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CONSTRUCTING MODULAR FORMS FROM HARMONIC MAASS JACOBI FORMS

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Abstract. We construct a family of modular forms from harmonic Maass Jacobi forms by considering their Taylor expansion and using the method of holomorphic projection. As an application we present a certain type Hurwitz class relations which can be viewed as a generalization of Mertens' result in M. H. Mertens (2016).

Keywords: modular form; harmonic Maass Jacobi form; holomorphic projection; Hurwitz class number

MSC 2020: 11F50, 11F37, 11F30

1. Introduction

Modular forms have played prominent roles in many mathematical and physical fields. Constructing large amount of modular forms is important. In [4], Cohen got a family of modular forms from two modular forms by the so called *Rankin-Cohen bracket*. In [5], Eichler and Zagier constructed modular forms from Jacobi forms by considering the Taylor expansion of Jacobi forms. In [9], Mertens constructed modular forms from harmonic Maass forms by using the Rankin-Cohen bracket and holomorphic projection. As applications Mertens gave new proofs for several class number relations including both classical and relatively new ones, and similar results for mock theta functions.

Recently the theory of harmonic Maass Jacobi forms has been extensively studied. In this article we construct modular forms from harmonic Maass Jacobi forms by

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considering their Taylor expansion and using the method, see [9]. This can be viewed as a generalization of the method of Eichler and Zagier, see [5].

Before we state the main result of this paper, we give some notations. For a complex number t we write $e(t) := \mathrm{e}^{2\pi i t}$. For the elements $\tau = x + \mathrm{i}y$ of the complex upper half plane $\mathbb{H}$ and z of $\mathbb{C}$, we set $q = e(\tau)$, $\zeta = e(z)$, respectively. We put $\Gamma = SL_2(\mathbb{Z})$. Also, for a positive integer $N > 1$, set the congruence groups

$$\Gamma_0(N) = \left\{\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma\colon\ c \equiv 0 \bmod N\right\},$$
$$\Gamma_1(N) = \left\{\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma\colon\ a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N\right\}.$$

For nonnegative integers a, b, introduce the homogenous polynomials $p_{2a}^{(b)}(X, Y)$, $P_{a,b}(X, Y)$ as

$$\frac{(a+b)!}{(2a)!\,(b-1)!} p_{2a}^{(b)}(X, Y) = \text{coefficients of } t^{2a} \text{ in } (1 - Xt + Yt^2)^{-b}$$

and

$$P_{a,b}(X, Y) := \sum_{0 \leqslant j \leqslant a-2} \binom{j+b-2}{j} X^j (X+Y)^{a-j-2},$$

respectively.

With the above notations the main theorem of this article follows.

Theorem 1.1. *Let $\varphi(\tau, z)$ be a harmonic Maass Jacobi form of weight $k \geqslant 2$ and index m on Γ whose Fourier expansion is*

$$\varphi(\tau, z) = \sum_{r^2 \leqslant 4mn} c(n, r) q^n \zeta^r + \frac{1}{k - \frac{3}{2}} \left(\frac{m}{\pi y}\right)^{k-3/2} \sum_{\substack{r \in \mathbb{Z} \\ 4m | r^2}} c^0\left(\frac{r^2}{4m}, r\right) q^{r^2/4m} \zeta^r$$
$$+ \sum_{r^2 > 4mn} (r^2 - 4mn)^{k-3/2} \Gamma\left(\frac{3}{2} - k, \frac{\pi(r^2 - 4mn)y}{m}\right) c^-(n, r) q^n \zeta^r.$$

Then for each nonnegative integer ν, the function

$$\sum_{n \geqslant 0} \left(\sum_{r^2 \leqslant 4mn} p_{2\nu}^{(k-1)}(r, mn) c(n, r)\right) q^n + f_{2\nu}^0(\tau) + f_{2\nu}^-(\tau)$$

is a modular form on Γ of weight $k+2\nu$ if $k+2\nu>2$ and a quasimodular form on Γ of weight 2 if $k+2\nu=2$, where $f_{2\nu}^0(\tau)$, $f_{2\nu}^-(\tau)$ are given by

$$
\begin{aligned}
f_{2\nu}^0(\tau) &= \frac{\nu!\,\Gamma(\frac{5}{2}-k)}{(k+2\nu-2)!\,(k-\frac{3}{2})} \sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} \sum_{0\leqslant\lambda\leqslant\nu} \binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda} \\
&\quad\times \frac{\Gamma(\frac{1}{2}+2\nu-\lambda)}{\Gamma(\frac{5}{2}-k-\lambda)} |r|^{2(k+\nu-3/2)} c^0\Big(\frac{r^2}{4m}, r\Big) q^{r^2/4m}, \\
f_{2\nu}^-(\tau) &= \nu!\,\Gamma\Big(\frac{3}{2}-k\Big) \sum_{n>0} \sum_{r^2>4mn} \sum_{0\leqslant\lambda\leqslant\nu} \binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda} \\
&\quad\times \big(|r|^{2(-\nu+1/2)} P_{k+2\nu,5/2-k-\lambda}(4mn, r^2-4mn) \\
&\quad - r^{2(\nu-\lambda)}(r^2-4mn)^{k+\lambda-3/2}\big) c^-(n,r) q^n.
\end{aligned}
$$

In particular, for weight $k=2$, the function

$$
\begin{aligned}
&\sum_{n\geqslant 0} \sum_{r^2\leqslant 4mn} p_{2\nu}^{(k-1)}(r, mn) c(n,r) q^n + \frac{2^{-2\nu}\sqrt{\pi}(2\nu)!}{\nu!} \\
&\times \Bigg(\sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} 2|r|^{2\nu+1} c_0\Big(\frac{r^2}{4m}, r\Big) q^{r^2/4m} + \sum_{n>0} \sum_{r^2>4mn} \big(|r| - \sqrt{r^2-4mn}\big)^{2\nu+1} c^-(n,r) q^n \Bigg)
\end{aligned}
$$

is a modular form on Γ of weight $2+2\nu$ if $\nu>0$ and a quasimodular form on Γ of weight 2 if $\nu=0$.

The classical example of a harmonic Maass Jacobi form is $E_{2,1}^*(\tau,z)$, i.e., the non-holomorphic Jacobi-Eisenstein series of weight 2 and index 1 which was constructed by Choie in [3]. Applying the Hecke type V-operator on $E_{2,1}^*(\tau,z)$ and using Theorem 1.1 for the weight k equal to 2 we obtain:

Theorem 1.2. *For positive integers m, l and nonnegative integers s, ν, the function*

$$
\begin{aligned}
&\sum_{n\geqslant 0} \sum_{r\equiv s \bmod l} p_{2\nu}^{(1)}(n, mr) \sum_{a|(n,m,r)} aH\Big(\frac{4mn-r^2}{a^2}\Big) q^n \\
&\qquad + \frac{(2\nu)!}{2\nu!} \sum_{n>0} \big(\lambda_{2\nu+1}^{(m,s,l)}(n) + \lambda_{2\nu+1}^{(m,-s,l)}(n)\big) q^n
\end{aligned}
$$

is a modular form (quasimodular form) of weight $2+2\nu$ for $\nu>0$ ($\nu=0$) on $\Gamma_0(l^2)\cap\Gamma_1(l)$ if $l\nmid s$ and on $\Gamma_0(l^2)$ otherwise. Here $H(n)$ denotes the Hurwitz class *number for nonnegative integer* $n \equiv 0, 3 \bmod 4$,

$$\lambda_{2\nu+1}^{(m,t,l)}(n) = \sum_{\substack{h|mn \\ h+mn/h \equiv t \bmod l}} \sigma_0\Big(\Big(m, n, h + \frac{mn}{h}\Big)\Big) \min\Big\{h, \frac{mn}{h}\Big\}^{1+2\nu}$$

with $(\cdot,\cdot,\cdot)$ *being the largest common divisor of integers in the parentheses.*

Remark 1.3. In [9] Mertens showed that

$$\sum_{n\geqslant 0} \sum_{r \equiv s \bmod l} p_{2\nu}^{(1)}(n,r) H(4n - r^2) q^n + \sum_{n \geqslant 0} a(n) q^n$$

are modular forms or quasimodular forms with $a(n)$ behaving like sums of smaller divisors, see [9], Proposition 7.2. Theorem 1.2 generalizes Mertens' result and, for the case $m = 1$, one can see it simplifies the related result of Mertens.

The outline of this article is as follows: In Section 2 we give a brief account of some basic facts about harmonic Maass forms, harmonic Maass Jacobi forms, and the holomorphic projection which is the key to prove the related results in [8]. After this we prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

2. Preliminaries

2.1. Harmonic Maass forms and harmonic Maass Jacobi forms. Harmonic Maass forms and harmonic Maass Jacobi forms have vast applications in many fields of mathematics and physics such as the partition theory, the theory of Lie-superalgebras, and the quantum black holes to name a few. In this subsection, we briefly review some basic facts about harmonic Maass forms and harmonic Maass Jacobi forms. Readers are referred to [1] for background materials.

Definition 2.1. Let $k \in \frac{1}{2}\mathbb{Z}$ and N be a positive integer. If $k \in \frac{1}{2} + \mathbb{Z}$ then we assume that $4 \mid N$. A smooth function $f\colon \mathbb{H} \to \mathbb{C}$ is called a *harmonic weak Maass form of weight* k, level N if it has the following properties:

(1) $(f|_k\gamma)(\tau) = f(\tau)$ for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The weight k slash operator is defined as

$$(f|_k\gamma)(\tau) := \begin{cases} (c\tau + d)^{-k} f\Big(\dfrac{a\tau + b}{c\tau + d}\Big) & \text{if } k \in \mathbb{Z}, \\ \Big(\dfrac{c}{d}\Big)\varepsilon_d\big(\sqrt{c\tau + d}\big)^{-2k} f\Big(\dfrac{a\tau + b}{c\tau + d}\Big) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

Here (c/d) is the Legendre symbol, $\sqrt{\tau}$ is the principal branch of the holomorphic square root, and

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ i & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

(2) $\Delta_k f \equiv 0$, where Δ_k is the k-hyperbolic Laplacian

$$\Delta_k := (\tau - \overline{\tau})^2 \partial_{\tau\overline{\tau}} + k(\tau - \overline{\tau})\partial_{\overline{\tau}}.$$

(3) f grows at most linearly exponentially approaching the cusps of $\Gamma_0(N)$.

The vector space of harmonic weak Maass forms of weight k, level N is denoted by $\mathcal{H}_k(N)$. By [1], Lemma 4.3, every f of $\mathcal{H}_k(N)$ has the Fourier expansion

$$f(\tau) = \sum_{n \gg -\infty} c_f^{+}(n) q^n + \frac{(4\pi y)^{1-k}}{k-1} \overline{c_f^{-}(0)} + \sum_{n=1}^{\infty} \overline{c_f^{-}(n)} n^{k-1} \Gamma(1-k, 4\pi n y) q^{-n},$$

where $\Gamma(\alpha; x) := \int_x^{\infty} t^{\alpha-1} \mathrm{e}^{-t}\, \mathrm{d}t$ is the incomplete gamma function.

In [2] Bringmann and Richter defined *harmonic Maass Jacobi forms.* Let $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$ be the Jacobi group, i.e., the semi-direct product of Γ and $\mathbb{Z}^2$ with the group law

$$(A_1, X_1)(A_2, X_2) = (A_1 A_2, X_1 A_2 + X_2).$$

For fixed integers k and m, define the Jacobi-slash operator of weight k, index m as

$$(\varphi|_{k,m} A)(\tau, z) := (c\tau + d)^{-k} e\Big(\frac{-mc(z + \lambda\tau + \mu)^2}{c\tau + d} + m\lambda^2\tau + 2m\lambda z\Big)\varphi(\tau, z) \tag{2.1}$$

with $A = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu)\right) \in \Gamma^J$. Let $\Gamma^J(\mathbb{R})$ be the real Jacobi group. One can extend (2.1) to an action of $\Gamma^J(\mathbb{R})$ on $\mathbb{C}^{\infty}(\mathbb{H} \times \mathbb{C})$. The center of the universal enveloping algebra of $\Gamma^J(\mathbb{R})$ is generated by a linear element and a cubic element which is called *Casimir element.* The action of the Casimir element under $|_{k,m}^{\mathbb{R}}$ is given by the operator

$$\begin{aligned} \mathcal{C}^{k,m} := {} & -2(\tau - \overline{\tau})^2 \partial_{\tau\overline{\tau}} - (2k-1)(\tau - \overline{\tau})\partial_{\overline{\tau}} + \frac{\tau - \overline{\tau}}{4\pi i m}\partial_{\overline{\tau} zz} + \frac{k(\tau - \overline{\tau})}{4\pi i m}\partial_{z\overline{z}} \\ & + \frac{(\tau - \overline{\tau})(z - \overline{z})}{4\pi i m}\partial_{zz\overline{z}} - 2(\tau - \overline{\tau})(z - \overline{z})\partial_{\tau\overline{z}} + (1-k)(z - \overline{z})\partial_{\overline{z}} \\ & + \frac{(\tau - \overline{\tau})}{4\pi i m}\partial_{\tau\overline{z}\overline{z}} + \Big(\frac{(z - \overline{z})^2}{2} + \frac{k(\tau - \overline{\tau})}{4\pi i m}\Big)\partial_{\overline{z}\overline{z}\overline{z}} + \frac{(\tau - \overline{\tau})(z - \overline{z})}{4\pi i m}\partial_{z\overline{z}\overline{z}}. \end{aligned}$$

Definition 2.2. Let $k \in \mathbb{Z}$ and m be a positive integer. A function $\varphi\colon \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is called a *harmonic Maass-Jacobi form of weight* k and index m on Γ if φ is real-analytic in $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ satisfies the following conditions:

(1) For all $A \in \Gamma^J$, $\varphi|_{k,m}A = \varphi$.
(2) We have that $\mathcal{C}^{k,m}(\varphi) = 0$.
(3) We have $\varphi(\tau, z) = O(\mathrm{e}^{ay}\mathrm{e}^{2\pi m v^2/y})$ for some $a > 0$, where y, u are the imaginary parts of τ, z, respectively.

Let $\widehat{\mathbb{J}}_{k,m}(\Gamma)$ denote the space of harmonic Maass-Jacobi forms of weight k and index m on Γ which are holomorphic in z. From the fact that it is annihilated by the Casimir operator and the growth condition, one can see that, for $\varphi(\tau, z) \in \widehat{\mathbb{J}}_{k,m}(\Gamma)$, φ has the Fourier expansion

$$\begin{aligned}(2.2)\quad \varphi(\tau, z) = &\sum_{r^2-4mn\ll\infty} c^+(n,r)q^n\zeta^r + \frac{1}{k-\frac{3}{2}}\Big(\frac{m}{\pi y}\Big)^{k-3/2}\sum_{4m|r^2} c^0\Big(\frac{r^2}{4m}, r\Big)q^{r^2/4m}\zeta^r \\ &+ \sum_{r^2-4mn>0} c^-(n,r)(r^2-4mn)^{k-3/2}\Gamma\Big(\frac{3}{2}-k, \frac{\pi(r^2-4mn)y}{m}\Big)q^n\zeta^r.\end{aligned}$$

If the Fourier expansion of (2.2) is only over $r^2 \leqslant 4mn$ and the non-holomorphic part vanishes then $\varphi(\tau, z)$ is a holomorphic Jacobi form of weight k and index m on Γ as in [5].

Let $\varphi(\tau, z) = \sum_{n,r} c(n,r,y)q^n\zeta^r \in \widehat{\mathbb{J}}_{k,1}(\Gamma)$. By [5], page 58, $\varphi(\tau, z)$ has the theta expansion

$$\varphi(\tau, z) = \sum_{u \bmod 2} h_u(\tau)\theta_{1,u}(\tau, z),$$

where

$$\theta_{1,u}(\tau, z) := \sum_{\substack{r\in\mathbb{Z}\\ r\equiv u \bmod 2}} q^{r^2/4}\zeta^r$$

and

$$h_u(\tau) := \sum_N C_u(N,y)q^{N/4} \quad \text{with } C_u(N) := c\Big(\frac{N+u^2}{4}, u, y\Big).$$

By [5], Theorem 5.4, the function $h(\tau) := \sum_{u \bmod 2} h_u(4\tau)$ transforms like a modular form of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$.

For $\varphi(\tau, z) \in \widehat{\mathbb{J}}_{k,1}(\Gamma)$ with the Fourier expansion (2.2), the function

$$\begin{aligned}(2.3)\qquad h(\tau) = \sum_{N\gg-\infty}\Big(&C^+(N)q^N + \frac{(4\pi y)^{3/2-k}}{k-\frac{3}{2}}C^0(0) \\ &+ N^{k-3/2}C^-(N)\Gamma\Big(\frac{3}{2}-k, 4\pi N y\Big)q^{-N}\Big)\end{aligned}$$

is a harmonic Maass form of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$ where $C_u^{\pm}(N) = C^{\pm}(r,n)$ with $r^2 \mp 4n = N$. For more relations between the harmonic Maass forms and harmonic Maass Jacobi forms, see [2].

2.2. Differential operators for modular forms and Jacobi forms. The derivative of a modular form is not a modular form. In [4], Cohen constructed a family of modular forms from two modular forms by using a certain type differential operator which is called the *Cohen bracket.* In the spirit of Cohen's idea, Eichler and Zagier got a series of modular forms from the Jacobi forms by using the *Taylor development operator* in [5]. Here we introduce these differential operators.

Definition 2.3. Let f, g be smooth functions defined on the complex upper half plane. Then for nonnegative integers k, l, ν, we define the νth Rankin-Cohen bracket of f and g as

$$[f,g]_\nu := \sum_{0\leqslant\mu\leqslant\nu} (-1)^\mu \binom{k+\nu-1}{\nu-\mu}\binom{l+\nu-1}{\mu} D^{(\mu)} f D^{(\nu-\mu)} g,$$

where $D^{(\mu)} := (1/2\pi \mathrm{i})^\mu \mathrm{d}^\mu/\mathrm{d}\tau^\mu$, and for $n, m \in \frac{1}{2}\mathbb{Z}$,

$$\binom{n}{m} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}.$$

It is well known that if f, g are smooth functions transforming like modular forms of weights k and l on a congruence subgroup, respectively, then for each nonnegative integer ν, the function $[f,g]_\nu$ satisfies the weight $k+l+2\nu$ modularity on the same group.

Definition 2.4. Let $\varphi(\tau,z)$ be a smooth function transforming like a Jacobi form of weight k and index m. For each nonnegative integer ν, define the (2ν)th Taylor development operator $\mathcal{D}_{2\nu}$ on $\varphi(\tau,z)$ by

$$(\mathcal{D}_{2\nu}\varphi)(\tau) = \frac{(2\nu)!}{(k+\nu-2)!} \sum_{0\leqslant\mu\leqslant\nu} \frac{(-m)^\mu (k+2\nu-\mu-2)!}{\mu!} (D^{(\mu)}\chi_{2\nu-2\mu})(\tau),$$

where for any nonnegative integer t, $\chi_t(\tau) = 1/t!\,(\partial^t\varphi/\partial z^t)(\tau,0)$.

In [5] Eichler and Zagier proved:

Theorem 2.5 ([5], Theorem 3.1)**.** *Let $\varphi(\tau,z) = \sum_{r^2\leqslant 4mn} c(n,r)q^n\zeta^r$ be a holomorphic Jacobi form of weight k and index m on Γ. Then*

$$(\mathcal{D}_{2\nu}\varphi)(\tau) = \sum_{n\geqslant 0}\bigg(\sum_{r^2\leqslant 4mn} p_{2\nu}^{(k-1)}(r,mn)c(n,r)\bigg)q^n$$

is a modular form of weight $k+2\nu$ on Γ.

Remark 2.6. For a smooth function $\varphi(\tau, z)$ with respect to τ satisfying the transformation law of the Jacobi form of weight k and index m on some congruence subgroup, the function $(\mathcal{D}_{2\nu}\varphi)(\tau)$ transforms under the same congruence subgroup like a modular form of weight $k+2\nu$. The Fourier coefficients of its holomorphic part are still as in Theorem 2.5. We discuss the Fourier expansion of its non-holomorphic part in Proposition 3.1.

Let $h(\tau)$ be as in (2.3) and let $\theta(\tau) := \sum_{r\in\mathbb{Z}} q^{r^2}$ be the usual theta function. For each nonnegative integer ν, the function $[h, \theta]_\nu(\tau)$ satisfies the weight $k + 2\nu$ modularity for $\Gamma_0(4m)$. In addition, if $\varphi(\tau, z) \in \widehat{\mathbb{J}_{k,1}}(\Gamma)$, then by [5], Theorem 5.5 we have $([h, \theta]_\nu | U(4))(\tau) = \nu!\,(\mathcal{D}_{2\nu}\varphi)(\tau)$, where the $U(N)$ operator is defined as $\sum_{n\in\mathbb{Z}} c(n, y)q^n \mid U(N) := \sum_{n\in\mathbb{Z}} c(Nn, y/N)q^n$.

2.3. Holomorphic projection. The holomorphic projection operator was introduced by Sturm in [10] and further developed by Gross and Zagier in [6]. In [7], Imamoglu-Raum-Richter extend this to the vector-valued modular form case. Applying this operator they found simple recursions for the Fourier coefficients of Ramanujan's mock theta functions. Motivated by Imamoglu-Raum-Richter, in [9] Mertens proved the Eichler-Selberg type relations for all harmonic Maass forms of weight $\frac{3}{2}$. Here we introduce the holomorphic projection briefly.

Definition 2.7. Let G be a congruence subgroup, $\kappa_1 := \mathrm{i}\infty, \ldots, \kappa_m$ the cusps of G and $\gamma_j\kappa_j = \mathrm{i}\infty$, where $\gamma_j \in \Gamma$ for $1 \leqslant j \leqslant m$. A smooth function $f = \sum_{n\in\mathbb{Z}} a_f(n, y)q^n$ transforms like a modular form of weight $k \geqslant 2$ on G such that for some positive integers δ, ε,

(1) $f(\gamma_j^{-1}\omega)(\mathrm{d}\tau/\mathrm{d}\omega)^{k/2} = c_0^{(j)} + O(\Im(\omega)^{-\delta})$ for all $1 \leqslant j \leqslant m$ and $\omega = \gamma_j\tau$;

(2) $a_f(n, y) = O(y^{1-k+\varepsilon})$ as $y \to 0$ for all $n > 0$.

Then we define the holomorphic projection of f by

$$(\pi_{\mathrm{hol}} f)(\tau) := (\pi_{\mathrm{hol}}^k f)(\tau) = c_0^{(1)} + \sum_{n=1}^{\infty}\biggl(\frac{(4\pi n)^{k-1}}{(k-2)!}\int_0^\infty a_f(n, y)\mathrm{e}^{-4\pi n y}y^{k-2}\,\mathrm{d}y\biggr)q^n.$$

Proposition 2.8 ([7], Theorem 3.3). *Let $f\colon \mathbb{H} \to \mathbb{C}$ be as in Definition 2.7.*

(1) *If f is holomorphic, then $\pi_{\mathrm{hol}} f = f$.*

(2) *We have that $\pi_{\mathrm{hol}} f$ is a modular form of weight k on G if $k > 2$ and $\pi_{\mathrm{hol}} f$ is a quasimodular form of weight 2 on G.*

3. Constructing modular forms from harmonic Maass Jacobi forms

In this section we prove Theorem 1.1. First we will study a harmonic Maass Jacobi form under the Taylor development operator. In the calculations we quote some related results of [9]. Readers can find further details in [8] or [9].

Proposition 3.1. *Let $\varphi(\tau,z)\in\widehat{\mathbb{J}}_{k,m}(\Gamma)$ whose Fourier expansion is*

$$\varphi(\tau,z)=\sum_{r^2\leqslant 4mn}c^+(n,r)q^n\zeta^r+\frac{1}{k-\frac{3}{2}}\Big(\frac{m}{\pi y}\Big)^{k-3/2}\sum_{4m|r^2}c^0\Big(\frac{r^2}{4m},r\Big)q^{r^2/4m}\zeta^r$$
$$+\sum_{r^2>4mn}(r^2-4mn)^{k-3/2}\Gamma\Big(\frac{3}{2}-k,\frac{\pi(r^2-4mn)y}{m}\Big)c^-(n,r)q^n\zeta^r.$$

Then for each nonnegative integer ν, the function

$$(\mathcal{D}_{2\nu}\varphi)(\tau)=\sum_{n\geqslant 0}\Big(\sum_{r^2\leqslant 4mn}p_{2\nu}^{(k-1)}(r,mn)c^+(n,r)\Big)q^n+F_{2\nu}^0(\tau)+F_{2\nu}^-(\tau)$$

transforms like a modular form of weight $k+2\nu$ on Γ, where

$$F_{2\nu}^0(\tau)=\frac{\nu!}{k-\frac{3}{2}}\sum_{\substack{r\in\mathbb{Z}\\4m|r^2}}c^0\Big(\frac{r^2}{4m},r\Big)\binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}\Big(\frac{\pi y}{m}\Big)^{3/2-k-\lambda}q^{r^2/4m},$$

$$F_{2\nu}^-(\tau)=\nu!\sum_{n\gg-\infty}\sum_{r^2>4mn}c^-(n,r)\sum_{0\leqslant\lambda\leqslant\nu}\frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}\binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}$$
$$\times r^{2(\nu-\lambda)}(r^2-4mn)^{k+\lambda-3/2}\Gamma\Big(\frac{3}{2}-k-\lambda,\frac{\pi(r^2-4mn)y}{m}\Big)q^n.$$

P r o o f. For $\varphi(\tau,z)\in\widehat{\mathbb{J}}_{k,m}(\Gamma)$, whose Fourier expansion is as in Proposition 3.1, we have

$$(\mathcal{D}_{2\nu}\varphi)(\tau)=\sum_{n\geqslant 0}\Big(\sum_{r^2\leqslant 4mn}p_{2\nu}^{(k-1)}(r,mn)c^+(n,r)\Big)q^n+F_{2\nu}^0(\tau)+F_{2\nu}^-(\tau),\tag{3.1}$$

where

$$F_{2\nu}^0(\tau)=\mathcal{D}_{2\nu}\Big(\frac{1}{k-\frac{3}{2}}\Big(\frac{m}{\pi}\Big)^{k-3/2}\sum_{\substack{r\in\mathbb{Z}\\4m|r^2}}c^0\Big(\frac{r^2}{4m},r\Big)q^{r^2/4m}\zeta^r\Big)$$
$$=\frac{(2\nu)!}{(k-\frac{3}{2})(k+\nu-2)!}\Big(\frac{m}{\pi}\Big)^{k-3/2}\sum_{\substack{r\in\mathbb{Z}\\4m|r^2}}c^0\Big(\frac{r^2}{4m},r\Big)$$
$$\times\sum_{0\leqslant\mu\leqslant\nu}\frac{(-m)^\mu(k+2\nu-\mu-2)!}{\mu!\,(2\nu-2\mu)!}r^{2(\nu-\mu)}D^{(\mu)}\Big(y^{3/2-k}c^0\Big(\frac{r^2}{4m},r\Big)q^{r^2/4m}\Big)$$

and

$$F^-_{2\nu}(\tau) = \mathcal{D}_{2\nu}\Big(\sum_{r^2>4mn}(r^2-4mn)^{k-3/2}\Gamma\Big(\frac{3}{2}-k,\frac{\pi(r^2-4mn)y}{m}\Big)c^-(n,r)q^n\zeta^r\Big)$$

$$= \frac{(2\nu)!}{(k+\nu-2)!}\sum_{r^2>4mn}(r^2-4mn)^{k-3/2}c^-(n,r)$$

$$\times \sum_{0\leqslant\mu\leqslant\nu}\frac{(-m)^\mu(k+2\nu-\mu-2)!}{\mu!\,(2\nu-2\mu)!}$$

$$\times D^{(\mu)}\Big(\Gamma\Big(\frac{3}{2}-k,\frac{\pi(r^2-4mn)y}{m}\Big)q^n\Big).$$

We calculate the Fourier expansion of the non-holomorphic part. Applying the Leibniz law and [8], Lemmas V.1.4, V.1.6, we get

$$\text{(3.2)}\quad D^{(\mu)}(y^{3/2-k}q^{r^2/4m})$$
$$= \sum_{0\leqslant\lambda\leqslant\mu}\binom{\mu}{\lambda}\frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)}\Big(-\frac{1}{4\pi}\Big)^\lambda\Big(\frac{r^2}{4m}\Big)^{\mu-\lambda}y^{3/2-k-\lambda}q^{r^2/4m}$$

and

$$\text{(3.3)}\quad D^{(\mu)}\Big(\Gamma\Big(\frac{3}{2}-k,\frac{\pi(r^2-4mn)y}{m}\Big)q^n\Big)$$
$$= D^{(\mu)}\Big(\Gamma\Big(\frac{3}{2}-k,4\pi\frac{(r^2-4mn)y}{4m}\Big)q^{-(r^2-4mn)/4m+r^2/4m}\Big)$$
$$= \sum_{0\leqslant\lambda\leqslant\mu}\binom{\mu}{\lambda}D^{(\lambda)}\Big(\Gamma\Big(\frac{3}{2}-k,4\pi\frac{(r^2-4mn)y}{4m}\Big)q^{-(r^2-4mn)/4m}\Big)D^{(\mu-\lambda)}q^{r^2/4m}$$
$$= \sum_{0\leqslant\lambda\leqslant\mu}\binom{\mu}{\lambda}\frac{(-1)^\lambda\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}\Big(\frac{r^2-4mn}{4m}\Big)^\lambda$$
$$\times\Gamma\Big(\frac{3}{2}-k-\lambda,\frac{\pi(r^2-4mn)y}{m}\Big)$$
$$\times q^{-(r^2-4mn)/4m}\Big(\frac{r^2}{4m}\Big)^{\mu-\lambda}q^{r^2/4m}$$
$$= \Big(\frac{1}{4m}\Big)^\mu\sum_{0\leqslant\lambda\leqslant\mu}\binom{\mu}{\lambda}\frac{(-1)^\lambda\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}r^{2(\mu-\lambda)}(r^2-4mn)^\lambda$$
$$\times\Gamma\Big(\frac{3}{2}-k-\lambda,\frac{\pi(r^2-4mn)y}{m}\Big)q^n.$$

Inserting (3.2) and (3.3) into (3.1), we obtain

$$
\begin{aligned}
(3.4)\quad F_{2\nu}^{0}(\tau) &= \frac{(2\nu)!}{(k+\nu-2)!(k-\frac{3}{2})}\Big(\frac{m}{\pi}\Big)^{k-3/2}\sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} r^{2(\nu-\mu)}c^0\Big(\frac{r^2}{4m},r\Big)q^{r^2/4m}\\
&\quad\times\sum_{0\leqslant\mu\leqslant\nu}\frac{(-m)^{\mu}(k+2\nu-\mu-2)!}{\mu!(2\nu-2\mu)!}\\
&\quad\times\sum_{0\leqslant\lambda\leqslant\mu}\binom{\mu}{\lambda}\frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)}\Big(-\frac{1}{4\pi}\Big)^{\lambda}\Big(\frac{r^2}{4m}\Big)^{\mu-\lambda}y^{3/2-k-\lambda}\\
&= \frac{(2\nu)!}{(k+\nu-2)!(k-\frac{3}{2})}\sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} r^{2(\nu-\mu)}c^0\Big(\frac{r^2}{4m},r\Big)q^{r^2/4m}\\
&\quad\times\sum_{0\leqslant\mu\leqslant\nu}\Big(-\frac{1}{4}\Big)^{\mu}\frac{(k+2\nu-\mu-2)!}{\mu!\,(2\nu-2\mu)!}\\
&\quad\times\sum_{0\leqslant\lambda\leqslant\mu}(-1)^{\lambda}\binom{\mu}{\lambda}\frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)}\Big(\frac{\pi y}{m}\Big)^{3/2-k-\lambda}\\
&= \frac{(2\nu)!}{(k+\nu-2)!\,(k-\frac{3}{2})}\sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} c^0\Big(\frac{r^2}{4m},r\Big)q^{r^2/4m}\\
&\quad\times\sum_{0\leqslant\lambda\leqslant\nu}\frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)}\Big(\frac{\pi y}{m}\Big)^{3/2-k-\lambda}r^{2(\nu-\lambda)}\\
&\quad\times\sum_{\lambda\leqslant\mu\leqslant\nu}\frac{(-1)^{\lambda}}{\lambda!}\Big(-\frac{1}{4}\Big)^{\mu}\frac{(k+2\nu-\mu-2)!}{(\mu-\lambda)!\,(2\nu-2\mu)!}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\quad F_{2\nu}^{-}(\tau) &= \frac{(2\nu)!}{(k+\nu-2)!}\sum_{n\gg-\infty}\sum_{r^2>4mn}c^-(n,r)q^n\\
&\quad\times\sum_{0\leqslant\mu\leqslant\nu}\frac{(-m)^{\mu}(k+2\nu-\mu-2)!}{\mu!(2\nu-2\mu)!}\Big(\frac{m}{4}\Big)^{\mu}\\
&\quad\times\sum_{0\leqslant\lambda\leqslant\mu}\binom{\mu}{\lambda}\frac{(-1)^{\lambda}\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}r^{2(\nu-\mu)}r^{2(\mu-\lambda)}(r^2-4mn)^{\lambda}\\
&\quad\times\Gamma\Big(\frac{3}{2}-k-\lambda,\frac{\pi(r^2-4mn)y}{m}\Big)
\end{aligned}
$$

$$= \frac{(2\nu)!}{(k+\nu-2)!} \sum_{n \gg -\infty} \sum_{r^2 > 4mn} c^-(n,r) q^n \sum_{0 \leqslant \mu \leqslant \nu} \Big(-\frac{1}{4}\Big)^{\mu} \frac{(k+2\nu-\mu-2)!}{\mu!\,(2\nu-2\mu)!}$$
$$\times \sum_{0 \leqslant \lambda \leqslant \mu} \binom{\mu}{\lambda} \frac{(-1)^{\lambda}\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)} \Big(r^2-4mn\Big)^{k+\lambda-3/2} r^{2(\nu-\lambda)}$$
$$\times \Gamma\Big(\frac{3}{2}-k-\lambda, \frac{\pi(r^2-4mn)y}{m}\Big)$$
$$= \frac{(2\nu)!}{(k+\nu-2)!} \sum_{n \gg -\infty} \sum_{r^2 > 4mn} c^-(n,r) q^n \sum_{0 \leqslant \lambda \leqslant \nu} \frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}$$
$$\times (r^2-4mn)^{k+\lambda-3/2} r^{2(\nu-\lambda)} \Gamma\Big(\frac{3}{2}-k-\lambda, \frac{\pi(r^2-4mn)y}{m}\Big)$$
$$\times \sum_{\lambda \leqslant \mu \leqslant \nu} \frac{(-1)^{\lambda}}{\lambda!} \Big(-\frac{1}{4}\Big)^{\mu} \frac{(k+2\nu-\mu-2)!}{(\mu-\lambda)!\,(2\nu-2\mu)!}.$$

We assert that

$$\frac{(-1)^{\lambda}(2\nu)!}{\lambda!\,(k+\nu-2)!} \sum_{\lambda \leqslant \mu \leqslant \nu} \Big(-\frac{1}{4}\Big)^{\mu} \frac{(k+2\nu-\mu-2)!}{(\mu-\lambda)!\,(2\nu-2\mu)!} = \nu! \binom{k+\nu-\frac{3}{2}}{\nu-\lambda} \binom{\nu-\frac{1}{2}}{\lambda}. \tag{3.6}$$

To show that the identity holds we use the relation between the Taylor development operator and the Rankin-Cohen bracket. For $\varphi(\tau,z) \in \widehat{\mathbb{J}_{k,1}}(\Gamma)$, $F^0_{2\nu}(\tau)$, $F^-_{2\nu}(\tau)$ become

$$F^0_{2\nu}(\tau) = \frac{(2\nu)!}{(k+\nu-2)!\,(k-\frac{3}{2})} \sum_{r \in \mathbb{Z}} c^0(r^2, 2r) q^{r^2} \tag{3.7}$$
$$\times \sum_{0 \leqslant \lambda \leqslant \nu} \frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)} (\pi y)^{3/2-k-\lambda} (2r)^{2(\nu-\lambda)}$$
$$\times \sum_{\lambda \leqslant \mu \leqslant \nu} \frac{(-1)^{\lambda}}{\lambda!} \Big(-\frac{1}{4}\Big)^{\mu} \frac{(k+2\nu-\mu-2)!}{(\mu-\lambda)!\,(2\nu-2\mu)!},$$

$$F^-_{2\nu}(\tau) = \frac{(2\nu)!}{(k+\nu-2)!} \sum_{n \gg -\infty} \sum_{r^2 > 4n} c^-(n,r) q^n \tag{3.8}$$
$$\times \sum_{0 \leqslant \lambda \leqslant \nu} \frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)} (r^2-4mn)^{k+\lambda-3/2} r^{2(\nu-\lambda)}$$
$$\times \Gamma\Big(\frac{3}{2}-k-\lambda, \frac{\pi(r^2-4mn)y}{m}\Big)$$
$$\times \sum_{\lambda \leqslant \mu \leqslant \nu} \frac{(-1)^{\lambda}}{\lambda!} \Big(-\frac{1}{4}\Big)^{\mu} \frac{(k+2\nu-\mu-2)!}{(\mu-\lambda)!\,(2\nu-2\mu)!},$$

respectively.

Let $h(\tau)$ be as stated in (2.3). By [8], Lemmas V.1.4, V.1.6, the non-holomorphic part of $[h,\theta]_\nu(\tau)$ is given by $H^0_\nu(\tau)+H^-_\nu(\tau)$, where

$$H^0_\nu(\tau)=\frac{4^{3/2-k}}{k-\frac{3}{2}}\sum_{r\in\mathbb{Z}}(\pi y)^{3/2-k-\lambda}r^{2(\nu-\lambda)}C^0(0)q^{r^2}\times\sum_{0\leqslant\lambda\leqslant\nu}\frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)}\binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}$$

and

$$H^-_\nu(\tau)=\sum_{n\gg-\infty}\sum_{r^2>4n}C^-(r^2-n)q^n\sum_{0\leqslant\lambda\leqslant\nu}\binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}\times\frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}r^{2(\nu-\lambda)}(r^2-n)^{k+\lambda-3/2}\times\Gamma\Big(\frac{3}{2}-k-\lambda,4\pi(r^2-n)y\Big).$$

We have that

$$(H_0|U(4))(\tau)=\frac{1}{k-\frac{3}{2}}\sum_{r\in\mathbb{Z}}(\pi y)^{3/2-k-\lambda}(2r)^{2(\nu-\lambda)}C^0(0)q^{r^2}\times\frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)}\sum_{0\leqslant\lambda\leqslant\nu}\binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}\tag{3.9}$$

and

$$(H^-_\nu|U(4))(\tau)=\sum_{n\gg-\infty}\sum_{r^2>4n}C^-(r^2-4n)q^n\sum_{0\leqslant\lambda\leqslant\nu}\binom{k+\nu-\frac{3}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}\times\frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)}r^{2(\nu-\lambda)}(r^2-4n)^{\lambda+1/2}\times\Gamma\Big(-\frac{1}{2}-\lambda,\pi(r^2-4n)y\Big).\tag{3.10}$$

Note that $C^0(0)=c^0(r^2,2r)$, $C^-(r^2-4n)=c^-(n,r)$ for any $r\in\mathbb{Z}$.

Since $\nu!\,([h,\theta]_\nu\mid U(4))(\tau)=(\mathcal{D}_{2\nu}\varphi)(\tau)$, we have that $F^0_{2\nu}(\tau)=(H_0\mid U(4))(\tau)$, $F^-_{2\nu}(\tau)=(H_-\mid U(4))(\tau)$. Combining the Fourier coefficients of (3.7) and (3.9), (3.8), (3.10), respectively, we get the proposed identity.

Inserting (3.6) into (3.4) and (3.5), we complete the proof of Proposition 3.1. □

P r o o f of Theorem 1.1. We calculate

$$f^0_{2\nu}(\tau)=(\pi_{\text{hol}}F^0_{2\nu})(\tau),\quad f^-_{2\nu}(\tau)=(\pi_{\text{hol}}F^-_{2\nu})(\tau).$$

We have that

$$
\begin{aligned}
f_{2\nu}^0(\tau) = (\pi_{\mathrm{hol}} F_{2\nu}^0)(\tau) &= \frac{\nu!}{k-\frac{3}{2}} \sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} c^0\Big(\frac{r^2}{4m}, r\Big) q^{r^2/4m} \\
&\quad \times \sum_{0\leqslant\lambda\leqslant\nu} \frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)} \binom{k+\nu-\frac{3}{2}}{\nu-\lambda} \binom{\nu-\frac{1}{2}}{\lambda} \Big(\frac{\pi}{m}\Big)^{3/2-k-\lambda} r^{2(\nu-\lambda)} \\
&\quad \times \frac{(4\pi r^2/4m)^{k+2\nu-1}}{(k+2\nu-2)!} \int_0^\infty \mathrm{e}^{-\pi r^2 y/m} y^{3/2-k-\lambda} y^{k+2\nu-2}\,\mathrm{d}y \\
&= \frac{\nu!}{(k+2\nu-2)!\,(k-\frac{3}{2})} \sum_{\substack{r\in\mathbb{Z}\\ 4m|r^2}} c^0\Big(\frac{r^2}{4m}, r\Big) q^{r^2/4m} \\
&\quad \times \sum_{0\leqslant\lambda\leqslant\nu} \frac{\Gamma(\frac{5}{2}-k)}{\Gamma(\frac{5}{2}-k-\lambda)} \binom{k+\nu-\frac{3}{2}}{\nu-\lambda} \binom{\nu-\frac{1}{2}}{\lambda} \Big(\frac{\pi}{m}\Big)^{2\nu-\lambda-1/2} r^{2(2\nu+k-\lambda)} \\
&\quad \times \int_0^\infty \mathrm{e}^{-\pi r^2 y/m} y^{2\nu-\lambda-1/2}\,\mathrm{d}y, \\
f_{2\nu}^-(\tau) = (\pi_{\mathrm{hol}} F_{2\nu}^-\Big)(\tau) &= \nu! \sum_{n\gg-\infty} \sum_{r^2>4mn} c^-(n,r) q^n \\
&\quad \times \sum_{0\leqslant\lambda\leqslant\nu} \frac{\Gamma(\frac{3}{2}-k)}{\Gamma(\frac{3}{2}-k-\lambda)} \binom{k+\nu-\frac{3}{2}}{\nu-\lambda} \binom{\nu-\frac{1}{2}}{\lambda} r^{2(\nu-\lambda)} (r^2-4mn)^{k+\lambda-3/2} \\
&\quad \times \frac{(4\pi n)^{k+2\nu-1}}{(k+2\nu-2)!} \int_0^\infty \Gamma\Big(\frac{3}{2}-k-\lambda, \frac{\pi(r^2-4mn)y}{m}\Big) \mathrm{e}^{-4\pi n y} y^{k+2\nu-2}\,\mathrm{d}y.
\end{aligned}
$$

By [8], Lemmas V.1.4, V.1.7 we have that

$$
\int_0^\infty \mathrm{e}^{-\pi r^2 y/m} y^{2\nu-\lambda-1/2}\,\mathrm{d}y = \Big(\frac{m}{\pi r^2}\Big)^{2\nu-\lambda+1/2} \Gamma\Big(2\nu-\lambda+\frac{1}{2}\Big)
$$

and

$$
\begin{aligned}
&\int_0^\infty \Gamma\Big(\frac{3}{2}-k-\lambda, \frac{\pi(r^2-4mn)y}{m}\Big) \mathrm{e}^{-4\pi n y} y^{k+2\nu-2}\,\mathrm{d}y \\
&= \int_0^\infty \Gamma\Big(\frac{3}{2}-k-\lambda, \frac{4\pi(r^2-4mn)y}{4m}\Big) \mathrm{e}^{-4\pi(r^2/4m-(r^2-4mn)/4m)y} y^{k+2\nu-2}\,\mathrm{d}y \\
&= -(4\pi)^{1-k-2\nu} \Big(\frac{r^2-4mn}{4m}\Big)^{3/2-k-\lambda} \frac{\Gamma(\frac{3}{2}-k-\lambda)(k+2\nu-2)!}{n^{k+2\nu-2}} \\
&\quad \times \bigg(\Big(\frac{r^2}{4m}\Big)^{\lambda-2\nu+1/2} P_{k+2\nu,5/2-k-\lambda}\Big(n, \frac{r^2-4mn}{4m}\Big) - \Big(\frac{r^2-4mn}{4m}\Big)^{k+\lambda-3/2}\bigg) \\
&= -(4\pi)^{1-k-2\nu} (r^2-4mn)^{3/2-k-\lambda} \frac{\Gamma(\frac{3}{2}-k-\lambda)(k+2\nu-2)!}{n^{k+2\nu-2}} \\
&\quad \times (r^{2(\lambda-2\nu+1/2)} P_{k+2\nu,5/2-k-\lambda}(n, r^2-4mn) - (r^2-4mn)^{k+\lambda-3/2}).
\end{aligned}
$$

Inserting this into the expression of $f^0_{2\nu}(\tau)\ f^-_{2\nu}(\tau)$ we prove the first statement of Theorem 1.1. The statement for $k=2$ of Theorem 1.1 follows from the identities, see [8], Lemma V.2.6, (ii) and Proposition V.2.7

$$\frac{2\Gamma(\frac{1}{2})}{(2\nu)!}\sum_{0\leqslant\lambda\leqslant\nu}\binom{\nu+\frac{1}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}\frac{\Gamma(\frac{1}{2}+2\nu-\lambda)}{\Gamma(\frac{1}{2}-\lambda)}=2^{1-2\nu}\sqrt{\pi}\binom{2\nu}{\nu}$$

and

$$\begin{aligned}\sum_{0\leqslant\lambda\leqslant\nu}&\binom{\nu+\frac{1}{2}}{\nu-\lambda}\binom{\nu-\frac{1}{2}}{\lambda}\\ &\times\left(r^{2(-\nu+1/2)}P_{2+2\nu,1/2-\lambda}(4mn,r^2-4mn)-r^{2(\nu-\lambda)}(r^2-4mn)^{\lambda+1/2}\right)\\ &=2^{-2\nu}\binom{2\nu}{\nu}\left(|r|-\sqrt{r^2-4mn}\right)^{2\nu+1}.\end{aligned}$$

Now we have completed the proof of Theorem 1.1. $\square$

4. An application in class number relations

In this section we prove Theorem 1.2 by applying Theorem 1.1. To prove Theorem 1.2 we need the following lemma.

Lemma 4.1. *Let $\varphi(\tau,z)=\sum\limits_{n,r}c(n,r,y)q^n\zeta^r\in\widehat{\mathbb{J}}_{k,m}(\Gamma)$. For $s\in\mathbb{Z}$, define the $S^J_{l,s}$ operator as*

$$(\varphi\mid S^J_{l,s})(\tau,z):=\sum_n\sum_{r\equiv s \bmod l}c(n,r,y)q^n\zeta^r.$$

Then $(\mathcal{D}_{2\nu}(\varphi\mid S^J_{l,s}))(\tau)$ satisfies the modular transformation law of weight $k+2\nu$ on the group $\Gamma_0(l^2)$ if $l\mid s$ and on $\Gamma_0(l^2)\cap\Gamma_1(l)$ otherwise.

Proof. Let $\varphi(\tau,z)=\sum\limits_{n,r}c(n,r,y)q^n\zeta^r\in\widehat{\mathbb{J}}_{k,m}(\Gamma)$. We have that

$$\begin{aligned}(\varphi\mid S^J_{l,s})(\tau,z)&=\frac{1}{l}\sum_{t \bmod l}e\Big(-\frac{st}{l}\Big)\sum_{n,r}c(n,r,y)e\Big(\frac{rt}{l}\Big)q^n\zeta^r\\ &=\frac{1}{l}\sum_{t \bmod l}e\Big(-\frac{st}{l}\Big)\varphi\Big(\tau,z+\frac{t}{l}\Big).\end{aligned}$$

We assert that $(\varphi \mid S^J_{l,s})(\tau, z)$ satisfies the same Jacobi transformation law but on the group $\Gamma_0(l^2) \cap \Gamma_1(l)$, since for each $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(l^2) \cap \Gamma_1(l)$ and $t \in \mathbb{Z}$,

$$\begin{aligned}\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d} + \frac{t}{l}\Big) &= \varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z + t(c\tau+d)/l}{c\tau+d}\Big)\\ &= (c\tau+d)^k e\Big(\frac{mc(z+t(c\tau+d)/l)^2}{c\tau+d}\Big)\varphi\Big(\tau, z + \frac{tc\tau}{l} + \frac{td}{l}\Big)\\ &= (c\tau+d)^k e\Big(\frac{mc(z+t(c\tau+d)/l)^2}{c\tau+d}\Big) e\Big(-m\Big(\Big(\frac{tc}{l}\Big)^2\tau + \frac{2tc}{l}z\Big)\Big)\varphi\Big(\tau, z + \frac{td}{l}\Big)\\ &= (c\tau+d)^k e\Big(\frac{mcz^2}{c\tau+d}\Big)\varphi\Big(\tau, z + \frac{t}{l}\Big),\end{aligned}$$

where the last identity follows from the fact that $d \equiv 1 \bmod l$. In particular, if $s \mid l$ then

$$(\varphi \mid S^J_{l,0})(\tau, z) = \frac{1}{l}\sum_{t \bmod l} \varphi\Big(\tau, z + \frac{t}{l}\Big).$$

One finds for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(l^2)$ that

$$\begin{aligned}\sum_{t \bmod l} \varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d} + \frac{t}{l}\Big) &= e\Big(\frac{mcz^2}{c\tau+d}\Big)\sum_{t \bmod l}\varphi\Big(\tau, z + \frac{td}{l}\Big)\\ &= e\Big(\frac{mcz^2}{c\tau+d}\Big)\sum_{t \bmod l}\varphi\Big(\tau, z + \frac{t}{l}\Big).\end{aligned}$$

Therefore $(\varphi \mid S^J_{l,s})(\tau, z)$ satisfies the same Jacobi transformation law on $\Gamma_0(l^2)$ for $s \mid l$. Now Lemma 4.1 follows from Theorem 2.5. $\square$

P r o o f of Theorem 1.2. By [3], we know that

$$\begin{aligned}E^*_{2,1}(\tau, z) = &\sum_{n\geqslant 0}\sum_{r\in\mathbb{Z}} H(4n - r^2) q^n \zeta^r + \frac{1}{4\pi\sqrt{y}}\sum_{r\in\mathbb{Z}} q^{r^2}\zeta^{2r}\\ &+ \frac{1}{4\sqrt{\pi}}\sum_{n>0}\sum_{r^2-4n \text{ is } \square^*} \sqrt{r^2-4n}\,\Gamma\Big(-\frac{1}{2}, \pi(r^2-4n)y\Big) q^n\zeta^r\end{aligned}$$

transforms like a Jacobi form of weight 2 and index 1, where $\square^*$ means the positive square. Moreover, one can see that $E^*_{2,1}(\tau, z)$ lies in the kernel of $\mathcal{C}^{2,1}$ by observing the Fourier expansion of its non-holomorphic part. Thus $E^*_{2,1}(\tau, z) \in \widehat{\mathbb{J}}_{2,1}(\Gamma)$.

Let V_m be the operator defined by

$$(\varphi|_{2,1}V_m)(\tau, z) = m \sum_{\substack{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma\backslash M_2(\mathbb{Z})\\ ad-bc=m}} (c\tau+d)^2 e\Big(-\frac{mcz^2}{c\tau+d}\Big)\varphi\Big(\frac{a\tau+b}{c\tau+d}, \frac{mz}{c\tau+d}\Big).$$

It is well-known that $(\varphi|_{2,1}V_m)(\tau,z)$ satisfies the Jacobi transformation law of weight 2 and index m, see [5], Theorem 4.1. By [5], Theorem 4.2 one calculates that

$$\begin{aligned}
E^*_{2,m}(\tau,z) &:= (E^*_{2,1}|_{2,1}V_m)(\tau,z)\\
&= \sum_{n\geqslant 0}\sum_{r\in\mathbb{Z}}\sum_{a|(n,m,r)} aH\Big(\frac{4mn-r^2}{a^2}\Big)q^n\zeta^r + \frac{\sqrt{m}}{4\pi\sqrt{y}}\sum_{4m|r^2}\sigma_0\Big(\Big(m,\frac{r^2}{4m},r\Big)\Big)q^{r^2/4m}\zeta^r\\
&\quad + \frac{1}{4\sqrt{\pi}}\sum_{n>0}\sum_{r^2-4mn \text{ is } \square^*}\sigma_0((m,n,r))\sqrt{r^2-4mn}\,\Gamma\Big(-\frac{1}{2},\frac{\pi(r^2-4mn)y}{m}\Big)q^n\zeta^r.
\end{aligned}$$

Thus $E^*_{2,m}(\tau,z)\in\widehat{\mathbb{J}}_{2,m}(\Gamma)$. Therefore, by Lemma 4.1, the function

$$\begin{aligned}
E^*_{2,m,s}(\tau,z) := &\sum_{n\geqslant 0}\sum_{r\equiv s \bmod l}\sum_{a|(m,n,r)} aH\Big(\frac{4mn-r^2}{a^2}\Big)q^n\zeta^r\\
&+ \frac{\sqrt{m}}{4\pi\sqrt{y}}\sum_{\substack{r\equiv s \bmod l\\ 4m|r^2}}\sigma_0\Big(\Big(m,\frac{r^2}{4m},r\Big)\Big)q^{r^2/4m}\zeta^r\\
&+ \frac{1}{4\sqrt{\pi}}\sum_{n\geqslant 0}\sum_{\substack{r\equiv s \bmod l\\ r^2-4mn \text{ is } \square^*}}\sigma_0((m,n,r))\sqrt{r^2-4mn}\\
&\times\Gamma\Big(-\frac{1}{2},\frac{\pi(r^2-4mn)y}{m}\Big)q^n\zeta^r
\end{aligned}$$

satisfies the $2+2\nu$ weight modular transform law on the group $\Gamma_0(l^2)$ if $l \mid s$ and on $\Gamma_0(l^2)\cap\Gamma_1(l)$ otherwise. Applying Theorem 1.1 one has

$$\begin{aligned}
(\pi_{\mathrm{hol}}(\mathcal{D}_{2\nu}E^*_{2,m,s}))(\tau) = &\sum_{n\geqslant 0}\sum_{r\equiv s \bmod l} p^{(1)}_{2\nu}(n,mr)\sum_{a|(n,m,r)} aH\Big(\frac{4mn-r^2}{a^2}\Big)q^n\\
&+ \frac{2^{-2(1+\nu)}(2\nu)!}{\nu!}\sum_{\substack{r\equiv s \bmod l\\ 4m|r^2}}\sigma_0\Big(\Big(m,\frac{r^2}{4m},r\Big)\Big)\frac{1}{2}|r|^{2\nu+1}q^{r^2/4m}\\
&+ \frac{(2\nu)!}{2\nu!}\sum_{n>0}\sum_{\substack{r\equiv s \bmod l\\ r^2-4mn \text{ is } \square^*}}\sigma_0((m,n,r))\Big(\frac{|r|-\sqrt{r^2-4mn}}{2}\Big)^{2\nu+1}q^n\\
= &\sum_{n\geqslant 0}\sum_{r\equiv s \bmod l} p^{(1)}_{2\nu}(n,mr)\sum_{a|(m,n,r)} aH\Big(\frac{4mn-r^2}{a^2}\Big)q^n\\
&+ \frac{(2\nu)!}{2\nu!}\sum_{n\geqslant 0}\sum_{\substack{r\equiv s \bmod l\\ r^2-4mn \text{ is } \square}}\sigma_0((m,n,r))\Big(1-\frac{1}{2}\delta_{1,r^2=4mn}\Big)\\
&\times\Big(\frac{|r|-\sqrt{r^2-4mn}}{2}\Big)^{2\nu+1}q^n,
\end{aligned}$$

where $\square$ means square and $\delta_{1,r^2=4mn}$ equals 1 if $r^2 = 4mn$ and 0 otherwise.

Fixing m, n, we have the 1-1 correspondence from the following one set to the other,

$$\Big\{h \mid mn\colon\ h^2 \leqslant mn,\ h + \frac{mn}{h} \equiv \pm s \bmod l\Big\} \to \{r \equiv s \bmod l\colon\ r^2 - 4mn \text{ is } \square\}\colon$$
$$h \mapsto \pm\Big(h + \frac{mn}{h}\Big).$$

Replacing r by $h + mn/h$, we obtain

$$\begin{aligned}(\pi_{\mathrm{hol}}(\mathcal{D}_{2\nu}E^*_{2,m,s}))(\tau) &= \sum_{n\geqslant 0}\sum_{r\equiv s \bmod l} p^{(1)}_{2\nu}(n, mr)\sum_{a\mid(n,m,r)} aH\Big(\frac{4mn-r^2}{a^2}\Big)q^n\\ &+ \frac{(2\nu)!}{2\nu!}\sum_{n>0}\sum_{\substack{h\mid mn\\ h+mn/h\equiv s \bmod l}} \sigma_0\Big(\Big(m, n, h + \frac{mn}{h}\Big)\Big)\min\Big\{h, \frac{mn}{h}\Big\}^{1+2\nu} q^n\\ &+ \frac{(2\nu)!}{2\nu!}\sum_{n>0}\sum_{\substack{h\mid mn\\ h+mn/h\equiv -s \bmod l}} \sigma_0\Big(\Big(m, n, h + \frac{mn}{h}\Big)\Big)\min\Big\{h, \frac{mn}{h}\Big\}^{1+2\nu} q^n.\end{aligned}$$

This completes the proof of Theorem 1.2. $\square$

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