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GENERALIZED SPECTRAL PERTURBATION AND THE BOUNDARY SPECTRUM

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Abstract. By considering arbitrary mappings ω from a Banach algebra A into the set of all nonempty, compact subsets of the complex plane such that for all $a \in A$, the set $\omega(a)$ lies between the boundary and connected hull of the exponential spectrum of a, we create a general framework in which to generalize a number of results involving spectra such as the exponential and singular spectra. In particular, we discover a number of new properties of the boundary spectrum.

Keywords: exponential spectrum; singular spectrum; boundary spectrum; boundary and hull; (strong) Riesz property; Mobius spectrum

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1. Introduction

Several properties of different types of spectra (e.g. the exponential, singular and boundary spectra) have been proven in the literature by utilizing the intrinsic characteristics of the particular spectrum. On the other hand, many properties of these spectra have been obtained (or recovered) by using the regularity theory of Müller, see [18]. In this note we take another approach by considering arbitrary mappings ω from a Banach algebra A into the set of all nonempty, compact subsets of the complex plane such that $\omega(a)$ lies between the boundary and connected hull of the exponential spectrum of a for every $a \in A$. Using results of Harte in [8] and Harte-Wickstead in [9] (e.g. the theory of Mobius spectra), we are able to generalize some of the above-mentioned spectral-theoretical results by providing a more general framework, and also prove a number of new properties of the boundary spectrum

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(which is in general *not* generated by a (semi)regularity, see [19], Examples 1.1 and 1.2 and [20], Examples 2.1 and 3.2).

Perturbation properties of the (usual) spectrum and the exponential spectrum have been studied in [5]. In Section 3 we prove an analogue of the authors' main result in this regard for the boundary spectrum (see Theorem 3.2 and, in particular, Corollary 3.4).

If the boundary and hull of any one of a certain set of spectra coincide, then all these spectra are the same. This has been established in [5] and [6]. In Section 4 we generalize these theorems. The main results in this regard are Theorems 4.5 and 4.8 (see also Corollaries 4.6 and 4.9 about the boundary spectrum).

Perturbation of the sets of accumulation points of the singular and exponential spectra have been studied in [10] and [11]. We conclude Section 4 by generalizing results in this area (see Theorem 4.10) and by presenting an analogue for the boundary spectrum (see Corollary 4.11).

If $T\colon A \to B$ is a bounded homomorphism with closed range between Banach algebras, then

$$\bigcap_{b\in \mathrm{N}(T)} \sigma(a+b, A) \subseteq \eta\sigma(Ta, B)$$

for all $a \in A$. This was proven by Harte in 1976 (see [7]), and in 1998 Lindeboom and Raubenheimer established analogous results for the exponential and singular spectra, see [10]. In Section 5 we develop more general forms of these results (see Theorems 5.3 and 5.4) and also provide an analogue for the boundary spectrum (see Corollary 5.5).

We first provide a complete account of all relevant definitions and basic results which will be needed for our study, and develop a number of auxiliary results. This is done in Section 2, after which the main results are presented in Sections 3, 4 and 5.

2. Preliminaries and auxiliary results

All our Banach algebras will be complex and unital (with unit 1). We denote the set of all invertible elements of a Banach algebra A by A^{-1} and elements of the form $\lambda 1$ in A by λ. The term "ideal" will always mean "proper two-sided ideal". The Banach algebra A is said to be *semisimple* if its radical $\mathrm{Rad}(A) = \{0\}$, where $\mathrm{Rad}(A)$ is the intersection of all maximal left ideals. (It is also the intersection of all maximal right ideals, and therefore it is an ideal.) If A and B are Banach algebras, then a linear operator $T\colon A \to B$ (not necessarily bounded) is called a *homomorphism* if $T(ab) = TaTb$ $(a, b \in A)$ and T maps the unit of A onto the unit of B. The null space (kernel) of T is denoted by $\mathrm{N}(T)$. The homomorphism T is said to be *bounded* *below* if there exists a constant $k > 0$ such that $\|x\| \leqslant k\|Tx\|$ for all $x \in A$. The following lemma, which will be applied in Section 5, is a consequence of the Bounded Inverse Theorem:

Lemma 2.1. *Let A and B be Banach algebras and $T\colon A \to B$ an injective bounded homomorphism with closed range. Then T is bounded below.*

For a compact set K in $\mathbb{C}$, the *connected hull* ηK of K is the compact set consisting of K together with its holes, where a *hole* of K is a bounded component of $\mathbb{C} \setminus K$. Recall that for any compact set K in $\mathbb{C}$ we have $K \subseteq \eta K = \eta(\eta K)$ (see [9]), and $\eta K_1 \subseteq \eta K_2$ whenever $K_1 \subseteq K_2$ holds for compact sets K_1 and K_2 in $\mathbb{C}$.

If a is an element of A, then the (usual) *spectrum* $\{\lambda \in \mathbb{C}\colon\ a - \lambda \notin A^{-1}\}$ of a in A will be denoted by $\sigma(a)$ (or by $\sigma(a, A)$ if necessary to avoid confusion). The symbols $\partial\sigma(a)$ and $\eta\sigma(a)$ will denote the topological boundary and the connected hull as mentioned previously, respectively, of $\sigma(a)$.

The following result is well known.

Theorem 2.2 ([4], Theorem 5.4, page 207)**.** *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$, and let $a \in B$.*

(1) *Then $\sigma(a, A) \subseteq \sigma(a, B)$, $\partial\sigma(a, B) \subseteq \partial\sigma(a, A)$ and hence $\eta\sigma(a, A) = \eta\sigma(a, B)$.*

(2) *If B is the Banach algebra generated by a, then $\sigma(a, B) = \eta\sigma(a, A)$.*

Corollary 2.3. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. If $\sigma(a, A) = \eta\sigma(a, A)$ for some $a \in B$, then $\sigma(a, A) = \sigma(a, B)$.*

The previous two results can now be used to establish the following corollary, which will be needed to prove Theorem 4.5.

Corollary 2.4. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. If either $\partial\sigma(a, A) = \eta\sigma(a, A)$ or $\partial\sigma(a, B) = \eta\sigma(a, B)$ for some $a \in B$, then*

$$\partial\sigma(a, B) = \partial\sigma(a, A) = \eta\sigma(a, A) = \eta\sigma(a, B).$$

P r o o f. If $\partial\sigma(a, A) = \eta\sigma(a, A)$, then $\sigma(a, A) = \eta\sigma(a, A)$, so the result follows from Corollary 2.3. By Theorem 2.2, $\partial\sigma(a, B) \subseteq \partial\sigma(a, A) \subseteq \eta\sigma(a, A) = \eta\sigma(a, B)$, so the result also follows if $\partial\sigma(a, B) = \eta\sigma(a, B)$. $\square$

The sets of isolated and accumulation points of a compact set K in $\mathbb{C}$ will be denoted by $\operatorname{iso} K$ and $\operatorname{acc} K$, respectively, and the interior of K by $\operatorname{int} K$.

The following results by Harte and Wickstead will be applied in Sections 4 and 5.

Theorem 2.5 ([8], Theorem 7.10.3, [9], Theorems 1.2 and 1.3, and (4.0.2)). *Let K, K_1 and K_2 be compact sets in $\mathbb{C}$. Then:*

(1) $\partial K_1 \subseteq K_2 \subseteq K_1 \Rightarrow \partial K_1 \subseteq \partial K_2$.

(2) $\partial K_1 \subseteq K_2 \Rightarrow K_1 \subseteq \eta K_2$.

(3) $\eta K = \eta \partial K$.

(4) $\eta K = \eta(\operatorname{acc} K) \cup \operatorname{iso} K$.

Corollary 2.6. *If $K \subseteq \mathbb{C}$ is a compact set, then* $\operatorname{acc}(\eta K) \subseteq \eta(\operatorname{acc} K)$.

Proof. Suppose that $\lambda_0 \in (\operatorname{acc}(\eta K)) \setminus \eta(\operatorname{acc} K)$. Then $\lambda_0 \in \eta K$ and there exists a sequence (λ_n) of different points in ηK such that $\lambda_n \to \lambda_0$ as $n \to \infty$. Since $\mathbb{C} \setminus \eta(\operatorname{acc} K)$ is an open set, we may assume that $\lambda_n \notin \eta(\operatorname{acc} K)$ for all $n \in \mathbb{N}$. It then follows from Theorem 2.5 (4) that $\lambda_n \in \operatorname{iso} K$ for all $n \in \mathbb{N}$. Therefore $\lambda_0 \in \operatorname{acc} K$, so $\lambda_0 \notin \operatorname{iso} K$, yielding a contradiction to Theorem 2.5 (4). $\square$

An ideal I in a Banach algebra A will be called *inessential* whenever $\operatorname{acc} \sigma(a) \subseteq \{0\}$ for all $a \in I$. For $a \in A$, a point $\lambda \in \operatorname{iso} \sigma(a)$ is said to be a *Riesz point* of $\sigma(a)$ (relative to I) if the spectral idempotent $p(a, \lambda)$ is an element of I. Given any element $a \in A$, we define the (compact) set $D(a)$ (or $D(a, I)$ when necessary to avoid confusion) as follows:

$$D(a) = \sigma(a) \setminus \{\lambda \in \sigma(a) \colon \lambda \text{ is a Riesz point of } \sigma(a)\}.$$

Aupetit obtained the following important result in 1986:

Theorem 2.7 ([2], Theorem 5.7.4). *Let I be an inessential ideal in a Banach algebra A and $a \in A$. Then $\sigma(a + \overline{I}) \subseteq D(a)$ and $\eta\sigma(a + \overline{I}) = \eta D(a)$.*

If $T\colon A \to B$ is a homomorphism and $\mathrm{N}(T)$ is an inessential ideal of A, then T is said to have the *Riesz property*. The homomorphism T has the *strong Riesz property* if $\partial\sigma(a) \subseteq \sigma(Ta) \cup \operatorname{iso} \sigma(a)$ for all $a \in A$. It follows from [9], Theorem 4.2, that T has the strong Riesz property if and only if $\sigma(a) \subseteq \eta\sigma(Ta) \cup \operatorname{iso}\sigma(a)$ for all $a \in A$. Clearly, the strong Riesz property implies the Riesz property. By [14], Corollary 7.9, T has the strong Riesz property whenever T has closed range and satisfies the Riesz property.

If I is a closed inessential ideal in A, then an element $b \in A$ is said to be *I-Riesz* if $\sigma(b + I) = \{0\}$. If $a, b \in A$, then b is called *a-inessential* (see [5]) if there exists a closed inessential ideal I in A such that either $b \in I$, or b is I-Riesz and $ab - ba \in I$.

The following result was established during the course of proving Theorem 3 in [5]. This theorem and the next one will be used to prove Theorems 4.8 and 4.10, respectively.

Theorem 2.8 ([5], proof of Theorem 3)**.** *Let A be a Banach algebra and $a, b \in A$ such that b is a-inessential and $\partial\sigma(a) = \eta\sigma(a)$. Then $\partial\sigma(a+b) = \eta\sigma(a+b)$.*

Theorem 2.9. *Let A be a Banach algebra and $a, b \in A$ with b an a-inessential element. Then $\operatorname{acc}\sigma(a+b) \subseteq \eta\sigma(a)$.*

P r o o f. If b is an a-inessential element, then there exists a closed inessential ideal I such that either $b \in I$, or b is I-Riesz and $ab - ba \in I$. For any $x \in A$ we have $D(x) \subseteq \eta\sigma(x+I)$ by Theorem 2.7, and hence $\operatorname{acc}\sigma(a+b) \subseteq D(a+b) \subseteq \eta\sigma(a+b+I)$.

If $b \in I$, then $\operatorname{acc}\sigma(a+b) \subseteq \eta\sigma(a+I) \subseteq \eta\sigma(a)$, and if b is I-Riesz and $ab - ba \in I$, then

$$\sigma(a+b+I) \subseteq \sigma(a+I) + \sigma(b+I) = \sigma(a+I),$$

so $\operatorname{acc}\sigma(a+b) \subseteq \eta\sigma(a+I) \subseteq \eta\sigma(a)$. □

The argument demonstrated in the above proof is familiar: it has been used in, e.g. the proofs of Theorems 3.3 and 3.5 in [10], Theorem 5.2 in [11] and Theorem 5.1 in [12]. The statement in Theorem 2.9 has been proven directly for the case when b is a rank one element in a semisimple Banach algebra, see [13], Theorem 2.4.

An element $a \in A$ is said to be a *topological divisor of zero* if there exist sequences (b_n) and (c_n) in A such that $\|b_n\| = 1 = \|c_n\|$ for all $n \in \mathbb{N}$, and $b_n a \to 0$ and $ac_n \to 0$ as $n \to \infty$. The *singular spectrum* of $a \in A$ is defined in [7] by

$$\tau(a) = \{\lambda \in \mathbb{C} \colon\ a - \lambda \text{ is a topological divisor of zero in } A\},$$

and the *boundary spectrum* of a in A (see [16]) by

$$S_\partial(a) = \{\lambda \in \mathbb{C} \colon\ a - \lambda \in \partial_A S_A\},$$

where $S_A = A \setminus A^{-1}$ and $\partial_A S_A$ is the topological boundary of S_A in A. Let $\operatorname{Exp} A$ denote the set $\{\mathrm{e}^{a_1} \ldots \mathrm{e}^{a_k} \colon\ k \in \mathbb{N},\ a_1, \ldots, a_k \in A\}$. Then $\operatorname{Exp} A$ equals the component $\operatorname{Comp}_A(1, A^{-1})$ of 1 in A^{-1} (see for instance [2], Theorem 3.3.7). The *exponential spectrum* of a in A is defined in [7] by

$$\varepsilon(a) = \{\lambda \in \mathbb{C} \colon\ a - \lambda \notin \operatorname{Exp} A\}.$$

If $a \in A$, then the singular spectrum, the boundary spectrum and the exponential spectrum of a are nonempty, compact subsets of the complex plane.

Let $K(\mathbb{C})$ denote the set of all nonempty, compact subsets of $\mathbb{C}$. If A is a Banach algebra, $\omega \colon A \to K(\mathbb{C})$ is a mapping and B is a closed subalgebra of A such that $1 \in B$, then we write $\omega(\cdot, A)$ for ω and $\omega(\cdot, B)$ for the restriction of ω to B. For any two mappings $\omega, \mu\colon A \to K(\mathbb{C})$ we write $\omega \subseteq \mu$ to indicate that $\omega(a, A) \subseteq \mu(a, A)$ for all $a \in A$ and $\omega(a, B) \subseteq \mu(a, B)$ for all $a \in B$, where B is any closed subalgebra of A such that $1 \in B$.

Similarly, if A_1 and A_2 are Banach algebras, $T\colon A_1 \to A_2$ a homomorphism and $\omega, \mu\colon A_1 \cup A_2 \to K(\mathbb{C})$ two mappings with $\omega(\cdot, A_i)$ and $\mu(\cdot, A_i)$ (or ω_{A_i} and μ_{A_i}) indicating the restrictions of ω and μ, respectively, to A_i, then $\omega_{A_i} \subseteq \mu_{A_i}$ will indicate that $\omega(a, A_i) \subseteq \mu(a, A_i)$ for all $a \in A_i$ $(i \in \{1, 2\})$.

A mapping $\omega\colon A \to K(\mathbb{C})$ is said to be a *Mobius spectrum* (see [9]) on A if $\omega(f(a)) \subseteq f(\omega(a))$ for all $a \in A$ and for all functions f of the form $f(z) = (\alpha z + \beta)/(\gamma z + \delta)$ (α, β, γ, δ constants) such that f is well-defined on $\omega(a) \cup \sigma(a)$. Note that σ, ε, $\partial\sigma$, $\eta\sigma$ in [9] and S_∂ in [17] are Mobius spectra on any Banach algebra.

The following theorem by Harte and Wickstead about Mobius spectra will be of importance in Sections 4 and 5.

Theorem 2.10 ([9], Theorem 3.1). *Let A be a Banach algebra. If ω_1 and ω_2 are Mobius spectra on A such that $\omega_2 \cup \sigma \subseteq \omega_1$, then the following conditions are equivalent:*

$$\partial\omega_1 \subseteq \omega_2, \quad \omega_1 \subseteq \eta\omega_2, \quad \partial\eta\omega_1 \subseteq \omega_2.$$

We have the following set of inclusions:

Theorem 2.11 ([7], [8], [16]). *Let A be a Banach algebra. Then*

$$\partial\varepsilon \subseteq \partial\sigma \subseteq S_\partial \subseteq \tau \subseteq \sigma \subseteq \varepsilon \subseteq \eta\varepsilon = \eta\sigma.$$

Proof. The last three inclusions and the equality are given in [7], Theorem 1, while the second and third inclusions are Proposition 2.1 and Corollary 2.5, respectively, in [16]. An application of Theorem 2.10, together with Theorem 2.5 (3), supplies the first inclusion (which, alternatively, can be obtained by Theorem 1 in [7] and Theorem 2.5 (1)). $\square$

If a is an element of a Banach algebra, then an application of Theorem 2.11 and Corollary 2.6 to $K = \sigma(a)$ yields $\operatorname{acc}\varepsilon(a) \subseteq \operatorname{acc}(\eta\varepsilon(a)) = \operatorname{acc}(\eta\sigma(a)) \subseteq \eta(\operatorname{acc}\sigma(a))$. Hence:

Corollary 2.12. *On any Banach algebra we have $\eta(\operatorname{acc}\varepsilon) = \eta(\operatorname{acc}\sigma)$.*

The following three results provide known properties of the usual and exponential spectra that will be needed in Section 5.

Theorem 2.13 ([15], Theorem 2.1, [14], Corollary 4.5)**.** *Let A and B be Banach algebras and $T\colon A \to B$ a homomorphism which is bounded below. Then*

$$\partial\sigma(a, A) \subseteq \partial\sigma(Ta, B) \quad \text{and} \quad \partial\varepsilon(a, A) \subseteq \partial\varepsilon(Ta, B).$$

Theorem 2.14 ([7], Theorem 2)**.** *Let A and B be Banach algebras and $T\colon A \to B$ a surjective bounded homomorphism. Then*

$$\bigcap_{b \in \mathrm{N}(T)} \varepsilon(a + b, A) = \varepsilon(Ta, B)$$

for all $a \in A$.

Theorem 2.15 ([7], Theorem 3, [22], Corollary 2.2)**.** *Let A and B be Banach algebras and $T\colon A \to B$ a homomorphism which is either bounded with closed range or has the strong Riesz property. Then*

$$\bigcap_{b \in \mathrm{N}(T)} \sigma(a + b, A) \subseteq \eta\sigma(Ta, B)$$

for all $a \in A$.

We now list a number of known properties of the singular and boundary spectra that will be applied.

Proposition 2.16 ([7], (1.6), [10], Proposition 2.1)**.** *Let A and B be Banach algebras and $T\colon A \to B$ a bounded homomorphism. If T is bounded below, then $\tau(a, A) \subseteq \tau(Ta, B)$ for all $a \in A$.*

Lemma 2.17 ([16], Lemma 2.6)**.** *Let A be a Banach algebra, $a \in \partial_A S_A$ and $d \in A^{-1}$. Then $ad, da \in \partial_A S_A$.*

Proposition 2.18 ([17], Proposition 2.5)**.** *Let a be an element of a Banach algebra A. Then $S_\partial(\lambda a) = \lambda S_\partial(a)$ and $S_\partial(a + \lambda) = S_\partial(a) + \lambda$ for all $a \in A$ and $\lambda \in \mathbb{C}$.*

Theorem 2.19 ([16], Corollary 2.12)**.** *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. If $a \in B$, then $S_\partial(a, B) \subseteq S_\partial(a, A)$.*

Analogous results to Theorem 2.19 for σ, ε and τ are given in Theorem 2.2 (1), Theorem 4.1 (a) in [6], and [21], page 398, see also Proposition 2.16. The inclusion in Theorem 2.19 is in general proper. This is illustrated in [20], Example 4.2.

If p is an idempotent in a Banach algebra A such that $p \neq 0$ and $p \neq 1$, then $B = pAp$ is a closed subalgebra of A with identity p, and $\sigma(a, A) = \sigma(a, B) \cup \{0\}$ for all $a \in B$, see [1], Lemma 6, page 4. For τ we have $\tau(a,B) \subseteq \tau(a,A)$ for all $a \in B$ (see [10], Proposition 2.3). If, in addition, the idempotent p commutes with every element in A, then $B = pAp = pA$ is an ideal in A, and then $\varepsilon(a,B) \subseteq \varepsilon(a,A)$ if $a \in B$ (see [11], Proposition 3.4 (ii)). Also, in [3], Proposition 5, it was shown that $\sigma(a+\alpha,A) = \sigma(a+\alpha p,B) \cup \{\alpha\}$ for all $a \in B$ and for all $\alpha \in \mathbb{C}$. The corresponding result for ε, and hence the property $\varepsilon(a,A) = \varepsilon(a,B) \cup \{0\}$, was obtained under the additional condition that the ideal B is inessential (see [11], Proposition 3.6), although the inclusion $\varepsilon(a,B) \cup \{0\} \subseteq \varepsilon(a,A)$ holds even when B is not inessential, see [11], Proposition 3.5. For S_∂ we have:

Proposition 2.20. *Let A be a Banach algebra and I an inessential ideal of A with an identity $0 \neq p \neq 1$ such that p commutes with every element of A. Then I is closed and $S_\partial(a+\alpha p,I) \cup \{\alpha\} \subseteq S_\partial(a+\alpha,A)$ for all $a \in I$ and for all $\alpha \in \mathbb{C}$.*

Proof. As shown in [11], Proposition 3.5, I equals pAp, so I is closed in A. If $a \in I$, then since I is inessential, we have that $\sigma(a,A) = S_\partial(a,A)$. Therefore, it follows from Proposition 5 in [3], and Proposition 2.18 that if $\alpha \in \mathbb{C}$, then

$$\begin{aligned} S_\partial(a+\alpha p,I) \cup \{\alpha\} \subseteq \sigma(a+\alpha p,I) \cup \{\alpha\} &= \sigma(a+\alpha,A) = \sigma(a,A)+\alpha \\ &= S_\partial(a,A)+\alpha = S_\partial(a+\alpha,A). \end{aligned}$$

□

Taking $\alpha = 0$ in the above proposition, we obtain $S_\partial(a,I) \cup \{0\} \subseteq S_\partial(a,A)$. Note that, indeed, $0 \in S_\partial(a,A)$ since I is inessential: $a \in I$ implies that $0 \in \sigma(a,A) = S_\partial(a,A)$.

In addition, we will need the following results related to the boundary spectrum in Section 3.

Lemma 2.21. *Let A and B be Banach algebras and $T\colon A \to B$ a bounded homomorphism. Then $T(\partial_A S_A) \cap S_B \subseteq \partial_B S_B$.*

Proof. Suppose that $a \in \partial_A S_A$ and $Ta \in S_B$. Then $a \in \partial_A A^{-1} \subseteq \overline{A^{-1}}$, so $Ta \in \overline{B^{-1}}$ because T is continuous. But since $\overline{B^{-1}} = B^{-1} \cup \partial_B B^{-1} = B^{-1} \cup \partial_B S_B$ and $Ta \in S_B$, it follows that $Ta \in \partial_B S_B$. □

Corollary 2.22. *Let A and B be Banach algebras and $T\colon A \to B$ a bounded homomorphism. Then*

$$\sigma(Ta,B) \cap \Big(\bigcup_{b \in \mathrm{N}(T)} S_\partial(a+b,A)\Big) \subseteq S_\partial(Ta,B).$$

The following result provides an analogue of Proposition 2.16 for S_∂ (see also Theorem 2.13 for $\partial\sigma$ and $\partial\varepsilon$):

Proposition 2.23. *Let A and B be Banach algebras and $T: A \to B$ a bounded homomorphism. If T is bounded below, then $S_\partial(a,A) \subseteq S_\partial(Ta,B)$ for all $a \in A$.*

Proof. Let $\lambda \in S_\partial(a,A)$. Then $a-\lambda \in \partial_A S_A$, and it follows from Theorem 2.11 and Proposition 2.16 that $Ta-\lambda$ is a topological divisor of zero in B, so $Ta-\lambda \in S_B$. Lemma 2.21 now implies that $Ta-\lambda = T(a-\lambda) \in \partial_B S_B$, i.e. $\lambda \in S_\partial(Ta,B)$. □

It is not difficult to check that Proposition 2.23 is, in fact, equivalent to Theorem 2.19.

Lemma 2.24. *Let A and B be Banach algebras, $T\colon A \to B$ a homomorphism and $a \in A$. If $S_\partial(a,A) = \sigma(a,A)$ and $Ta \in \partial_B S_B$, then $a \in \partial_A S_A$.*

Proof. Since $Ta \in \partial_B S_B \subseteq S_B$, we have $a \notin A^{-1}$, so $0 \in \sigma(a,A) = S_\partial(a,A)$. Hence $a \in \partial_A S_A$. □

3. Perturbation properties of the boundary spectrum

In [5] the authors studied the behaviour of the usual and exponential spectra of an element a under perturbation by an a-inessential element b. In this section we provide an analogue of their result for the boundary spectrum, see Corollary 3.4. Most of the boundary spectrum-results mentioned in Section 2 will be used in this development.

Lemma 3.1. *Let A, B and D be Banach algebras, $T\colon A \to B$ and $S\colon A \to D$ homomorphisms and $b, c \in A$. Suppose that at least one of the following conditions holds:*

(A) *S has the Riesz property and $Sb = 0$.*

(B) *S has the strong Riesz property, $\sigma(Sb,D) = \{0\}$ and $S(bc-cb) = 0$.*

If $c+b \in A^{-1}$ and $Tc \in \partial_B S_B$, then $c \in \partial_A S_A$.

Proof. Suppose that $u = c+b \in A^{-1}$, and let $w = u^{-1} \in A^{-1}$ and $v = wb$. Then $1-v = wc$.

(A) If S has the Riesz property and $Sb = 0$, then $v \in \mathrm{N}(S)$, so $\operatorname{acc}\sigma(v,A) \subseteq \{0\}$. Therefore $\partial\sigma(v,A) = \sigma(v,A)$, so $S_\partial(v,A) = \sigma(v,A)$.

(B) If $S(bc-cb) = 0$, then Sb commutes with Sc, and hence Sb commutes with Sw. If $\sigma(Sb,D) = \{0\}$ as well, then it follows that $\sigma(Sv,D) \subseteq \sigma(Sw,D)\sigma(Sb,D) = \{0\}$.

If, in addition, S has the strong Riesz property, then $\operatorname{acc}\sigma(v,A) \subseteq \eta\sigma(Sv,D) = \{0\}$, so once again, $S_\partial(v,A) = \sigma(v,A)$.

Now suppose that $Tc \in \partial_B S_B$. Since $Tw \in B^{-1}$, it follows from Lemma 2.17 that $T(1-v) = TwTc \in \partial_B S_B$. But, by Proposition 2.18, $S_\partial(1-v,A) = 1 - S_\partial(v,A) = 1 - \sigma(v,A) = \sigma(1-v,A)$. Hence $1 - v \in \partial_A S_A$, by Lemma 2.24. It follows that $wc \in \partial_A S_A$, so $c \in \partial_A S_A$, by Lemma 2.17. $\square$

Theorem 3.2. *Let A, B and D be Banach algebras, $T\colon A \to B$ and $S\colon A \to D$ homomorphisms and $a, b \in A$. Suppose that at least one of the following conditions holds:*

(A) *S has the Riesz property and $Sb = 0$.*

(B) *S has the strong Riesz property, $\sigma(Sb,D) = \{0\}$ and $S(ab - ba) = 0$.*

Then $S_\partial(Ta,B) \subseteq S_\partial(a,A) \cup \sigma(a+b,A)$. Hence:

(1) *$S_\partial(Ta,B) \setminus \sigma(a+b,A) \subseteq S_\partial(a,A) \setminus \sigma(T(a+b),B)$.*

(2) *If T is bounded as well as bounded below, then*

$$S_\partial(Ta,B) \setminus \sigma(a+b,A) = S_\partial(a,A) \setminus \sigma(a+b,A).$$

P r o o f. If $\lambda \in S_\partial(Ta,B) \setminus (S_\partial(a,A) \cup \sigma(a+b,A))$, then $Ta - \lambda \in \partial_B S_B$, $a - \lambda \notin \partial_A S_A$ and $a + b - \lambda \in A^{-1}$. By replacing c by $a - \lambda$ in Lemma 3.1, we obtain a contradiction.

(1) This follows from the part already proven and the fact that $\sigma(Tx,B) \subseteq \sigma(x,A)$ for all $x \in A$.

(2) This is a consequence of (1) and Proposition 2.23. $\square$

Corollary 3.3. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. If $a, b \in B$, with b an a-inessential element, then $S_\partial(a,A) \subseteq S_\partial(a,B) \cup \sigma(a+b,B)$. Hence*

$$S_\partial(a,A) \setminus \sigma(a+b,B) = S_\partial(a,B) \setminus \sigma(a+b,B).$$

P r o o f. Let $T\colon B \to A$ be the inclusion map. Then T is bounded as well as bounded below. Since b is a-inessential, there exists a closed inessential ideal I in B such that either $b \in I$, or $\sigma(b+I, B/I) = \{0\}$ and $ab - ba \in I$. Therefore, the canonical homomorphism $S\colon B \to B/I$ has the Riesz property, and S also has the strong Riesz property, since S is onto. If $b \in I$, then $Sb = 0$, and if $\sigma(b+I, B/I) = \{0\}$ and $ab - ba \in I$, then $\sigma(Sb, B/I) = \{0\}$ and $S(ab - ba) = 0$. Therefore, the results follow from Theorem 3.2. $\square$

By replacing a and b in Corollary 3.3 by $a+b$ and $-b$, respectively, we obtain the following result, which can be seen as an analogue for the boundary spectrum of [5], (3.1), page 442.

Corollary 3.4. *Let A be a Banach algebra, B a closed subalgebra of A such that $1 \in B$ and $a, b \in B$, with b an a-inessential element. Then:*
(1) $S_\partial(a+b, A) \setminus \sigma(a, B) = S_\partial(a+b, B) \setminus \sigma(a, B)$.
(2) *If $\sigma(a, B) = \sigma(a, A)$, then*

$$S_\partial(a+b, A) \setminus \sigma(a, A) = S_\partial(a+b, B) \setminus \sigma(a, B).$$

See also Corollary 4.9 and the remarks thereafter.

4. Boundaries, hulls and accumulation points

In [5] and [6] it was shown that if the boundary and hull of any one of a certain set of spectra coincide, then all these spectra are the same. In this section we generalize these results, obtaining statements involving the boundary spectrum as well (see Theorems 4.5 and 4.8, Corollaries 4.6 and 4.9.

The following lemma will be used several times.

Lemma 4.1. *Let A be a Banach algebra and $\omega\colon A \to K(\mathbb{C})$ a mapping. If $\partial\varepsilon \subseteq \omega \subseteq \eta\varepsilon$, then*
(1) $\eta\sigma = \eta\omega$,
(2) $\operatorname{acc}\omega \subseteq \eta(\operatorname{acc}\varepsilon)$.

Proof. (1) This holds since $\eta\sigma = \eta\varepsilon = \eta\partial\varepsilon$, by Theorems 2.11 and 2.5 (3).

(2) Just apply Corollary 2.6 to $K = \varepsilon(a)$ for $a \in A$. $\square$

In particular, if B is a closed subalgebra of A such that $1, a \in B$, then together with Theorem 2.2 (1) we have

$$\eta S_\partial(a, B) = \eta\sigma(a, B) = \eta\sigma(a, A) = \eta S_\partial(a, A). \tag{4.1}$$

Theorem 2.2 (1) and Lemma 4.1 (1) also imply Corollary 4.2 in [6]. In addition, we note that Lemma 4.1 (1) is (a slightly stronger version of) the implication (1) $\Rightarrow$ (3) in [12], Theorem 4.2. Finally, Lemma 4.1 (1) yields a corollary that includes Proposition 4.8 and Corollary 4.9 in [6].

Corollary 4.2. *Let A be a Banach algebra, $a \in A$ and B the Banach algebra generated by a. Let $\omega\colon A \to K(\mathbb{C})$ be a mapping such that $\sigma \subseteq \omega \subseteq \eta\varepsilon$. Then $\omega(a, B) = \eta\omega(a, A) = \sigma(a, B)$.*

P r o o f. Lemma 4.1(1) implies that $\eta\sigma = \eta\omega$, and hence, by Theorem 2.2 (1), $\eta\omega(a, B) = \eta\sigma(a, A) = \eta\omega(a, A)$. But by Theorem 2.2 (2), $\eta\sigma(a, A) = \sigma(a, B)$ and therefore

$$\omega(a, B) \subseteq \eta\omega(a, B) = \sigma(a, B) \subseteq \omega(a, B).$$

□

The second part of Lemma 4.1 will be needed in order to prove Theorem 4.10.

Lemma 4.3. *Let A be a Banach algebra and $\omega\colon A \to K(\mathbb{C})$ a Mobius spectrum such that $\sigma \subseteq \omega \subseteq \eta\varepsilon$. Then $\partial\omega \subseteq \partial\sigma$.*

P r o o f. We have from Lemma 4.1 (1) and Theorem 2.5 (3) that $\omega \subseteq \eta\omega = \eta\sigma = \eta\partial\sigma$. Since $\partial\sigma \cup \sigma = \sigma \subseteq \omega$ and both $\partial\sigma$ and ω are Mobius spectra, the result follows from Theorem 2.10. □

From Lemma 4.3 we have the following result, which (partly) includes Theorem 4.1 in [6]:

Corollary 4.4. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. Let $\omega\colon A \to K(\mathbb{C})$ be a mapping such that $\sigma \subseteq \omega \subseteq \eta\varepsilon$ and ω is a Mobius spectrum on B. If $a \in B$ such that $\omega(a, A) \subseteq \omega(a, B)$, then $\partial\omega(a, B) \subseteq \partial\omega(a, A)$, and hence $\eta\omega(a, A) = \eta\omega(a, B)$.*

P r o o f. Let $\lambda \in \partial\omega(a, B)$. Since $\operatorname{int}\omega(a, A) \subseteq \operatorname{int}\omega(a, B)$, we only need to show that $\lambda \in \omega(a, A)$. By Lemma 4.3, $\partial\omega \subseteq \partial\sigma$ on B, and therefore $\lambda \in \partial\sigma(a, B)$. Hence the assumption $\sigma \subseteq \omega$ together with Theorem 2.11 and Proposition 2.16 imply that $\lambda \in \omega(a, A)$. The last statement follows by using Theorem 2.5 (3). □

We also use Lemmas 4.1 and 4.3, together with Corollary 2.4, to obtain our first main result in this section:

Theorem 4.5. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. Let $\omega\colon A \to K(\mathbb{C})$ be a mapping such that either $\partial\sigma \subseteq \omega \subseteq \sigma$ or ω is a Mobius spectrum on both A and B satisfying $\sigma \subseteq \omega \subseteq \eta\sigma$. If either $\partial\omega(a, A) = \eta\omega(a, A)$ or $\partial\omega(a, B) = \eta\omega(a, B)$ for some $a \in B$, then*

$$\partial\sigma(a, A) = \partial\sigma(a, B) = \omega(a, A) = \omega(a, B) = \eta\sigma(a, A) = \eta\sigma(a, B).$$

P r o o f. If ω is a Mobius spectrum satisfying $\sigma \subseteq \omega \subseteq \eta\sigma$, then it follows from Theorem 2.11, together with Lemmas 4.3 and 4.1 (1), that $\partial\omega \subseteq \partial\sigma \subseteq \eta\sigma = \eta\omega$. Hence, if $\partial\omega(a, X) = \eta\omega(a, X)$ with $X \in \{A, B\}$, then $\partial\sigma(a, X) = \eta\sigma(a, X)$, so the result follows from Corollary 2.4 and the fact that $\partial\sigma \subseteq \omega \subseteq \eta\sigma$.

If instead $\partial\sigma \subseteq \omega \subseteq \sigma$, then it follows from Theorem 2.11 and Lemma 4.1 (1) that $\partial\omega \subseteq \omega \subseteq \sigma \subseteq \eta\sigma = \eta\omega$. Hence if $\partial\omega(a, X) = \eta\omega(a, X)$ with $X \in \{A, B\}$, then $\omega(a, X) = \sigma(a, X)$ and so $\partial\sigma(a, X) = \eta\sigma(a, X)$. As before, the result follows from Corollary 2.4. $\square$

Corollary 4.6. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. If $a \in B$ and $\partial K = \eta K$ for any*

$$K \in \{S_\partial(a, A),\ S_\partial(a, B),\ \tau(a, A),\ \tau(a, B),\ \sigma(a, A),\ \sigma(a, B),\ \varepsilon(a, A),\ \varepsilon(a, B)\},$$

then all these sets coincide.

The above corollary implies Theorem 4.10 in [6]. If the condition $\partial K = \eta K$ for at least one K in the set of subsets of $\mathbb{C}$ in the above result does not hold, then these sets may not all coincide. This is illustrated in the following example.

Example 4.7. With $\mathbb{T}$ the unit circle and $\mathbb{D}$ the open unit disk, let $A = C(\mathbb{T})$, the Banach algebra of all continuous complex valued functions on $\mathbb{T}$, $B = \mathcal{A}(\mathbb{D})$, the subalgebra of A consisting of all elements of A which can be extended to a continuous function on $\overline{\mathbb{D}}$ which is analytic on $\mathbb{D}$, and $f(z) = z$ for all $z \in \overline{\mathbb{D}}$. Consider the set $X = \{S_\partial(f, A),\ S_\partial(f, B),\ \tau(f, A),\ \tau(f, B),\ \sigma(f, A),\ \sigma(f, B),\ \varepsilon(f, A),\ \varepsilon(f, B)\}$. Then $\partial K \neq \eta K$ for each $K \in X$ and the sets in X do not all coincide.

Proof. By [21], Problem 9, page 399, $\sigma(f, A) = \mathbb{T}$ and $\sigma(f, B) = \overline{\mathbb{D}}$, and by [6], Example 3.5, $\tau(f, A) = \mathbb{T} = \tau(f, B)$. Since $\partial\sigma \subseteq S_\partial \subseteq \sigma$, we have $S_\partial(f, A) = \mathbb{T}$, and it follows from Theorem 2.19 that $\mathbb{T} = \partial\sigma(f, B) \subseteq S_\partial(f, B) \subseteq S_\partial(f, A) = \mathbb{T}$, so $S_\partial(f, B) = \mathbb{T}$. Since $\sigma \subseteq \varepsilon \subseteq \eta\sigma$, we have $\overline{\mathbb{D}} = \sigma(f, B) \subseteq \varepsilon(f, B) \subseteq \eta\sigma(f, B) = \overline{\mathbb{D}}$, so $\varepsilon(f, B) = \overline{\mathbb{D}}$. Finally, it follows from [6], Theorem 4.1 that $\mathbb{T} = \sigma(f, A) \subseteq \varepsilon(f, A) \subseteq \varepsilon(f, B) \subseteq \eta\sigma(f, B) = \overline{\mathbb{D}}$, so $\partial K = \eta K$ does not hold for any $K \in X$. $\square$

Theorems 2.8 and 4.5 yield our next main result in this section, which concerns a-inessential elements:

Theorem 4.8. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. Let $\omega\colon A \to K(\mathbb{C})$ be a mapping such that either $\partial\sigma \subseteq \omega \subseteq \sigma$ or ω is a Mobius spectrum on both A and B satisfying $\sigma \subseteq \omega \subseteq \eta\sigma$. Suppose either $\partial\omega(a, A) = \eta\omega(a, A)$ or $\partial\omega(a, B) = \eta\omega(a, B)$ for some $a \in B$. If $b \in B$ is a-inessential (in B), then*

$$\partial\sigma(c, A) = \partial\sigma(c, B) = \omega(c, A) = \omega(c, B) = \eta\sigma(c, A) = \eta\sigma(c, B),$$

where $c = a + b$.

P r o o f. By Theorem 4.5 we may assume that $\partial\sigma(a,B) = \eta\sigma(a,B)$. Then by Theorem 2.8 we have $\partial\sigma(a+b,B) = \eta\sigma(a+b,B)$. The result follows by recalling that $\partial\sigma \subseteq \omega \subseteq \eta\sigma$ and by replacing ω and a by σ and $a+b$, respectively, in Theorem 4.5. $\square$

Corollary 4.9. *Let A be a Banach algebra and B a closed subalgebra of A such that $1 \in B$. If $a \in B$, $\partial K = \eta K$ for any*

$$K \in \{S_\partial(a,A),\ S_\partial(a,B),\ \tau(a,A),\ \tau(a,B),\ \sigma(a,A),\ \sigma(a,B),\ \varepsilon(a,A),\ \varepsilon(a,B)\}$$

and $b \in B$ is a-inessential (in B), then the sets $S_\partial(c,A)$, $S_\partial(c,B)$, $\tau(c,A)$, $\tau(c,B)$, $\sigma(c,A)$, $\sigma(c,B)$, $\varepsilon(c,A)$, $\varepsilon(c,B)$, where $c = a+b$, all coincide.

The above corollary strengthens, see [5], Theorem 3. Note that Corollary 3.4 (2) states that if $\sigma(a,B) = \sigma(a,A)$, then $S_\partial(a+b,A) \setminus \sigma(a,A) = S_\partial(a+b,B) \setminus \sigma(a,B)$, whenever $a,b \in B$ with b an a-inessential element. It now follows from Corollary 4.9 (among other things) that if, in addition, either $\partial S_\partial(a,A) = \eta S_\partial(a,A)$ or $\partial S_\partial(a,B) = \eta S_\partial(a,B)$, then we have equality $S_\partial(a+b,A) = S_\partial(a+b,B)$.

In [10] and [11] the authors presented properties regarding perturbation of the sets of accumulation points of the singular and exponential spectra, respectively, of an element a by an a-inessential element b. We now generalize those results to obtain our final main result in this section (see Theorem 4.10), and conclude by offering an analogue for the boundary spectrum (see Corollary 4.11).

Theorem 4.10. *Let A be a Banach algebra and $\omega\colon A \to K(\mathbb{C})$ a mapping. If $\partial\varepsilon \subseteq \omega \subseteq \eta\varepsilon$, then $\operatorname{acc}\omega(a+b) \subseteq \eta\omega(a)$ for all $a,b \in A$ such that b is a-inessential.*

P r o o f. Using Lemma 4.1, Corollary 2.12 and Theorem 2.9, it follows that

$$\operatorname{acc}\omega(a+b) \subseteq \eta(\operatorname{acc}\varepsilon(a+b)) = \eta(\operatorname{acc}\sigma(a+b)) \subseteq \eta\sigma(a) = \eta\omega(a).$$

$\square$

If A is a semisimple Banach algebra, then the closure of the socle $\operatorname{Soc}A$ of A is a closed, inessential ideal which contains all the rank one elements. Therefore, Theorem 4.10 applies to any rank one element b in a semisimple Banach algebra A (with any $a \in A$). Theorem 4.10 implies Theorems 3.2, 3.3 and 3.5 in [10], as well as Theorems 5.1 and 5.2 in [11]. See also [12], Theorem 5.1 for an analogous result which was proven in the context of regularities. In addition we have:

Corollary 4.11. *Let A be a Banach algebra and let $a,b \in A$ with b an a-inessential element. Then*

$$\operatorname{acc} S_\partial(a+b) \subseteq \eta S_\partial(a).$$

An example used in [10], page 355, can be utilized to show that the condition that $ab - ba$ belongs to the relevant closed inessential ideal in the definition of "a-inessential element" cannot be omitted in the above corollary.

Example 4.12. There exists a Banach algebra A containing a closed inessential ideal I and elements a and b such that $b \notin I$ and $\sigma(b+I) = \{0\}$, but $ab - ba \notin I$, and $\operatorname{acc} S_\partial(a+b) \nsubseteq \eta S_\partial(a)$.

P r o o f. Let $U \in \mathcal{L}(l^2)$ be the right shift operator, where $\mathcal{L}(l^2)$ denotes the Banach algebra of bounded linear operators on the sequence space l^2, $A = \mathcal{L}(l^2 \oplus l^2)$ and $T, S \in A$ the operators defined by $T(x,y) = (Uy, x)$ and $S(x,y) = (0,x)$ for all $(x,y) \in l^2 \oplus l^2$. Represented as 2×2 operator matrices, let

$$a = T = \begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then by [5], Example 1, $\sigma(b+I) = \{0\}$, where I is the closed inessential ideal of all compact operators on $l^2 \oplus l^2$, $\sigma(a) = \overline{\mathbb{D}}$ and $\sigma(a+b) = \sqrt{2}\overline{\mathbb{D}}$. Since S is not a compact operator, we have $b \notin I$. Also, $TS - ST = \begin{pmatrix} U & 0 \\ 0 & -U \end{pmatrix}$, and since U is not a compact operator, neither is $TS - ST$. Hence $ab - ba \notin I$. Finally, from (4.1) we note that $\eta S_\partial(a) = \eta\sigma(a) = \overline{\mathbb{D}}$, and $\sqrt{2}\mathbb{T} = \partial\sigma(a+b) \subseteq S_\partial(a+b)$, so $\sqrt{2}\mathbb{T} \subseteq \operatorname{acc} S_\partial(a+b)$, since $\sqrt{2}\mathbb{T}$ is a circle. Therefore $\sqrt{2}\mathbb{T} \subseteq (\operatorname{acc} S_\partial(a+b)) \setminus \eta S_\partial(a)$. $\square$

5. A generalization of a theorem by Harte

In [7] (see Theorem 2.15 in this note) Harte proved that the *Weyl spectrum*

$$\bigcap_{b \in \mathrm{N}(T)} \sigma(a+b, A)$$

of an element $a \in A$ relative to a bounded homomorphism $T\colon A \to B$ with closed range is contained in the connected hull of the *Fredholm spectrum* $\sigma(Ta, B)$ of a. Analogues of this result for the exponential and singular spectra were obtained by Lindeboom and Raubenheimer in [10]. A version of Harte's theorem has been proven for homomorphisms having closed range and satisfying the Riesz property in [14], Corollary 7.6, and this result has been extended to homomorphisms having the strong Riesz property in [22], Corollary 2.2. In this section we generalize these results by considering abstract mappings (see Theorems 5.3 and 5.4) and establish an analogue for the boundary spectrum (see Corollary 5.5).

Theorem 5.1. *Let A and B be Banach algebras, $T\colon A \to B$ a homomorphism that is bounded below and $\omega\colon A \cup B \to K(\mathbb{C})$ a mapping such that ω_A is a Mobius spectrum on A. If $\sigma_A \subseteq \omega_A \subseteq \eta\sigma_A$ and $\sigma_B \subseteq \omega_B$, then*

$$\omega(a, A) \subseteq \eta\omega(Ta, B)$$

for all $a \in A$.

Proof. Applying Theorem 2.10 with $\omega_1 = \omega_A$ and $\omega_2 = \sigma_A$, we get that $\partial\omega_A \subseteq \sigma_A$. From $\sigma_A \subseteq \omega_A$ we also have that $\operatorname{int}\sigma_A \subseteq \operatorname{int}\omega_A$, so $\partial\omega_A \subseteq \partial\sigma_A$. Theorem 2.13 now implies that $\partial\omega(a, A) \subseteq \sigma(Ta, B)$ for all $a \in A$. Since $\sigma_B \subseteq \omega_B$, it follows that $\partial\omega(a, A) \subseteq \omega(Ta, B)$ for all $a \in A$, and hence the result follows from Theorem 2.5 (2). $\square$

Theorem 5.1 implies, among other things, Lemma 2.4 in [10], and Corollary 4.6 in [14], where $\omega = \varepsilon$.

Theorem 5.2. *Let A and B be Banach algebras, $T : A \to B$ a surjective bounded homomorphism and $\omega\colon A \cup B \to K(\mathbb{C})$ a mapping. If $\omega \subseteq \varepsilon$ and $\partial\varepsilon_B \subseteq \omega_B$, then*

$$\bigcap_{b \in \mathrm{N}(T)} \omega(a + b, A) \subseteq \eta\omega(Ta, B)$$

for all $a \in A$.

Proof. Let $a \in A$. Since $\omega_A \subseteq \varepsilon_A$, it follows from Theorem 2.14 that $\bigcap_{b \in \mathrm{N}(T)} \omega(a + b, A) \subseteq \varepsilon(Ta, B)$. By Theorem 2.11 $\varepsilon(Ta, B) \subseteq \eta\sigma(Ta, B)$, and since $\partial\varepsilon_B \subseteq \omega_B \subseteq \varepsilon_B$, we have from Lemma 4.1 (1) that $\eta\sigma(Ta, B) = \eta\omega(Ta, B)$, which yields the result. $\square$

We now obtain our first main result in this section by applying both Theorems 5.1 and 5.2.

Theorem 5.3. *Let A and B be Banach algebras, $T\colon A \to B$ a bounded homomorphism with closed range, $J = \mathrm{N}(T)$ and $\omega\colon A \cup B \cup A/J \to K(\mathbb{C})$ a map such that $\omega_{A/J}$ is a Mobius spectrum on A/J. If $\omega_A \subseteq \varepsilon_A$, $\sigma_B \subseteq \omega_B$ and $\sigma_{A/J} \subseteq \omega_{A/J} \subseteq \varepsilon_{A/J}$, then*

$$\bigcap_{b \in \mathrm{N}(T)} \omega(a + b, A) \subseteq \eta\omega(Ta, B)$$

for all $a \in A$.

P r o o f. Consider the homomorphisms $\pi\colon A \to A/J$, $\pi a = a + J$, and $\widehat{T}\colon A/J \to B$, $\widehat{T}(a+J) = Ta$. Then π is surjective and bounded, and $\widehat{T}$ is bounded below, by Lemma 2.1. Since $\omega_A \subseteq \varepsilon_A$, $\omega_{A/J} \subseteq \varepsilon_{A/J}$ and $\partial\varepsilon_{A/J} \subseteq \sigma_{A/J} \subseteq \omega_{A/J}$, an application of Theorem 5.2 to the homomorphism $\pi\colon A \to A/J$ with $\mathrm{N}(\pi) = J = \mathrm{N}(T)$ yields

$$\bigcap_{b\in \mathrm{N}(T)} \omega(a+b, A) \subseteq \eta\omega(a+J, A/J) \tag{5.1}$$

for all $a \in A$. Since $\omega_{A/J}$ is a Mobius spectrum on A/J, $\sigma_{A/J} \subseteq \omega_{A/J} \subseteq \varepsilon_{A/J} \subseteq \eta\sigma_{A/J}$ and $\sigma_B \subseteq \omega_B$, an application of Theorem 5.1 to the homomorphism $\widehat{T}\colon A/J \to B$ yields

$$\omega(a+J, A/J) \subseteq \eta\omega(\widehat{T}(a+J), B) = \eta\omega(Ta, B) \tag{5.2}$$

for all $a \in A$. The result follows by combining (5.1) and (5.2). $\square$

Theorem 5.3 generalizes the first part of Theorem 2.15 and Theorem 2.5 in [10] (for the case ε). By utilizing Theorem 2.15 itself, we can generalize its second part to obtain our second main result in this section.

Theorem 5.4. *Let A and B be Banach algebras and $T\colon A \to B$ a homomorphism which is either bounded with closed range or satisfies the strong Riesz property, and $\omega\colon A\cup B \to K(\mathbb{C})$ a mapping such that $\omega_A \subseteq \sigma_A$ and $\partial\varepsilon_B \subseteq \omega_B \subseteq \sigma_B$. Then*

$$\bigcap_{b\in \mathrm{N}(T)} \omega(a+b, A) \subseteq \eta\omega(Ta, B)$$

for all $a \in A$.

P r o o f. Let $a \in A$. Since $\partial\varepsilon_B \subseteq \omega_B \subseteq \sigma_B \subseteq \eta\varepsilon_B$, it follows from Lemma 4.1 (1) that $\eta\sigma(Ta, B) = \eta\omega(Ta, B)$. Using $\omega_A \subseteq \sigma_A$ and applying Theorem 2.15, we obtain

$$\bigcap_{b\in \mathrm{N}(T)} \omega(a+b, A) \subseteq \bigcap_{b\in \mathrm{N}(T)} \sigma(a+b, A) \subseteq \eta\sigma(Ta, B) = \eta\omega(Ta, B).$$

$\square$

The approach to proving Theorem 5.4 is similar to that used in obtaining Theorem 5.4 in [12], although we note that the latter involved regularities. Theorem 5.4 generalizes Corollary 7.6 in [14], Corollary 2.2 in [22] and Theorem 2.5 in [10] (for the case τ), and also yields the following for the boundary spectrum.

Corollary 5.5. *Let A and B be Banach algebras and $T\colon A \to B$ a homomorphism which is either bounded with closed range or satisfies the strong Riesz property. Then*

$$\bigcap_{b \in \mathrm{N}(T)} S_\partial(a+b, A) \subseteq \eta S_\partial(Ta, B)$$

for all $a \in A$.

The above result, (4.1) and Corollary 2.22 imply the following:

Corollary 5.6. *Let A and B be Banach algebras and $T\colon A \to B$ a bounded homomorphism with closed range. If $a \in A$ and $\sigma(Ta, B)$ is simply connected, then*

$$\bigcap_{b \in \mathrm{N}(T)} S_\partial(a+b, A) \subseteq S_\partial(Ta, B).$$

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