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CONSENSUS OF A TWO-AGENT SYSTEM WITH NONLINEAR DYNAMICS AND TIME-VARYING DELAY

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Abstract. To explore the impacts of time delay on nonlinear dynamics of consensus models, we incorporate time-varying delay into a two-agent system to study its long-time behaviors. By the classical 3/2 stability theory, we establish a sufficient condition for the system to experience unconditional consensus. Numerical examples show the effectiveness of the proposed protocols and present possible Hopf bifurcations when the time delay changes.

Keywords: consensus; multi-agent system; nonlinear dynamics; time-varying delay; Hopf bifurcation

MSC 2020: 34A34, 34D05, 34K25

1. Introduction

Recent years have witnessed an increasing number of studies concerned with the consensus problem of multi-agent systems due to its vast potential in applications, including flocking theory [2], [14], sensor networks [9], [20], distributed decision-making [11], and UAV systems [16], [26]. The main objective of the problem is to design the control law, also called the consensus protocol or algorithm, to drive a group of autonomous agents to achieve consensus in which states of all agents agree upon a common assessment or certain quantity of interest. To reach consensus, every individual evolves by comparing its current state with the information coming from its neighbors.

Consensus was theoretically introduced by Olfati-Saber and Murray in [15], where the authors established a systematic framework of some agreement problems for networked agents. This pioneering work has then been studied from different perspec-

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tives in the past decade, leading to many fascinating results and questions (see [1], [6], [8], [10], [17], [25], [27] and the references therein). Among them, delay-induced consensus problem has always drawn some researchers' attention, since time delay is ubiquitous at the moment of information transmission or processing between agents. For example, Atay [3] studied discrete and continuous time consensus problems on networks in the presence of distributed time delay. Bliman and Ferrari-Trecate [5] studied average consensus problems for undirected networks of dynamic agents having communication delay. Lin and Jia [13] investigated average consensus problem in directed networks of agents with both switching topology and time delay. Xiao and Wang [23] presented state consensus problems for discrete-time multi-agent systems with changing communications topologies and bounded time-varying communication delay. However, it should be pointed out that all these works considered time delay systems through linear consensus protocols. As we know, most practical systems have nonlinear dynamics, but few results regarding them have appeared in the literature. The difficulty mainly arises from the existence of time delay in nonlinear dynamics so that some existing methods, such as the Nyquist stability criterion and graph theory, become invalid. Meanwhile, as the delay changes, some complex behaviors may occur. On the other hand, convergence rate is a vital performance index for a proposed consensus protocol. Although the aforementioned algorithms are available for solving asymptotic consensus, they cannot control the convergence speed flexibly. From a practical point of view, fast convergence is more desirable in theories and applications when the control accuracy is crucial. Motivated by the above analysis, this paper sets out to solve the delay-induced consensus for nonlinear dynamics and its main feature is twofold. Firstly, to shed light on the impacts of time delay on consensus, we incorporate time-varying delay into a nonlinear consensus model with two agents. The resulting dynamics setting may provide us some insights regarding our further study of coupled systems with a larger number of agents. Making use of the classical 3/2 stability theory for scalar delay differential equations [19], [24], we establish sufficient conditions to guarantee that the proposed model converges to a consensus. Secondly, we add nonlinear protocol functions to the proposed protocol, which can increase the amplitude of the control input to improve the convergence speed.

The rest of this paper is as follows. The problem to be addressed is mathematically described in Section 2. Section 3 presents the delay-induced consensus results, which is numerically illustrated in Section 4. Conclusions and future research directions end the paper in Section 5.

2. Problem formulation

Consider a group of N autonomous agents whose evolution is governed by the following dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, N. \tag{2.1}$$

Here, $x_i(t) \in \mathbb{R}^d$ denotes the ith agent's state (position, opinion, voltage, incremental cost, etc.) at time t, which changes under the interaction with other agents in a manner described by the control input $u_i(t) \in \mathbb{R}^d$, called the *consensus protocol*. With a given protocol $u_i(t)$, the closed-loop system (2.1) is said to *achieve consensus* if for any $i = 1, 2, \ldots, N$ there exists $c \in \mathbb{R}^d$ such that

$$\lim_{t\to\infty} x_i(t) = c. \tag{2.2}$$

In particular, if the consensus value $c = N^{-1} \sum\limits_{k=1}^{N} x_k(0)$, then system (2.1) is said to *reach average consensus*.

Remark 2.1. One can distinguish between two main classes of consensus behaviors in system (2.1). It is referred to as unconditional consensus if (2.2) holds for arbitrary initial conditions. In contrast, conditional consensus emerges if (2.2) is limited to certain types of initial conditions.

This work considers a two-agent system with nonlinear dynamics and time-varying delay as follows:

$$\begin{cases} \dot{x}_1(t) = \frac{1}{2}\alpha a_{12}(t)(H(\widetilde{x}_2(t)) - H(\widetilde{x}_1(t))), \\ \dot{x}_2(t) = \frac{1}{2}\alpha a_{21}(t)(H(\widetilde{x}_1(t)) - H(\widetilde{x}_2(t))), \end{cases} \tag{2.3}$$

where we denote by $\widetilde{x}_i(t)$ the same quantity evaluated at time $t-\tau(t)$, i.e., $\widetilde{x}_i(t) = x_i(t-\tau(t))$ and $\alpha > 0$ measures the overall interaction strength. For $i = 1, 2$, the so-called protocol function $H(\widetilde{x}_i) = [h(\widetilde{x}_{i1}), h(\widetilde{x}_{i2}), \ldots, h(\widetilde{x}_{id})]^\top$, where $h(w)\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $0 < \dot{h}(w) \leqslant 1$. The time delay function is assumed to be bounded, namely, for $t > 0$, there exists $\overline{\tau} > 0$ such that

$$0 < \tau(t) \leqslant \overline{\tau}.$$

We take the influence function

$$a_{ij}(t) = f(\|\widetilde{x}_j(t) - \widetilde{x}_i(t)\|) \tag{2.4}$$

as a strictly decreasing function of distance between agents with a prototype example $f(r) = 1/(1+r^2)^\beta$ for $\beta > 0$, meaning that two agents have the same impact on each other. Actually, the symmetry is the cornerstone for the following analysis of long-time behaviors of system (2.3), because it implies that the total linear momentum in the system is conserved in the sense that $(\dot{x}_1(t)+\dot{x}_2(t))/2=0$, which makes (2.3) an average consensus protocol candidate.

Remark 2.2. It is reasonable to assume that the mutual influence is a function of distance between agents. We can refer to [7] where the authors built a celebrated framework (the CS model) to describe the emergence of flocking behavior by introducing a symmetric pairwise influence function as in (2.4) under the assumption that the closer two agents are, the more they tend to align with each other.

3. Consensus analysis

In this section, we study the emergence of consensus for system (2.3). Setting $x_0(t)=x_1(t)-x_2(t)$, then (2.3) simplifies to

$$\dot{x}_0(t)=-\alpha f(\|\widetilde{x}_0(t)\|)(H(\widetilde{x}_1(t))-H(\widetilde{x}_2(t))).$$

For computational convenience, we consider its component-wise form

$$\dot{x}_{0k}(t)=-\alpha f(\|\widetilde{x}_0(t)\|)(h(\widetilde{x}_{1k}(t))-h(\widetilde{x}_{2k}(t))),\quad k=1,2,\dots,d.$$

Since $h(w)$ is continuously differentiable, according to the Lagrange mean value theorem, there exists at least one constant $\xi\in(w_1,w_2)$ such that $h(w_2)-h(w_1)=\dot{h}(\xi)(w_2-w_1)$. Using this, we obtain

$$\dot{x}_{0k}(t)=-\alpha\dot{h}(\xi_k(t))f(\|\widetilde{x}_0(t)\|)\widetilde{x}_{0k}(t) \tag{3.1}$$

and the associated initial condition reads as

$$x_{0k}(\theta)=\varphi_{0k}(\theta),\quad \theta\in[-\tau(0),0]. \tag{3.2}$$

Definition 3.1. Consensus in the two-agent system (2.3) is said to *be admitted* if

$$\lim_{t\to\infty}\|x_1(t)-x_2(t)\|=0,$$

which is equivalent to, for all $k=1,2,\dots,d$, the solution $x_{0k}(t)$ of the delay differential equation (3.1) subject to (3.2) satisfies

$$\lim_{t\to\infty}x_{0k}(t)=0. \tag{3.3}$$

Definition 3.2. Let $x_{0k}(t)$ be a solution of system (3.1). Then $x_{0k}(t)$ is said to be an *eventually positive (negative) solution* if there exists $T>0$ sufficiently large so that $x_{0k}(t)\geqslant 0$ ($x_{0k}(t)\leqslant 0$) for $t>T$. Otherwise, $x_{0k}(t)$ is said to be an *oscillatory solution.*

The main result of this section is given in the following theorem.

Theorem 3.1. *Assume that*

$$\alpha\overline{\tau} < \frac{3}{2}, \tag{3.4}$$

then the solution $x_{0k}(t)$ *of system* (3.1) *satisfies* (3.3).

We are now going to prove this theorem. The proof is divided into two lemmas [19].

Lemma 3.1. *Let* $x_{0k}(t)$ *be a nonoscillatory solution of* (3.1). *Then* (3.3) *holds unconditionally.*

Proof. Suppose that $x_{0k}(t)$ is eventually positive. Then by (3.1) we find that $\dot{x}_{0k}(t) \leqslant 0$ eventually, which implies the existence of $\lim\limits_{t\to\infty} x_{0k}(t)$. Furthermore, the limit must satisfy $\lim\limits_{t\to\infty} x_{0k}(t) = x_{0k}^* = 0$. In fact, if $x_{0k}^* > 0$, it follows from (3.1) that $\lim\limits_{t\to\infty} \dot{x}_{0k}(t) = -\alpha \lim\limits_{t\to\infty} \dot{h}(\xi_k(t))f(\|\widetilde{x}_0(t)\|)x_{0k}^* < 0$, which is impossible. The same reasoning applies to the case where $x_{0k}(t)$ is eventually negative. □

Lemma 3.2. *Let* $x_{0k}(t)$ *be an oscillatory solution of* (3.1). *Then* (3.3) *is true provided that* (3.4) *holds.*

Proof. We first prove that $x_{0k}(t)$ is bounded. On the contrary, there must exist $T > 0$ sufficiently large and $t^* \geqslant T + \tau$ such that $|x_{0k}(t)| < |x_{0k}(t^*)|$ for $t \in [0, t^*]$ and $\dot{x}_{0k}(t^*) = 0$ which implies $\widetilde{x}_{0k}(t^*) = 0$ by (3.1). Now we consider two cases.

Case I: $x_{0k}(t^*) > 0$. Then there must exist $t_0 \in [t^* - \tau(t^*), t^*)$ such that $x_{0k}(t_0) = 0$ and $x_{0k}(t) > 0$ for $t \in (t_0, t^*]$. For $s \in [0, t_0]$, by integrating

$$\dot{x}_{0k}(t) \leqslant \alpha\dot{h}(\xi_k(t))f(\|\widetilde{x}_0(t)\|)|\widetilde{x}_{0k}(t)| < \alpha x_{0k}(t^*) \tag{3.5}$$

from s to t_0, we obtain

$$-x_{0k}(s) < \alpha x_{0k}(t^*)(t_0 - s). \tag{3.6}$$

Substituting (3.6) into the right hand of (3.1) with $s = t - \tau(t)$ yields

$$\dot{x}_{0k}(t) < \alpha^2 x_{0k}(t^*)(t_0 - t + \tau(t)) = \alpha x_{0k}(t^*) \int_{t-\tau(t)}^{t_0} \alpha \, \mathrm{d}\sigma.$$

From this and (3.5), we have the estimation

$$\dot{x}_{0k}(t) < \min\left\{\alpha x_{0k}(t^*), \alpha x_{0k}(t^*) \int_{t-\tau(t)}^{t_0} \alpha \, \mathrm{d}\sigma\right\}, \quad t \in [t_0, t^*]. \tag{3.7}$$

Three cases need to be considered subsequently to illustrate that $x_{0k}(t)$ is bounded.

(i) $\alpha\overline{\tau} < 1$. By integrating (3.7) from t_0 to t^*, we have

$$x_{0k}(t^*) < \int_{t_0}^{t^*} \alpha x_{0k}(t^*)\,\mathrm{d}t < \alpha\tau(t^*)x_{0k}(t^*) < x_{0k}(t^*),$$

which is impossible.

(ii) $1 \leqslant \alpha\overline{\tau} < 3/2$ and $\alpha(t^* - t_0) \leqslant 1$. Again by (3.7), we get

$$\begin{aligned}
x_{0k}(t^*) &< x_{0k}(t^*)\int_{t_0}^{t^*} \alpha \int_{t-\tau(t)}^{t_0} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t \\
&= x_{0k}(t^*)\int_{t_0}^{t^*} \alpha\Big(\int_{t-\tau(t)}^{t} \alpha\,\mathrm{d}\sigma - \int_{t_0}^{t} \alpha\,\mathrm{d}\sigma\Big)\,\mathrm{d}t \\
&= x_{0k}(t^*)\Big(\alpha\int_{t_0}^{t^*} \alpha\tau(t)\,\mathrm{d}t - \int_{t_0}^{t^*} \alpha \int_{t_0}^{t} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t\Big) \\
&\leqslant x_{0k}(t^*)\Big(\alpha\overline{\tau}\int_{t_0}^{t^*} \alpha\,\mathrm{d}t - \frac{1}{2}\Big(\int_{t_0}^{t^*} \alpha\,\mathrm{d}t\Big)^2\Big).
\end{aligned}$$

Since $\alpha\overline{\tau}x - x^2/2$ is nondecreasing for $x \leqslant 1$, it follows that

$$x_{0k}(t^*) < x_{0k}(t^*)\Big(\alpha\overline{\tau} - \frac{1}{2}\Big) < x_{0k}(t^*),$$

which is also a contradiction.

(iii) $1 \leqslant \alpha\overline{\tau} < 3/2$ and $\alpha(t^* - t_0) > 1$. In this way, there must exist $\bar{t} \in (t_0, t^*)$ such that $\alpha(t^* - \bar{t}) = 1$. Then

$$\begin{aligned}
x_{0k}(t^*) &< x_{0k}(t^*)\int_{t_0}^{\bar{t}} \alpha\,\mathrm{d}t + x_{0k}(t^*)\int_{\bar{t}}^{t^*} \alpha \int_{t-\tau(t)}^{t_0} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t \\
&= x_{0k}(t^*)\int_{\bar{t}}^{t^*} \alpha \int_{t_0}^{\bar{t}} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t + x_{0k}(t^*)\int_{\bar{t}}^{t^*} \alpha \int_{t-\tau(t)}^{t_0} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t \\
&= x_{0k}(t^*)\int_{\bar{t}}^{t^*} \alpha \int_{t-\tau(t)}^{\bar{t}} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t \\
&= x_{0k}(t^*)\int_{\bar{t}}^{t^*} \alpha\Big(\int_{t-\tau(t)}^{t} \alpha\,\mathrm{d}\sigma - \int_{\bar{t}}^{t} \alpha\,\mathrm{d}\sigma\Big)\,\mathrm{d}t \\
&\leqslant x_{0k}(t^*)\Big(\alpha\overline{\tau}\int_{\bar{t}}^{t^*} \alpha\,\mathrm{d}t - \int_{\bar{t}}^{t^*} \alpha \int_{\bar{t}}^{t} \alpha\,\mathrm{d}\sigma\,\mathrm{d}t\Big) \\
&= x_{0k}(t^*)\Big(\alpha\overline{\tau}\int_{\bar{t}}^{t^*} \alpha\,\mathrm{d}t - \frac{1}{2}\Big(\int_{\bar{t}}^{t^*} \alpha\,\mathrm{d}t\Big)^2\Big) \\
&= x_{0k}(t^*)\Big(\alpha\overline{\tau} - \frac{1}{2}\Big) < x_{0k}(t^*),
\end{aligned}$$

which again leads to a contradiction.

Case II: $x_{0k}(t^*) < 0$. In this case, there should exist $t_0 \in [t^* - \tau(t^*), t^*)$ such that $x_{0k}(t_0) = 0$ and $x_{0k}(t) < 0$ for $t \in (t_0, t^*]$. Then one obtains from (3.1) that

$$|\dot{x}_{0k}(t)| < \alpha|\widetilde{x}_{0k}(t)| < -\alpha x_{0k}(t^*),$$

which implies that

$$-\dot{x}_{0k}(t) < -\alpha x_{0k}(t^*). \tag{3.8}$$

Integrating (3.8) from s to t_0 yields

$$x_{0k}(s) < -\alpha x_{0k}(t^*)(t_0 - s). \tag{3.9}$$

Substituting (3.9) into the right hand of (3.1) with $s = t - \tau(t)$ yields

$$\dot{x}_{0k}(t) > \alpha^2 \dot{h} f(\|\widetilde{x}_0(t)\|) x_{0k}(t^*)(t_0 - t + \tau(t)) > \alpha x_{0k}(t^*) \int_{t-\tau(t)}^{t_0} \alpha \,\mathrm{d}\sigma,$$

which implies

$$-\dot{x}_{0k}(t) < \alpha x_{0k}(t^*) \int_{t-\tau(t)}^{t_0} \alpha \,\mathrm{d}\sigma.$$

From this and (3.8), we have

$$-\dot{x}_{0k}(t) < \min\left\{ -\alpha x_{0k}(t^*), -\alpha x_{0k}(t^*) \int_{t-\tau(t)}^{t_0} \alpha \,\mathrm{d}\sigma \right\}, \quad t \in [t_0, t^*]. \tag{3.10}$$

Using the same argument as in the proof of Case I, we can also derive a contradiction with (3.10).

Hence, $x_{0k}(t)$ is bounded. We next show that $\lim\limits_{t\to\infty} x_{0k}(t) = 0$. Now we put

$$m = \limsup_{t\to\infty} x_{0k}(t), \quad n = \liminf_{t\to\infty} x_{0k}(t).$$

Then $-\infty < n \leqslant 0 \leqslant m < \infty$. For our purpose, it suffices to show that $m = n = 0$. For any $\eta > 0$, there exists $S > 0$ sufficiently large such that

$$n_1 = n - \eta < x_{0k}(t) < m + \eta = m_1, \quad t - \tau(t) \geqslant S.$$

Combining this with (3.1), for $t \geqslant S$, one has

$$\dot{x}_{0k}(t) < -\alpha n_1, \tag{3.11}$$

$$\dot{x}_{0k}(t) > -\alpha m_1. \tag{3.12}$$

Now let $\{t_p\}$ be an increasing infinite sequence subject to $t_p > S$ such that $\lim\limits_{p\to\infty} t_p = \infty$, $\lim\limits_{p\to\infty} x_{0k}(t_p) = m$. Without loss of generality, we take t_p satisfying $x_{0k}(t_p) > 0$ and $\dot{x}_{0k}(t_p) = 0$. By (3.1), we have $\widetilde{x}_{0k}(t_p) = 0$. Then there exists $\zeta_p \in [t_p - \tau(t_p), t_p)$ such that $x_{0k}(\zeta_p) = 0$ and $x_{0k}(t) > 0$ for $t \in (\zeta_p, t_p)$.

For $S \leqslant s \leqslant \zeta_p$, by integrating (3.11) from s to ζ_p, we get

$$-x_{0k}(s) < -\alpha n_1(\zeta_p - s).$$

Then by (3.1), we obtain

$$\dot{x}_{0k}(t) < -\alpha^2 n_1(\zeta_p - t + \tau(t)) = -\alpha n_1 \int_{t-\tau(t)}^{\zeta_p} \alpha \, \mathrm{d}\sigma.$$

Hence, we have the following estimation

$$\dot{x}_{0k}(t) < \min\Big\{ -\alpha n_1, -\alpha n_1 \int_{t-\tau(t)}^{\zeta_p} \alpha \, \mathrm{d}\sigma \Big\}, \quad t \in (\zeta_p, t_p). \tag{3.13}$$

There are also three cases to consider.

(i) $\alpha\overline{\tau} < 1$. By integrating (3.13) from ζ_p to t_p, we have

$$x_{0k}(t_p) < \int_{\zeta_p}^{t_p} (-\alpha n_1) \, \mathrm{d}t = -\alpha n_1 (t_p - \zeta_p) \leqslant -\alpha\overline{\tau} n_1.$$

Letting $p \to \infty$ and $\eta \to 0$, we obtain

$$m \leqslant -\alpha\overline{\tau} n. \tag{3.14}$$

(ii) $1 \leqslant \alpha\overline{\tau} < 3/2$ and $\alpha(t_p - \zeta_p) \leqslant 1$. By (3.13), one has

$$\begin{aligned}
x_{0k}(t_p) <& -n_1 \int_{\zeta_p}^{t_p} \alpha \int_{t-\tau(t)}^{\zeta_p} \alpha \, \mathrm{d}\sigma \, \mathrm{d}t \\
=& -n_1 \int_{\zeta_p}^{t_p} \alpha \Big(\int_{t-\tau(t)}^{t} \alpha \, \mathrm{d}\sigma - \int_{\zeta_p}^{t} \alpha \, \mathrm{d}\sigma \Big) \, \mathrm{d}t \\
\leqslant& -n_1 \Big(\alpha\overline{\tau} \int_{\zeta_p}^{t_p} \alpha \, \mathrm{d}t - \int_{\zeta_p}^{t_p} \alpha \int_{\zeta_p}^{t} \alpha \, \mathrm{d}\sigma \, \mathrm{d}t \Big) \\
=& -n_1 \Big(\alpha\overline{\tau} \int_{\zeta_p}^{t_p} \alpha \, \mathrm{d}t - \frac{1}{2} \Big(\int_{\zeta_p}^{t_p} \alpha \, \mathrm{d}t \Big)^2 \Big) \\
\leqslant& -n_1 \Big(\alpha\overline{\tau} - \frac{1}{2} \Big).
\end{aligned}$$

Letting $p \to \infty$ and $\eta \to 0$ leads to

$$m \leqslant -n \Big(\alpha\overline{\tau} - \frac{1}{2} \Big). \tag{3.15}$$

(iii) $1 \leqslant \alpha\overline{\tau} < 3/2$ and $\alpha(t_p - \zeta_p) > 1$. Then there exists $\overline{z} \in (\zeta_p, t_p)$ such that $\alpha(t_p - \overline{z}) = 1$ and

$$\begin{aligned}
x_{0k}(t_p) <& -n_1 \int_{\zeta_p}^{\overline{z}} \alpha \,\mathrm{d}t - n_1 \int_{\overline{z}}^{t_p} \alpha \int_{t-\tau(t)}^{\zeta_p} \alpha \,\mathrm{d}\sigma \,\mathrm{d}t \\
=& -n_1 \int_{\overline{z}}^{t_p} \alpha \int_{\zeta_p}^{\overline{z}} \alpha \,\mathrm{d}\sigma \,\mathrm{d}t - n_1 \int_{\overline{z}}^{t_p} \alpha \int_{t-\tau(t)}^{\zeta_p} \alpha \,\mathrm{d}\sigma \,\mathrm{d}t \\
=& -n_1 \int_{\overline{z}}^{t_p} \alpha \int_{t-\tau(t)}^{\overline{z}} \alpha \,\mathrm{d}\sigma \,\mathrm{d}t = -n_1 \int_{\overline{z}}^{t_p} \alpha \Big(\int_{t-\tau(t)}^{t} \alpha \,\mathrm{d}\sigma - \int_{\overline{z}}^{t} \alpha \,\mathrm{d}\sigma \Big) \,\mathrm{d}t \\
\leqslant& -n_1 \Big(\alpha\overline{\tau} \int_{\overline{z}}^{t_p} \alpha \,\mathrm{d}t - \int_{\overline{z}}^{t_p} \alpha \int_{\overline{z}}^{t} \alpha \,\mathrm{d}\sigma \,\mathrm{d}t \Big) \\
=& -n_1 \Big(\alpha\overline{\tau} \int_{\overline{z}}^{t_p} \alpha \,\mathrm{d}t - \frac{1}{2} \Big(\int_{\overline{z}}^{t_p} \alpha \,\mathrm{d}t \Big)^2 \Big) = -n_1 \Big(\alpha\overline{\tau} - \frac{1}{2} \Big),
\end{aligned}$$

which implies (3.15).

Similar arguments apply to the case where we consider the inferior limit part. It is not difficult to show that $n \geqslant -\alpha\overline{\tau}m$ when $\alpha\overline{\tau} < 1$ and $n \geqslant -m(\alpha\overline{\tau} - 1/2)$ when $1 \leqslant \alpha\overline{\tau} < 3/2$. Combining these with (3.14) and (3.15) finally yields $m = n = 0$. $\square$

This completes the proof of Theorem 3.1. By summarizing what we have discussed, the final conclusion on the consensus of system (2.3) follows immediately.

Theorem 3.2. *Two agents interacting by system* (2.3) *can reach unconditional average consensus if* (3.4) *holds true.*

Remark 3.1. To improve convergence rate, we incorporate the protocol function $H(x)$ into the classical consensus protocol. Therefore, the presented protocol (2.3) facilitates design by providing more alternative functions $H(x)$ to improve consensus performance and satisfy control requirements, which will be validated in the following simulations.

Remark 3.2. Note that we did not really have to use $f(r) = 1/(1 + r^2)^\beta$ explicitly during consensus analysis. We only need the property $0 < f < 1$, which means our result can be applied to more consensus problems. Furthermore, we are able to draw another conclusion of mathematical significance as a byproduct: that for any one-dimensional delay differential equation in the form of

$$\dot{x}(t) = -\alpha f(t, x(t), x(t-\tau))x(t-\tau)$$

with $0 < f < 1$, the zero solution is asymptotically stable for all initial configurations as long as $\alpha\tau < 3/2$.

4. Numerical simulations

In what follows, we perform numerical simulations in one-dimensional case ($d = 1$) of system (2.3). In Section 3 we established a sufficient condition for the emergence of consensus of system (2.3) that the coupling strength α and the time delay $\tau(t)$ satisfy a technical restriction, $\alpha\overline{\tau} < 3/2$. This section aims to verify the condition and explore other possible dynamics. Meanwhile, we choose two concrete protocol functions $H(x) = \sin(x/3) + x/2$ and $H(x) = \tanh(4x/5)$ to make comparisons with the existing $H(x) = x$ in terms of convergence rate. For demonstration purposes, we fix $\alpha = \beta = 1$, $\tau(t) = \tau$ and take a series of stochastic constants as initial data throughout this section.

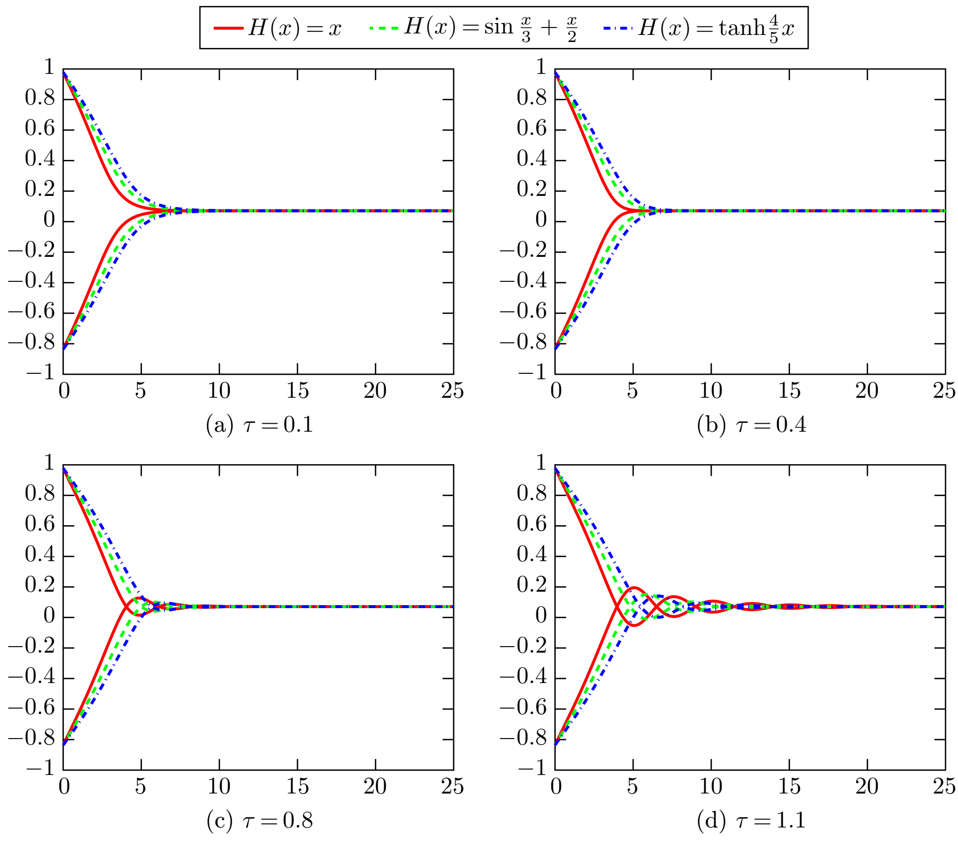


Figure 1. Consensus of system (2.3) with different $H(x)$ and τ (horizontal axes show time, vertical axes show states).

To begin, we take $\tau = 0.1$ in Figure 1(a), $\tau = 0.4$ in Figure 1(b), $\tau = 0.8$ in Figure 1(c), and $\tau = 1.1$ in Figure 1(d) to show the response of agents' states, where we can see that all the three protocols enable the states of two agents to reach average consensus provided that $\alpha\tau < 3/2$. Moreover, the convergence rate changes with τ and $H(x)$. As can be seen from Figure 2, generally speaking, the more time an agent needs for processing the information it receives, the slower the consensus will be to arise. One also observes that the protocols with $H(x) = \sin(x/3) + x/2$ and $H(x) = \tanh(4x/5)$ achieve consensus faster than the protocol with $H(x) = x$.

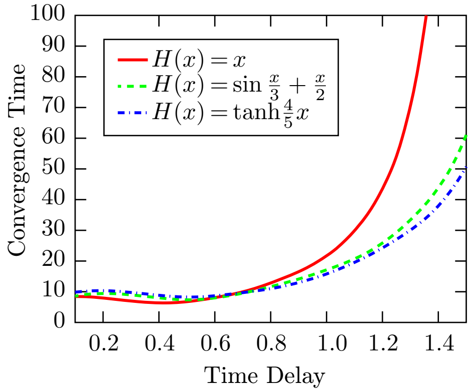


Figure 2. Convergence time versus time delay.

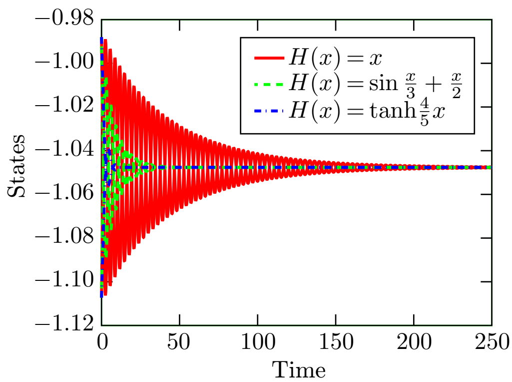


Figure 3. Consensus of system (2.3) can also arise when $\alpha\tau = 1.501$.

However, we should point out that the restriction of $\alpha\tau < 3/2$ is not a necessary condition. In Figure 3, it is clear that consensus can also appear when $\tau = 1.501$ ($\alpha\tau = 1.501 > 1.5$). This interesting phenomenon motivates our further study and intensive efforts to improve the condition. In fact, as τ increases, a sequence of Hopf bifurcations will occur at the equilibrium $x = 0$, which is illustrated in Figure 4(a) with $\tau = \pi/2 + 0.1$ and in Figure 4(b) with $\tau = \pi/2 + 0.5$. It is also shown that the three protocols lead to different bifurcation values (critical values for bifurcation occurrence). The two nontrivial protocols have bigger bifurcation values, that is, they have better tolerances to time delay than $H(x) = x$. Finally, motivated by [21], we can locate the bifurcation values for the case with $H(x) = x$. Then the associated system (3.1) takes

$$\dot{x}(t) = -\alpha f(x(t-\tau))x(t-\tau). \tag{4.1}$$

Linearizing system (4.1) at $x = 0$ yields $\dot{x}(t) = -\alpha x(t-\tau)$, whose characteristic equation is

$$\lambda = -\alpha \mathrm{e}^{-\lambda\tau}. \tag{4.2}$$

For $\tau > 0$, system (4.2) has a pair of purely imaginary roots $\lambda = \pm \mathrm{i}\omega$ $(\omega > 0)$ if and only if

$$\mathrm{i}\omega = -\alpha(\cos\omega\tau - \mathrm{i}\sin\omega\tau).$$

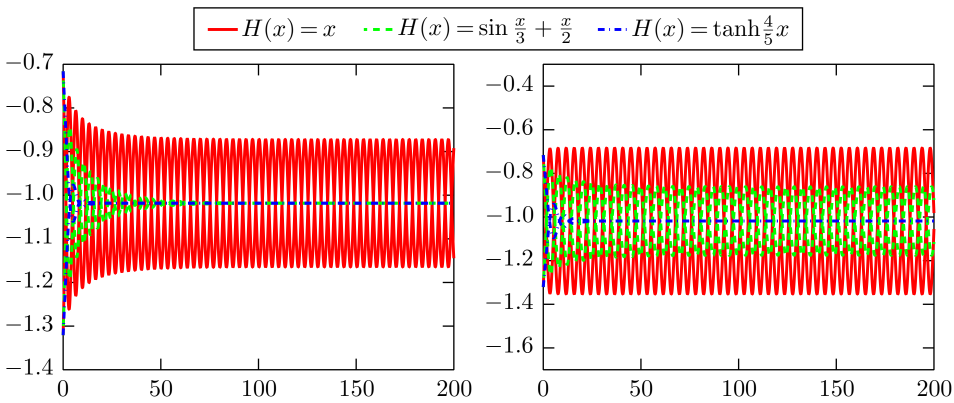
Separating the real and imaginary parts, we obtain

$$0 = -\alpha\cos\omega\tau, \quad \omega = \alpha\sin\omega\tau,$$

which leads to $\omega = \alpha$ and

$$\tau = \frac{\pi}{2} + 2k\pi, \quad k = 0, 1, 2, \ldots \tag{4.3}$$

This indicates that Hopf bifurcations of the protocol with $H(x) = x$ will occur when (4.3) holds. We emphasize that these bifurcations are local, which means they are also likely (but not guaranteed) to occur when (4.3) is not satisfied. Therefore, it cannot be easily asserted that the sharpest condition for system (2.3) with $H(x) = x$ to achieve consensus is $\alpha\tau < \pi/2$.



(a) $\tau = \frac{\pi}{2} + 0.1$ (b) $\tau = \frac{\pi}{2} + 0.5$

Figure 4. As τ increases, Hopf bifurcations may occur (horizontal axes show time, vertical axes show states).

5. Conclusions and discussions

We have incorporated time-varying delay into a two-agent system with nonlinear dynamics. The resulting model defines an infinitely dimensional dynamical system and admits complex long-time behaviors. A sufficient condition that the coupling strength α and the time delay $\tau(t)$ satisfy $\alpha\overline{\tau} < 3/2$ is obtained for the system to ensure the existence of unconditional consensus by the classical 3/2 stability results for scalar delay differential equations.

Our simulations, firstly, have validated the sufficient condition and explored how the time delay influences consensus behaviors. Generally speaking, when τ is relatively too small, it has little effect on consensus behaviors. As τ continues increasing, however, the consensus of system (2.3) turns to be significantly slower and even undergoes bifurcations. This phenomenon can be shown in real life in that it is difficult for teammates to agree with each other if everyone takes too much time for thinking and as time goes, some divergent opinions may arise. It is worth noting that our sufficient condition for system (2.3) to reach consensus is $\alpha\tau < 3/2$, which is numerically verified but not the sharpest. When $\alpha\tau \geqslant 3/2$, consensus may also occur. In fact, the 3/2 stability result for some scalar delay differential equations has attracted wide attention. Typically, in 1955, Wright [22] proved that the zero solution of the delay differential equation (the celebrated Wright's equation)

$$\dot{x}(t) = -\alpha x(t-1)(1+x(t)), \quad \alpha > 0,$$

is asymptotically stable for $\alpha < 3/2$ and conjectured that this is even true for $\alpha < \pi/2$. Bánhelyi et al. [4] subsequently provided a computer-assisted proof for $\alpha \in [1.5, 1.5705]$. However, more precise results still remain to be checked in theory. These, together with some analogous conclusions in references [19], [24], [12], [18] and bifurcation analysis in Section 4, naturally motivate us to propose the following open conjecture.

Conjecture 5.1. *Two agents interacting by system* (2.3) *can reach unconditional average consensus if* $\alpha\overline{\tau} < \pi/2$*. For the case with* $H(x) = x$*, this is also the necessary condition.*

For the case with $H(x) \neq x$, a much better condition is to be discovered by further experiments. Secondly, by selecting two nontrivial protocol functions, our figures have revealed that the proposed protocol can really provide controller-designers with more flexibility to improve consensus performance. However, consensus can never be formed within a finite time because the protocols are Lipschitz continuous. This motivates our further investigations, in order to present a finite-time consensus protocol for time delay systems.

On the other hand, by virtue of its complexity, our focus is restricted to the case where the dynamics have a symmetric influence function and two agents only. Nevertheless, technical analysis may provide some insights for our further study of the case where the model involves asymmetric mutual communication and multiple agents.

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