{\sqrt[4]{4\eta}} \\ 0, & |\xi| > \dfrac{4}{\sqrt[4]{4\eta}} \end{cases} \quad \text{be smooth cut-off function.}$$

Denote $\widehat{G}_L = x(\xi)\widehat{G}(\xi,t)$, $\widehat{G}_H = (1 - x(\xi))\widehat{G}(\xi,t)$.

Theorem 2.1. *For any multi-index α, integer $2 \leqslant p \leqslant \infty$, there exists positive constant $C_{\alpha,p}$ which is dependent on α, p such that*

$$\|\partial_x^\alpha G_L(\cdot,t)\|_{L_p} \leqslant C_{\alpha,p}(1+t)^{-|\alpha|/4 - n(1-1/p)/4},$$
$$\|\partial_x^\alpha \partial_t G_L(\cdot,t)\|_{L_p} \leqslant C_{\alpha,p}(1+t)^{-|\alpha|/4 - n(1-1/p)/4}.$$

Proof. When $4\eta|\xi|^4 \leqslant \frac{1}{2}$, we have

$$\lambda_+ \leqslant \frac{1}{2\eta}(-1 + 1 - 2\eta|\xi|^4) = -|\xi|^4, \tag{2.6}$$

$$\lambda_- \leqslant -\frac{1}{2\eta}, \tag{2.7}$$

$$\lambda_+ - \lambda_- = \frac{1}{\eta}\sqrt{1 - 4\eta|\xi|^4} \geqslant \frac{1}{\eta}\frac{\sqrt{2}}{2}. \tag{2.8}$$

When $\frac{1}{2} < 4\eta|\xi|^4 \leqslant 1$,

$$\lambda_+ = \frac{1}{2\eta}\frac{1 - (1 - 4\eta|\xi|^4)}{-1 - \sqrt{1 - 4\eta|\xi|^4}} = \frac{-2|\xi|^4}{1 + \sqrt{1 - 4\eta|\xi|^4}} \leqslant -|\xi|^4. \tag{2.9}$$

When $1 < 4\eta|\xi|^4 < 4^4$,

$$\lambda_+ = \frac{-2|\xi|^4}{1 + \sqrt{1 - 4\eta|\xi|^4}},$$

thus

$$R_e\lambda_+ = \frac{-2|\xi|^4}{4\eta|\xi|^4 - 1 + 1} = -\frac{1}{2\eta}. \tag{2.10}$$

When $2 \leqslant p \leqslant \infty$, we have

$$(2.11)\quad \begin{aligned} \|\partial_x^\alpha G_L\|_{L_p} &\leqslant C_{\alpha,p}\|\xi^\alpha \widehat{G}_L\|_{L_q} \\ &\leqslant C_{\alpha,p}\Big(\int_{|\xi|^4\leqslant 1/(8\eta)} |\xi|^{q|\alpha|}\frac{|\mathrm{e}^{\lambda_+ qt}|}{|\lambda_+-\lambda_-|^q}\,\mathrm{d}\xi\Big)^{1/q} \\ &\quad + C_{\alpha,p}\Big(\int_{|\xi|^4\leqslant 1/(8\eta)} |\xi|^{q|\alpha|}\frac{|\mathrm{e}^{\lambda_- qt}|}{|\lambda_+-\lambda_-|^q}\,\mathrm{d}\xi\Big)^{1/q} \\ &\quad + C_{\alpha,p}\Big(\int_{1/(8\eta)<|\xi|^4<64/\eta} |\xi|^{q|\alpha|}|\mathrm{e}^{\lambda_+ qt}|\Big|\frac{\mathrm{e}^{(\lambda_--\lambda_+)t}-1}{\lambda_+-\lambda_-}\Big|^q\,\mathrm{d}\xi\Big)^{1/q} \\ &= I_1+I_2+I_3. \end{aligned}$$

From (2.4), (2.6), (2.7), (2.8) we obtain

$$(2.12)\quad \begin{aligned} I_1+I_2 &\leqslant C_{\alpha,p}\Big(\int_{|\xi|^4\leqslant 1/(8\eta)} |\xi|^{q|\alpha|}(\mathrm{e}^{-|\xi|^4 t\cdot q}+\mathrm{e}^{-qt/(2\eta)})\,\mathrm{d}\xi\Big)^{1/q} \\ &\leqslant C_{\alpha,p}(1+t)^{-|\alpha|/4-n(1-1/p)/4}. \end{aligned}$$

Because

$$\Big|\frac{\mathrm{e}^{(\lambda_--\lambda_+)t}-1}{\lambda_+-\lambda_-}\Big| \leqslant C$$

for $1/(8\eta) < |\xi|^4 \leqslant 64/\eta$, from (2.9), (2.10) we also get

$$(2.13)\quad I_3 \leqslant C_{\alpha,p}(1+t)^{-|\alpha|/4-n(1-1/p)/4}.$$

From (2.11), (2.12), (2.13) we get the estimate for $\|\partial_x^\alpha G_L\|_{L_p}$.

From (2.3) we deduce

$$\partial_t\widehat{G} = \frac{\lambda_+\mathrm{e}^{\lambda_+ t}-\lambda_-\mathrm{e}^{\lambda_- t}}{\lambda_+-\lambda_-} = \frac{1-\mathrm{e}^{(\lambda_--\lambda_+)t}}{\lambda_+-\lambda_-}\cdot\lambda_+\mathrm{e}^{\lambda_+ t}+\mathrm{e}^{\lambda_- t},$$

using the same method as that of estimate for $\|\partial_x^\alpha G_L\|_{L_p}$, we can also get the decay rate for $\|\partial_x^\alpha \partial_t G_L\|_{L_p}$. □

Theorem 2.2. *Suppose $u \in H^l$, $0 \leqslant \tau \leqslant t$. For any multi-index α, $|\alpha| \leqslant l$, there exists constant $C_0 > 0$ such that*

$$\begin{aligned} \|\partial_x^\alpha G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_2} &\leqslant C_0\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}, \\ \|\partial_x^\alpha G_H(\cdot,t-\tau)*\Delta u(\cdot,\tau)\|_{L_2} &\leqslant C_0\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}, \\ \|\partial_x^\alpha\partial_t G_H(\cdot,t-\tau)*u\|_{L_2} &\leqslant C_0\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u\|_{L_2}. \end{aligned}$$

If $1 \leqslant n \leqslant 3$, *we can go further and get*

$$\|\partial_x^\alpha G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_\infty} \leqslant C_0\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}, \quad |\alpha| \leqslant l,$$
$$\|\partial_x^\beta\partial_t G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_\infty} \leqslant C_0\mathrm{e}^{-(t-\tau)/(2\eta)}\|\Delta\partial_x^\beta u\|_{L_2}, \qquad |\beta| \leqslant l-2.$$

Proof. When $4\eta|\xi|^4 \geqslant 2$, from (2.4) there exist positive constants C_1, C_2, C_3, C_4 such that

$$\frac{1}{\lambda_+-\lambda_-} = \frac{\eta}{\sqrt{1-4\eta|\xi|^4}} \leqslant \frac{C_1}{|\xi|^2} \leqslant C_2, \quad R_e\lambda_\pm = -\frac{1}{2\eta}.$$

Thus, we can deduce

$$\begin{aligned}
\|\partial_x^\alpha G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_2} &= \|\widehat{G}_H(\cdot,t-\tau)\cdot\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&\leqslant C_2\mathrm{e}^{-t-\tau/(2\eta)}\|\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&= C_2\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2},\\
\|\partial_x^\alpha G_H(\cdot,t-\tau)*\Delta u(\cdot,\tau)\|_{L_2} &= \|\widehat{G}_H(\cdot,t-\tau)\cdot|\xi|^2\cdot\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&\leqslant C_1\mathrm{e}^{-(t-\tau)/(2\eta)}\|\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&= C_1\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2},\\
\|\partial_x^\alpha\partial_t G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_2} &= \|\partial_t\widehat{G}_H(\cdot,t-\tau)\cdot\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&\leqslant C_3\mathrm{e}^{-(t-\tau)/(2\eta)}\|\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&= C_3\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}.
\end{aligned}$$

When $1 \leqslant n \leqslant 3$, we have

$$\begin{aligned}
\|\partial_x^\alpha G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_\infty} &\leqslant \int|\widehat{G}_H||\xi^\alpha\hat{u}(\cdot,\tau)|\,\mathrm{d}\xi\\
&\leqslant \|\widehat{G}_H\|_{L_2}\|\xi^\alpha\hat{u}(\cdot,\tau)\|_{L_2}\\
&\leqslant C_1\Big(\int\frac{1}{|\xi|^4}\mathrm{e}^{-(t-\tau)/\eta}\,\mathrm{d}\xi\Big)^{1/2}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}\\
&\leqslant C_4\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2},\\
\|\partial_x^\beta\partial_t G_H(\cdot,t-\tau)*u(\cdot,\tau)\|_{L_\infty} &\leqslant \int|\partial_t\widehat{G}_H||\xi^\beta\hat{u}|\xi|^2|\frac{1}{|\xi|^2}\,\mathrm{d}\xi\\
&\leqslant C_3\Big(\int\frac{1}{|\xi|^4}\mathrm{e}^{-(t-\tau)/\eta}\,\mathrm{d}\xi\Big)^{1/2}\|\partial_x^\beta\Delta u\|_{L_2}\\
&\leqslant C_4\mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\beta\Delta u\|_{L_2}.
\end{aligned}$$

Choosing constant $C_0 \geqslant \max\{C_1,C_2,C_3,C_4\}$, we get our theorem. □

After accurate analysis of G, we will use fixed-point theorem to get global existence of the solution.

3. Global existence

First, we need some lemmas. They are necessary in the process of dealing with the nonlinear term of (1.1). The proof of the lemmas can be found in [3], [16], [17].

Lemma 3.1. *Suppose s, θ are positive integers, $p, q, r \in [1, \infty]$ be such that $1/r = 1/p + 1/q$. Suppose $F(v)$ is a function of C^s that satisfies*

$$|\partial_v^l F(v)| \leqslant C|v|^{\theta+1-l}, \quad 0 \leqslant l \leqslant s,\ l < \theta + 1,$$

and

$$|\partial_v^l F(v)| \leqslant C, \quad l \leqslant s,\ \theta + 1 \leqslant l.$$

If $v \in w^{k,q} \cap L^p \cap L^\infty$, where $0 \leqslant k \leqslant s$, we have

$$\begin{aligned} \|F(v)\|_{w^{k,r}} &\leqslant C\|v\|_{w^{k,q}}\|v\|_{L_p}\|v\|_{L_\infty}^{\theta-1}, \\ \|\partial_x^\alpha F(v)\|_{L_r} &\leqslant C\|\partial_x^\alpha v\|_{L_q}\|v\|_{L_p}\|v\|_{L_\infty}^{\theta-1}. \end{aligned}$$

The constant C is dependent on p, q, r and $\|v\|_{L_\infty}$.

Lemma 3.2. *Suppose $s, \theta, p, q, r, k, v, F(v)$ satisfy the assumptions of Lemma 3.1. Moreover, assume that*

$$|\partial_v^s F(v_1) - \partial_v^s F(v_2)| \leqslant C(|v_1| + |v_2|)^{\max(\theta-s,\theta)}|v_1 - v_2|.$$

If $v_1, v_2 \in w^{k,q} \cap L^p \cap L^\infty$, we have

$$\begin{aligned} \|F(v_1) - F(v_2)\|_{w^{k,r}} \leqslant{}& C(\|v_1\|_{w^{k,q}} + \|v_2\|_{w^{k,q}})\|v_1 - v_2\|_{L_p}(\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})^{\theta-1} \\ &+ C(\|v_1\|_{L_p} + \|v_2\|_{L_p})(\|v_1 - v_2\|_{w^{k,q}})(\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})^{\theta-1}, \end{aligned}$$

where constant C is dependent on p, q, r, $\|v_1\|_{L_\infty}$, $\|v_2\|_{L_\infty}$.

Let $s \geqslant 1 + [\frac{n}{2}]$, $E = \max\{\|u_0\|_{H^{s+2}}, \|u_1\|_{H^s}, \|u_0\|_{L_1}, \|u_1\|_{L_1}\}$, where u_0, u_1 are the initial data of (1.1). Define a space $X_{s,E}$ as follows:

$$X_{s,E} = \{u = u(x,t);\ D_{s,E}(u) \leqslant C_1 E\},$$

where

$$D_{s,E}(u) = \sup_{t\geqslant 0}\{(1+t)^{n/8}\|u(\cdot,t)\|_{H^s}\},$$

and C_1 is a fixed constant which will be defined later. The metric in space $X_{s,E}$ is induced by the norm $D_{s,E}(u)$:

$$\varrho(v_1, v_2) = D_{s,E}(v_1 - v_2) \quad \forall\, v_1, v_2 \in X_{s,E}.$$

It is obvious that $(X_{s,E}, D_{s,E}(\cdot))$ is a non-empty Banach space. If $u \in X_{s,E}$, we have

$$\|\partial_x^\alpha u(\cdot,t)\|_{L_2} \leqslant C_1 E(1+t)^{-n/8} \quad \text{for } |\alpha| \leqslant s. \tag{3.1}$$

Define

$$T(u) = G(x,t) * (u_0 + u_1) + \eta \partial_t G(\cdot,t) * u_0 + \int_0^t G(\cdot, t-\tau) * \Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau. \tag{3.2}$$

We next prove that T is a contraction map from $(X_{s,E}, D_{s,E}(\cdot))$ to itself.

Theorem 3.3. *The function T is from $(X_{s,E}, D_{s,E}(\cdot))$ to itself.*

Proof. From Theorem 2.1, for any multi-index α we have

$$\begin{aligned} &\|\partial_x^\alpha G_L(x,t) * (u_0 + u_1)\|_{L_2} + \|\eta \partial_x^\alpha \partial_t G_L(\cdot,t) * u_0\|_{L_2} \\ &\quad \leqslant \|\partial_x^\alpha G_L\|_{L_2}(\|u_0\|_{L_1} + \|u_1\|_{L_1}) + \eta \|\partial_x^\alpha \partial_t G_L\|_{L_2}\|u_0\|_{L_1} \\ &\quad \leqslant C_{\alpha,p} E(1+t)^{-|\alpha|/4 - n/8}. \end{aligned} \tag{3.3}$$

From Theorem 2.2 for $|\alpha| \leqslant s$ we have

$$\begin{aligned} &\|\partial_x^\alpha G_H(\cdot,t) * (u_0 + u_1)(\cdot)\|_{L_2} + \|\eta \partial_x^\alpha \partial_t G_H(\cdot,t) * u_0(\cdot)\|_{L_2} \\ &\quad \leqslant C_0 \mathrm{e}^{-t/(2\eta)}(\|\partial_x^\alpha u_0\|_{L_2} + \|\partial_x^\alpha u_1\|_{L_2}) \leqslant C_0 E \mathrm{e}^{-t/(2\eta)}. \end{aligned} \tag{3.4}$$

Take $r = 1$, $p = q = 2$ in Lemma 3.1. Then there exists constant $C(\|u\|_{L_\infty})$ which is dependent on $\|u\|_{L_\infty}$, thus on C_1 such that

$$\|\partial_x^\alpha f(u)\|_{L_1} \leqslant C(C_1)\|\partial_x^\alpha u\|_{L_2}\|u\|_{L_2}\|u\|_{L_\infty}^{\theta-1}. \tag{3.5}$$

If $s \geqslant 1 + [\frac{n}{2}]$, from (3.1) and the Sobolev embedding inequality we have

$$\|u\|_{L_\infty} \leqslant C(n)\|u\|_{H^s} \leqslant C(n) \cdot C_1 E(1+t)^{-n/8}. \tag{3.6}$$

From Theorem 2.1 and (3.1), (3.6), (3.5) for $|\alpha| \leqslant s$ we have

$$
\begin{aligned}
(3.7)\quad & \left\| \partial_x^\alpha \int_0^t G_L(\cdot, t-\tau) * \Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau \right\|_{L_2} \\
& \leqslant \left(\int_0^t \|\partial_x^\alpha \Delta G_L(\cdot, t-\tau)\|_{L_2}^2 \|f(u)(\cdot,\tau)\|_{L_1}^2 \,\mathrm{d}\tau \right)^{1/2} \\
& \leqslant C_{\alpha,p} \cdot C^{\theta-1}(C_1) \cdot C_1^2 \cdot C(n) \cdot E^{\theta+1} \\
& \quad \times \left[\int_0^t (1+t-\tau)^{(-(|\alpha|+2)/4-n/8)\cdot 2} (1+\tau)^{-n/8\cdot 2\cdot 2} (1+\tau)^{-n/8\cdot 2\cdot(\theta-1)} \,\mathrm{d}\tau \right]^{1/2} \\
& \leqslant C_{\alpha,p} \cdot C^{\theta-1}(C_1) \cdot C_1^2 \cdot C(n) \cdot E^{\theta+1} (1+t)^{-n/8-1/4}.
\end{aligned}
$$

Take $r = 2 = q$, $p = \infty$ in Lemma 3.1. Then there exists a constant $C_2(\|u\|_{L_\infty})$ which is dependent on $\|u\|_{L_\infty}$, thus on C_1 such that

$$
(3.8)\qquad \|\partial_x^\alpha f(u)\|_{L_2} \leqslant C_2(C_1) \|\partial_x^\alpha u\|_{L_2} \|u\|_{L_\infty}^\theta.
$$

If $u \in X_{s,E}$, then from Theorem 2.2, (3.8), (3.6) for $|\alpha| \leqslant s$ we obtain

$$
\begin{aligned}
(3.9)\quad & \left\| \partial_x^\alpha \int_0^t G_H(\cdot, t-\tau) * \Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau \right\|_{L_2} \\
& \leqslant \left[\int_0^t \|\Delta G_H(\cdot, t-\tau) * \partial_x^\alpha f(u)(\cdot,\tau)\|_{L_2}^2 \,\mathrm{d}\tau \right]^{1/2} \\
& \leqslant C_0 \left[\int_0^t \mathrm{e}^{-(t-\tau)/\eta} \|\partial_x^\alpha f(u)(\cdot,\tau)\|_{L_2}^2 \,\mathrm{d}\tau \right]^{1/2} \\
& \leqslant C_1 \cdot C_2^\theta(C_1) \cdot C_0 \left[\int_0^t \mathrm{e}^{-(t-\tau)/\eta} \|\partial_x^\alpha u\|_{L_2}^2 \|u\|_{L_\infty}^{2\theta} \,\mathrm{d}\tau \right]^{1/2} \\
& \leqslant C_1 \cdot C_2^\theta(C_1) \cdot C_0 \left[\int_0^t \mathrm{e}^{-(t-\tau)/\eta} (1+\tau)^{-n/4} (1+\tau)^{-n/8\cdot 2\theta} \,\mathrm{d}\tau \right]^{1/2} E^{\theta+1} \\
& \leqslant C_1 \cdot C_2^\theta(C_1) \cdot C_0 \cdot E^{\theta+1} (1+t)^{-n/8-1/8}.
\end{aligned}
$$

From (3.2), (3.3), (3.4), (3.7), (3.9) for $|\alpha| \leqslant s$. We have

$$
\begin{aligned}
\|\partial_x^\alpha T(u)\|_{L_2} & \leqslant \max\{C_0, C_{\alpha,p}\} E (1+t)^{-n/8} \\
& \quad + \max\{C_{\alpha,p} \cdot C^{\theta-1}(C_1) \cdot C_1^2 \cdot C(n), C_1 \cdot C_2^\theta(C_1) \cdot C_0\} E^{\theta+1} (1+t)^{-n/8-1/8}.
\end{aligned}
$$

Take t large enough, choose

$$
C_1 \geqslant \max\{C_0, C_{\alpha,p}\} + \max\{C_{\alpha,p} \cdot C^{\theta-1}(C_1) \cdot C_1^2 \cdot C(n), C_1 \cdot C_2^\theta(C_1) \cdot C_0\} \cdot E^\theta (1+t)^{-1/8},
$$

we can deduce $T(u) \in X_{s,E}$. The theorem is proved. □

Theorem 3.4. *The function $T(u)$ is a contraction map from $(X_{s,E}, D_{s,E}(\cdot))$ to itself.*

Proof. For $v_1, v_2 \in X_{s,E}$, take $k = s$, $r = 1$, $p = q = 2$ in Lemma 3.2. Then we have

$$
\begin{aligned}
(3.10)\qquad & \|f(v_1) - f(v_2)\|_{W^{s,1}} \\
& \leqslant C((\|v_1\|_{W^{s,2}} + \|v_2\|_{W^{s,2}}) \cdot \|v_1 - v_2\|_{L_2})(\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})^{\theta-1} \\
& \quad + C((\|v_1\|_{L_2} + \|v_2\|_{L_2}) \cdot \|v_1 - v_2\|_{W^{s,2}})(\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})^{\theta-1}.
\end{aligned}
$$

For $|\alpha| \leqslant s$, from Theorem 2.1, (3.10), (3.1), (3.6), there exists a constant C such that

$$
\begin{aligned}
(3.11)\qquad & \left\|\partial_x^\alpha \int_0^t G_L(\cdot, t-\tau) * \Delta(f(v_1)(\cdot,\tau) - f(v_2)(\cdot,\tau))\,\mathrm{d}\tau\right\|_{L_2} \\
& \leqslant \left[\int_0^t \|\partial_x^\alpha \Delta G_L(\cdot, t-\tau)\|_{L_2}^2 \|f(v_1) - f(v_2)\|_{L_1}^2\,\mathrm{d}\tau\right]^{1/2} \\
& \leqslant \left[\int_0^t (1+t-\tau)^{(-2/4-n/8)\cdot 2}(1+\tau)^{-n/8\cdot 2}(1+\tau)^{-n/8\cdot 2}\right. \\
& \qquad \left.\times \|v_1 - v_2\|_{W^{s,2}}^2 (1+\tau)^{n/8\cdot 2}(1+\tau)^{-n/8\cdot 2(\theta-1)}\,\mathrm{d}\tau\right]^{1/2} E^\theta \\
& \leqslant C \cdot E^\theta \varrho(v_1 - v_2)\left[\int_0^t (1+t-\tau)^{-n/4-1}(1+\tau)^{-n/2-n(\theta-1)/4}\,\mathrm{d}\tau\right]^{1/2} \\
& \leqslant C \cdot E^\theta \varrho(v_1 - v_2)(1+t)^{-n/8-1/8}.
\end{aligned}
$$

Take $k = s$, $r = 2$, $q = 2$, $p = \infty$ in Lemma 3.2. Then we have

$$
\begin{aligned}
(3.12)\qquad & \|f(v_1) - f(v_2)\|_{H^s} \\
& \leqslant C((\|v_1\|_{H^s} + \|v_2\|_{H^s})\|v_1 - v_2\|_{L_\infty})(\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})^{\theta-1} \\
& \quad + C((\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})\|v_1 - v_2\|_{H^s})(\|v_1\|_{L_\infty} + \|v_2\|_{L_\infty})^{\theta-1}.
\end{aligned}
$$

For $|\alpha| \leqslant s$, from Theorem 2.2, (3.12), (3.1), (3.6) we get

$$
\begin{aligned}
(3.13)\qquad & \left\|\partial_x^\alpha \int_0^t G_H(\cdot, t-\tau) * \Delta(f(v_1) - f(v_2))(\cdot,\tau)\,\mathrm{d}\tau\right\|_{L_2} \\
& \leqslant \left[\int_0^t \|\partial_x^\alpha G_H(\cdot, t-\tau) * \Delta(f(v_1) - f(v_2))(\cdot,\tau)\|_{L_2}^2\,\mathrm{d}\tau\right]^{1/2} \\
& \leqslant \left[\int_0^t \mathrm{e}^{-(t-\tau)/\eta}\|\partial_x^\alpha(f(v_1) - f(v_2))(\cdot,\tau)\|_{L_2}^2\,\mathrm{d}\tau\right]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \Bigl[\int_0^t \mathrm{e}^{-(t-\tau)/\eta}(1+\tau)^{-n/8\cdot 2}E^2\cdot\|v_1-v_2\|_{H^s}^2 \\
&\qquad\qquad\qquad \times E^{(\theta-1)\cdot 2}(1+\tau)^{-n/8\cdot 2(\theta-1)}\,\mathrm{d}\tau\Bigr]^{1/2} \\
&\leqslant C\cdot E^\theta\varrho(v_1-v_2)\Bigl[\int_0^t \mathrm{e}^{-(t-\tau)/\eta}(1+\tau)^{-n/4-n/4}(1+\tau)^{-n(\theta-1)/4}\,\mathrm{d}\tau\Bigr]^{1/2} \\
&\leqslant C\cdot E^\theta\varrho(v_1-v_2)(1+t)^{-n/4}.
\end{aligned}
$$

From (3.2), (3.11), (3.12) we have

$$\|\partial_x^\alpha(T(v_1)-T(v_2))\|_{L_2}\leqslant C\cdot E^\theta(1+t)^{-n/8-1/8}\varrho(v_1-v_2).$$

Take t large enough such that $CE^\theta(1+t)^{-1/8}<\frac{1}{2}$. Then

$$\varrho(T(v_1),T(v_2))<C\cdot E^\theta(1+t)^{-1/8}\varrho(v_1-v_2)<\frac{1}{2}\varrho(v_1-v_2),$$

the theorem is proved.

Together with Theorem 3.3, Theorem 3.4, (2.5), we get our global existence theorem. □

Theorem 3.5. *Let*

$$E=\max\{\|u_0\|_{H^{s+2}},\|u_1\|_{H^s},\|u_0\|_{L_1},\|u_1\|_{L_1}\},$$

where $s\geqslant 1+[\frac{n}{2}]$. *Cauchy problem* (1.1) *admits a unique, global classical solution* $u(x,t)\in L_\infty(0,\infty;H^s(\mathbb{R}^n))$ *and*

$$\|u\|_{H^s}\leqslant CE(1+t)^{-n/8}.$$

Next, we want to get more accurate estimate for $u\in X_{s,E}$.

4. Decay rate

Set $M(t)=\sup\limits_{t\geqslant\tau\geqslant 0}(1+\tau)^{n/8+|\alpha|/4}\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}$, $|\alpha|\leqslant s$. Then

$$\|\partial_x^\alpha u(\cdot,\tau)\|_{L_2}\leqslant M(t)(1+\tau)^{-n/8-|\alpha|/4}. \tag{4.1}$$

We want to prove that $M(t)$ is bounded.

Theorem 4.1. *Suppose that* $s\geqslant 1+[\frac{n}{2}]$, $u_0\in H^{s+2}\cap L_1$, *and* $u_1\in H^s\cap L_1$. *Then the solution* u *of* (1.1) *satisfies*

$$\|\partial_x^\alpha u\|_{L_2}\leqslant C(1+\tau)^{-|\alpha|/4-n/8}\quad\text{for } |\alpha|\leqslant s.$$

P r o o f. From Theorems 2.1 and 3.5, (3.5), (4.1), we have

$$
\begin{aligned}
(4.2)\quad &\left\|\partial_x^\alpha \int_0^t G_L(\cdot,t-\tau) * \Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau\right\|_{L_2}^2 \\
&\leqslant \left[\int_0^{t/2} \|\partial_x^\alpha \Delta G_L(\cdot,t-\tau)\|_{L_2}^2 \|f(u)(\cdot,\tau)\|_{L_1}^2\,\mathrm{d}\tau\right]^{1/2} \\
&\quad + \left[\int_{t/2}^t \|\Delta G_L(\cdot,t-\tau)\|_{L_2}^2 \|\partial_x^\alpha f(u)(\cdot,\tau)\|_{L_1}^2\,\mathrm{d}\tau\right]^{1/2} \\
&\leqslant \left[\int_0^{t/2} (1+t-\tau)^{(-(|\alpha|+2)/4-n/8)\cdot 2}(1+\tau)^{-n/8\cdot 2\cdot 2}(1+\tau)^{-n/8\cdot 2(\theta-1)}\,\mathrm{d}\tau\right]^{1/2} \\
&\quad + \left[\int_{t/2}^t (1+t-\tau)^{(-2/4-n/8)\cdot 2} M^2(t)(1+\tau)^{(-n/8-|\alpha|/4)\cdot 2} \right. \\
&\qquad \left. \times (1+\tau)^{-n/8\cdot 2}(1+\tau)^{-n/8\cdot 2(\theta-1)}\,\mathrm{d}\tau\right]^{1/2} \\
&\leqslant C(1+t)^{-n/8-|\alpha|/4-1/4} + CM(t)(1+t)^{-n/8-|\alpha|/4-1/8}.
\end{aligned}
$$

From Theorems 2.2 and 3.5, (3.8), (4.1), we have

$$
\begin{aligned}
(4.3)\quad &\left\|\partial_x^\alpha \int_0^t G_H(\cdot,t-\tau) * \Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau\right\|_{L_2} \\
&\leqslant \left[\int_0^t \|\partial_x^\alpha G_H(\cdot,t-\tau) * \Delta f(u)(\cdot,\tau)\|_{L_2}^2\,\mathrm{d}\tau\right]^{1/2} \\
&\leqslant \left[\int_0^t \mathrm{e}^{-(t-\tau)/\eta} \|\partial_x^\alpha f(u)(\cdot,\tau)\|_{L_2}^2\,\mathrm{d}\tau\right]^{1/2} \\
&\leqslant \left[\int_0^t \mathrm{e}^{-(t-\tau)/\eta} M^2(t)(1+\tau)^{(-|\alpha|/4-n/8)\cdot 2}(1+\tau)^{-n/8\cdot 2\cdot\theta}\,\mathrm{d}\tau\right]^{1/2} \\
&\leqslant CM(t)(1+t)^{-n/8-|\alpha|/4-1/8}.
\end{aligned}
$$

From (2.5), (3.3), (3.4), (4.2), (4.3), we get

$$\|\partial_x^\alpha u\|_{L_2} \leqslant CE(1+t)^{-n/8-|\alpha|/4} + C(1+t)^{-n/8-|\alpha|/4-1/4} + CM(t)(1+t)^{-n/8-|\alpha|/4-1/8},$$

thus $M(t) \leqslant CE + C(1+t)^{-1/4} + CM(t)(1+t)^{-1/8}$, we can deduce $M(t) \leqslant CE$. The theorem is proved. □

Theorem 4.2. *Suppose* $s \geqslant 1 + [\frac{n}{2}]$, $u_0 \in H^{s+2} \cap L_1$, $u_1 \in H^s \cap L_1$, $1 \leqslant n \leqslant 3$, $n + n\theta - 4 \geqslant 0$. *The solution* u *of* (1.1) *satisfies*

$$\|\partial_x^\alpha u\|_{L_\infty} \leqslant C(1+t)^{-|\alpha|/4-n/4} \quad \textit{for } |\alpha| \leqslant s-2.$$

P r o o f. From Theorems 2.1 and 3.5, (3.5), we have

$$
\begin{aligned}
(4.4)\qquad &\Big\|\partial_x^\alpha \int_0^t G_L(\cdot,t-\tau)*\Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau\Big\|_{L_\infty} \\
&\leqslant \int_0^{t/2} \|\partial_x^\alpha \Delta G_L(\cdot,t-\tau)\|_{L_\infty}\|f(u)(\cdot,\tau)\|_{L_1}\,\mathrm{d}\tau \\
&\quad + \int_{t/2}^t \|\Delta G_L(\cdot,t-\tau)\|_{L_\infty}\|\partial_x^\alpha f(u)(\cdot,\tau)\|_{L_1}\,\mathrm{d}\tau \\
&\leqslant \int_0^{t/2} (1+t-\tau)^{-n/4-(|\alpha|+2)/4}\cdot C(1+\tau)^{-n/4}(1+\tau)^{-n/8\cdot(\theta-1)}\,\mathrm{d}\tau \\
&\quad + \int_{t/2}^t (1+t-\tau)^{-n/4-2/4}\cdot(1+\tau)^{-|\alpha|/4-n/8-n/8}(1+\tau)^{-n/8\cdot(\theta-1)}\,\mathrm{d}\tau \\
&\leqslant C(1+t)^{-n/4-|\alpha|/4-n\theta/8+1/2-n/8}.
\end{aligned}
$$

From Theorems 2.2 and 3.5, (3.8), (3.6), when $1\leqslant n\leqslant 3$, $|\alpha|\leqslant s-2$, we obtain

$$
\begin{aligned}
(4.5)\qquad &\Big\|\partial_x^\alpha \int_0^t G_H(\cdot,t-\tau)*\Delta f(u)(\cdot,\tau)\,\mathrm{d}\tau\Big\|_{L_\infty} \\
&\leqslant \int_0^t \mathrm{e}^{-(t-\tau)/(2\eta)}\|\partial_x^\alpha \Delta f(u)(\cdot,\tau)\|_{L_2}\,\mathrm{d}\tau \\
&\leqslant \int_0^t \mathrm{e}^{-(t-\tau)/(2\eta)}(1+\tau)^{-(|\alpha|+2)/4-n/8}(1+\tau)^{-n/8\cdot\theta}\,\mathrm{d}\tau \\
&\leqslant C(1+t)^{-n/4-|\alpha|/4-1/2}.
\end{aligned}
$$

From Theorem 2.1, for $|\alpha|\leqslant s$ we get

$$
\begin{aligned}
(4.6)\qquad &\|\partial_x^\alpha G_L(\cdot,t)*(u_0+u_1)(\cdot)\|_{L_\infty}+\|\eta\partial_x^\alpha\partial_t G_L(\cdot,t)*u_0(\cdot)\|_{L_\infty} \\
&\leqslant \|\partial_x^\alpha G_L\|_{L_\infty}(\|u_0\|_{L_1}+\|u_1\|_{L_1})+\eta\|\partial_x^\alpha\partial_t G_L\|_{L_\infty}\|u_0\|_{L_1} \\
&\leqslant C(1+t)^{-|\alpha|/4-n/4}.
\end{aligned}
$$

From Theorem 2.2, for $|\alpha|\leqslant s$, if $1\leqslant n\leqslant 3$, we find that

$$
\begin{aligned}
(4.7)\qquad &\|\partial_x^\alpha G_H(\cdot,t)*(u_0+u_1)(\cdot)\|_{L_\infty}+\|\eta\partial_x^\alpha\partial_t G_H(\cdot,t)*u_0(\cdot)\|_{L_\infty} \\
&\leqslant C\mathrm{e}^{-t/(2\eta)}(\|\partial_x^\alpha u_0\|_{L_2}+\|\partial_x^\alpha u_1\|_{L_2}+\|\partial_x^\alpha\Delta u_0\|_{L_2})\leqslant C\mathrm{e}^{-t/(2\eta)}.
\end{aligned}
$$

From (2.5), (4.4), (4.5), (4.6), (4.7) for $|\alpha|\leqslant s-2,\ n+n\theta-4\geqslant 0$, we have

$$
\begin{aligned}
\|\partial_x^\alpha u\|_{L_\infty} &\leqslant C(1+t)^{-|\alpha|/4-n/4}+C(1+t)^{-n/4-|\alpha|/4-1/2} \\
&\quad + C(1+t)^{-n/4-|\alpha|/4-n\theta/8+1/2-n/8} \\
&\leqslant C(1+t)^{-|\alpha|/4-n/4}.
\end{aligned}
$$

The theorem is proved. □

By interpolation inequality, from Theorem 4.1, Theorem 4.2, we obtain the following theorem.

Theorem 4.3. *Suppose* $s \geqslant 1 + [\frac{n}{2}]$, $1 \leqslant n \leqslant 3$, $u_0 \in H^{s+2} \cap L_1$, $u_1 \in H^s \cap L_1$, $n + n\theta - 4 \geqslant 0$. *The solution* u *of* (1.1) *satisfies* $\|\partial_x^\alpha u\|_{L_p} \leqslant C(1+t)^{-|\alpha|/4 - n(1-1/p)/4}$ *for* $|\alpha| \leqslant s-2$, $2 \leqslant p \leqslant \infty$.

Together with Theorems 4.1 and 4.3, we get our main result Theorem 1.1.

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