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MAXIMUM NUMBER OF LIMIT CYCLES FOR GENERALIZED LIÉNARD POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We consider limit cycles of a class of polynomial differential systems of the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(g_{21}(x)y^{2\alpha+1} + f_{21}(x)y^{2\beta}) - \varepsilon^2(g_{22}(x)y^{2\alpha+1} + f_{22}(x)y^{2\beta}), \end{cases}$$

where β and α are positive integers, g_{2j} and f_{2j} have degree m and n, respectively, for each $j = 1, 2$, and ε is a small parameter. We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first and second order.

Keywords: polynomial differential system; limit cycle; averaging theory

MSC 2020: 34C07, 34C23, 37G15

1. Introduction

One of the main problems in the qualitative theory of real planar differential equations is to determine the number of limit cycles for a given planar differential system. As we all know, this is a very difficult problem for a general polynomial system. Therefore, many mathematicians study some systems with special conditions. To obtain as many limit cycles as possible for a planar differential system, we usually take into consideration the bifurcation theory. In recent decades, many new results have been obtained (see [9], [10]).

The number of medium amplitude limit cycles bifurcating from the linear center $\dot{x} = y$, $\dot{y} = -x$ for the following three kind of generalized polynomial Liénard

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differential systems, were studied in the papers [2], [4], [5], [6], [1] and [14], [15], respectively:

$$\begin{aligned} &\dot{x} = y, && \dot{y} = -x - g_2(x) + f_2(x)y, \\ &\dot{x} = y - g_1(x), && \dot{y} = -x - g_2(x) + f_2(x)y, \\ &\dot{x} = y - f_1(x)y, && \dot{y} = -x - g_2(x) + f_2(x)y. \end{aligned}$$

In [13], Llibre and Valls studied the polynomial differential systems

$$\begin{cases} \dot{x} = y - \varepsilon(g_{11}(x) + f_{11}(x)y) - \varepsilon^2(g_{12}(x) + f_{12}(x)y), \\ \dot{y} = -x - \varepsilon(g_{21}(x) + f_{21}(x)y) - \varepsilon^2(g_{22}(x) + f_{22}(x)y), \end{cases} \tag{1.1}$$

where g_{1i}, f_{1i}, g_{2i}, f_{2i} have degree l, k, m, n, respectively, for each $i = 1, 2$, and ε is a small parameter. They proved an sharp upper bound of the maximum number of limit cycle that (1.1) can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order.

In 2014, Garca, Llibre and Pérez del Río (see [7]) using the averaging theory studied the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential system of the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(h_1(x) + p_1(x)y + q_1(x)y^2) - \varepsilon^2(h_2(x) + p_2(x)y + q_2(x)y^2), \end{cases} \tag{1.2}$$

where h_1, h_2, p_1, q_1, p_2 and q_2 have degree n and ε is a small parameter. More precisely, they found the maximum number of medium amplitude limit cycles which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ perturbed as in (1.2).

In [11], the authors proved that the maximum number of limit cycles of the following generalized Liénard polynomial differential system

$$\begin{cases} \dot{x} = y^{2p-1}, \\ \dot{y} = -x^{2q-1} - \varepsilon f(x) y^{2n-1} \end{cases}$$

is at most $[\frac{1}{2}m]$, where p, q and n are positive integers, m is the degree of the polynomial $f(x)$.

In this paper, first we consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(g_{21}(x)y^{2\alpha+1} + f_{21}\,(x)\,y^{2\beta}), \end{cases} \tag{1.3}$$

where β and α are positive integers, g_{21} and f_{21} have degree m and n, respectively, and ε is a small parameter. We find the maximum number of limit cycle that (1.3) can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first order.

Let $[x]$ denote the integer part function of $x \in \mathbb{R}$. Our main result is the following one.

Theorem 1. *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial differential systems* (1.3) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first order is $[\frac{1}{2}m]$.*

The proof of the above theorem is given in Section 3.

Now we consider the system

$$\tag{1.4} \begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon(g_{21}(x)y^{2\alpha+1} + f_{21}(x)y^{2\beta}) - \varepsilon^2(g_{22}(x)y^{2\alpha+1} + f_{22}(x)y^{2\beta}), \end{cases}$$

where β and α are positive integers, g_{2j} and f_{2j} have degree m and n, respectively, for each $j = 1, 2$, and ε is a small parameter. We find the maximum number of limit cycle that (1.4) can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order. Our main result is the following one.

Theorem 2. *For $|\varepsilon|$ sufficiently small and $[\frac{1}{2}m] \geqslant \beta - 1$, the maximum number of limit cycles of the polynomial differential systems* (1.4) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order is*

$$\lambda = \max\Big\{ \Big[\frac{m}{2}\Big], \Big[\frac{n}{2}\Big] + \Big[\frac{m-1}{2}\Big] + \beta \Big\}.$$

The proof of the above theorem is given in Section 4.

2. Preliminaries

The averaging theory of first and second orders. In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper. The averaging theory up to second order for studying specifically periodic orbits was developed in [13], [3], [12]. It is summarized as follows.

Consider the differential system

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

where $F_1, F_2\colon \mathbb{R}\times D\to\mathbb{R}$, $R\colon \mathbb{R}\times D\times(-\varepsilon_f,\varepsilon_f)\to\mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses hold.

(i) $F_1(t,\cdot)\in C^2(D)$, $F_2(t,\cdot)\in C^1(D)$ for all $t\in\mathbb{R}$, F_1, F_2, R are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0}\colon D\to\mathbb{R}$ for $k=1,2$ as

$$F_{10}(x)=\frac{1}{T}\int_0^T F_1(s,x)\,\mathrm{d}s,$$
$$F_{20}(x)=\frac{1}{T}\int_0^T\left((D_xF_1(s,x))\int_0^s F_1(t,x)\,\mathrm{d}t+F_2(s,x)\right)\mathrm{d}s.$$

(ii) For $V\subset D$ an open and bounded set and for each $\varepsilon\in(-\varepsilon f,\varepsilon f)\setminus\{0\}$, there exists $a_\varepsilon\in V$ such that $F_{10}(a_\varepsilon)+\varepsilon F_{20}(a_\varepsilon)=0$ and $d_B(F_{10}+\varepsilon F_{20},V,a_\varepsilon)\neq 0$.

Then for $|\varepsilon|>0$ sufficiently small there exists a T-periodic solution $\varphi(\cdot,\varepsilon)$ of the system such that $\varphi(0,\varepsilon)\to a_\varepsilon$ when $\varepsilon\to 0$.

The expression $d_B(F_{10}+\varepsilon F_{20},V,a_\varepsilon)\neq 0$ means that the Brouwer degree of the function $F_{10}+\varepsilon F_{20}\colon V\to\mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for this inequality to hold is that the Jacobian of the function $F_{10}+\varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the averaging theory of first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10}+\varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the averaging theory of second order.

Descartes theorem. In order to confirm the number of zeros of certain real polynomial, we will make use of the following Descartes theorem (see [11]).

Theorem 3. *Consider the real polynomial $p(x)=a_{i_1}x^{i_1}+a_{i_2}x^{i_2}+\ldots+a_{i_k}x^{i_k}$ with $0\leqslant i_1<i_2<\ldots<i_k$ and $a_{i_j}\neq 0$ real constants for $j\in\{1,2,\ldots,k\}$. When $a_{i_j}a_{i_{j+1}}<0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $k-1$ positive real roots.*

3. Proof of Theorem 1

For the proof we shall use the first order averaging theory as it was stated in Section 2. We write

$$g_{21}(x) = \sum_{i=0}^{m} c_i x^i, \quad f_{21}(x) = \sum_{i=0}^{n} d_i x^i. \tag{3.1}$$

Then in polar coordinates (r, θ) given by $x = r\cos\theta$ and $y = r\sin\theta$, the differential system (1.4) becomes

$$\begin{cases} \dot{r} = -\varepsilon G_1(r,\theta), \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r} G_2(r,\theta), \end{cases}$$

where

$$\begin{aligned} G_1(r,\theta) &= \sum_{i=0}^{n} d_i h_{i,2\beta+1}(\theta) r^{2\beta+i} + \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1}, \\ G_2(r,\theta) &= \sum_{i=0}^{n} d_i h_{i+1,2\beta}(\theta) r^{2\beta+i} + \sum_{i=0}^{m} c_i h_{i+1,2\alpha+1}(\theta) r^{2\alpha+i+1}, \end{aligned}$$

where $h_{i,j}(\theta) = \cos^i\theta \sin^j\theta$. Taking θ as the new independent variable, system (1.4) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \varepsilon F_1(r,\theta) + O(\varepsilon^2), \tag{3.2}$$

where

$$F_1(r,\theta) = G_1(r,\theta). \tag{3.3}$$

First, we shall study the limit cycles of the differential equation (3.2) using the averaging theory of first order. Therefore, by Section 2 we must study the simple positive zeros of the function $F_{10}(r) = \frac{1}{2}\pi \int_0^{2\pi} F_1(r,\theta)\,\mathrm{d}\theta$. For each of these zeros we will have a limit cycle of the polynomial differential system (1.3).

Taking into account the expression of (3.3), in order to obtain $F_{10}(r)$ it is necessary to evaluate the integrals of the form $\int_0^{2\pi} h_{i,j}(\theta)\,\mathrm{d}\theta$, where $h_{i,j}(\theta) = \cos^i\theta\sin^j\theta$.

In the following lemma we compute these integrals.

Lemma 4. *Let $h_{i,j}(\theta) = \cos^i\theta\sin^j\theta$ and $M_{i,j}(\theta) = \int_0^{\theta} h_{i,j}(s)\,\mathrm{d}s$. Then*

$$M_{i,j}(2\pi) = \begin{cases} 0 & \text{if } i \text{ is odd or } j \text{ is odd}, \\ \xi_{i,j}\pi & \text{if } i \text{ and } j \text{ are even}, \end{cases} \tag{3.4}$$

where

$$\xi_{i,j}=\frac{(j-1)(j-3)\dots 1}{(j+i)(j+i-2)\dots(i+2)}\frac{1}{2^{i-1}}\binom{i}{\frac{1}{2}i}\quad \text{and}\quad \binom{i}{\frac{1}{2}i}=\frac{i!}{(\frac{1}{2}i!)^2}.$$

Proof. Using integrals (5.3) and (5.4) given in the Appendix with $\theta=2\pi$ and taking into account that $h_{i,j}(2\pi)=0$ if $j\neq 0$, we have that

$$M_{i,2\alpha}(2\pi)=\frac{(2\alpha-1)(2\alpha-3)\dots 1}{(2\alpha+i)(2\alpha+i-2)\dots(i+2)}M_{i,0}(2\pi),\quad M_{i,2\alpha+1}(2\pi)=0. \tag{3.5}$$

Again, using integrals (5.1) and (5.2) given in the Appendix, with $\theta=2\pi$ we have that $M_{2i,0}(2\pi)=2\pi(2i-1)(2i-3)\dots 1/(2^i i!)$ and $M_{2i+1,0}(2\pi)=0$. Substituting $M_{2i,0}(2\pi)$ and $M_{2i+1,0}(2\pi)$ given as above into (3.5) we obtain (3.4). □

Using this lemma we shall obtain in the next proposition the integral of the function $F_{10}(r)$.

Proposition 5. *We have*

$$2\pi F_{10}(r)=r^{2\alpha+1}\sum_{i=0}^{[m/2]}c_{2i}M_{2i,2\alpha+2}(2\pi)r^{2i}. \tag{3.6}$$

Proof. Since

$$2\pi F_{10}(r)=\sum_{i=0}^{n}d_i r^{2\beta+i}\int_0^{2\pi}h_{i,2\beta+1}(\theta)+\sum_{i=0}^{m}c_i r^{2\alpha+i+1}\int_0^{2\pi}h_{i,2\alpha+2}(\theta)\,\mathrm{d}\theta,$$

taking into account that $\int_0^{2\pi}h_{i,2\alpha+2}(\theta)\,\mathrm{d}\theta=0$ if i is odd and $\int_0^{2\pi}h_{i,2\beta+1}(\theta)\,\mathrm{d}\theta=0$, for all $i,\beta\in\mathbb{N}$ (see Lemma 4), we have that

$$\begin{aligned}2\pi F_{10}(r)&=\int_0^{2\pi}\sum_{\substack{i=0\\ i\text{ even}}}^{m}c_i h_{i,2\alpha+2}(\theta)\,\mathrm{d}\theta r^{2\alpha+i+1}=\sum_{i=0}^{[m/2]}r^{2i+2\alpha+1}\int_0^{2\pi}c_{2i}h_{2i,2\alpha+2}(\theta)\,\mathrm{d}\theta\\&=r^{2\alpha+1}\sum_{i=0}^{[m/2]}c_{2i}M_{2i,2\alpha+2}(2\pi)r^{2i}.\end{aligned}$$

This completes the proof of Proposition 5. □

Proof of Theorem 1. From Proposition 5, the polynomial $F_{10}(r)$ has at most $\lambda_1=\{[\frac{1}{2}m]\}$ positive roots, and we can choose c_{2i} in a way that $F_{10}(r)$ has exactly λ_1 simple positive roots, hence Theorem 1 is proved. □

4. Proof of Theorem 2

Now using the results stated in Section 2 we shall apply the second order averaging theory to the previous differential equation. We write $g_{21}(x)$ and $f_{21}(x)$ as in (3.1), and $g_{22}(x) = \sum_{i=0}^{m} C_i x^i$, $f_{22}(x) = \sum_{i=0}^{n} D_i x^i$. Then in polar coordinates (r, θ) given by $x = r\cos\theta$ and $y = r\sin\theta$, the differential system (1.4) becomes

$$\begin{cases} \dot{r} = -\varepsilon G_1(r,\theta) - \varepsilon^2 H_1(r,\theta), \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r} G_2(r,\theta) - \dfrac{\varepsilon^2}{r} H_2(r,\theta), \end{cases}$$

where

$$\begin{aligned} G_1(r,\theta) &= \sum_{i=0}^{n} d_i h_{i,2\beta+1}(\theta) r^{2\beta+i} + \sum_{i=0}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1}, \\ H_1(r,\theta) &= \sum_{i=0}^{n} D_i h_{i,2\beta+1}(\theta) r^{2\beta+i} + \sum_{i=0}^{m} C_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1}, \\ G_2(r,\theta) &= \sum_{i=0}^{n} d_i h_{i+1,2\beta}(\theta) r^{2\beta+i} + \sum_{i=0}^{m} c_i h_{i+1,2\alpha+1}(\theta) r^{2\alpha+i+1}, \\ H_2(r,\theta) &= \sum_{i=0}^{n} D_i h_{i+1,2\beta}(\theta) r^{2\beta+i} + \sum_{i=0}^{m} C_i h_{i+1,2\alpha+1}(\theta) r^{2\alpha+i+1}, \end{aligned}$$

where $h_{i,i}(\theta) = \cos^i\theta \sin^j\theta$. Taking θ as a new independent variable, system (1.4) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + O(\varepsilon^3),$$

where

$$F_1(r,\theta) = G_1(r,\theta), \quad F_2(r,\theta) = H_1(r,\theta) - \frac{1}{r} G_1(r,\theta) G_2(r,\theta).$$

If $F_{10}(r)$ is identically zero, applying the theory of averaging of second order (see again Section 2), every simple positive zero of the function

$$F_{20}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\mathrm{d}}{\mathrm{d}r} F_1(r,\theta) \left(\int_0^{\theta} F_1(r,s)\,\mathrm{d}s \right) + F_2(r,\theta) \right) \mathrm{d}\theta \tag{4.1}$$

will provide a limit cycle of the polynomial differential system (1.4).

In order to compute $F_{20}(r)$, we need F_{10} to be identically zero. Then from (3.6) in what follows we must take $c_{2i} = 0$, for all $i \in \mathbb{N}$.

We must study the simple positive zeros of the function $F_{20}(r)$. We split the computation of the function $F_{20}(r)$ in two pieces, i.e. we define

$$2\pi F_{20}(r) = L(r) + J(r),$$

where

$$L(r) = \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}r} F_1(r,\theta)\Big(\int_0^{\theta} F_1(r,s)\,\mathrm{d}s\Big)\,\mathrm{d}\theta, \quad J(r) = \int_0^{2\pi} F_2(r,\theta)\,\mathrm{d}\theta.$$

Proposition 6. *If* $F_{10}(r) \equiv 0$, *then*

$$J(r) = 2r^{2\beta+2\alpha+1} \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} d_{2i} c_{2p+1} M_{2p+2+2i,2\alpha+2\beta+2}(2\pi) r^{2p+2i} + r^{2\alpha+1} \sum_{i=0}^{[m/2]} C_{2i} M_{2i,2\alpha+2}(2\pi) r^{2i}.$$

P r o o f. Taking into account the expression of $F_2(r,\theta)$, first we shall compute the function $\int_0^{2\pi} H_1(r,\theta)\,\mathrm{d}\theta$. Using the expression of $H_1(r,\theta)$ and taking into account that $\int_0^{2\pi} h_{i,2\alpha+2}(\theta)\,\mathrm{d}\theta = 0$ if i is odd and $\int_0^{2\pi} h_{i,2\beta+1}(\theta)\,\mathrm{d}\theta = 0$ (see Lemma 4), we have

$$\int_0^{2\pi} H_1(r,\theta)\,\mathrm{d}\theta = \sum_{\substack{i=0 \\ i \text{ even}}}^{m} C_i r^{i+2\alpha+1} \int_0^{2\pi} h_{i,2\alpha+2}(\theta)\,\mathrm{d}\theta = r^{2\alpha+1} \sum_{i=0}^{[m/2]} C_{2i} M_{2i,2\alpha+2}(2\pi) r^{2i}.$$

Next, we shall study the contribution of the second part $\int_0^{2\pi} r^{-1} G_1(r,\theta) G_2(r,\theta)\,\mathrm{d}\theta$ of $F_2(\theta,r)$ to $F_{20}(r)$. Using the expression of $G_1(r,\theta)$ and $G_2(r,\theta)$ and taking into account that $c_{2i} = 0$, for all $i \in \mathbb{N}$, we have

$$\begin{aligned} G_1(r,\theta) &= \sum_{i=0}^{n} d_i h_{i,2\beta+1}(\theta) r^{2\beta+i} + \sum_{\substack{i=0 \\ i \text{ odd}}}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1} \\ &= \sum_{i=0}^{[(n-1)/2]} d_{2i+1} h_{2i+1,2\beta+1}(\theta) r^{2i+2\beta+1} + \sum_{i=0}^{[n/2]} d_{2i} h_{2i,2\beta+1}(\theta) r^{2i+2\beta} \\ &\quad + \sum_{i=0}^{[(m-1)/2]} c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2i+2\alpha+2} \end{aligned}$$

and

$$G_2(r,\theta) = \sum_{p=0}^{n} d_p h_{p+1,2\beta}(\theta) r^{2\beta+p} + \sum_{\substack{p=0 \\ p \text{ odd}}}^{m} c_p h_{p+1,2\alpha+1}(\theta) r^{2\alpha+p+1}$$

$$= \sum_{p=0}^{[n/2]} d_{2p} h_{2p+1,2\beta}(\theta) r^{2p+2\beta} + \sum_{p=0}^{[(n-1)/2]} d_{2p+1} h_{2p+2,2\beta}(\theta) r^{2p+2\beta+1}$$
$$+ \sum_{p=0}^{[(m-1)/2]} c_{2p+1} h_{2p+2,2\alpha+1}(\theta) r^{2p+2\alpha+2}.$$

By using Lemma 4, from the 9 main products of $\int_0^{2\pi} r^{-1} G_1(r,\theta) G_2(r,\theta)\, \mathrm{d}\theta$ only the following 2 are not zero when we integrate them between 0 and 2π. So the terms of $\int_0^{2\pi} r^{-1} G_1(r,\theta) G_2(r,\theta)\, \mathrm{d}\theta$ which will contribute to $F_{20}(r)$ are

$$\int_0^{2\pi} \frac{1}{r} G_1(r,\theta) G_2(r,\theta)\, \mathrm{d}\theta$$
$$= \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} d_{2i} c_{2p+1} M_{2p+2+2i,2\alpha+2\beta+2}(2\pi) r^{2p+2\alpha+2i+2\beta+1}$$
$$+ \sum_{i=0}^{[(m-1)/2]} \sum_{p=0}^{[n/2]} c_{2i+1} d_{2p} M_{2p+2i+2,2\beta+2\alpha+2}(2\pi) r^{2p+2\beta+2i+2\alpha+1}$$
$$= 2r^{2\alpha+2\beta+1} \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} d_{2i} c_{2p+1} M_{2p+2+2i,2\alpha+2\beta+2}(2\pi) r^{2p+2i}.$$

This completes the proof of Proposition 6. □

In order to complete the computation of $F_{20}(r)$ we must determine the function $L(r)$. First we compute the integrals $\int_0^{2\pi} M_{i,j}(\theta) h_{p,q}(\theta)\, \mathrm{d}\theta$. In the following lemma we compute these integrals.

Lemma 7. *Let $\varphi_{i,j}^{p,q}(2\pi) = \int_0^{2\pi} M_{i,j}(\theta) h_{p,q}(\theta)\, \mathrm{d}\theta$. Then the following equalities hold:*

(a) *The integral $\varphi_{2i+1,0}^{p,q}(2\pi)$ is zero if p is odd or q is even, and equal to*

$$\frac{1}{2i+1}\left(M_{2i+p,q+1}(2\pi) + \sum_{l=0}^{i-1} \frac{2^{l+1} i(i-1)\dots(i-l)}{(2i-1)(2i-3)\dots(2i-2l-1)} M_{2i+p+2l-2,q+1}(2\pi) \right)$$

if p is even and q is odd.

(b) *The integral $\varphi_{2i,2j+1}^{p,q}(2\pi)$ is zero if p is even or q is odd, and equal to*

$$-\frac{1}{2j+2i+1} \sum_{l=1}^{j-1} \frac{2^l j(j-1)\dots(j-l+1)}{(2j+2i-1)(2j+2i-3)\dots(2j+2i-2l+1)}$$
$$\times M_{2i+p+1,2j-2l+q}(2\pi) - \frac{1}{2j+2i+1} M_{2i+p+1,2j+q}(2\pi)$$

if p is odd and q is even.

(c) *The integral* $\varphi^{p,q}_{2i+1,2j}(2\pi)$ *is zero if* p *is odd or* q *is even, and equal to*

$$\frac{(2j-1)(2j-3)\dots 1}{(2j+2i+1)(2j+2i-1)\dots(2i+3)}\varphi^{p,q}_{2i+1,0}(2\pi) \\ - \frac{1}{2j+2i+1}M_{2i+p+2,2j+q+1}(2\pi) + \frac{1}{2j+2i+1} \\ \times \sum_{l=1}^{j-1} \frac{(2j-1)(2j-3)\dots(2j-2l+1)}{(2j+2i-1)(2j+2i-3)\dots(2j+2i-2l+1)} M_{2i+2+p,2j-2l+q-1}(2\pi)$$

if p *is even and* q *is odd.*

Proof. Using integral (5.2) from Appendix and taking into account

$$h_{i,j}(\theta)h_{p,q}(\theta) = h_{i+p,j+q}(\theta),$$

we have

$$\varphi^{p,q}_{2i+1,0}(2\pi) = \frac{1}{2i+1}\int_0^{2\pi}\sum_{l=0}^{i-1}\frac{2^{l+1}i(i-1)\dots(i-l)}{(2i-1)(2i-3)\dots(2i-2l-1)}h_{2i-2l+p-2,q+1}(\theta)\,\mathrm{d}\theta \\ + \frac{1}{2i+1}\int_0^{2\pi} h_{2i+p,q+1}(\theta)\,\mathrm{d}\theta.$$

Statement (a) now follows from Lemma 4. Using integrals from Appendix and taking into account $h_{i,j}(\theta)h_{p,q}(\theta) = h_{i+p,j+q}(\theta)$, we have

$$\varphi^{p,q}_{2i+1,2\alpha}(2\pi) \\ = \frac{(2\alpha-1)(2\alpha-3)\dots 1}{(2\alpha+2i+1)(2\alpha+2i-1)\dots(2i+3)}\varphi^{p,q}_{2i+1,0}(2\pi) \\ - \frac{1}{2\alpha+2i+1} \\ \times \sum_{l=1}^{\alpha-1}\frac{(2\alpha-1)(2\alpha-3)\dots(2\alpha-2l+1)}{(2\alpha+2i-1)(2\alpha+2i-3)\dots(2\alpha+2i-2l+1)}M_{2i+p+2,2\alpha-2l+q-1}(2\pi) \\ + \frac{1}{2\alpha+2i+1}M_{2i+p+2,2\alpha+q+1}(2\pi)$$

and

$$\varphi^{p,q}_{i,2\alpha+1}(2\pi) = \frac{-1}{2\alpha+i+1} \\ \times \sum_{l=1}^{\alpha-1}\frac{2^l\alpha(\alpha-1)\dots(\alpha-l+1)}{(2\alpha+i-1)(2\alpha+i-3)\dots(2\alpha+i-2l+1)}M_{i+p+1,2\alpha-2l+q}(2\pi) \\ + \frac{1}{2\alpha+i+1}M_{i+p+1,2\alpha+q}(2\pi),$$

Statements (b) and (c) now follow again from Lemma 4. □

Proposition 8. *If $F_{10}(r) \equiv 0$, then*

$$L(r) = \sum_{i=0}^{[(m-1)/2]} \sum_{p=0}^{[n/2]} 2(i+\alpha+1) c_{2i+1} d_{2p} \varphi_{2p,2\beta+1}^{2i+1,2\alpha+2}(2\pi) r^{2p+2\beta+2i+2\alpha+1} + \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} 2(i+\beta) d_{2i} c_{2p+1} \varphi_{2p+1,2\alpha+2}^{2i,2\beta+1}(2\pi) r^{2p+2\alpha+2i+2\beta+1}.$$

P r o o f. Since

$$L(r) = \int_0^{2\pi} \left(\left(\frac{\mathrm{d}}{\mathrm{d}r} F_1(r,\theta) \right) \int_0^\theta F_1(r,s)\,\mathrm{d}s \right) \mathrm{d}\theta,$$

using the expression of $F_1(r,\theta)$ and taking into account that $c_{2i} = 0$, for all $i \in \mathbb{N}$, we have

$$\begin{aligned} F_1(r,\theta) &= \sum_{i=0}^{n} d_i h_{i,2\beta+1}(\theta) r^{2\beta+i} + \sum_{\substack{i=0\\ i \text{ odd}}}^{m} c_i h_{i,2\alpha+2}(\theta) r^{2\alpha+i+1} \\ &= \sum_{i=0}^{[(m-1)/2]} c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2i+2\alpha+2} + \sum_{i=0}^{[n/2]} d_{2i} h_{2i,2\beta+1}(\theta) r^{2i+2\beta} \\ &\quad + \sum_{i=0}^{[(n-1)/2]} d_{2i+1} h_{2i+1,2\beta+1}(\theta) r^{2i+2\beta+1}. \end{aligned}$$

Next we calculate the terms of this integral. First we have that

$$\begin{aligned} \frac{\mathrm{d}F_1(r,\theta)}{\mathrm{d}r} = &\sum_{i=0}^{[(m-1)/2]} 2(i+\alpha+1) c_{2i+1} h_{2i+1,2\alpha+2}(\theta) r^{2i+2\alpha+1} \\ &+ \sum_{i=0}^{[(n-1)/2]} (2i+2\beta+1) d_{2i+1} h_{2i+1,2\beta+1}(\theta) r^{2i+2\beta} \\ &+ \sum_{i=0}^{[n/2]} 2(i+\beta) d_{2i} h_{2i,2\beta+1}(\theta) r^{2i+2\beta-1} \end{aligned}$$

and

$$\begin{aligned} \int_0^\theta F_1(r,s)\,\mathrm{d}s = &\sum_{i=0}^{[(m-1)/2]} c_{2i+1} M_{2i+1,2\alpha+2}(\theta) r^{2i+2\alpha+2} + \sum_{i=0}^{[n/2]} d_{2i} M_{2i,2\beta+1}(\theta) r^{2i+2\beta} \\ &+ \sum_{i=0}^{[(n-1)/2]} d_{2i+1} M_{2i+1,2\beta+1}(\theta) r^{2i+2\beta+1}. \end{aligned}$$

By using Lemma 7, from the 9 main products of $L(r)$ only the following 2 are not zero when we integrate them between 0 and 2π. So the terms of $L(r)$ which will contribute to $F_{20}(r)$ are

$$\begin{aligned} L(r) = & \sum_{i=0}^{[(m-1)/2]} \sum_{p=0}^{[n/2]} 2(i+\alpha+1) c_{2i+1} d_{2p} \varphi_{2p,2\beta+1}^{2i+1,2\alpha+2}(2\pi) r^{2p+2\beta+2i+2\alpha+1} \\ & + \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} 2(i+\beta) d_{2i} c_{2p+1} \varphi_{2p+1,2\alpha+2}^{2i,2\beta+1}(2\pi) r^{2p+2\alpha+2i+2\beta+1}. \end{aligned}$$

This completes the proof of Proposition 8. $\square$

Proposition 9. *If $F_{10}(r) \equiv 0$, then the function $F_{20}(r)$ defined in* (4.1) *can be expressed as $\frac{1}{2} r^{2\alpha+1} \pi^{-1} P(r)$, where*

$$\begin{aligned} P(r) = & \sum_{i=0}^{[(m-1)/2]} \sum_{p=0}^{[n/2]} 2(i+\alpha+1) c_{2i+1} d_{2p} \varphi_{2p,2\beta+1}^{2i+1,2\alpha+2}(2\pi) r^{2p+2i+2\beta} \\ & + \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} 2(i+\beta) d_{2i} c_{2p+1} \varphi_{2p+1,2\alpha+2}^{2i,2\beta+1}(2\pi) r^{2p+2i+2\beta} \\ & + 2 \sum_{i=0}^{[n/2]} \sum_{p=0}^{[(m-1)/2]} d_{2i} c_{2p+1} M_{2p+2+2i,2\alpha+2\beta+2}(2\pi) r^{2p+2i+2\beta} \\ & + \sum_{i=0}^{[m/2]} C_{2i} M_{2i,2\alpha+2}(2\pi) r^{2i}. \end{aligned} \tag{4.2}$$

P r o o f. The proof of the proposition follows immediately from the results of Proposition 6 and Proposition 8. $\square$

P r o o f of Theorem 2. Taking into account the above arguments and $[\frac{1}{2}m] \geqslant \beta - 1$, we deduce that according to the Descartes theorem stated in Section 2, we can choose the appropriate coefficients c_i, d_i, C_i and D_i in order that the simple positive roots number $F_{20}(r) = \frac{1}{2} r^{2\alpha+1} \pi^{-1} P(r)$ can have at most $\lambda = \max\{[\frac{1}{2}m], [\frac{1}{2}n] + [\frac{1}{2}(m-1)] + \beta\}$ simple positive zeros. This completes the proof of Theorem 2. $\square$

E x a m p l e 10. We consider the differential system (1.4) with $m = 2$, $\alpha = 1$, $n = 0$ and $\beta = 2$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon\Big(xy^3 + \dfrac{1}{7}y^4\Big) - \varepsilon^2\Big(\Big(-\dfrac{1}{16} + 2x + \dfrac{1}{2}x^2\Big)y^3 + 3y^4\Big). \end{cases} \tag{4.3}$$

An easy computation shows that $F_{10}(r) \equiv 0$ and

$$F_{20}(r) = -\frac{1}{128} r^3 (r^2 - 1)(r^2 - 3).$$

Therefore from the periodic orbits of radius 1 and 3 of the linear center $\dot{x} = y$, $\dot{y} = -x$, it bifurcates two limit cycles.

Example 11. We consider the differential system (1.4) with $n = 2$, $\alpha = 2$, $\beta = 1$ and $m = 3$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - \varepsilon\Big(\Big(-\dfrac{1}{10}x + \dfrac{1}{29}x^3\Big)y^5 + x^2y^2\Big) \\ \qquad - \varepsilon^2\Big(\Big(\dfrac{1}{160} - \dfrac{11}{120}x^2 - x^3\Big)y^5 + (x - x^2)y^2\Big). \end{cases}$$

An easy computation shows that $F_{10}(r) \equiv 0$ and

$$F_{20}(r) = -\frac{1}{3072} r^5 (r^2 - 1)(r^2 - 2)(r^2 - 3).$$

Therefore from the periodic orbits of radius 1, 2 and 3 of the linear center $\dot{x} = y$, $\dot{y} = -x$, it bifurcates three limit cycles.

Remark 12. The function $P(r)$ defined in (4.2) can be expressed as $P(r) = \sum\limits_{j=0}^{\lambda} A_i r^{2j}$, where $\lambda = \max\{[\frac{1}{2}m], [\frac{1}{2}n] + [\frac{1}{2}(m-1)] + \beta\}$. Then if $[\frac{1}{2}m] \geqslant \beta - 1$ we can choose arbitrary values for c_i, d_i, C_i and D_i and, in addition, these coefficients appear multiplied by nonzero constants, it is possible to reach this upper bound and if $[\frac{1}{2}m] < \beta - 1$, the coefficients $A_i = 0$, for $[\frac{1}{2}m] < i < \beta$, then according to the Descartes theorem it is not possible to reach this upper bound.

5. Appendix

Here we list some important formulas used in this article; for more details see [8]. For $i \geqslant 0$ and $j \geqslant 0$ we have

$$\begin{aligned} (5.1)\quad \int_0^\theta \cos^i s \sin^\alpha s \,\mathrm{d}s &= \frac{\cos^{i-1}\theta \sin^{\alpha+1}\theta}{i+\alpha} + \frac{i-1}{i+\alpha}\int_0^\theta \cos^{i-2} s \sin^\alpha s \,\mathrm{d}s \\ &= -\frac{\cos^{i+1}\theta \sin^{\alpha-1}\theta}{i+\alpha} + \frac{\alpha-1}{i+\alpha}\int_0^\theta \cos^i s \sin^{\alpha-2} s \,\mathrm{d}s, \end{aligned}$$

$$(5.2)\qquad \int_0^\theta \cos^{2i} s\,\mathrm{d}s = \frac{\sin\theta}{2i}\sum_{l=1}^{i-1}\frac{(2i-1)(2i-3)\dots(2i-2l+1)}{2^l(i-1)(i-2)\dots(i-l)}\cos^{2i-2l-1}\theta + \frac{\sin\theta}{2i}\cos^{2i-1}\theta + \frac{(2i-1)(2i-3)\dots 1}{2^i i!}\theta$$
$$= \frac{1}{2^{2i-1}}\sum_{l=0}^{i-1}\binom{2i}{l}\frac{\sin 2(i-l)\theta}{2(i-l)} + \frac{1}{2^{2i}}\binom{2i}{i}\theta,$$

$$(5.3)\qquad \int_0^\theta \cos^{2i+1} s\,\mathrm{d}s = \frac{\sin\theta}{2i+1}\sum_{l=0}^{i-1}\frac{2^{l+1}i(i-1)\dots(i-l)}{(2i-1)(2i-3)\dots(2i-2l-1)}\cos^{2i-2l-2}\theta + \frac{\sin\theta}{2i+1}\cos^{2i}\theta$$
$$= \frac{1}{2^{2i}}\sum_{l=0}^{i-1}\binom{2i+1}{l}\frac{\sin(2i-2l+1)\theta}{2i-2l+1},$$

where $\binom{2i}{p} = (2i)!/p!(2i-p)!$;

$$(5.4)\qquad \int_0^\theta \cos^i s\sin^{2\alpha} s\,\mathrm{d}s$$
$$= -\frac{\cos^{i+1}\theta}{2\alpha+i}\sum_{l=1}^{\alpha-1}\frac{(2\alpha-1)(2\alpha-3)\dots(2\alpha-2l+1)}{(2\alpha+i-2)(2\alpha+i-4)\dots(2\alpha+i-2l)}\sin^{2\alpha-2l-1}\theta + \frac{(2\alpha-1)(2\alpha-3)\dots 1}{(2\alpha+i)(2\alpha+i-2)\dots(i+2)}\int_0^\theta \cos^i s\,\mathrm{d}s - \frac{\cos^{i+1}\theta}{2\alpha+i}\sin^{2\alpha+1}\theta.$$

$$(5.5)\qquad \int_0^\theta \cos^i s\sin^{2\alpha+1} s\,\mathrm{d}s$$
$$= -\frac{\cos^{i+1}\theta}{2\alpha+i+1}\sum_{l=1}^{\alpha-1}\frac{2^l\alpha(\alpha-1)\dots(\alpha-l+1)}{(2\alpha+i-1)(2\alpha+i-3)\dots(2\alpha+i-2l+1)}\sin^{2\alpha-2l}\theta - \frac{\cos^{i+1}\theta}{2\alpha+i+1}\sin^{2\alpha}\theta.$$

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