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# AN IMPROVED REGULARITY CRITERIA FOR THE MHD SYSTEM BASED ON TWO COMPONENTS OF THE SOLUTION

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*Abstract.* As observed by Yamazaki, the third component $b_3$ of the magnetic field can be estimated by the corresponding component $u_3$ of the velocity field in $L^\lambda$ $(2 \leqslant \lambda \leqslant 6)$ norm. This leads him to establish regularity criterion involving $u_3, j_3$ or $u_3, \omega_3$. Noticing that $\lambda$ can be greater than 6 in this paper, we can improve previous results.

*Keywords*: MHD equations; regularity criteria

*MSC 2020*: 35B65, 35Q35, 76D03

## 1. Introduction

In this paper, we investigate the following three-dimensional (3D) magnetohydrodynamic (MHD) equations:

$$(1)\qquad \begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} - \triangle \boldsymbol{u} + \nabla \Pi = \boldsymbol{0}, \\ \partial_t \boldsymbol{b} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{b} = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \nabla\cdot\boldsymbol{b} = 0, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0,\ \boldsymbol{b}|_{t=0} = \boldsymbol{b}_0, \end{cases}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, $\boldsymbol{b} = (b_1, b_2, b_3)$ is the magnetic field, $\Pi$ is a scalar pressure, and $\boldsymbol{u}_0, \boldsymbol{b}_0$ are the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0 = \nabla\cdot\boldsymbol{b}_0 = 0$ in the distributional sense. From the physical point of view,

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(1) governs the dynamics of the velocity and magnetic fields in electrically conducting fluids, such as plasmas, liquid metals, and salt water. Moreover, $(1)_1$ reflects the conservation of momentum, $(1)_2$ is the induction equation, and $(1)_3$ specifies the conservation of mass.

Besides its physical applications, the MHD system (1) is also mathematically significant. Duvaut-Lions [5] (see also Sermange-Temam [14]) showed that (1) possesses at least one global weak solution for initial data with finite energy. However, the issue of regularity and uniqueness of weak solutions remains a challenging open problem in mathematical fluid dynamics. Many interesting and important sufficient conditions (see e.g., [1], [3], [4], [6], [7], [8], [9], [10], [11], [12], [15], [18], [20], [21], [24], [25] and the references therein) were derived to guarantee the regularity of the weak solution. In this paper, we are interested in regularity criteria involving only two components of the solution, in view of the fact that there exists no such result via one component. In 2010, Ji-Lee [7] showed that if

$$u_1, u_2 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=\frac{1}{2}, \quad 6\leqslant q\leqslant\infty, \tag{2}$$

then the solution is smooth on $(0,T)$. The method is to establish first a regularity criterion involving $u_1$, $u_2$, $b_1$, $b_2$, and then to control $b_1$, $b_2$ in terms of $u_1$, $u_2$. For later developments, let us denote by

$$\nabla\times\boldsymbol{u}=\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3), \quad \nabla\times\boldsymbol{b}=\boldsymbol{j}=(j_1,j_2,j_3)$$

the vorticity and the current density respectively. In 2014, Yamazaki [15] found a fine structure of the horizontal convective terms

$$\begin{aligned}(3)\qquad &\int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\triangle_h\boldsymbol{u}\,\mathrm{d}x-\int[(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\triangle_h\boldsymbol{u}\,\mathrm{d}x\\ &\quad+\int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\triangle_h\boldsymbol{b}\,\mathrm{d}x-\int[(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\triangle_h\boldsymbol{b}\,\mathrm{d}x\\ &\leqslant C\int(|u_3|+|b_3|)\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})|\,\mathrm{d}x+C\int|\nabla_h(\boldsymbol{u},\boldsymbol{b})|^2\cdot|j_3|\,\mathrm{d}x,\end{aligned}$$

where $\triangle_h=\partial_1\partial_1+\partial_2\partial_2$ is the horizontal Laplacian. With (3) and the following bound of $b_3$ in terms of $u_3$,

$$\begin{aligned}(4)\qquad &\sup_{t_1\leqslant t\leqslant t_2}\|b_3(t)\|_{L^\lambda}^2\\ &\qquad\leqslant\|b_3(t_1)\|_{L^2}^2+C\int_{t_1}^{t_2}\|\nabla\boldsymbol{b}(t)\|_{L^2}^2\|u_3(t)\|_{L^{6\lambda/(6-\lambda)}}^2\,\mathrm{d}t\quad(2\leqslant\lambda\leqslant6),\end{aligned}$$

he was able to prove the following regularity condition

$$
\begin{aligned}
&u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad j_3 \in L^r(0,T;L^s(\mathbb{R}^3)),\\
&\frac{2}{p}+\frac{3}{q}=\frac{1}{3}+\frac{1}{2q}, \quad \frac{15}{2}<q\leqslant\infty; \quad \frac{2}{r}+\frac{3}{s}=2, \quad \frac{3}{2}<s\leqslant\infty.
\end{aligned} \tag{5}
$$

By treating (4) via the multiplicative Sobolev inequality

$$
\|f\|_{L^6}\leqslant C\|\partial_1 f\|_{L^2}^{1/3}\|\partial_2 f\|_{L^2}^{1/3}\|\partial_3 f\|_{L^2}^{1/3}, \tag{6}
$$

and estimating the nonlinear terms in a flexible way, Zhang [21], [22] refined (5) to be

$$
\begin{aligned}
&u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad j_3 \in L^r(0,T;L^s(\mathbb{R}^3)),\\
&\frac{2}{p}+\frac{3}{q}=\frac{4}{9}, \quad \frac{27}{4}\leqslant q\leqslant\infty; \quad \frac{2}{r}+\frac{3}{s}=2, \quad \frac{3}{2}<s\leqslant\infty.
\end{aligned} \tag{7}
$$

In 2016, Yamazaki [18] observed one another new nice structure

$$
\begin{aligned}
&\int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\triangle_h\boldsymbol{u}\,\mathrm{d}x-\int[(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot\triangle_h\boldsymbol{u}\,\mathrm{d}x\\
&\qquad+\int[(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot\triangle_h\boldsymbol{b}\,\mathrm{d}x-\int[(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot\triangle_h\boldsymbol{b}\,\mathrm{d}x\\
&\quad\leqslant C\int(|u_3|+|b_3|)\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})|\,\mathrm{d}x\\
&\qquad+C\int|(-\triangle_h)^{-1}\nabla_h^2\omega_3|\cdot|\nabla_h\boldsymbol{b}|^2\,\mathrm{d}x\\
&\qquad+C\int|(-\triangle_h)^{-1}\nabla_h^2u_3|\cdot|\nabla_h\boldsymbol{b}|\cdot|\nabla\nabla_h\boldsymbol{b}|\,\mathrm{d}x,
\end{aligned} \tag{8}
$$

whose proof relies on the fact that $u_1,u_2$ (resp. $b_1,b_2$) can be expressed explicitly in terms of $\omega_3$ and $\partial_3u_3$ (resp. $j_3$ and $\partial_3b_3$), see also [13], [23] (and [2], [19], [17] for its ultility in the well-posedness theory). This structure (8) enabled Yamazaki to show the following regularity criterion:

$$
\begin{aligned}
&u_3 \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \omega_3 \in L^r(0,T;L^s(\mathbb{R}^3)),\\
&\frac{2}{p}+\frac{3}{q}=\frac{4}{9}-\frac{1}{3q}, \quad \frac{15}{2}\leqslant q\leqslant\infty; \quad \frac{2}{r}+\frac{3}{s}=2, \quad \frac{3}{2}<s\leqslant\infty.
\end{aligned} \tag{9}
$$

The scaling $\frac{4}{9}-\frac{1}{3}q^{-1}$ has been improved to be $\frac{4}{9}$ in [22].

The purpose of the present paper is to improve the scaling $\frac{4}{9}$ in (7) a bit further. The key is that we can dominate the $L^\lambda$ ($\lambda>6$) norm of $b_3$ via $u_3$. Precisely, we have

**Theorem 1.** *Let $(\boldsymbol{u}_0,\boldsymbol{b}_0)\in H^2(\mathbb{R}^3)$ with $\nabla\cdot\boldsymbol{u}_0=\nabla\cdot\boldsymbol{b}_0=0$, and $T>0$. Assume that $(\boldsymbol{u},\boldsymbol{b})$ is the unique strong solution of the MHD system* (1) *with initial data $(\boldsymbol{u}_0,\boldsymbol{b}_0)$ on $[0,\Gamma^*)$, where $\Gamma^*$ is the maximal existence time. If for some $T>\Gamma^*$ one of the following conditions holds for the weak solution $(\boldsymbol{u},\boldsymbol{b})$,*

(i)

$$u_3\in L^p(0,T;L^q(\mathbb{R}^3)),\quad j_3\in L^r(0,T;L^s(\mathbb{R}^3)),$$
$$\frac{2}{p}+\frac{3}{q}=\frac{7}{15},\quad \frac{45}{7}\leqslant q\leqslant\infty;\quad \frac{2}{r}+\frac{3}{s}=2,\quad \frac{3}{2}<s\leqslant\infty; \tag{10}$$

(ii)

$$u_3\in L^p(0,T;L^q(\mathbb{R}^3)),\quad \omega_3\in L^r(0,T;L^s(\mathbb{R}^3)),$$
$$\frac{2}{p}+\frac{3}{q}=\frac{7}{15},\quad \frac{45}{7}\leqslant q\leqslant\infty;\quad \frac{2}{r}+\frac{3}{s}=2,\quad \frac{3}{2}<s\leqslant\infty, \tag{11}$$

*then the strong solution can be smoothly extended beyond time $\Gamma^*$.*

## 2. Proof of Theorem 1

In this section, we provide the proof of Theorem 1 under condition (10). The proof of Theorem 1 under condition (11) can be similarly treated via the structure (8), once the following consequence of the boundedness of two-dimensional Riesz transformation in $L^\alpha$ $(1<\alpha<\infty)$ is noticed:

$$\begin{aligned}\|(-\triangle_h)^{-1}\nabla_h^2 f\|_{L^\alpha}^\alpha &= \int_{\mathbb{R}^3}|(-\triangle_h)^{-1}\nabla_h^2 f|^\alpha\,\mathrm{d}x\\ &= \int_{\mathbb{R}}\Big[\int_{\mathbb{R}^2}|(-\triangle_h)^{-1}\nabla_h^2 f|^\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2\Big]\,\mathrm{d}x_3\\ &\leqslant C\int_{\mathbb{R}}\Big[\int_{\mathbb{R}^2}|f|^\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2\Big]\,\mathrm{d}x_3 = C\int_{\mathbb{R}^3}|f|^\alpha\,\mathrm{d}x.\end{aligned} \tag{12}$$

As is well-known, it suffices to show that $\|\nabla(\boldsymbol{u},\boldsymbol{b})(t)\|_{L^2}$ remains bounded as $t\nearrow\Gamma^*$. To this end, for $0<\varepsilon,\ \delta\ll 1$ to be determined later on, we may find a $\Gamma\in[0,\Gamma^*)$ such that

$$\nabla(\boldsymbol{u},\boldsymbol{b})(\Gamma)\in L^2(\mathbb{R}^3),\quad \boldsymbol{b}(\Gamma)\in L^{10}(\mathbb{R}^3),\quad \int_\Gamma^{\Gamma^*}\|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau\leqslant\varepsilon, \tag{13}$$
$$\int_\Gamma^{\Gamma^*}\|j_3(\tau)\|_{L^s}^{2s/(2s-3)}\,\mathrm{d}\tau\leqslant\delta.$$

Taking the inner product of $(1)_1$ with $-\triangle_h \boldsymbol{u}$, $(1)_2$ with $-\triangle_h \boldsymbol{b}$ in $L^2(\mathbb{R}^3)$, respectively, we deduce from (3) that

$$\begin{aligned}
(14)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 \\
&\leqslant C\int |u_3|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})|\,\mathrm{d}x \\
&\quad + C\int |b_3|\cdot|\nabla(\boldsymbol{u},\boldsymbol{b})|\cdot|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})|\,\mathrm{d}x + C\int |\nabla_h(\boldsymbol{u},\boldsymbol{b})|^2\cdot|j_3|\,\mathrm{d}x \\
&\equiv K_1 + K_2 + K_3.
\end{aligned}$$

For $K_1$,

$$\begin{aligned}
(15)\qquad K_1 &\leqslant C\|u_3\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^{2q/(q-2)}}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2} \\
&\qquad\qquad \text{(by the Hölder inequality)} \\
&\leqslant C\|u_3\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(q-3)/q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^6}^{3/q}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2} \\
&\qquad\qquad \text{(by the interpolation inequality)} \\
&\leqslant C\|u_3\|_{L^q}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(q-3)/q}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/q}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(q+2)/q} \\
&\qquad\qquad \text{(by (6) and classical elliptic estimates)} \\
&\leqslant C\|u_3\|_{L^q}^{2q/(q-2)}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{2(q-3)/(q-2)}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{2/(q-2)} \\
&\quad + \frac{1}{6}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.
\end{aligned}$$

Similarly, $K_2, K_3$ can be bounded as

$$\begin{aligned}
(16)\quad K_2 &\leqslant C\|b_3\|_{L^{10}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^{5/2}}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2} \\
&\leqslant C\|b_3\|_{L^{10}}\cdot\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/10}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^6}^{3/10}\cdot\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2} \\
&\leqslant C\|b_3\|_{L^{10}}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/10}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/5}\|\nabla^2(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/10}\cdot\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2} \\
&\leqslant C\|b_3\|_{L^{10}}^{5/2}\|\nabla(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{7/4}\|\triangle(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{1/4} + \frac{1}{6}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2. \\
(17)\quad K_3 &\leqslant C\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^{2s/(s-1)}}^2\|j_3\|_{L^s} \\
&\leqslant C\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{(2s-3)/s}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^{3/s}\cdot\|j_3\|_{L^s} \\
&\leqslant C\|j_3\|_{L^s}^{2s/(2s-3)}\|\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2 + \frac{1}{6}\|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})\|_{L^2}^2.
\end{aligned}$$

Gathering (15), (16), and (17) into (14), and denoting by

$$(18)\qquad \begin{aligned}
\mathcal{K}^2(t) &= \sup_{\Gamma\leqslant\tau\leqslant t}\|\nabla_h(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2 + \int_\Gamma^t \|\nabla\nabla_h(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau, \\
\mathcal{M}^2(t) &= \sup_{\Gamma\leqslant\tau\leqslant t}\|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2 + \int_\Gamma^t \|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau,
\end{aligned}$$

we obtain

$$
\begin{aligned}
(19)\quad \mathcal{K}^2(t) \leqslant{}& \|\nabla_h(\boldsymbol{u},\boldsymbol{b})(\Gamma)\|_{L^2}^2 \\
&+ C\int_\Gamma^t \|u_3(\tau)\|_{L^q}^{2q/(q-2)} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{2(q-3)/(q-2)} \|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{2/(q-2)}\,\mathrm{d}\tau \\
&+ C\int_\Gamma^t \|b_3(\tau)\|_{L^{10}}^{5/2} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{7/4} \|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{1/4}\,\mathrm{d}\tau \\
&+ C\int_\Gamma^t \|j_3(\tau)\|_{L^s}^{2s/(2s-3)} \|\nabla_h(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\
\leqslant{}& C + \mathcal{K}_1(t) + \mathcal{K}_2(t) + \mathcal{K}_3(t).
\end{aligned}
$$

We bound the terms one by one as

$$
\begin{aligned}
\mathcal{K}_1(t) \leqslant{}& C \sup_{\Gamma\leqslant\tau\leqslant t} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{(3q-10)/(2q-4)} \\
&\times \int_\Gamma^t \|u_3(\tau)\|_{L^q}^{2q/(q-2)} \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{1/2} \|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^{2/(q-2)}\,\mathrm{d}\tau \\
\leqslant{}& C\mathcal{M}^{(3q-10)/(2q-4)}(t)\cdot\left(\int_\Gamma^t \|u_3(\tau)\|_{L^q}^{8q/(3q-10)}\,\mathrm{d}\tau\right)^{(3q-10)/(4q-8)} \\
&\times \left(\int_\Gamma^t \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{1/4}\cdot\left(\int_\Gamma^t \|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{1/(q-2)} \\
\leqslant{}& C\mathcal{M}^{(3q-10)/(2q-4)}(t) \\
&\times \left\{\int_\Gamma^t \Big[1+\|u_3(\tau)\|_{L^q}^{30q/(7q-45)}\Big]\,\mathrm{d}\tau\right\}^{(3q-10)/(4q-8)}\cdot\varepsilon^{1/4}\mathcal{M}^{2/(q-2)}(t) \\
\leqslant{}& C\varepsilon^{1/4}\mathcal{M}^{3/2}(t), \\
\mathcal{K}_2(t) \leqslant{}& C \sup_{\Gamma\leqslant\tau\leqslant t} \|b_3(\tau)\|_{L^{10}}^{5/2}\cdot\left(\int_\Gamma^t \|\nabla(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{7/8} \\
&\times \left(\int_\Gamma^t \|\triangle(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right)^{1/8} \\
\leqslant{}& C \sup_{\Gamma\leqslant\tau\leqslant t} \|b_3(\tau)\|_{L^{10}}^{5/2}\cdot\varepsilon^{7/8}\mathcal{M}^{1/4}(t), \\
\mathcal{K}_3(t) \leqslant{}& C \sup_{\Gamma\leqslant\tau\leqslant t} \|\nabla_h(\boldsymbol{u},\boldsymbol{b})(\tau)\|_{L^2}^2\cdot\int_\Gamma^t \|j_3(\tau)\|_{L^s}^{2s/(2s-3)}\,\mathrm{d}\tau \\
\leqslant{}& C\delta\mathcal{K}^2(t).
\end{aligned}
$$

Collecting the above estimates into (19), and taking $\delta$ to be sufficiently small, we deduce that

$$
(20)\qquad \mathcal{K}^2(t) \leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\sup_{\Gamma\leqslant\tau\leqslant t} \|b_3(\tau)\|_{L^{10}}^{5/2}\cdot\varepsilon^{7/8}\mathcal{M}^{1/4}(t).
$$

To proceed further, we dominate $b_3$ in terms of $u_3$ via the following lemma, which be can tracked back to [16].

**Lemma 1.** *For $q \geqslant 5$, we have*

$$\|b_3(t)\|_{L^{10}} \leqslant \Big[\|b_3(\Gamma)\|_{L^{10}}^{10q/(4q-15)} + C\int_\Gamma^t \|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)} \times \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\,\mathrm{d}\tau\Big]^{(4q-15)/10q} \quad \text{for all } \Gamma \leqslant t < \Gamma^*. \tag{21}$$

P r o o f. Multiplying the equation of $b_3$:

$$\partial_t b_3 + (\boldsymbol{u}\cdot\nabla)b_3 - (\boldsymbol{b}\cdot\nabla)u_3 - \triangle b_3 = 0$$

with $10b_3^9$, integrating over $\mathbb{R}^3$, we deduce by suitable integration by parts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b_3^5\|_{L^2}^2 + \frac{18}{5}\|\nabla(b_3^5)\|_{L^2}^2 = 10\int [(\boldsymbol{b}\cdot\nabla)u_3]\cdot b_3^9\,\mathrm{d}x \equiv L. \tag{22}$$

The term $L$ may be dominated as

$$\begin{aligned} L = &-10\int_{\mathbb{R}^3} \boldsymbol{b}u_3\cdot\nabla(b_3^9)\,\mathrm{d}x = -18\int_{\mathbb{R}^3}\boldsymbol{b}u_3(b_3^5)^{4/5}\cdot\nabla(b_3^5)\,\mathrm{d}x \\ &\leqslant C\|\boldsymbol{b}\|_{L^6}\|u_3\|_{L^q}\|b_3^5\|_{L^{12q/(5q-15)}}^{4/5}\|\nabla(b_3^5)\|_{L^2} \\ &\leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{2/3}\|\nabla\boldsymbol{b}\|_{L^2}^{1/3}\cdot\|u_3\|_{L^q}\cdot(\|b_3^5\|_{L^2}^{3(q-5)/4q}\|\nabla(b_3^5)\|_{L^2}^{(q+15)/4q})^{4/5}\|\nabla(b_3^5)\|_{L^2} \\ &\leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{20q/(12q-45)}\|\nabla\boldsymbol{b}\|_{L^2}^{10q/(12q-45)}\|u_3\|_{L^q}^{10q/(4q-15)}\|b_3^5\|_{L^2}^{6(q-5)/(4q-15)} \\ &\quad + \frac{9}{5}\|\nabla(b_3^5)\|_{L^2}^2. \end{aligned} \tag{23}$$

Putting (23) into (22), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b_3\|_{L^{10}}^{10} \leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{20q/(12q-45)}\|\nabla\boldsymbol{b}\|_{L^2}^{10q/(12q-45)}\|u_3\|_{L^q}^{10q/(4q-15)}\|b_3\|_{L^{10}}^{30(q-5)/(4q-15)}.$$

Dividing by $\|b_3\|_{L^{10}}^{30(q-5)/(4q-15)}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b_3\|_{L^{10}}^{10q/(4q-15)} \leqslant C\|\nabla_h\boldsymbol{b}\|_{L^2}^{20q/(12q-45)}\|\nabla\boldsymbol{b}\|_{L^2}^{10q/(12q-45)}\|u_3\|_{L^q}^{10q/(4q-15)}.$$

Integrating in time, we obtain (21) as desired. □

Now, substituting (21) into (20), we obtain

$$\begin{aligned}\mathcal{K}^2(t) &\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\bigg[C + C\int_\varGamma^t \|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)} \\ &\quad \times \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\,\mathrm{d}\tau\bigg]^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t) \\ &\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\bigg[\sup_{\varGamma\leqslant\tau\leqslant t} \|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)} \\ &\quad \times \int_\varGamma^t \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\,\mathrm{d}\tau\bigg]^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t) \\ &\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\mathcal{K}^{5/3}(t) \\ &\quad \times \bigg(\int_\varGamma^t \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{10q/(12q-45)} \|u_3(\tau)\|_{L^q}^{10q/(4q-15)}\,\mathrm{d}\tau\bigg)^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t).\end{aligned}$$

Here, we push all $\|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{20q/(12q-45)}$ out of the integral, rather than $\|\nabla_h \boldsymbol{b}(\tau)\|_{L^2}^{s}$ for suitable smaller $s$ as in [15], [18], [21], [22]. This ensures the optimality of the present method. Estimation further by the Young inequality yields

$$\begin{aligned}\mathcal{K}^2(t) &\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\mathcal{K}^{5/3}(t) \\ &\quad \times \bigg(\int_\varGamma^t \|\nabla \boldsymbol{b}(\tau)\|_{L^2}^{2} + \|u_3(\tau)\|_{L^q}^{30q/(7q-45)}\,\mathrm{d}\tau\bigg)^{(4q-15)/4q} \cdot \varepsilon^{7/8}\mathcal{M}^{1/4}(t) \\ &\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + C\mathcal{K}^{5/3}(t)\cdot\varepsilon^{7/8}\mathcal{M}^{1/4}(t) \\ &\leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + \frac{5}{6}\mathcal{K}^2(t) + \frac{1}{6}C^6\varepsilon^{21/4}\mathcal{M}^{3/2}(t).\end{aligned}$$

Consequently,

$$\mathcal{K}^2(t) \leqslant C + C\varepsilon^{1/4}\mathcal{M}^{3/2}(t) + C\varepsilon^{7/8}\mathcal{M}^{1/4}(t) + \frac{1}{6}C^6\varepsilon^{21/4}\mathcal{M}^{3/2}(t). \tag{24}$$

Now, we can argue as in [21], Step 3, in particular, equation (22) to obtain

$$\mathcal{M}^2(t) \leqslant C + C\varepsilon\mathcal{K}^2(t)\cdot\mathcal{M}^{1/2}(t).$$

Putting (24) into this above inequality, and choosing $\varepsilon$ sufficiently small, we see $\mathcal{M}(t)$ is uniformly bounded for $t \in [\varGamma, \varGamma^*)$, as desired.

The proof of Theorem 1 under condition (10) is completed. □

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