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EXACT SOLUTION OF THE TIME FRACTIONAL VARIANT BOUSSINESQ-BURGERS EQUATIONS

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Abstract. In the present article, we consider a nonlinear time fractional system of variant Boussinesq-Burgers equations. Using Lie group analysis, we derive the infinitesimal groups of transformations containing some arbitrary constants. Next, we obtain the system of optimal algebras for the symmetry group of transformations. Afterward, we consider one of the optimal algebras and construct similarity variables, which reduces the given system of fractional partial differential equations (FPDEs) to fractional ordinary differential equations (FODEs). Further, under the invariance condition we construct the exact solution and the physical significance of the solution is investigated graphically. Finally, we study the conservation law of the system of equations.

Keywords: fractional variant Boussinesq equation; symmetry analysis; exact solution

MSC 2020: 35R11, 76M60, 35D99

1. Introduction

The fractional differential equation (FDEs) and it applications for modeling of nonlinear physical phenomena has been the topic of great interest among the scientific researchers in the recent year. The concept of fractional order derivatives and it development starts with the letter of L'Hopital to Leibniz asking, for the nth order derivative of the linear function $f(x) = x$ is $D^n x/Dx^n$, what would be the result if $n = 1/2$? In reality, the description of a complex nonlinear physical phenomenon and it dynamic processes depends not only on its current state but also on its historical

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states (nonlocal property), which can be successfully modelled using the theory of derivatives and integrals of fractional order (see [16], [9], [26], [2]).

The variant Boussinesq type of equations are the class of nonlinear evolution equations with many applications in physical sciences and engineering fields. For example, in plasma physics, these equations give rise to the ion acoustic solitons [24], [21]; in geophysical fluid dynamics, they describe a long wave in shallow seas and deep oceans [20], etc. In particular, the time fractional variant Boussinesq-Burger system is an evolution equations for surface water waves with weak nonlinearity and weak dispersion. This model with the structure of horizontal and vertical flow velocity is very helpful in the study of coastal hydrodynamics.

In order to understand the complete physical phenomena in a better way, it is very essential to solve them exactly. On the contrary, it is extremely hard to get the exact solution. During the last decade, many mathematicians and physicists have devoted considerable effort to find the numerical as well as analytical solutions for the FPDEs. In order to obtain the numerical solutions, different techniques are used by various researchers; for example, the homotopy analysis method [19], the homotopy perturbation method [6], the Adomian decomposition method [29], [30], and the variational iteration method [10].

Although many scientific researchers have studied the variant Boussinesq-Burger system in different prospectives, the group invariance properties of the time fractional variant Boussinesq-Burger system are much less understood and scientific researchers are actively working in this field. Lie group analysis is one of the most powerful and systematic techniques to handle such problems. The symmetry group of transformations and invariance properties of FPDEs is discussed in [4], [8]. On the contrary, the authors in [17], [3], [18], [28], [11] studied the group classification, symmetry reductions and exact solutions of FPDEs arising in many physical phenomena. For the explicit and exact solutions of the variant Boussinesq system, the reader is referred to [25], [27], [7]. The authors in [15], [12] discussed the lump, breather and solitary wave solutions, while for the optical solitons of various physical models, we referred to [14], [13].

The work in this article is represented as follows: in Section 2, from the application of Lie group analysis, the symmetric group of transformations under which the given equation remains invariant are derived. Next, depending on the parameters in the transformations, a set of optimal algebras is obtained in Section 3. In Section 4, the symmetry reduction is presented, whereas derivation of the exact solution of the FPDEs through one of the optimal algebras and the discussion of the nonlinear property of the solution with respect to fractional order α with the help of $2D$ and $3D$-plots is placed in Section 5. Further, the conservation laws of a given equation are studied in Section 6 and finally we give our brief conclusion in Section 7.

2. Symmetry group of transformations

The time fractional variant Boussinesq-Burgers equations [19], which were derived by Sachs [23] in the year 1988 as a model for water waves is considered as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + uv_x + vu_x + v_{xxx} = 0, \quad \frac{\partial^\alpha v}{\partial t^\alpha} + u_x + vv_x = 0, \tag{2.1}$$

where $u(x,t)$ denotes the height of the water surface above the horizontal level at the bottom, $v(x,t)$ horizontal velocity and α is the parameter standing for the order of the fractional time derivative and satisfying $0 < \alpha < 1$.

The Lie group of point transformations (see, [8], [3], [22]) which leaves the system of FPDEs (2.1) invariant and is useful to reduce FPDEs to FODEs as well as to find invariant solutions is given by

$$\begin{aligned} x^* &= x + \varepsilon \xi^x(x,t,u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon \xi^t(x,t,u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon \eta^u(x,t,u) + O(\varepsilon^2), \\ v^* &= v + \varepsilon \eta^v(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{2.2}$$

where ξ^x, ξ^t, η^u and η^v in (2.2) are the infinitesimals Lie group of transformations and they are to be determined.

From straightforward analysis and tedious calculation, we obtain the infinitesimals as follows:

$$\xi^x = \alpha C_2 x + C_3, \quad \xi^t = 2C_2 t + C_1, \quad \eta^u = -2\alpha C_2 u, \quad \eta^v = -\alpha C_2 v,$$

where C_1, C_2, and C_3 are arbitrary constants. The infinitesimal generators associated with C_1, C_2, and C_3 are given as follows:

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \alpha x \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - 2\alpha u \frac{\partial}{\partial u} - \alpha v \frac{\partial}{\partial v}, \quad V_3 = \frac{\partial}{\partial x}.$$

The form of the operators V_i, $i = 1,2,3$, suggests their signification, as V_1 and V_3 generate the symmetry of time and space translations, respectively, whereas V_2 is associated with the scaling transformations.

3. Optimal algebra

3.1. Commutator table. In order to calculate the invariant, we first construct the commutative table for the vector fields such that $[V_i, V_j] = V_iV_j - V_jV_i$, where i and j represent the row and column, respectively.

$*$	V_1	V_2	V_3
V_1	0	$2V_1$	0
V_2	$-2V_1$	0	$-\alpha V_3$
V_3	0	αV_3	0

Considering $X = \sum_{i=1}^{3} a_iV_i$ and $Y = \sum_{j=1}^{3} b_jV_j$, we write the adjoint operator as

$$\mathrm{Ad}_{\exp(\varepsilon Y)}X = (a_1V_1 + a_2V_2 + a_3V_3) - \varepsilon(\theta_1V_1 + \theta_2V_2 + \theta_3V_3) + O(\varepsilon^2).$$

3.2. Adjoint table. The adjoint representation of the vector operators X and Y is given by

$$\mathrm{Ad}_{\exp(\varepsilon Y)}X = \mathrm{e}^{-\varepsilon Y}X\mathrm{e}^{\varepsilon Y} = X - \varepsilon[Y, X] + \frac{1}{2!}\varepsilon^2[Y, [Y, X]] - \dots \tag{3.1}$$

From equation (3.1), we obtain the following adjoint table:

Adj	V_1	V_2	V_3
V_1	V_1	$V_2 - 2\varepsilon V_1$	V_3
V_2	$\mathrm{e}^{2\varepsilon}V_1$	V_2	$\mathrm{e}^{\varepsilon\alpha}V_3 - X_4$
V_3	V_1	$V_2 - 2\varepsilon^2V_3$	V_3

Then we take the adjoint action of V_i on V_i and construct the following matrices. Let us take

$$\begin{aligned}\mathrm{Ad}_{\exp(\varepsilon_1V_1)}V_i &= \mathrm{Ad}_{\exp(\varepsilon_1V_1)}(a_1V_1 + a_2V_2 + a_3V_3)\\ &= (a_1 - 2a_2\varepsilon_1)X_1 + a_2X_2 + a_3X_3.\end{aligned} \tag{3.2}$$

The matrix corresponding to equation (3.2) is given by:

$$A_1 = \begin{pmatrix} 1 & 0 & 0\\ -2\varepsilon_1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \tag{3.3}$$

Similarly, one can construct the matrices A_2 and A_3 for the adjoint action of V_2 and V_3 on V_i, respectively, as shown below:

$$A_2 = \begin{pmatrix} e^{2\varepsilon_2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\alpha\varepsilon_2} \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_3^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Further, the general adjoint transformation matrix A is written as

$$A = A_1A_2A_3 = \begin{pmatrix} e^{2\varepsilon_2} & 0 & 0 \\ -2\varepsilon_1 e^{2\varepsilon_2} & 1 & -\varepsilon_3^2 \\ 0 & 0 & e^{\alpha\varepsilon_2} \end{pmatrix}.$$

3.3. Optimal algebra. We classify the optimal by using the adjoint transformation equations as:

$$(a_1^*, a_2^*, a_3^*) = (a_1, a_2, a_3)A. \tag{3.4}$$

Solving equation (3.4) for different case, finally we get that the optimal algebras are V_1, V_2, and V_3.

4. Symmetry reduction

For the similarity reduction of equation (2.1), let us consider the optimal algebra V_2 and we get the corresponding characteristic equations as

$$\frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}t}{2t} = -\frac{\mathrm{d}u}{2\alpha u} = -\frac{\mathrm{d}v}{\alpha v},$$

and the corresponding similarity variables are obtained as

$$\eta = xt^{-\alpha/2}, \quad u = t^{-\alpha}U(\eta), \quad v = t^{-\alpha/2}V(\eta). \tag{4.1}$$

Using the similarity variables from (4.1) and in the sense of Riemann Liouville derivatives, the equation (2.1) can be reduced to a nonlinear ODE of fractional order through the following theorem.

Theorem 4.1. *The transformation* (4.1) *reduces* (2.1) *to the following nonlinear ordinary differential equation of fractional order:*

$$\begin{aligned} (P_{2/\alpha}^{1-2\alpha,n-\alpha}U)(\eta) + UV_\eta + VU_\eta + V_{\eta\eta\eta} &= 0, \\ (P_{2/\alpha}^{1-3\alpha/2,n-\alpha}V)(\eta) + U_\eta + VV_\eta &= 0, \end{aligned} \tag{4.2}$$

with the Erdélyi-Kober fractional differential operator $P_\beta^{\tau,\alpha}$ of order:

$$(P_\beta^{\tau,\alpha}) = \prod_{j=0}^{n-1}\Big(\tau + j - \frac{1}{\beta}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\Big)(K_\beta^{\tau+\alpha,n-\alpha}U)(\eta), \tag{4.3}$$

$$n = \begin{cases} [\alpha]+1, & \alpha \notin N \\ \alpha, & \alpha \in N, \end{cases}$$

where

$$(K_\beta^{\tau,\alpha}U)(\eta) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}U(\eta u^{1/\beta})\,\mathrm{d}u, & \alpha > 0 \\ U(\eta), & \alpha = 0, \end{cases}$$

and

$$(K_\beta^{\tau,\alpha}V)(\eta) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\int_1^\infty (v-1)^{\alpha-1}v^{-(\tau+\alpha)}V(\eta v^{1/\beta})\,\mathrm{d}v, & \alpha > 0 \\ V(\eta), & \alpha = 0, \end{cases}$$

is the Erdélyi-Kober fractional integral operator.

P r o o f. Let α be such that $\alpha \in (n, n+1)$, where $n \in N$. Then using the Riemann Liouville derivative, we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\Big[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{-\alpha}U(xs^{-\alpha/2})\,\mathrm{d}s\Big]. \tag{4.4}$$

Consider $h = t/s$, from which it follows that $\mathrm{d}s = -th^{-2}\,\mathrm{d}h$ and hence equation (4.4) can be written as

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n}\Big[t^{n-2\alpha}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (h-1)^{n-\alpha-1}h^{-(n-2\alpha+1)}U(\eta h^{\alpha/2})\,\mathrm{d}h\Big] \\ &= \frac{\partial^n}{\partial t^n}[t^{n-2\alpha}(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta)],\end{aligned} \tag{4.5}$$

where

$$(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta) = \frac{1}{\Gamma(n-\alpha)}\int_1^\infty (h-1)^{n-\alpha-1}h^{-(n-2\alpha+1)}U(\eta h^{\alpha/2})\,\mathrm{d}h.$$

From the relation $\eta = xt^{-\alpha/2}$, one can obtain

$$t\frac{\partial}{\partial t}\varphi(\eta) = t\frac{\partial\eta}{\partial t}\varphi'(\eta) = tx\Big(\frac{-\alpha}{2}\Big)t^{-\alpha/2-1}\varphi'(\eta) = \frac{-\alpha}{2}\eta\varphi'(\eta). \tag{4.6}$$

Using (4.6) in (4.5), we obtain

$$\begin{aligned}\frac{\partial^n}{\partial t^n}[t^{n-2\alpha}(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta)] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}(t^{n-2\alpha}(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta))\Big] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-2\alpha}\Big(n-2\alpha-\frac{\alpha}{2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta)\Big].\end{aligned}$$

One can repeat the above procedure to get the following:

$$\begin{aligned}\frac{\partial^n}{\partial t^n}[t^{n-2\alpha}(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta)] &= \frac{\partial^{n-1}}{\partial t^{n-1}}\Big[\frac{\partial}{\partial t}(t^{n-2\alpha}(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta))\Big] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\Big[t^{n-2\alpha}\Big(n-2\alpha-\frac{\alpha}{2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta)\Big] \\ &= t^{-2\alpha}\prod_{j=0}^{n-1}\Big(n-2\alpha+j-\frac{\alpha}{2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta),\end{aligned}$$

which yields

$$\frac{\partial^n}{\partial t^n}[t^{n-2\alpha}(K_{\alpha/2}^{1-\alpha,n-\alpha}U)(\eta)] = t^{-2\alpha}(P_{2/\alpha}^{1-2\alpha,n-\alpha}U)(\eta).$$

and hence, we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-2\alpha}(P_{2/\alpha}^{1-2\alpha,n-\alpha}U)(\eta). \tag{4.7}$$

Similarly, for the second equation following the above procedure, we get

$$\frac{\partial^\alpha v}{\partial t^\alpha} = t^{-3\alpha/2}(P_{2/\alpha}^{1-3\alpha/2,n-\alpha}V)(\eta). \tag{4.8}$$

Finally, using (4.8), (4.7), and (4.1), in (2.1) one can obtain the reduced system of fractional order ODEs (4.2). □

5. Exact solution

In this section, we discuss the analytical solution of the time-fractional variant Boussinesq-Burger's equations (2.1). For that, let us derive the fractional derivative operators for the given system of fractional order PDEs by considering $c = \alpha/2$, $d = \alpha = \beta$ (for details see [5]) and hence, we obtain

$$\frac{\partial^\beta u}{\partial t^\beta} = t^{-2\alpha}\Big[\Big(1-d-\beta-c\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)(F_\beta^{-d,c}U)(\eta)\Big]. \tag{5.1}$$

Similarly for $v = Vt^{-c}$, and $\eta = xt^{-c}$ where $c = \alpha/2$ and by carrying out the analysis as above, we get

$$\frac{\partial^\beta v}{\partial t^\beta} = t^{-3\alpha/2}\Big[\Big(1 - c - \beta - c\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)\big(F_\beta^{-c,c}U\big)(\eta)\Big]. \tag{5.2}$$

Further, the reduced system of fractional order ODEs can be written as

$$\begin{aligned} &\Big[\Big(1 - 2\alpha - \frac{\alpha}{2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)(F_\beta^{-\alpha,\alpha/2}U)(\eta)\Big] + UV_\eta + VU_\eta + V_{\eta\eta\eta} = 0, \\ &\Big[\Big(1 - \frac{3\alpha}{2} - \frac{\alpha}{2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\Big)(F_\beta^{-\alpha/2,\alpha/2}U)(\eta)\Big] + U_\eta + VV_\eta = 0. \end{aligned} \tag{5.3}$$

For the similarity solution let us first introduce functions as below

$$U(\eta) = A\eta^a, \quad V(\eta) = B\eta^b, \quad \eta = xt^{-c}, \tag{5.4}$$

where the parameters A, B, a, and b are arbitrary real constants which are to be determined and $c = \alpha/2$. If we substitute (5.4) in (5.3), we get

$$\begin{aligned} &\frac{\Gamma(1-\alpha-\frac{1}{2}a\alpha)}{\Gamma(2-2\alpha-\frac{1}{2}a\alpha)}A\eta^a + ABb\eta^{a+b-1} + ABa\eta^{a+b-1} + Bb(b-1)(b-2)\eta^{b-3} = 0, \\ &\frac{\Gamma(1-\frac{1}{2}(1+b)\alpha)}{\Gamma(2-\alpha-\frac{1}{2}(1+b)\alpha)}B\eta^b + Aa\eta^{a-1} + B^2b\eta^{2b-1} = 0. \end{aligned} \tag{5.5}$$

The exact group invariant solutions will exist iff the equation (5.5) remains invariant with respect to η. The second equation of the above system of equations will remain invariant with respect to η if $a = 2b$ and the first equation will remain invariant with respect to η if $b = 1$. Hence we can obtain the particular solution of (5.5), for $a = 2$ and $b = 1$.

Using $a = 2$ and $b = 1$ in (5.5), we get

$$\begin{aligned} A &= -\frac{\Gamma(1-2\alpha)}{6\Gamma(2-3\alpha)}\Big[\frac{\Gamma(1-2\alpha)}{3\Gamma(2-3\alpha)} - \frac{\Gamma(1-\alpha)}{\Gamma(2-2\alpha)}\Big], \\ B &= -\Big[\frac{\Gamma(1-2\alpha)}{3\Gamma(2-3\alpha)}\Big]. \end{aligned}$$

Hence, the particular exact solution of (2.1) can be given as

$$\begin{aligned} u(x,t) &= -\frac{\Gamma(1-2\alpha)}{6\Gamma(2-3\alpha)}\Big[\frac{\Gamma(1-2\alpha)}{3\Gamma(2-3\alpha)} + \frac{\Gamma(1-\alpha)}{\Gamma(2-2\alpha)}\Big]\frac{x^2}{t^{2\alpha}}, \\ v(x,t) &= -\Big[\frac{\Gamma(1-2\alpha)}{3\Gamma(2-3\alpha)}\Big]\frac{x}{t^\alpha}. \end{aligned} \tag{5.6}$$

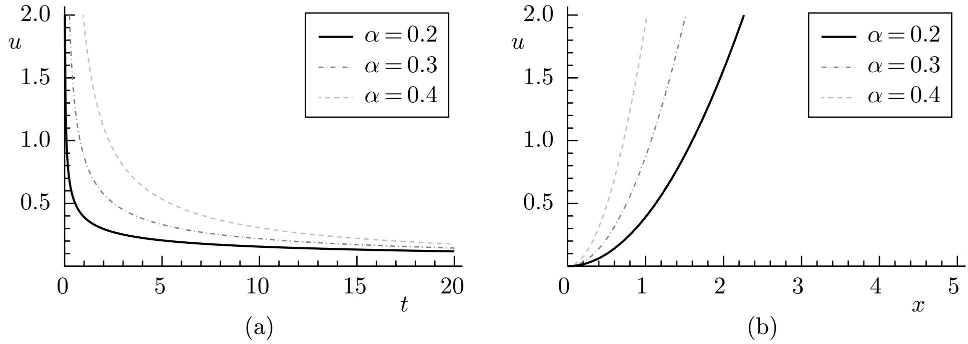

Figure 1. Behavior of $u(x,t)$ for $0 < \alpha < 1$ and (a) fixed x, (b) fixed t.

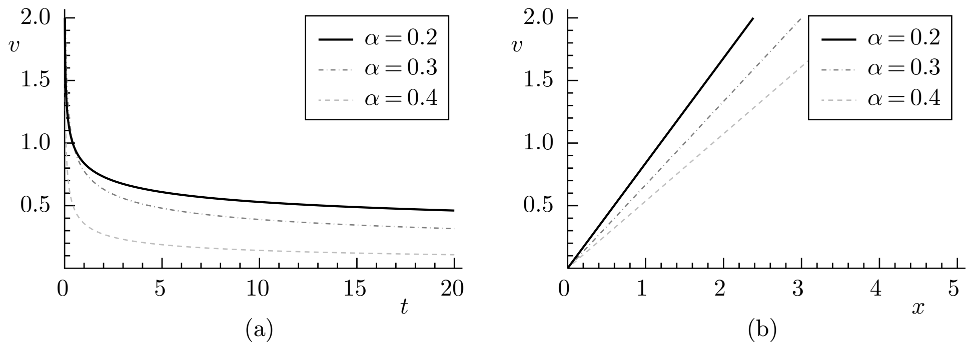

Figure 2. Behavior of $v(x,t)$ for $0 < \alpha < 1$ and (a) fixed x, (b) fixed t.

From Figures 1 (a)–2 (a), it is noticed that both $u(x,t)$ and $v(x,t)$ are valid for $t > 0$ and $x \in [0,\infty)$. This solution is called a very singular solution or dipole solution. Figure 1 (a) illustrates that with increasing α, the shock formation time for the solution $u(x,t)$ further delays whereas Figure 2 (a) shows that with increasing α, the water wave formation time for the solution $v(x,t)$ speeds up for fixed x, indicating the height of water surface and it horizontal velocity complementary in nature with respect to the fractional order α. However, from the Figure 1 (b) it is noticed that for fixed t the shock formation time for $u(x,t)$ decreases with increasing α and $v(x,t)$ is complimentary to it with linearity. Moreover, from the Figures 1 (a)–2 (b), it can be observed that a change in the noninteger order derivative value affects solution behavior in a fundamental way [1], which suggests that the noninteger order derivative can be used to modulate the shape of the water waves. Therefore, we can say that the noninteger order derivative can be used to modify the shape of the water wave without changing the nonlinearity and the dissipative effect in the medium.

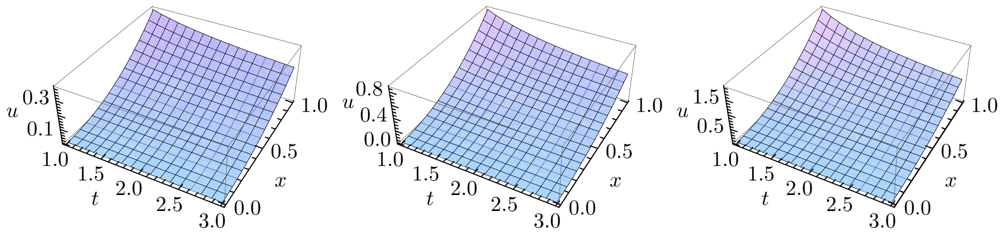

Figure 3. $3D$-plot of $u(x,t)$ for $\alpha = 0.2$, $\alpha = 0.3$ and $\alpha = 0.4$.

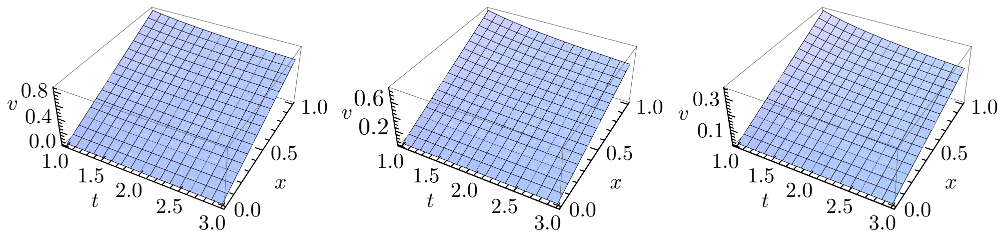

Figure 4. $3D$-plot of $v(x,t)$ for $\alpha = 0.2$, $\alpha = 0.3$ and $\alpha = 0.4$.

6. Conservation laws

In this section, conservation laws of time fractional Boussinesq-Burgers equations are derived. For that, we find the Lagrangian of (2.1) as follows:

$$\mathcal{L} = \gamma(x,t)\Big(\frac{\partial^\alpha u}{\partial t^\alpha} + uv_x + vu_x + v_{xxx}\Big) + \lambda(x,t)\Big(\frac{\partial^\alpha v}{\partial t^\alpha} + u_x + vv_x\Big), \tag{6.1}$$

where γ and λ are new dependent variables of x and t. The action integral of (5.5) can be given by

$$\int_0^t \int_\Omega \mathcal{L}\Big(x, t, u, v, \gamma, \lambda, \frac{\partial^\alpha u}{\partial t^\alpha}, u_x, \frac{\partial^\alpha v}{\partial t^\alpha}, v_x, v_{xxx}\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{6.2}$$

The Euler-Lagrange operators are given by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial(D_t^\alpha u)} - D_x \frac{\partial}{\partial u_x}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} + (D_t^\alpha)^* \frac{\partial}{\partial(D_t^\alpha v)} - D_x \frac{\partial}{\partial v_x} - D_x^3 \frac{\partial}{\partial v_{xxx}}, \end{aligned} \tag{6.3}$$

where $(D_t^\alpha)^*$ is the adjoint operator of (D_t^α). Now the Lagrange equation of (2.1) can be written as

$$\frac{\partial \mathcal{L}}{\partial u} = 0, \quad \frac{\partial \mathcal{L}}{\partial v} = 0.$$

As there are two independent variables and two dependent variables we have:

$$\widetilde{X} + D_t(\tau)I + D_x(\xi)I = W_1\frac{\partial}{\partial u} + W_2\frac{\partial}{\partial v} + D_tC^1 + D_xC^t, \tag{6.4}$$

where $\widetilde{X}$ is the extended prolongation and C^1, C^2 are conserved vectors are to be found,

$$W_1 = -2\alpha u - \alpha x u_x - 2tu_t,\ W_2 = -\alpha v - \alpha x v_x - 2tv_t. \tag{6.5}$$

Then C^1 and C^2 are calculated as

$$\begin{aligned} C^1 &= \tau\mathcal{L} +_0 D_t^{\alpha-1}(W_1)\frac{\partial\mathcal{L}}{\partial_0(D_t^\alpha)}u + J\Big(W_1, \frac{\partial\mathcal{L}}{\partial_0({}_0D_t^\alpha)}u\Big) \\ &\quad +_0 D_t^{\alpha-1}(W_2)\frac{\partial\mathcal{L}}{\partial_0(D_t^\alpha)}v + J\Big(W_2, \frac{\partial\mathcal{L}}{\partial_0({}_0D_t^\alpha)}v\Big) \end{aligned}$$

and

$$\begin{aligned} C^2 &= \xi\mathcal{L} + W_1\Big[\frac{\partial\mathcal{L}}{\partial u_x}\Big] + W_2\Big[\frac{\partial\mathcal{L}}{\partial v_x} + D_xD_x\frac{\partial\mathcal{L}}{\partial v_{xxx}}\Big] \\ &\quad + D_x(W_2)\Big[-D_x\frac{\partial\mathcal{L}}{\partial v_{xxx}}\Big] + D_xD_x(W_2)\frac{\partial\mathcal{L}}{\partial v_{xxx}}, \end{aligned}$$

where $J(\cdot)$ is defined by

$$J(f,g) = \frac{1}{\Gamma(m-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)}^{\alpha+1-m}\,\mathrm{d}\mu\,\mathrm{d}\tau.$$

7. Conclusion

In the present article, we consider the time fractional variant Boussinesq-Burgers equations which arise as a model for water waves. Using Lie group analysis, a symmetric group of transformations as well as the associated optimal systems are derived. The given system of FPDEs is reduced to a system of FODEs and a particular exact solution for the given system of FPDEs are discovered. Further, the effect of the fractional order α on the behavior of the solution is studied graphically. From the Figures 1 (a) and 2 (a), for fixed x, it is noticed that the solution is very singular in nature and valid only for $t > 0$. In Figure 1 (b), it is also clear that for fixed t, the solution $u(x,t)$ is very singular in nature. From Figures 3–4, it is also observed that a change in noninteger order derivative value affects soliton behavior in a fundamental way and modulates the shape of wave without changing the nonlinearity and the dissipative effect in the medium. To the best of our knowledge, one of the short coming of the proposed method is in handling the time fractional initial and boundary value problems. In the future, we are planing to consider some problems like higher dimensional FDEs and space-time FDEs arising in many physical phenomena and using the proposed method we will solve them.

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