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Abstract. Completely regular semigroups equipped with the unary operation of inversion within their maximal subgroups form a variety, denoted by $\mathscr{CR}$. The lattice of subvarieties of $\mathscr{CR}$ is denoted by $\mathcal{L}(\mathscr{CR})$.

For each variety in an $\bigcap$-subsemilattice Γ of $\mathcal{L}(\mathscr{CR})$, we construct at least one basis of identities, and for some important varieties, several. We single out certain remarkable types of bases of general interest. As an application for the local relation L, we construct $\mathbf{L}$-classes of all varieties in Γ. Two figures illustrate the theory.

Keywords: semigroup; completely regular; variety; basis; local relation

Classification: 20M07, 20M10

1. Introduction and summary

Completely regular semigroups (alias unions of groups) provided with the unary operation of inversion within their maximal subgroups form a variety $\mathscr{CR}$. The lattice of subvarieties of $\mathscr{CR}$ is denoted by $\mathcal{L}(\mathscr{CR})$. The paper [2], published in 1982, contains a compendium of bases of identities of the varieties commonly known at that time. As a result, we arrived at a diagram of varieties with at least one basis of identities of each. As time passed, new varieties emerged, most of which with their bases, and their position in the lattice $\mathcal{L}(\mathscr{CR})$ as well. As a result, we now have a number of simpler bases, and bases for new ones, as well as their relationship under inclusion, in the upper part of the original diagram.

In order to update the process initiated in the paper [2], we have constructed in [6] an $\bigcap$-subsemilattice of $\mathcal{L}(\mathscr{CR})$ which includes all presently known varieties suitable to be encompassed into a system.

This paper is a follow up of [6], and represents the second part of a trilogy. The third part [7] will cover kernel, trace, etc., and classes of all the varieties covered in the first two papers. We recommend browsing through [6] while watching for [7]. The idea in [6] is to extract an $\bigcap$-subsemilattice of $\mathcal{L}(\mathscr{CR})$, as large as feasible (with a sublattice again as large as possible), then providing each of its members with a basis of identities (present paper), and classes of these varieties of certain (equivalence) relations (third paper).

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It is often useful to have several bases for the varieties which play an important role in the theory. For they may have different form thereby giving away more information about the nature of the variety. Moreover, different bases may prove adequate in diverse situations.

Given a finite number of identities, we may construct a single identity in a trivial manner. But we generally want an identity as simple as feasible with as few variables as possible, which may take some effort.

By sections, the paper has the following content. In Section 2, we briefly discuss some notation and terminology used throughout the paper. This is followed by a few citations from the literature in Section 3 with certain useful convention. Section 4 contains Diagram 1 of the varieties under study. Sections 5–7 contain the bases of varieties in Diagram 1 by sorting them into intervals of the lattice $\mathcal{L}(\mathscr{CR})$. In Section 8, we discuss certain types of bases. We conclude the paper with Section 9 by applying the bases constructed to determine **L**-classes of all varieties under consideration exhibited in Diagram 2.

2. Notation and terminology

For notation, terminology and results, the paper depends heavily on the monograph [10], which we will often use without explicit reference. Throughout the paper, S *represents an arbitrary completely regular semigroup*, unless stated otherwise.

A finite meet (that is, the intersection of varieties) will generally be represented by juxtaposition of their acronyms. By duality, we mean interchanging left and right for words and concepts. Acronyms for most varieties studied can be found in [10, pages 470 and 471]. The juxtaposition of acronyms is provided with parentheses for easy and unambiguous identification of the variety.

Except in the last section, we write operators on the left and compose them from right to left of the argument. From [10, Section II.7], we have the H-, L-, and C-operators. We may derive from [8] the **T**-operator with $T\theta$ being the greatest element of the **T**-class of θ (**T** for trace).

Recall that $E(S)$ denotes the set of all idempotents of S, and $C(S)$ the subsemigroup of S generated by $E(S)$, called the *core* of S.

Let $\mathscr{A}$ be the variety of abelian groups, and $\mathscr{O}$ the variety of orthogroups. We will often encounter the following varieties:

- $H\mathscr{A}$: which coincides with the overabelian completely regular semigroups (that is, having all subgroups abelian);
- $L\mathscr{O}$: locally orthodox completely regular semigroups;
- $CH\mathscr{A}$: completely regular semigroups whose core is overabelian;

- $\mathscr{C}$: central completely regular semigroups in which the product of any two idempotents lies in the center of the maximal subgroup containing it.

If w is a word, then $h(w)$ and $t(w)$ denote the first and last variables in w, called the *head* and *tail* of w, respectively. Recall that we write $w^0 = ww^{-1} = w^{-1}w$.

Note that $\mathscr{BG}$ is the variety of cryptogroups (alias bands of groups whence the acronym). Correspondingly, $\mathscr{BA}$ stands for bands of abelian groups.

3. Citation and conventions

We list here a minimum of citations from the literature which are either used frequently or are not available in the form we need.

Fact 3.1. *Let* $\mathscr{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathcal{L}(\mathscr{CR})$.

(i) $T\mathscr{V} = [(xu_\alpha y)^0 = (xv_\alpha y)^0]$ *if* $h(u_\alpha) = h(v_\alpha)$ *and* $t(u_\alpha) = t(v_\alpha)$ *for all* $\alpha \in A$.

(ii) $C\mathscr{V} = \{S \in \mathscr{CR} \colon C(S) \in \mathscr{V}\}$.

(iii) $T\mathscr{O} = C(\mathscr{BG})$.

(iv) $(T\mathscr{O})CH\mathscr{A} = C(\mathscr{BA})$.

(v) *Let* $a, b, c \in S$ *be such that* $D_a = D_b \leq D_c$. *Then* $(ab)^0 = (acb)^0$.

Proof: (i) See [11, Theorem 3.9].

(ii) See [9, Theorem 3.1].

(iii) See [11, Proposition 3.5].

(iv) Using part (iii) and [10, Proposition II.7.6 (i)], we get

$$(T\mathscr{O})CH\mathscr{A} = C\mathscr{BG} \cap CH\mathscr{A} = C(\mathscr{BG} \cap H\mathscr{A}) = C(\mathscr{BA}).$$

(v) This is straightforward. $\square$

The following convention will save much space and possibly make the proofs more transparent.

Convention 3.2. In parts of a theorem which contain the equality of certain statements, we will tacitly denote them by letters $\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots$ starting with the first one, continuing with the second, etc., and write the proof in some order, say $\mathsf{A} \subseteq \mathsf{B}$, $\mathsf{B} \subseteq \mathsf{A}, \ldots$ or $\mathsf{A} \subseteq \mathsf{B}$, $\mathsf{B} \subseteq \mathsf{C}$, etc.

4. A diagram of varieties

Part of Diagram 1 stems from [6]. It represents an $\bigcap$-subsemilattice Γ of $\mathcal{L}(\mathscr{CR})$ containing the sublattice encompassed by heavy lines. Three sections of the paper contain one or more bases of all varieties in Γ. This is done by decomposing Γ into

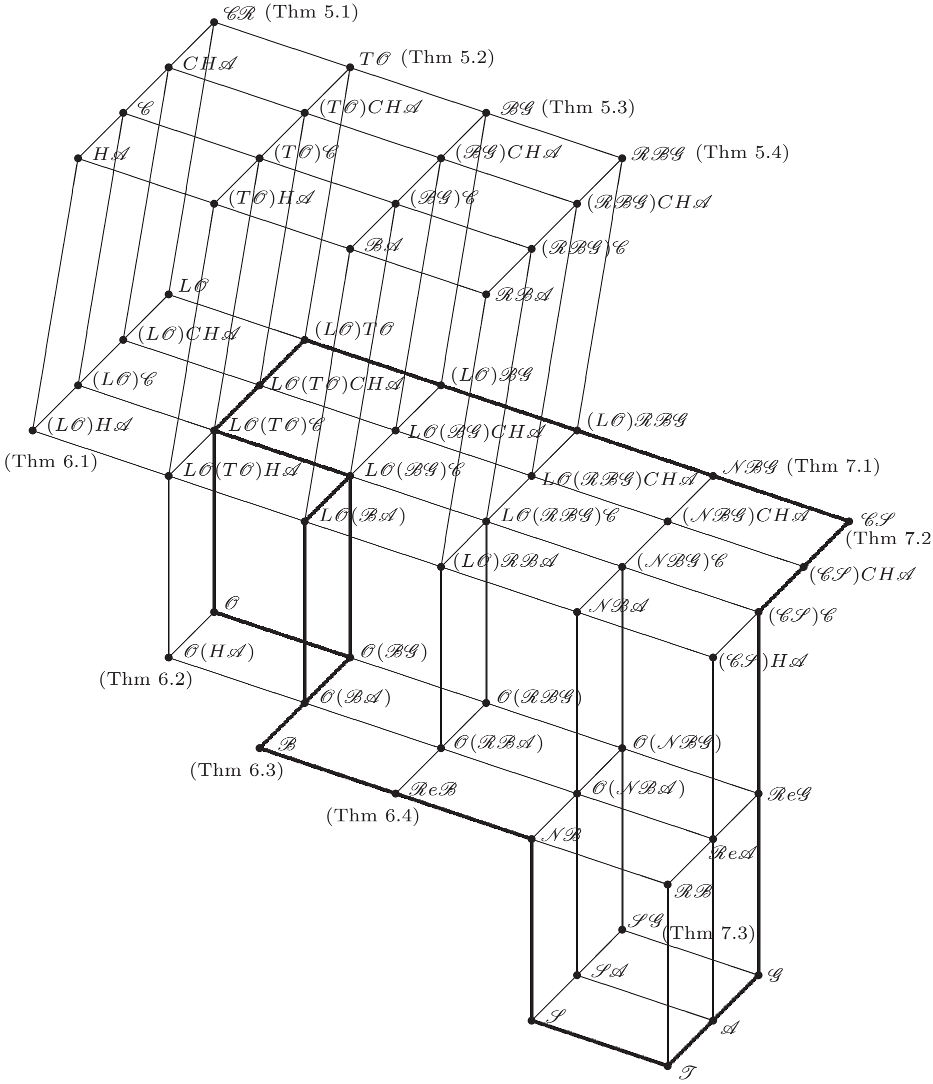

DIAGRAM 1. $\bigcap$-subsemilattice Γ with inclosed sublattice of $\mathcal{L}(\mathscr{CR})$.

non-overlapping intervals of the following form. In Diagram 1 they are pictured from lower left to upper right, and are maximal as such. Some of them are marked by the reference to the theorem in which they are treated. In the proofs, they depend on their labeling since we will refer to the acronyms without further specification.

5. The interval $[\mathscr{RBA}, \mathscr{CR}]$

Starting with the top of Diagram 1, bases of identities of $\mathscr{CR}$, $CH\mathscr{A}$, $\mathscr{C}$, and $H\mathscr{A}$ are given in Theorem 5.1. Theorem 5.2 provides bases for the meets of $T\mathscr{O}$ with each of the four varieties in Theorem 5.1. We repeat this pattern with the variety $\mathscr{BG}$ of cryptogroups in Theorem 5.3, and with the variety $\mathscr{RBG}$ of regular cryptogroups in Theorem 5.4. This makes up the content of the present section.

Theorem 5.1.

(i) $\mathscr{CR} = [ab = a(ba)^0 b] = [ab = a(b^0 a)^0 b] = [ab = a(ba^0)^0 b]$
$= [ab = a(b^0 a^0)^0 b]$.

(ii) $CH\mathscr{A} = [ax^0 a^0 y^0 a = ay^0 a^0 x^0 a]$ *where* $a = wxyw$
$= [(aba)ab(aba) = (aba)ba(aba)]$
where $a = (xy)^0 (wz)^0$ *and* $b = (yx)^0 (zw)^0$.

(iii) $\mathscr{C} = [a^0 b^0 a (b^0 a^0)^0 = (a^0 b^0)^0 a b^0 a^0]$
$= [(aba)ab(aba) = (aba)ba(aba)]$
where $a = (xy)^0 (wz)^0$ *and* $b = yx(zw)^0$.

(iv) $H\mathscr{A} = [(aba)^0 ba = ab(aba)^0] = [(aba)ab(aba) = (aba)ba(aba)]$
$= [(ab)^0 ba = ab(ba)^0] = [(ab)^2 ba = ab(ba)^2]$.

Proof: (i) $\mathsf{A} \subseteq \mathsf{B}$. Using [10, Lemma II. 2.3], we obtain

$$ab = ab(ab)^{-1}ab = ab(ab)^0 b^{-1}(ba)^0 a^{-1}(ab)^0 ab$$
$$= abb^{-1}(ba)^0 a^{-1}ab = a(ba)^0 b.$$

$\mathsf{B} \subseteq \mathsf{A}$. This is trivial.

$\mathsf{A} = \mathsf{C} = \mathsf{D} = \mathsf{E}$. This follows similarly.

(ii) $\mathsf{A} = \mathsf{B}$. This follows directly from [10, Theorem II.6.5].

$\mathsf{A} \subseteq \mathsf{C}$. Let $x = (i, g, \lambda)$, $y = (j, h, \mu)$, $w = (k, s, \sigma)$, and $z = (l, t, \tau)$. Then

$$a = (xy)^0 (wz)^0 = (i, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1}, \tau),$$
$$b = (yx)^0 (zw)^0 = (j, p_{\lambda j}^{-1} p_{\lambda l}\, p_{\sigma l}^{-1}, \sigma),$$
$$ab = (i, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1} p_{\tau j}\, p_{\lambda j}^{-1} p_{\lambda l}\, p_{\sigma l}^{-1}, \sigma),$$
$$ba = (j, p_{\lambda j}^{-1} p_{\lambda l}\, p_{\sigma l}^{-1} p_{\sigma i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1}, \tau),$$
$$aba = (i, u, \tau) \quad \text{for some suitable } u \in G,$$

(1) $$(aba)ab(aba) = (i, up_{\tau i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1} p_{\tau j}\, p_{\lambda j}^{-1} p_{\lambda l}\, p_{\sigma l}^{-1} p_{\sigma i} u, \tau),$$

(2) $$(aba)ba(aba) = (i, up_{\tau j}\, p_{\lambda j}^{-1} p_{\lambda l}\, p_{\sigma l}^{-1} p_{\sigma i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1} p_{\tau i} u, \tau).$$

By [10, Proposition III. 6.1], the entries of P commute which by (1) and (2) gives C.

$\mathsf{C} \subseteq \mathsf{A}$. The given identity implies that the right sides of (1) and (2) are equal. For $j = k = \mu = \sigma = 1$, we get $p_{\tau i}\, p_{\lambda l} = p_{\lambda l}\, p_{\tau i}$, and by the above reference we obtain $C\mathcal{H}\mathscr{A}$.

(iii) $\mathsf{A} = \mathsf{B}$. See [10, Theorem II. 6.4] or the original [1, Theorem 5.1 (a)].

$\mathsf{A} = \mathsf{C}$. The preamble here is the same as in part (ii).

$\mathsf{A} \subseteq \mathsf{C}$. We have the expression for a above. Now

$$\begin{aligned}
a &= (xy)^0(wz)^0 = (i, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1}, \tau),\\
b &= yx(zw)^0 = (j, hp_{\mu i} g p_{\lambda k}\, p_{\tau k}^{-1}, \tau),\\
ab &= (i, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1} p_{\tau j} h p_{\mu i} g p_{\lambda k}\, p_{\tau k}^{-1} p_{\tau i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1}, \tau),\\
ba &= (j, hp_{\mu i} g p_{\lambda k}\, p_{\tau k}^{-1} p_{\tau i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1}, \tau),\\
aba &= (i, v, \tau) \quad \text{for some suitable } v \in G,
\end{aligned}$$

$$(aba)ab(aba) = (i, vp_{\tau i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1} p_{\tau j} h p_{\mu i} g p_{\lambda k}\, p_{\tau k}^{-1} p_{\tau i} v, \tau), \tag{3}$$

$$(aba)ba(aba) = (i, vp_{\tau j} h p_{\mu i} g p_{\lambda k}\, p_{\tau k}^{-1} p_{\tau i}\, p_{\mu i}^{-1} p_{\mu k}\, p_{\tau k}^{-1} p_{\tau i} v, \tau). \tag{4}$$

By [10, Proposition III. 6.2], the entries of P are in the center of G, and thus (3) and (4) are equal.

$\mathsf{C} \subseteq \mathsf{A}$. The given identity implies that the right sides of (3) and (4) are equal. Set $i = j = \mu = \tau = 1$ and replace g by $gp_{\lambda i}^{-1}$. From the equality of (3) and (4), we get

$$p_{\mu k}\, p_{\tau k}^{-1} g p_{\lambda k}^{-1} p_{\lambda k}\, p_{\tau k}^{-1} = g p_{\lambda k}^{-1} p_{\lambda k}\, p_{\tau k}^{-1} p_{\mu k}\, p_{\tau k}^{-1},$$

whence $p_{\mu k} g = g p_{\mu k}$. The last reference now gives C.

(iv) $\mathsf{A} = \mathsf{B}$. See [10, Proposition II. 7.2 (i)].

$\mathsf{B} \subseteq \mathsf{C}$. Pre- and postmultiply the identity in B by aba.

$\mathsf{C} \subseteq \mathsf{D}$. Exchange a and b getting $bab^2abab = babab^2ab$. Now premultiplying by $(ab)^{-2}a$ and postmultiplying by $a(ba)^{-2}$, we get

$$(ab)^{-2}abab^2ababa(ba)^{-2} = (ab)^{-2}ababab^2aba(ba)^{-2},$$

that is, $(ab)^0 ba = ab(ba)^0$.

$\mathsf{D} \subseteq \mathsf{A}$. In any group, the given identity implies commutativity.

$\mathsf{D} = \mathsf{E}$. This is straightforward. □

The equality $\mathsf{A} = \mathsf{B}$ in Theorem 5.1 (i) is due to J. A. Gerhard (private communication). We will generally use the basis of the variety D in part (iv) when dealing with $H\mathscr{A}$.

In the next theorem, we treat $T\mathscr{O}$ and its meet with varieties in Theorem 5.1. Recall Fact 3.1 (iii).

Theorem 5.2.

(i) $T\mathscr{O} = [(xa^0b^0y)^0 = (x(a^0b^0)^0y)^0]$.
(ii) $T\mathscr{O}(CH\mathscr{A}) = [(xa^0b^0y)^0u^0y^0v^0y = (x(a^0b^0)^0y)^0v^0y^0u^0y]$ *where* $y = wuvw$.
(iii) $(T\mathscr{O})\mathscr{C} = [(xa^0b^0y)^0z^0y(z^0y^0)^0 = (x(a^0b^0)^0y)^0(y^0z^0)^0yz^0y^0]$.
(iv) $(T\mathscr{O})H\mathscr{A} = [(xa^0b^0y)^0(yz)^0zy = (x(a^0b^0)^0y)^0yz(zy)^0]$.

Proof: (i) This follows directly from Fact 3.1 (i) since $\mathscr{O} = [(a^0b^0)^0 = a^0b^0]$.

(ii) $\mathsf{A} \subseteq \mathsf{B}$. By Theorem 5.1 (ii) and part (i) above, we obtain

$$\begin{aligned}(xa^0b^0y)^0u^0y^0v^0y &= (x(a^0b^0)^0y)^0y^{-1}yu^0y^0v^0y \\ &= (xa^0b^0y)^0y^{-1}yv^0y^0u^0y \\ &= (x(a^0b^0)^0y)^0v^0y^0u^0y.\end{aligned}$$

$\mathsf{B} \subseteq \mathsf{A}$. Using the same references for $u = w = y^0$, $v = y$ we get $T\mathscr{O}$, and for $a = b = y^0$, we obtain $CH\mathscr{A}$.

(iii) This follows similarly from Theorem 5.1 (iii).

(iv) This follows similarly from Theorem 5.1 (iv). □

The variety $\mathscr{BG}$ of all cryptogroups is one of the principal subvarieties of $\mathscr{CR}$. In view of its importance, it is only fitting to have several bases for it. For each of them sheds different light on $\mathscr{BG}$, even though for application, only the first one below would suffice.

Theorem 5.3.

(i) $\mathscr{BG} = [(ab)^0 = (a^0b^0)^0] = [(a^0b)^0 = (ab^0)^0] = [(axb)^0 = (ax^0b)^0]$
$= [(aba)^0 = (a^0ba^0)^0] = [a(aba)^0 = (aba)^0a] = [a^0(ba)^0 = (ab)^0a^0]$
$= [(a^0(ba)^0)^0 = ((ab)^0a^0)^0] = [(a^0(bc)^0)^0 = ((ab)^0c^0)^0]$.
(ii) $(\mathscr{BG})CH\mathscr{A} = [ax^0a^0y^0(a^0b^0)^0 = ay^0a^0x^0(ab)^0]$ *where* $a = wxyw$.
(iii) $(\mathscr{BG})\mathscr{C} = [a^0b^0a = ab^0a^0] = [a^0x^0y^0a = ax^0y^0a^0] = [aba^0b^0 = a^0b^0ab]$.
(iv) $\mathscr{BA} = [aba^0 = a^0ba]$.

Proof: (i) $\mathsf{A} = \mathsf{B}$. See [10, Theorem II. 8.1].

$\mathsf{A} = \mathsf{C} = \mathsf{D} = \mathsf{E}$. This is straightforward.

$\mathsf{A} \subseteq \mathsf{F}$. Indeed,

$$\begin{aligned}a(aba)^0 &= a(a^0ba)^0 = a(a^0ba)(a^0ba)^{-1} = aba(a^0ba)^{-1} \\ &= (aba)^0(aba)(a^0ba)^{-1} = (aba)^0a(a^0ba)(a^0ba)^{-1} \\ &= (aba)^0a(a^0ba)^0 = (aba)^0a(aba)^0\end{aligned}$$

and similarly $(aba)^0a = (aba)^0a(aba)^0$, which imply that $a(aba)^0 = (aba)^0a$.

$\mathsf{F} \subseteq \mathsf{A}$. For any $S \in \mathsf{F}$, we have

$$ab = abab(ab)^{-1} = (aba)^0 ab = a(aba)^0 b \in a^2 bS,$$
$$a^2 b = a(ab) = a(aba)^0 ab = (aba)^0 a(ab) \in abS,$$

so that $a^2bS = abS$, and dually, $Sab^2 = Sab$, which by [10, Theorem II. 8.1] gives $S \in \mathscr{BG} = \mathsf{A}$.

$\mathsf{A} \subseteq \mathsf{G}$. Indeed,

$$\begin{aligned} a^0(ba)^0 &= a^0(b^0a^0)^0 = a^0b^0a^0(b^0a^0)^{-1} \\ &= (a^0b^0)^0a^0b^0a^0(b^0a^0)^{-1} = (a^0b^0)^0a^0(b^0a^0)^0 \end{aligned}$$

and similarly, $(ab)^0a^0 = (a^0b^0)^0a^0(b^0a^0)^0$, which imply that $a^0(ba)^0 = (ab)^0a^0$.

$\mathsf{G} \subseteq \mathsf{A}$. The substitution $b \to a^0b$ gives $a^0(a^0ba)^0 = (aa^0b)^0a^0$, so that $(a^0ba)^0 = (ab)^0a^0$. For any $S \in \mathsf{G}$, the last identity yields $a^0ba \in (ab)^0a^0S$. Postmultiplying by $a^{-1}b(a^0b)^{-1}$, we get $a^0b \in abS$. Also $(ab)^0a^0 \in a^0bS$ and postmultiplying by ab we obtain $ab \in a^0bS$, whence $a^0bS = abS$. It follows that $abS = a^2bS$ and dually, $Sab = Sab^2$. The above reference yields $S \in \mathscr{BG} = \mathsf{A}$.

$\mathsf{B} \subseteq \mathsf{I}$. Indeed,

$$(a^0(bc)^0)^0 = (a^0(b^0c^0)^0)^0 = (a^0b^0c^0)^0 = ((a^0b^0)^0c^0)^0 = ((ab)^0c^0)^0.$$

$\mathsf{I} \subseteq \mathsf{H}$. Set $c = a$.

$\mathsf{H} \subseteq \mathsf{A}$. The argument here is similar to that for $\mathsf{G} \subseteq \mathsf{A}$.

(ii) $\mathsf{A} \subseteq \mathsf{B}$. This follows directly from Theorem 5.1 (ii) and part (i) above.

$\mathsf{B} \subseteq \mathsf{A}$. With the same references as above, letting $x = y = a$, we get $(a^0b^0)^0 = (ab)^0$, that is $\mathscr{BG}$, and for $b = a$, we obtain $ax^0a^0y^0a = ay^0a^0x^0a$ whence $CH\mathscr{A}$.

(iii) $\mathsf{A} = \mathsf{B}$. See [1, Theorem 5.1 (b)].

$\mathsf{A} = \mathsf{C}$. See [2, Lemma 6.6].

$\mathsf{B} = \mathsf{D}$ See [2, Lemma 6.8].

(iv) See [2, Lemma 6.5]. □

We will generally use the first basis in Theorem 5.3 (i). Next we consider regular cryptogroups.

Theorem 5.4.

(i) $\mathscr{RBG} = [(axya)^0 = (axaya)^0] = [(ax^0y^0a)^0 = (axaya)^0]$
$= [(axyxa)^0 = (axayaxa)^0]$.

(ii) $(\mathscr{RBG})CH\mathscr{A} = [(axya)^0u^0a^0v^0a = (axaya)^0v^0a^0u^0a]$ *where* $a = wuvw$.

(iii) $(\mathscr{RBG})\mathscr{C} = [b(axya)^0b^0 = b^0(axaya)^0b]$.

(iv) $\mathscr{RBA} = [(axya)^0ba = (axaya)^0aba^0]$.

PROOF: (i) $\mathsf{A} = \mathsf{B}$. See [10, Proposition V.4.4].

$\mathsf{B} \subseteq \mathsf{C}$. In particular, $(ax^0y^0a)^0 = (ax^0ay^0a)^0$. Since $\mathsf{A} = \mathsf{B}$, it follows that $\mathscr{BG}$ implies $(ax^0ay^0a)^0 = (axaya)^0$ and thus $(ax^0y^0a)^0 = (axaya)^0$.

$\mathsf{C} \subseteq \mathsf{B}$. We first show that $\mathscr{BG}$ holds. For any $S \in \mathsf{C}$, we have

$$(ab)^0 = (abab)^0 \in (aba^0)S = (a^0aa^0ba^0)S = (a^0a^0b^0a^0)S \subseteq a^0b^0S,$$

and conversely,

$$(a^0b^0)^0 \in (a^0b^0a^0)S = (a^0a^0b^0a^0)S = (a^0aa^0ba^0)S \subseteq (ab)^0S.$$

It follows that $(ab)^0\mathscr{R}(a^0b^0)^0$ and dually, $(ab)^0\mathscr{L}(a^0b^0)^0$, whence $(ab)^0 = (a^0b^0)^0$, that is $\mathscr{BG}$. Part B follows.

$\mathsf{B} \subseteq \mathsf{D}$. The substitution $x \to yx$ implies $(ayxya)^0 = (ayxaya)^0$, and interchanging x and y, we obtain

$$(axyxa)^0 = (axyaxa)^0. \tag{5}$$

In the given identity, the substitution $y \to yx$ gives

$$(axyxa)^0 = (axayxa)^0. \tag{6}$$

Now (6), (5) and $\mathscr{BG}$, which holds since $\mathsf{A} = \mathsf{B}$, and Fact 3.1 (v) yield

$$\begin{aligned}(axyxa)^0 &= (axayxa)^0(axyaxa)^0 = ((axayxa)^0(axyaxa)^0)^0\\ &= (axayxaaxyaxa)^0 = (axayaxa)^0.\end{aligned}$$

$\mathsf{D} \subseteq \mathsf{A}$. For $x = a^{-1}$, the given identity yields $(a^0ya^0)^0 = (aya)^0$ which by Theorem 5.3 (i) gives $\mathscr{BG}$.

Recall from [10] the notation $i(w)$ and $f(w)$ and let

$$u = axyxa \quad \text{and} \quad v = axayaxa.$$

In a band, we have $i(u) = i(v)$ and $f(u) = f(v)$, which by [10, Theorem V.1.9 (xii)] implies that $[u^0 = v^0] \cap \mathscr{B} = \mathscr{R}e\mathscr{B}$. For any $S \in \mathsf{B}$, we get $S/\mathscr{H} \in \mathscr{R}e\mathscr{B}$ and thus $S \in \mathscr{RBG} = \mathsf{A}$.

(ii) $\mathsf{A} \subseteq \mathsf{B}$. This follows easily from Theorem 5.1 (ii) and part (i) above.

$\mathsf{B} \subseteq \mathsf{A}$. Using the same references and setting $u = v = a$, we obtain $\mathscr{RBG}$, and letting $x = y = a$, we get $CH\mathscr{A}$.

(iii) This follows from Theorem 5.3 (iii) and part (i) above.

(iv) This follows from Theorem 5.3 (iv) and part (i) above. □

The first basis in Theorem 5.4 (i) is the most useful one.

6. The interval $[\mathscr{R}e\mathscr{B}, L\mathscr{O}]$

The interval $[\mathscr{R}e\mathscr{B}, L\mathscr{O}]$ is the disjoint union of the intervals $[(L\mathscr{O})\mathscr{R}\mathscr{B}\mathscr{A}, L\mathscr{O}]$ and $[\mathscr{R}e\mathscr{B}, \mathscr{O}]$. The former interval is obtained by forming the meets of varieties in $[\mathscr{R}\mathscr{B}\mathscr{A}, \mathscr{C}\mathscr{R}]$ with $L\mathscr{O}$. Recall that for any word w, the expression $w \in E$ stands for the identity $w^2 = w$.

Theorem 6.1. *Let $z = (xa)^0(ya)^0$.*

(i) $L\mathscr{O} = [(ax)^0(ay)^0 \in E] = [z \in E]$.
(ii) $(L\mathscr{O})CH\mathscr{A} = [zu^0a^0v^0a = zv^0a^0u^0a]$ *where* $a = wuvw$.
(iii) $(L\mathscr{O})\mathscr{C} = [zb^0a(b^0a^0)^0 = z^0(a^0b^0)^0ab^0a^0]$.
(iv) $(L\mathscr{O})H\mathscr{A} = [z(ab)^0ba = z^0ab(ba)^0]$.

Proof: (i) $\mathsf{A} = \mathsf{B}$. See [10, Corollary II. 7.5].

$\mathsf{B} = \mathsf{C}$. This follows by duality in view of the definition of $L\mathscr{O}$.

(ii)–(iv) This follows easily from Theorem 5.1 (ii)–(iv) and part (i) above. □

Next we combine Theorems 5.2 and 6.1.

Theorem 6.2.

(i) $(L\mathscr{O})T\mathscr{O} = [(xz)^0(xa^0b^0y)^0 = ((xz)^0(x(a^0b^0)^0y)^0)^0]$.
(ii) $L\mathscr{O}(T\mathscr{O})CH\mathscr{A} = [(xz)^0(xa^0b^0y)^0u^0y^0v^0y = (xz)^0(x(a^0b^0)^0y)^0v^0y^0u^0y]$ *where* $y = wuvw$.
(iii) $L\mathscr{O}(T\mathscr{O})\mathscr{C} = [(xz)^0(xa^0b^0y)^0c^0y(c^0y^0)^0 = (xz)^0(x(a^0b^0)^0y)^0(y^0c)^0yc^0y^0]$.
(iv) $L\mathscr{O}(T\mathscr{O})H\mathscr{A} = [(xz)^0(xa^0b^0y)^0(yc)^0cy = (xz)^0(x(a^0b^0)^0y)^0yc(cy)^0]$.
(v) $\mathscr{O} = [a^0b^0 = (a^0b^0)^0] = [ab = ab^0a^0b]$
$= [(ab)^0 = (ab)^0(ba)^0(ab)^0] = [(aba)^0 = (ab)^0(ba)^0]$.
(vi) $\mathscr{O}(H\mathscr{A}) = [(ab)^0ba = ab^0a^0b(ba)^0] = [ab = (ab)^0a^0bab^0(ab)^0]$.

Proof: (i) $\mathsf{A} \subseteq \mathsf{B}$. In view of Theorems 6.1 (i) and 5.2 (i), we obtain

$$(xz)^0(xa^0b^0y)^0 = (xz)^0(x(a^0b^0)^0y)^0 = ((xz)^0(x(a^0b^0)^0y)^0)^0.$$

$\mathsf{B} \subseteq \mathsf{A}$. Using the same references for $a = b = x$ we get $L\mathscr{O}$, and for $z = x$, we obtain $T\mathscr{O}$.

(ii)–(iv) This follows easily from Theorem 5.1 (ii)–(iv) and part (i) above.

(v) $\mathsf{A} = \mathsf{B}$. This is obvious.

$\mathsf{A} \subseteq \mathsf{C}$. Indeed,

$$ab = aa^0b^0b = aa^0b^0a^0b^0b = ab^0a^0b.$$

$\mathsf{C} \subseteq \mathsf{A}$. Trivially $a^0b^0 = a^0b^0a^0b^0$ whence $\mathscr{O}$.

$\mathsf{A} \subseteq \mathsf{D}$. Let $S \in \mathsf{A}$. Then $S = (Y; S_\alpha)$ is a semilattice Y of rectangular groups S_α. Hence $ab, ba \in S_\alpha$ for some $\alpha \in Y$. Thus $(ab)^0(ba)^0 \in E(S_\alpha)$ and $(ab)^0 = (ab)^0(ba)^0(ab)^0$ since $E(S_\alpha)$ is a rectangular band.

$\mathsf{D} \subseteq \mathsf{A}$. Let $S \in \mathsf{D}$ and $e, f \in E(S)$. Then $(ef)^0 = (ef)^0(fe)^0(ef)^0$ whence by [10, Lemma II. 4.4 (ii)], we have

$$ef = ef(fe)^0(ef)^0 = e(fe)^0(ef)^0 = (ef)^0 e(ef)^0 = (ef)^0$$

and $\mathscr{O}$ follows.

$\mathsf{A} \subseteq \mathsf{E}$. Let $S \in \mathsf{A}$ and $a, b \in S$. Then

$$(ab)^0(ba)^0 = (abab)^0(baba)^0 = (aba)^0(ab)^0(ba)^0(aba)^0$$

where $(aba)^0$, $(ab)^0$ and $(ba)^0$ are $\mathscr{D}$-related, hence they are contained in a rectangular band. Therefore $(ab)^0(ba)^0 = (aba)^0$.

$\mathsf{E} \subseteq \mathsf{A}$. In view of [10, Theorem II. 5.3], it suffices to consider $S \in \mathsf{E} \cap \mathscr{CS}$. Hence let $a = (i, g, \lambda)$, $b = (j, h, \mu) \in \mathcal{M}(I, G, \Lambda; P) = S$. By hypothesis, we have $(i, p_{\lambda i}^{-1}, \lambda) = (i, p_{\mu i}^{-1}, \mu)(j, p_{\lambda j}^{-1}, \lambda)$, whence $p_{\lambda i}^{-1} = p_{\mu i}^{-1} p_{\mu i} p_{\lambda j}^{-1}$. Suppose that P is normalized. For $j = \mu = 1$, the normalization gives $p_{\lambda i}^{-1} = 1$. Therefore $p_{\lambda i} = 1$ for all $i \in I$, $\lambda \in \Lambda$, and S is a rectangular group.

(vi) $\mathsf{A} \subseteq \mathsf{B}$. This follows at once from Theorem 5.1 (iv) and part (v) above.

$\mathsf{B} \subseteq \mathsf{A}$. Let $S \in \mathsf{B}$ and $e, f \in E(S)$. Then $(ef)^0 fe = efef(fe)^0$ whence $(ef)^0 e = efe(fe)^0 = efe$. Postmultiplying by f, we get $ef = (ef)^2$ giving $\mathscr{O}$. In a group, the given identity implies commutativity.

$\mathsf{A} = \mathsf{C}$. See [10, Proposition II. 7.2 (ii)]. $\square$

We can use any basis in Theorem 6.2 (v) to form a basis of $T\mathscr{O}$ in Theorem 5.2. Next we treat locally orthodox cryptogroups.

Theorem 6.3.

(i) $(L\mathscr{O})\mathscr{BG} = [(axaya)^0 = (ax)^0 a^0 (ya)^0] = [(axay)^0 = (ax)^0(ay)^0]$
$= [(xaya)^0 = (xa)^0(ya)^0]$.

(ii) $L\mathscr{O}(\mathscr{BG})CH\mathscr{A} = [(uava)^0 y^0 a^0 x^0 a = (ua)^0(va)^0 x^0 a^0 y^0 a]$
where $a = wxyw$.

(iii) $L\mathscr{O}(\mathscr{BG})\mathscr{C} = [b^0(axaya)^0 b = b(ax)^0 a^0 (ya)^0 b^0]$.

(iv) $(L\mathscr{O})\mathscr{BA} = [axay = (ax)^0 ayxa(ay)^0]$.

(v) $\mathscr{O}(\mathscr{BG}) = [(ab)^0 = a^0 b^0] = [a^0(bc)^0 = (ab)^0 c^0] = [(axya)^0 = (ax)^0(ya)^0]$.

(vi) $\mathscr{O}(\mathscr{BA}) = [ab = a^0 bab^0] = [(ab)^2 = a^2 b^2]$.

(vii) $\mathscr{B} = [a = a^0] = [a = a^2]$.

Proof: (i) $\mathsf{A} = \mathsf{C}$. See [10, Corollary II. 8.6].

$\mathsf{A} = \mathsf{D}$. First, A is self-dual, and thus $\mathsf{A} = \mathsf{D}$ follows from $\mathsf{A} = \mathsf{C}$.

$\mathsf{A} \subseteq \mathsf{B}$. Using $\mathsf{A} = \mathsf{C} = \mathsf{D}$, we obtain

$$\begin{aligned}(axaya)^0 &= (ax)^0(aya)^0 = (ax)^0(a^0aya)^0 \\ &= (ax)^0(a^0a)^0(ya)^0 = (ax)^0 a^0 (ya)^0.\end{aligned}$$

$\mathsf{B} \subseteq \mathsf{A}$: Let $S \in \mathsf{B}$ and $e, f, g \in E(S)$ satisfy $e \geq f$ and $e \geq g$. Then $(efege)^0 = (ef)^0 e(ge)^0$ whence $(fg)^0 = fg$, so that $S \in L\mathscr{O}$. For $y = a^{-1}$, we get $(axa)^0 = (ax)^0 a^0$, and for $x = a^{-1}$, we obtain $(aya)^0 = a^0(ya)^0$ whence $(ax)^0 a^0 = a^0(xa)^0$. Theorem 5.3 (i) now implies that $S \in \mathscr{BG} = \mathsf{A}$.

(ii) This easily follows from Theorem 5.1 (ii) and part (i) above.

(iii) $\mathsf{A} \subseteq \mathsf{B}$. This follows from Theorem 5.3 (ii) and part (i) above.

$\mathsf{B} \subseteq \mathsf{A}$. With the same references for $b = a^0$ we get $(L\mathscr{O})\mathscr{BG}$, and for $x = y = a$, we obtain $(\mathscr{BG})\mathscr{C}$.

(iv) $\mathsf{A} \subseteq \mathsf{B}$. By Theorem 5.3 (iv) and part (i) above, we get

$$axay = (axay)^0 axay = (ax)^0(ay)^0(ax)(ay) = (ax)^0 ayax(ay)^0.$$

$\mathsf{B} \subseteq \mathsf{A}$. Let $S \in \mathsf{B}$. We first prove that $S \in \mathscr{BG}$. The substitution $x \to a^{-2}$, $y \to ya$ in the given identity yields

$$a^0 ya = a^0 aya^2 a^{-2}(aya)^0 = ay(aya)^0. \tag{7}$$

Using (7) twice, we obtain

$$\begin{aligned} a^0 y = a^0 y(a^0 y)^0 = (a^0 ya)a^{-1}(a^0 y)^0 = ay(aya)^0 a^{-1}(a^0 y)^0 \in ayS, \\ ay = ay(ayay)^0 = ay(aya)^0(ay)^0 = a^0 ya(ay)^0 \in a^0 yS \end{aligned}$$

and thus $a^0 y \mathscr{R} ay$.

With the substitution $x \to a^{-1}x$, $y \to a^0$ in the given identity, we get

$$a^0 xa = (a^0 x)^0 a^0 xa.$$

Using this twice, we obtain

$$\begin{aligned} xa^0 = (xa^0)^0 xa^0 = (xa)^0 a^{-1}(axa)^0 = (xa)^0 a^{-1}(ax)^0 \in Sax(axa^0) \\ \subseteq Sa^0 x(axa)^0 \subseteq S(a^0 x)^0 axa^0 = Sa^0 xa \subseteq Sxa, \\ xa = (xa)^0 xa = (xa)^0 a^0 xa = (xa)^0(a^0 x)^0 axa^0 \in Sxa^0, \end{aligned}$$

whence $xa^0 \mathscr{L} xa$. Since $\mathscr{R}$ is a left congruence and $\mathscr{L}$ is a right congruence, we deduce that $\mathscr{H}$ is a congruence, that is, $S \in \mathscr{BG}$.

Now by (7), we obtain

$$a^0 xa = ax(axa)^0 = axa^0(axa^0)^0 = axa^0$$

and by Theorem 5.3 (iv), we get $\mathscr{BA}$. Using this, we obtain

$$\begin{aligned} (ax)^0(ay)^0 = (ax)^{-1} axay(ay)^{-1} = (ax)^{-1}[(ax)^0(ay)(ax)](ay)^0(ay)^{-1} \\ = (ax)^{-1}(ax)[(ay)(ax)^0(ay)^0](ay)^{-1} = (ax)^0(ay)^0(ax)^0(ay)^0 \end{aligned}$$

and $(ax)^0(ay)^0$ is an idempotent. Next

$$axay = (ax)^0[(ay)(ax)(ay)^0] = (ax)^0(ay)^0 axay$$

whence $(axay)^0 = (ax)^0(ay)^0(axay)^0$. Similarly

$$axay = [(ax)^0(ay)(ax)](ay)^0 = (ax)(ay)(ax)^0(ay)^0$$

and thus $(axay)^0 = (axay)^0(ax)^0(ay)^0$. Since $(ax)^0(ay)^0$ is an idempotent, it follows that $(axay)^0 \leq (ax)^0(ay)^0$. But $(axay)^0 \mathscr{D} (ax)^0(ay)^0$ and therefore $(axay)^0 = (ax)^0(ay)^0$, that is, $(L\mathscr{O})\mathscr{B}\mathscr{G}$. This together with $\mathscr{B}\mathscr{A}$ yields $(L\mathscr{O})\mathscr{B}\mathscr{A} = \mathsf{A}$.

(v) $\mathsf{A} = \mathsf{B}$. See [10, Theorem II. 8.5].

$\mathsf{A} \subseteq \mathsf{C}$. Clearly both a band and a Clifford semigroup satisfy the identity in C. The claim now follows from [10, Theorem II. 8.5].

$\mathsf{C} \subseteq \mathsf{B}$. For $c = b^0$, the given identity yields $a^0(bb^0)^0 = (ab^0)^0 b^0$ whence $a^0b^0 = (ab)^0$.

$\mathsf{B} \subseteq \mathsf{D}$. This is obvious.

$\mathsf{D} \subseteq \mathsf{A}$. For $x = y$, by Theorem 6.2 (v), we get $\mathscr{O}$. Next

$$a^0(ba)^0 = (aa^0)^0(ba)^0 = (aa^0ba)^0 = (aba)^0$$

and dually $(ab)^0a^0 = (aba)^0$ and thus $a^0(ba)^0 = (ab)^0a^0$ which by Theorem 5.3 (i) yields $\mathscr{B}\mathscr{G}$. Obviously D implies $\mathscr{O}$.

(vi) $\mathsf{A} \subseteq \mathsf{B}$. Using Theorems 6.2 (v) and 5.3 (iv), we have

$$ab = ab^0a^0b = a^0b^0ab = a^0bab^0.$$

$\mathsf{B} \subseteq \mathsf{C}$. Premultiplying by a and postmultiplying by b, we get $a^2b^2 = (ab)^2$.

$\mathsf{C} \subseteq \mathsf{A}$. We first prove $\mathscr{B}\mathscr{G}$. For $x = a$ and $y = (aba)^0$, the given identity implies $a(aba)^0a(aba)^0 = aa(aba)^0(aba)^0$ and premultiplying by a^{-1}, we get $(aba)^0a(aba)^0 = a(aba)^0$. Similarly, for $x = (aba)^0$ and $y = a$, the same identity yields $(aba)^0a(aba)^0 = (aba)^0a$ and thus $a(aba)^0 = (aba)^0a$. Now Theorem 5.3 (i) gives $\mathscr{B}\mathscr{G}$. The given identity obviously implies $\mathscr{O}$ and $H\mathscr{A}$, whence also $\mathscr{O}(\mathscr{B}\mathscr{A})$.

(vii) This is trivial. □

Theorem 6.3 (vi) answers [2, Problem 7.3] in the affirmative. Our next result concerns locally orthodox regular cryptogroups.

Theorem 6.4.

(i) $(L\mathscr{O})\mathscr{R}\mathscr{B}\mathscr{G} = [(axya)^0 = (ax)^0a^0(ya)^0]$.

(ii) $L\mathscr{O}(\mathscr{R}\mathscr{B}\mathscr{G})CH\mathscr{A} = [(axya)^0u^0a^0v^0a = (ax)^0a^0(ya)^0v^0a^0u^0a]$ where $a = wuvw$.

(iii) $L\mathscr{O}(\mathscr{RBG})\mathscr{C} = [(axya)^0b^0a = (ax)^0a^0(ya)^0ab^0a^0]$
$= [b^0(axya)^0b = b(ax)^0a^0(ya)^0b^0]$.

(iv) $(L\mathscr{O})\mathscr{RBA} = [(axya)^0ba = (ax)^0a^0(ya)^0aba^0]$.

(v) $(\mathscr{RO})\mathscr{BG} = [a(xy)^0a = ax^0a^0y^0a] = [a^0x^0y^0a^0 = (axaya)^0]$
$= [axya = (axaya)^0a(xy^0x^0y)a(axaya)^0]$.

(vi) $\mathscr{R}e\mathscr{B} = [axya = axaya]$.

Proof: (i) $\mathsf{A} \subseteq \mathsf{B}$. This follows immediately from Theorems 5.4 (i) and 6.3 (i).

$\mathsf{B} \subseteq \mathsf{A}$. Let $S \in \mathsf{B}$ and $e, f, g \in E(S)$ satisfy $e \geq f$ and $e \geq g$. Then $(efge)^0 = (ef)^0e^0(ge)^0$ implies $fg \in E(S)$ and thus $\mathsf{B} \subseteq L\mathscr{O}$. Next,

$$(axa)^0 = (axa^0a)^0 = (ax)^0a^0(a^0a)^0 = (ax)^0a^0$$

and thus

$$(a(xa)ya)^0 = (axa)^0a^0(ya)^0 = (ax)^0a^0(ya)^0,$$

whence $(axya)^0 = (axaya)^0$. Therefore by Theorem 5.4 (i), we get $\mathscr{RBG}$.

(ii), (iii), (iv) This follows easily from Theorem 5.3 parts (ii), (iii), and (iv), respectively, and part (i) above.

(v) $\mathsf{A} = \mathsf{B}$. See [10, Theorem V. 5.3].

$\mathsf{A} \subseteq \mathsf{C}$. This follows from Theorems 5.4 (i) and 6.3 (v).

$\mathsf{C} \subseteq \mathsf{A}$. Let $S \in \mathsf{C}$ and $e, f \in E(S)$. For $e = a = x$ and $f = y$, by hypothesis, we have $efe = (efe)^0$. Therefore $efe = efe(efe)^0 = efefe$, whence $(efe)^2 = (efe)^3$, which implies $ef = (ef)^2$ and thus $S \in \mathscr{O}$. By the above references, we conclude that $S \in \mathscr{RBG}$.

$\mathsf{A} \subseteq \mathsf{D}$. By Theorems 5.4 (i) and 6.2 (v), we obtain

$$axya = (axya)^0axya(axya)^0 = (axaya)^0axy^0x^0ya(axaya)^0.$$

$\mathsf{D} \subseteq \mathsf{A}$. The given identity implies that $(axya)^0 \leq (axaya)^0$ and since $(axya)^0\mathscr{D}(axaya)^0$, it yields $(axya)^0 = (axaya)^0$, and Theorem 5.4 (i) gives $\mathscr{RBG}$. For $a = (xy)^0$, the given identity implies

$$xy = (xy)^0xy^0x^0y(xy)^0 = xy^0x^0y,$$

which by Theorem 6.2 (v) yields $\mathscr{O}$.

(vi) $\mathsf{A} \subseteq \mathsf{B}$. This follows directly from the definition of $\mathscr{R}e\mathscr{B}$.

$\mathsf{B} \subseteq \mathsf{A}$. For $x = y = a^0$, we get $a^2 = a^3$, whence $a = a^2$. □

For a different proof of $\mathsf{A} = \mathsf{B}$ in Theorem 6.4 (i), see [3, Lemma 6.3]. Observe the resemblance of the first basis in Theorem 6.3 (i) and the basis in Theorem 6.4 (i).

7. The interval $[\mathscr{T}, \mathscr{NBG}]$

This is a somewhat simpler subject than in the preceding two sections. It is more familiar in view of the better known structure thanks to the Rees theorem and strong semilattices thereof. Hence we may take a faster pace in the statements and proofs. Our next subject is the case of normal cryptogroups.

Theorem 7.1.

(i) $\mathscr{NBG} = [(axya)^0 = (ayxa)^0] = [(axyb)^0 = (ayxb)^0]$
$= [axa(aya)^0 = (aya)^0axa] = [a^0x(aya)^0 = (aya)^0xa^0]$.

(ii) $(\mathscr{NBG})CH\mathscr{A} = [ax^0a^0y^0a = ay^0a^0x^0a]$.

(iii) $(\mathscr{NBG})\mathscr{C} = [ax^0a^0ya = aya^0x^0a] = [b^0(axya)^0b = b(ayxa)^0b^0]$.

(iv) $\mathscr{NBA} = [axaya = ayaxa] = [axa^0ya = aya^0xa] = [axay^0a = ay^0axa]$.

(v) $\mathscr{O}(\mathscr{NBG}) = [axy^0a = ay^0xa] = [axy^0b = ay^0xb]$
$= [ax^0y^0a = ay^0x^0a] = [ax^0y^0b = ay^0x^0b]$
$= [a(xy)^0a = a(yx)^0a] = [a(xy)^0b = a(yx)^0b]$.

(vi) $\mathscr{O}(\mathscr{NBA}) = [axya = ayxa] = [axyb = ayxb]$.

(vii) $\mathscr{NB} = [axy^0a = ayxa] = [axy^0b = ayxb]$.

Proof: (i) $\mathsf{A} = \mathsf{B}$. See [10, Theorem IV. 1.6].

$\mathsf{A} \subseteq \mathsf{C}, \mathsf{D}, \mathsf{E}$. By the same reference, any semigroup in A is a strong semilattice of completely simple semigroups, which reduces the problem to Rees matrix semigroups. We omit the details.

$\mathsf{C} \subseteq \mathsf{B}$. This is trivial.

$\mathsf{D}, \mathsf{E} \subseteq \mathsf{A}$. In both cases, simple verification will show that they satisfy $\mathscr{D}$-majorization, which by the first cited reference shows that A holds.

(ii) See [2, Lemma 4.6].

(iii) $\mathsf{A} = \mathsf{B}$. See [2, Lemma 4.5].

$\mathsf{A} \subseteq \mathsf{C}$. This follows directly from Theorem 5.3 (iii) and part (i) above.

$\mathsf{C} \subseteq \mathsf{A}$. By the same reference for $b = a^0$ we get $\mathscr{NBG}$, and for $x = y = a$ we obtain $(\mathscr{BG})\mathscr{C}$.

(iv) $\mathsf{A} = \mathsf{B}$. See [2, Lemma 4.4].

$\mathsf{B} \subseteq \mathsf{C}, \mathsf{D}$. This is trivial.

$\mathsf{C}, \mathsf{D} \subseteq \mathsf{A}$. Let $S \in \mathsf{C}$ and $e, f, g \in E(S)$ satisfy $e \geq f$, $e \geq g$, and $f\mathscr{D}g$. Then $efege = egefe$ whence $fg = gf$ and thus $f = g$. By [10, Theorem IV.1.6], we get $S \in \mathscr{NBG}$. The given identity in a group implies commutativity.

(v) This follows in a straightforward manner using [10, Theorem IV. 2.7].

(vi) This is similar to part (v).

(vii) This is very easy. $\square$

For completely simple case, we have a spectrum of different bases as follows.

Theorem 7.2.

(i) $\mathscr{CS} = [a^0 = (aba)^0] = [a^0 = (ab)^0 a] = [(ab)^0 = (axb)^0]$.

(ii) $(\mathscr{CS})CH\mathscr{A} = [a^0 = ax^0a^0y^0a(ay^0a^0x^0a)^{-1}]$
$= [ax^0a^0y^0a = ay^0(aba)^0x^0a^0]$.

(iii) $(\mathscr{CS})\mathscr{C} = [a^0 = ab^0(a^0b^0a)^{-1}] = [a^0 = ax^0a^0ya(aya^0x^0a)^{-1}]$
$= [a = a^0b^0a(ab)^0(ba)^0] = [ab^0a^0 = a^0(bxb)^0a]$.

(iv) $(\mathscr{CS})H\mathscr{A} = [a^0 = ab(a^0ba)^{-1}] = [a^0 = axay(ayxa)^{-1}]$
$= [a = (ab)^0bab^{-1}(ba)^0] = [a^0ba = ab(axb)^0a^0]$.

(v) $\mathscr{R}e\mathscr{G} = [a^0 = a^0b^0a^0] = [ab = ax^0b]$.

(vi) $\mathscr{R}e\mathscr{A} = [a^0 = axya(ayxa)^{-1}] = [a = a(ab)^{-1}ba] = [axya = ayb^0xa]$.

(vii) $\mathscr{RB} = [a = aba] = [ab = axb]$.

Proof: (i) See [10, Proposition III. 1.1].

(ii) $\mathsf{A} = \mathsf{B}$. See [10, Proposition III. 6.7 (i)].

$\mathsf{A} \subseteq \mathsf{C}$. Let $S \in \mathsf{A}$. In view of [10, Proposition III. 6.1], we may set $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized and its entries commute. Let

$$a = (i, g, \lambda),\ x = (j, b, \mu),\ y = (k, t, \nu),\ b = (l, s, \theta) \in S$$

so that

$$ax^0a^0y^0a = (i, gp_{\lambda j}\, p_{\mu j}^{-1} p_{\mu i}\, p_{\lambda i}^{-1} p_{\lambda k}\, p_{\nu k}^{-1} p_{\nu i}\, g, \lambda), \tag{8}$$

$$ay^0a^0x^0a = (i, gp_{\lambda k}\, p_{\nu k}^{-1} p_{\nu i}\, p_{\lambda i}^{-1} p_{\lambda j}\, p_{\mu j}^{-1} p_{\mu i}\, g, \lambda). \tag{9}$$

Comparing (8) and (9), we conclude that these two elements are equal.

$\mathsf{C} \subseteq \mathsf{A}$. Since the given identities are heterotypical, C is a completely simple variety. Comparing (8) and (9), and setting $\lambda = i = 1$, we get $p_{\mu j}^{-1} p_{\nu k}^{-1} = p_{\nu k}^{-1} p_{\mu j}^{-1}$ whence $p_{\nu k} p_{\mu j} = p_{\mu j} p_{\nu k}$. It remains to apply [10, Proposition III. 6.1].

(iii) $\mathsf{A} = \mathsf{B}$. This follows from [10, Proposition III. 6.7 (ii)].

$\mathsf{A} = \mathsf{C}$. See [2, Lemma 3.5].

$\mathsf{A} \subseteq \mathsf{D}$. Let $S \in \mathsf{A}$. In view of [10, Proposition III. 6.2], we may set $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized and its entries lie in the center of G. For $a = (i, g, \lambda)$, $b = (j, h, \mu) \in S$, we have

$$a^0b^0a(ab)^0(ba)^0 = (i, p_{\lambda i}^{-1} p_{\lambda j}\, p_{\mu j}^{-1} p_{\mu i}\, gp_{\lambda i}\, p_{\mu i}^{-1} p_{\mu j}\, p_{\lambda j}^{-1}, \lambda) = (i, g, \lambda) = a. \tag{10}$$

$\mathsf{D} \subseteq \mathsf{A}$. Since the given identity is heterotypical, any $S \in \mathsf{D}$ is completely simple. By (10), we have

$$g = p_{\lambda i}^{-1} p_{\lambda j}\, p_{\mu j}^{-1} p_{\mu i}\, gp_{\lambda i}\, p_{\mu i}^{-1} p_{\mu j}\, p_{\lambda j}^{-1}$$

and setting $\mu = j = 1$, we get $g = p_{\lambda i}^{-1} gp_{\lambda i}$ whence $p_{\lambda i} g = gp_{\lambda i}$. Now apply [10, Proposition III. 6.2].

$\mathsf{A} = \mathsf{E}$. The argument required here is very similar to that in the proof of $\mathsf{A} \subseteq \mathsf{C}$ and $\mathsf{C} \subseteq \mathsf{A}$ in part (i) using [10, Proposition III. 6.2] and is somewhat simpler.

(iv) $\mathsf{A} = \mathsf{B}$. This follows from [10, Proposition III. 6.7 (iii)].

$\mathsf{A} = \mathsf{C}$. See [2, Lemma 3.4].

$\mathsf{A} = \mathsf{D}$. The argument here is similar to the proof of $\mathsf{A} = \mathsf{D}$ in part (iii) by using [10, Proposition III. 6.3].

$\mathsf{A} \subseteq \mathsf{E}$. Let $S \in \mathsf{A}$. In view of [10, Proposition III. 6.3], we may set $S = \mathcal{M}(I, G, \Lambda; P)$ where P is normalized and G is abelian. For

$$a = (i, g, \lambda),\ b = (j, h, \mu),\ x = (k, t, \nu) \in S,$$

we have

$$\begin{aligned} a^0ba &= (i, p_{\lambda i}^{-1} p_{\lambda j}\, h p_{\mu i}\, g, \lambda), \\ ab(axb)^0a^0 &= (i, g p_{\lambda j}\, h p_{\mu i}\, p_{\mu i}^{-1} p_{\mu i}\, p_{\lambda i}^{-1}, \lambda) \end{aligned}$$

and these two elements are equal.

$\mathsf{E} \subseteq \mathsf{A}$. The given identity is heterotypical, so the variety is completely simple. This identity in a group implies commutativity.

(v) This is well known.

(vi) $\mathsf{A} = \mathsf{B}$. See [2, Lemma 3.3].

$\mathsf{A} \subseteq \mathsf{C}$. Any rectangular band and any abelian group satisfy the identity in C.

$\mathsf{C} \subseteq \mathsf{A}$. Let $S \in \mathsf{C}$ and $e, f \in E(S)$. Then $e = e(ef)^{-1}fe = (ef)^{-1}e$ whence $ef = (ef)^{-1}ef = (ef)^0$ and $E(S)$ is a subsemigroup of S. But then $e = e(ef)e = efe$ and $E(S)$ is a rectangular band. In a group, the identity in C implies commutativity.

$\mathsf{A} = \mathsf{D}$. This is straightforward.

(viii) This is well known. □

Observe that parts (ii)–(iv) and (vi) of Theorem 7.2 are related to the same parts of Theorem 7.1, and also to Theorem 5.3 (iv).

We complete the list of bases with the following simple cases.

Theorem 7.3.

(i) $\mathscr{S}\mathscr{G} = [ab^0 = b^0a] = [a^0b^0 = b^0a^0] = [(ab)^0 = (ba)^0]$.

(ii) $\mathscr{S}\mathscr{A} = [ab = ba]$.

(iii) $\mathscr{S} = [ab^0 = ba]$.

(iv) $\mathscr{G} = [a^0 = b^0]$.

(v) $\mathscr{A} = [a = b^{-1}ab]$.

PROOF: (i) $\mathsf{A} \subseteq \mathsf{B} \subseteq \mathsf{C}$. This follows from [10, Theorem IV. 2.4].

$\mathsf{C} \subseteq \mathsf{D}$. This is straightforward.

$\mathsf{D} \subseteq \mathsf{A}$. If we write S as a semilattice Y of completely simple semigroups S_α, then the given identity immediately implies that all S_α are groups. Now apply the cited reference.

(ii)–(v) This is straightforward. □

Given an identity $u = v$ such that $a \in c(u) = c(v)$, $\{a\} \neq c(u)$, where $c(w)$ is the content of w, we can form the variety $[a = uv^{-1}]$ which is completely simple. Another way of forming a completely simple variety is by letting $b \notin c(u)$ and inserting b into the word v thereby obtaining a word v'. Then $[u = v']$ is a completely simple variety. In order to retain some properties of the variety $[u = v]$, we may require that restricted to $\mathcal{L}(\mathscr{CS})$, we have $v = v'$. The first method introduces the superscript " $^{-1}$ ", the second, one more variable. We have examples of both in Theorem 7.2.

8. Comments and bases

Here we discuss two phenomena that came to light in the material dealt with so far.

1. The first is related to Theorem 5.1 (iv) and amounts to a curious property of any $\mathscr{H}$-class of a completely regular semigroup, which even though quite elementary, has escaped notice so far.

The second basis of the variety in Theorem 5.1 (iv) can be written as

$$(aba)(ababa) = (ababa)(aba).$$

The reason for this identity forming a basis of $H\mathscr{A}$ is a consequence of the following result which seems to be of general interest.

Lemma 8.1. *Let $S \in \mathscr{CR}$ and $a, b \in S$. Then $a\mathscr{H}b$ if and only if $a = xyx$ and $b = xyxyx$ for some $x, y \in S$.*

Proof: Let $a\mathscr{H}b$, $x = ab^{-1}a$ and $y = a^{-1}ba^{-1}ba^{-1}$. Then

$$xyx = (ab^{-1}a)a^{-1}ba^{-1}ba^{-1}(ab^{-1}a) = a,$$
$$xyxyx = ayx = a(a^{-1}ba^{-1}ba^{-1})ab^{-1}a = b.$$

Conversely, we obtain

$$xyx = (xy)^{-1}xyxyx = xyxyx(xy)^{-1},$$
$$xyxyx = (xy)xyx = xyx(yx)$$

and thus $a\mathscr{H}b$. □

2. The second item deals with bases consisting of a single identity of the form $uv = vu$ for some words u and v. Next we extract from Section 5 several examples.

For brevity, we write $u \rightleftarrows v$ to indicate the case that u and v commute, that is, $uv = vu$. An identity of the form $u^0 = v^0$ where u and v contain the same set of variables, is equivalent to $u^0v^0 = v^0u^0$. We omit such bases. For the remaining cases, we state

(a) the acronym,
(b) the pair of words,
(c) the theorem, its part, and the letter which indicates its order within that part of the theorem.

Variety	Basis	Theorem
$H\mathscr{A}$	$aba \rightleftarrows ababa$	5.1 (iv) C
$\mathscr{B}\mathscr{G}$	$a \rightleftarrows (aba)^0$	5.3 (i) F
$(\mathscr{B}\mathscr{G})\mathscr{C}$	$a \rightleftarrows a^0b^0a^0$	5.3 (iii) B
$\mathscr{B}\mathscr{A}$	$a \rightleftarrows a^0ba^0$	5.3 (iv) B
$\mathscr{N}\mathscr{B}\mathscr{G}$	$axa \rightleftarrows (aya)^0$	7.1 (i) D
$\mathscr{N}\mathscr{B}\mathscr{G}$	$a^0xa^0 \rightleftarrows (aya)^0$	7.1 (i) E
$(\mathscr{N}\mathscr{B}\mathscr{G})CH\mathscr{A}$	$a^0x^0a^0 \rightleftarrows a^0y^0a^0$	7.1 (ii) B
$(\mathscr{N}\mathscr{B}\mathscr{G})\mathscr{C}$	$a^0x^0a^0 \rightleftarrows a^0ya^0$	7.1 (iii) B
$\mathscr{N}\mathscr{B}\mathscr{A}$	$a^0xa^0 \rightleftarrows a^0ya^0$	7.1 (iv) C
$\mathscr{S}\mathscr{G}$	$a \rightleftarrows b^0$	7.3 (i) B
$\mathscr{S}\mathscr{G}$	$a^0 \rightleftarrows b^0$	7.3 (i) C
$\mathscr{S}\mathscr{A}$	$a \rightleftarrows b$	7.3 (ii) B

It is an interesting query: what do all these varieties have in common? A much more difficult question would be to characterize varieties with such a type of basis.

9. L-classes of varieties in Γ

Recall from [9] that the **L**-relation on $\mathcal{L}(\mathscr{C}\mathscr{R})$ is defined by

$$\mathscr{U}\,\mathbf{L}\,\mathscr{V} \quad \Longleftrightarrow \quad \mathscr{U} \cap \mathscr{M} = \mathscr{V} \cap \mathscr{M},$$

where $\mathscr{M}$ stands for the class of all completely regular monoids. Clearly **L** is an equivalence relation all of whose classes are intervals. Hence for any $\mathscr{V} \in \mathcal{L}(\mathscr{C}\mathscr{R})$, we may denote its **L**-class by $\mathscr{V}\mathbf{L} = [\mathscr{V}_L, \mathscr{V}^L]$. Recall that $L\mathscr{O}$ is the variety of locally orthogroups, which may now be written as $\mathscr{O}^L$, but we will still mostly use the notation $L\mathscr{O}$ for it.

As an application of the bases we found in Sections 5–7, we present in this section the computation of **L**-classes of all varieties in the set Γ. Why this is now possible will be clear from the next result.

Fact 9.1. *Let* $\mathscr{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathcal{L}(\mathscr{CR})$.

(i) *A basis for* $\mathscr{V}_L$ *is obtained as follows. For each* $\alpha \in A$, *delete all variables in a proper (possibly empty) subset of variables for all choices.*

(ii) $\mathscr{V}^L = [u_\alpha(x^0 x_i x^0) = v_\alpha(x^0 x_i x^0)]_{\alpha \in A}$ *where* $x \notin \bigcup_{\alpha \in A} c(u_\alpha v_\alpha)$.

(iii) *The mapping* $\mathscr{V} \to \mathscr{V}_L$ $(\mathscr{V} \in \mathcal{L}(\mathscr{CR}))$ *is a complete endomorphism of* $\mathcal{L}(\mathscr{CR})$.

(iv) *The mapping* $\mathscr{V} \to \mathscr{V}^L$ $(\mathscr{V} \in \mathcal{L}(\mathscr{CR}))$ *is a complete* $\bigcap$*-endomorphism of* $\mathcal{L}(\mathscr{CR})$.

Proof: (i) See [9, Proposition 5.5] which deals with the varieties in $[\mathscr{S}, \mathscr{CR}]$. It is easy to check that for every $\mathscr{V} \in \mathcal{L}(\mathscr{CS})$, we have $\mathscr{V}_L = \mathscr{V} \cap \mathscr{G}$, and is thus generated by submonoids of $\mathscr{V}$, so that $\mathscr{V}_L$ has the required form.

(ii) This is one of the two choices for a basis of $\mathscr{V}^L$ in [10, Proposition II. 7.3 (iii)].

(iii) See [9, Theorem 5.3] or [10, Proposition IX. 8.5].

(iv) See [10, Proposition II. 7.3 (i)]. □

Fact 9.1 parts (iii) and (iv) will be very useful when we consider meets of varieties. For $\mathscr{U}, \mathscr{V} \in \Gamma$, where $\mathscr{U} \subseteq \mathscr{V}$, the notation $[\mathscr{U}, \mathscr{V}]$ will have two meanings, namely in Γ and in $\mathcal{L}(\mathscr{CR})$ since $\mathscr{U}, \mathscr{V} \in \mathcal{L}(\mathscr{CR})$. It should be obvious from the context which meaning is assigned to it in various contexts. For example, $[\mathscr{U}, \mathscr{V}]$ has in Γ only a finite number of elements, while $[\mathscr{U}, \mathscr{V}] \subseteq \mathcal{L}(\mathscr{CR})$ generally will not. Occasionally we will add $\bigcap \Gamma$ when we refer to the first meaning. In the second case, we will sometimes assert that $[\mathscr{U}, \mathscr{V}]$ is an **L**-class qua sublattice of $\mathcal{L}(\mathscr{CR})$.

The purpose of this section is to identify the set of **L**-classes of varieties in Γ. The result is incomplete, for we are unable to identify $\mathscr{V}^L$ for a few $\mathscr{V}$ even though Fact 9.1 (ii) provides a basis for it. It is definitely illuminating to follow where these varieties are in Diagram 1.

We begin by providing three lemmas.

Lemma 9.2. *If* $\mathscr{V} \in [\mathscr{RBA}, \mathscr{CR}] \cap \Gamma$, *then* $\mathscr{V}_L = \mathscr{V}$.

Proof: In view of Fact 9.1 parts (iii) and (iv), it suffices to consider the varieties

$$H\mathscr{A},\ \mathscr{C},\ CH\mathscr{A},\ \mathscr{CR},\ T\mathscr{O},\ \mathscr{BG},\ \mathscr{RBG}.$$

In order to use Fact 9.1 (i), we must refer to Section 5 for a possibly simple basis for each of these varieties.

$H\mathscr{A}$: in any basis in Theorem 5.1 (iv), if we drop one variable, we get a trivial identity. Hence $(H\mathscr{A})_L = H\mathscr{A}$.

$\mathscr{C}$: ditto in Theorem 5.1 (iii).

$CH\mathscr{A}$: the first basis in Theorem 5.1 (ii) has three variables. Hence we must delete up to two variables each time.

Omit x: $a = wyw$ and $ay^0a = ay^0a$,

omit y: this is dual,

omit w: $a = xy$ whence

$$xyx^0(xy)^0y^0xy = xyy^0(xy)^0x^0xy$$

which holds trivially,

omit x, y: trivial,

omit w, x: $a = y$, $yy^0y^0y = yy^0y^0y$ which is trivial,

omit w, y: this is dual,

$\mathscr{CR} = [x = x]$: this follows by default,

$T\mathscr{O}$: has only one basis in Theorem 5.2 (i),

omit x: $(a^0b^0y)^0 = ((a^0b^0)^0y)^0$ can be obtained by setting $x = a^0$ in the basis for $T\mathscr{O}$,

omit y: this is dual,

omit a: $(xb^0y)^0 = (xb^0y)^0$ which is trivial,

omit b: this is dual,

omit x, y: $(a^0b^0)^0 = (a^0b^0)^0$ which is trivial,

omit x, a: $(b^0y)^0 = (b^0y)^0$ which is trivial,

omit x, b or y, a or y, b: this is similar,

$\mathscr{BG}$: all but two bases in Theorem 5.3 (i) have two variables, so any omission results in triviality,

$\mathscr{RBG}$: from the first basis in Theorem 5.4 (i), we get

omit a: trivial,

$$\left.\begin{array}{l}\text{omit } x\text{: } (aya)^0 = (a^2ya)^0 \\ \text{omit } y\text{: } (axa)^0 = (axa^2)^0\end{array}\right\} \text{these two together imply } \mathscr{BG} \text{ and } \mathscr{RBG} \subseteq \mathscr{BG},$$

omit a, x: $(ya)^0 = (ya)^0$ which is trivial,

omit a, y: this is dual,

omit x, y: $(a^2)^0 = (a^3)^0$ which is trivial. □

Lemma 9.3. *If $\mathscr{V} \in \{H\mathscr{A}, \mathscr{BG}, \mathscr{BA}, \mathscr{B}, \mathscr{NBG}, \mathscr{CS}, \mathscr{CR}\}$, then $\mathscr{V}^L = \mathscr{V}$.*

Proof: We will use Fact 9.1 (ii) several times.

$H\mathscr{A}$: $(H\mathscr{A})^L = [x^0ax^0bx^0(x^0bx^0ax^0)^0 = (x^0ax^0bx^0)^0x^0bx^0ax^0]$. In a group, this identity implies commutativity. Hence $(H\mathscr{A})^L \subseteq H\mathscr{A}$ and equality prevails.

$\mathscr{BG}$: The equality $(\mathscr{BG})^L = \mathscr{BG}$ follows from [10, Proposition II. 8.4].

$\mathscr{BA}$: Fact 9.1 (iv) yields

$$(\mathscr{BA})^L = (\mathscr{BG} \cap H\mathscr{A})^L = \mathscr{BG}^L \cap (H\mathscr{A})^L = \mathscr{BG} \cap H\mathscr{A} = \mathscr{BA}.$$

$\mathscr{B}$: By the first reference, we have

$$\mathscr{B}^L = [a = a^0]^L = [x^0 a x^0 = (x^0 a x^0)^0]$$

and for $x \to a$, we get $\mathscr{B}^L = \mathscr{B}$.

$\mathscr{NBG}$: By the equivalence of parts (i) and (ii) in [10, Theorem IV. 1.6], we have $\mathscr{SG}^L = \mathscr{NBG}$ whence $\mathscr{NBG}^L = \mathscr{NBG}$.

$\mathscr{CS}$: The formula in Fact 9.1 (ii) for $\mathscr{V}^L$ shows that if $\mathscr{V}$ is heterotypical, then so is $\mathscr{V}^L$; this evidently implies that $\mathscr{CS}^L = \mathscr{CS}$.

$\mathscr{CR}$: The equality $\mathscr{CR}^L = \mathscr{CR}$ is trivial. □

In the next lemma, we will use the notation in [4, Section 2]. Let

$$\mathcal{R} = \mathscr{I}_3 \cap \overline{\mathscr{I}_3} \qquad \text{and} \qquad \mathcal{Z} = \mathscr{H}_4 \cap \overline{\mathscr{H}_4}.$$

Lemma 9.4.

(i) $\mathscr{RBG} = \mathcal{R} \cap \mathscr{BG}$.

(ii) $[\mathcal{R}, \mathcal{Z}]$ *is an* **L**-*class.*

(iii) $\mathscr{RBG}^L = \mathcal{Z}\mathscr{BG}$ and $\mathscr{RBA}^L = \mathcal{Z}\mathscr{BA}$.

(iv) $\mathscr{SG}^L = \mathscr{NBG}$ and $\mathscr{SA}^L = \mathscr{NBA}$.

(v) $\mathscr{G}^L = \mathscr{CS}$ and $\mathscr{A}^L = (\mathscr{CS})H\mathscr{A}$.

Proof: (i) By [4, Theorem 5.1], we have $\mathcal{R} = [(axya)^0 = (axa^0ya)^0]$, and thus by Theorem 5.4 (i), we get $\mathcal{R} \cap \mathscr{BG} \subseteq \mathscr{RBG}$. Again by these references, we obtain $\mathscr{RBG} \subseteq \mathcal{R}$ and trivially $\mathscr{RBG} \subseteq \mathscr{BG}$.

(ii) This follows from [5, Theorem 12.3]. (In this paper, on mid page 93, Figure 2 refers to Diagram 2 on the (wrong) page 68.)

(iii) We can now write

$$\mathscr{RBG}^L = (\mathcal{R} \cap \mathscr{BG})^L = \mathcal{R}^L \cap \mathscr{BG}^L = \mathcal{Z}\mathscr{BG},$$
$$\mathscr{RBA}^L = (\mathscr{RBG} \cap H\mathscr{A})^L = \mathscr{RBG}^L \cap (H\mathscr{A})^L = \mathcal{Z}\mathscr{BG}(H\mathscr{A}) = \mathcal{Z}\mathscr{BA}.$$

(iv) The first part follows directly from [10, Theorem IV. 1.6]. Next

$$\mathscr{SA}^L = (\mathscr{SG} \cap H\mathscr{A})^L = \mathscr{NBG} \cap (H\mathscr{A})^L = \mathscr{NBA}.$$

(v) This follows easily from part (iv) and [10, Corollary IV. 1.11]. □

We are now ready for the only theorem in this section.

Theorem 9.5.

(i) *The following intervals are* **L**-*classes.*
1. $[\mathscr{T},\mathscr{RB}]$, $[\mathscr{A},(\mathscr{CS})H\mathscr{A}]$, $[\mathscr{G},\mathscr{CS}]$,
2. $[\mathscr{S},\mathscr{NB}]$, $[\mathscr{SA},\mathscr{NBA}]$, $[\mathscr{SG},\mathscr{NBG}]$,
3. $[\mathscr{ReB},\mathscr{ZB}]$, $[\mathscr{O}(\mathscr{RBA}),(L\mathscr{O})\mathscr{ZBA}]$, $[\mathscr{O}(\mathscr{RBG}),(L\mathscr{O})\mathscr{ZBG}]$,
   $[\mathscr{RBA},\mathscr{ZBA}]$, $[\mathscr{RBG},\mathscr{ZBG}]$,
   $[\mathscr{OBA},(L\mathscr{O})\mathscr{BA}]$, $[\mathscr{OBG},(L\mathscr{O})\mathscr{BG}]$,
   $[\mathscr{O}(H\mathscr{A}),(L\mathscr{O})H\mathscr{A}]$, $[\mathscr{O},L\mathscr{O}]$,
4. $\{\mathscr{B}\}$, $\{\mathscr{BA}\}$, $\{\mathscr{BG}\}$, $\{H\mathscr{A}\}$, $\{\mathscr{CR}\}$.

(ii) *If $\mathscr{V}$ is any of the remaining varieties in $[\mathscr{RBA},\mathscr{CR})\cap\Gamma$, then $\mathscr{V}_L=\mathscr{V}$, and its $\mathscr{V}^L$ can be evaluated using Fact* 9.1 *parts* (ii) *and* (iv).

(iii) *Parts* (i) *and* (ii) *saturate* Γ.

(iv) *The* **L**-*classes in part* (i) *which are not contained in* Γ *are:*

$$[\mathscr{ReB},\mathscr{ZB}]\cap\Gamma=\{\mathscr{ReB}\},$$
$$[\mathscr{O}(\mathscr{RBA}),(L\mathscr{O})\mathscr{ZBA}]\cap\Gamma=[\mathscr{O}(\mathscr{RBA}),(L\mathscr{O})\mathscr{RBA}],$$
$$[\mathscr{O}(\mathscr{RBG}),(L\mathscr{O})\mathscr{ZBG}]\cap\Gamma=[\mathscr{O}(\mathscr{RBG}),(L\mathscr{O})\mathscr{RBG}],$$
$$[\mathscr{RBA},\mathscr{ZBA}]\cap\Gamma=\{\mathscr{RBA}\},$$
$$[\mathscr{RBG},\mathscr{ZBG}]\cap\Gamma=\{\mathscr{RBG}\}.$$

Proof: (i) We will freely use Fact 9.1 and Lemmas 9.2–9.4 without specific reference. For each interval $[\mathscr{U},\mathscr{V}]$ we will prove that $\mathscr{U}^L=\mathscr{V}$ and $\mathscr{V}_L=\mathscr{U}$. The argument proceeds by numbered line in the statement of the theorem and the order within the line.

(1, 1): $\mathscr{T}^L=\mathscr{RB}$, $\mathscr{RB}_L=\mathscr{T}$, this is straightforward;
(1, 3): $\mathscr{G}^L=\mathscr{CS}$, $\mathscr{CS}_L=\mathscr{G}$, by the Rees theorem;
(1, 2): $\mathscr{A}^L=(\mathscr{G}\cap H\mathscr{A})^L=\mathscr{G}^L\cap(H\mathscr{A})^L=(\mathscr{CS})H\mathscr{A}$,
$((\mathscr{CS})H\mathscr{A})_L=\mathscr{CS}_L\cap(H\mathscr{A})_L=\mathscr{G}(H\mathscr{A})=\mathscr{A}$;
(2, 1): $\mathscr{S}^L=\mathscr{NB}$, $\mathscr{NB}_L=\mathscr{S}$;
(2, 3): $[\mathscr{SG},\mathscr{NBG}]$ is an **L**-class by [10, Theorem IV. 1.6];
(2, 2): $\mathscr{SA}^L=(\mathscr{SG}\cap H\mathscr{A})^L=\mathscr{SG}^L\cap(H\mathscr{A})^L=\mathscr{NBG}\cap H\mathscr{A}=\mathscr{NBA}$,
$\mathscr{NBA}=(\mathscr{NBG}\cap H\mathscr{A})_L=\mathscr{NBG}_L\cap(H\mathscr{A})_L=\mathscr{SG}\cap H\mathscr{A}=\mathscr{SA}$;
(3, 1): $\mathscr{ReB}^L=(\mathcal{R}\cap\mathscr{B})^L=\mathcal{R}^L\cap\mathscr{B}^L=\mathcal{Z}\mathscr{B}$,
$(\mathcal{Z}\mathscr{B})_L=(\mathcal{Z}\cap\mathscr{B})_L=\mathcal{Z}_L\cap\mathscr{B}_L=\mathcal{R}\cap\mathscr{B}=\mathscr{ReB}$.

For example, $\mathscr{BG}^L=\mathscr{BG}$ by [10, Proposition II. 8.4] and by Theorem 5.1 (iv), $H\mathscr{A}$ has a basis with only two variables whence $(H\mathscr{A})_L=H\mathscr{A}$ (also used above).

The remaining cases follow either similarly or are easy.

(ii) This forms part of Lemma 9.2.

(iii) This requires simple inspection of Diagram 2.

(iv) This requires straightforward inspection and checking. □

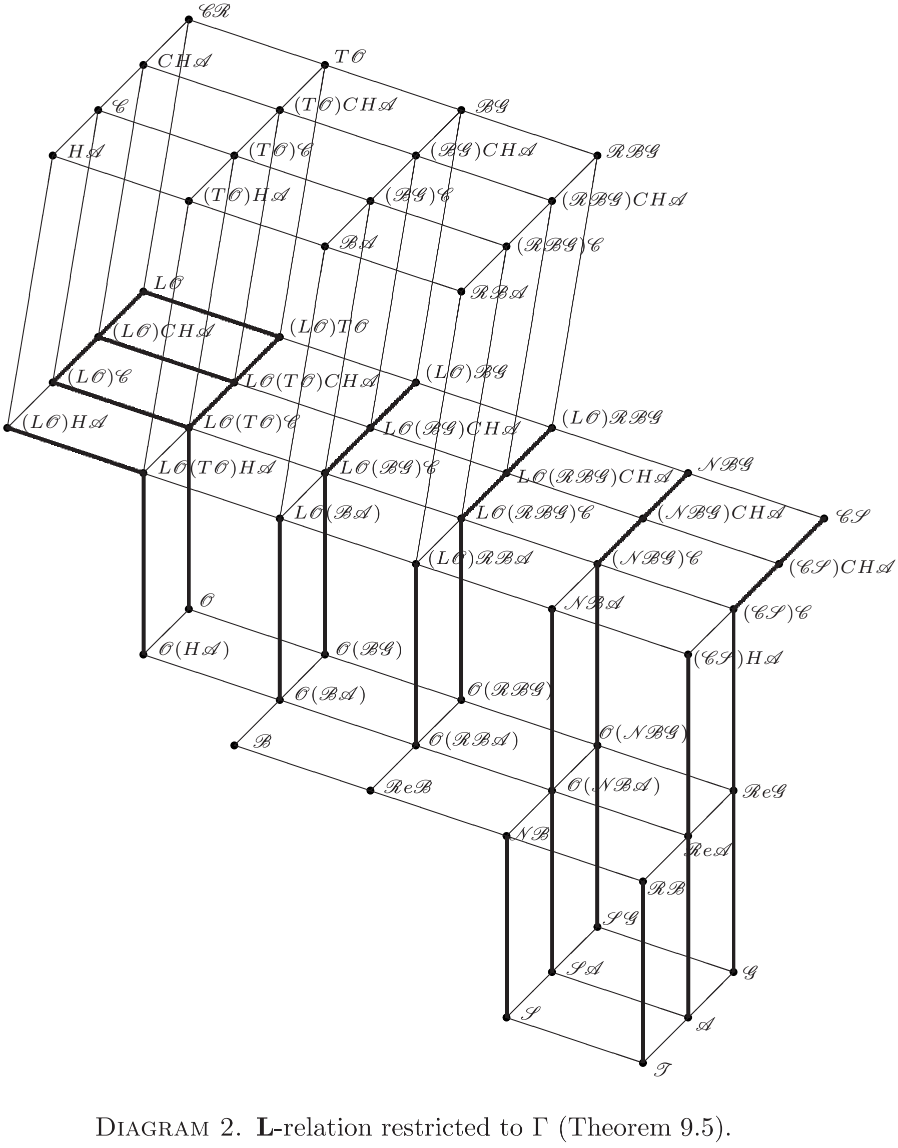

DIAGRAM 2. **L**-relation restricted to Γ (Theorem 9.5).

This section takes care of **L**-classes of varieties in Γ. The kernel, trace, core, $\mathbf{B}^{\wedge}$ and $\mathbf{B}^{\vee}$-classes are relegated to a separate publication. See [8] and [9] for generalities.

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