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REGULARITY CRITERION FOR A NONHOMOGENEOUS INCOMPRESSIBLE GINZBURG-LANDAU-NAVIER-STOKES SYSTEM

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Abstract. We prove a regularity criterion for a nonhomogeneous incompressible Ginzburg-Landau-Navier-Stokes system with the Coulomb gauge in $\mathbb{R}^3$. It is proved that if the velocity field in the Besov space satisfies some integral property, then the solution keeps its smoothness.

Keywords: Ginzburg-Landau; Navier-Stokes; regularity criterion

MSC 2020: 35Q30, 35Q56, 76D03, 82D55

1. Introduction

In this work, we consider the following nonhomogeneous incompressible Ginzburg-Landau-Navier-Stokes system with the Coulomb gauge [6]:

$$\operatorname{div} u = 0, \tag{1.1}$$

$$\partial_t \varrho + u\cdot\nabla\varrho = 0, \tag{1.2}$$

$$\varrho\partial_t u + \varrho u\cdot\nabla u + \nabla\pi - \Delta u = |\psi|^2\nabla h, \tag{1.3}$$

$$\eta\partial_t\psi + \mathrm{i}\eta k\varphi\psi + u\cdot\nabla\psi + \Big(\frac{\mathrm{i}}{k}\nabla + A\Big)^2\psi + (|\psi|^2-1)\psi = 0, \tag{1.4}$$

$$\partial_t A + \nabla\varphi - \Delta A + \mathrm{Re}\Big\{\Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}\Big\} = 0, \tag{1.5}$$

$$\operatorname{div} A = 0 \quad \text{in } \mathbb{R}^3\times(0,\infty), \tag{1.6}$$

$$(u,\psi,A)(\cdot,0) = (u_0,\psi_0,A_0)(\cdot) \quad \text{in } \mathbb{R}^3, \tag{1.7}$$

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where ϱ is the density, u is the velocity, π is the pressure, ψ is the complex order parameter, A is the vector potential and φ is the electric potential, respectively, η and k are the positive Ginzburg-Landau constants, $\overline{\psi}$ is the complex conjugate of ψ, $\operatorname{Re}\psi := (\psi + \overline{\psi})/2$ is the real part of ψ, $|\psi|^2 := \psi\overline{\psi}$ is the density of superconductivity carriers and i is the imaginary unit. The function $h := h(x)$ denotes a potential function. We will assume that h is a smooth function.

When h is a constant, system (1.1), (1.2), and (1.3) reduces to the nonhomogeneous incompressible Navier-Stokes equations. Choe and Kim [3] showed that if the data ϱ_0 and u_0 satisfy

$$(1.8)\qquad 0 \leqslant \varrho_0 \in L^{3/2} \cap H^2, \quad u_0 \in \dot{H}^1 \cap \dot{H}^2 \quad \text{and} \quad -\Delta u_0 + \nabla \pi_0 = \sqrt{\varrho_0}\, g$$

for some $(\pi_0, g) \in \dot{H}^1 \times L^2$, then there exists a positive time T_* and a unique strong solution (ϱ, u) to the problem such that

$$(1.9)\qquad \begin{gathered} \varrho \in C([0,T_*]; L^{3/2} \cap H^2), \quad u \in C([0,T_*]; \dot{H}^1 \cap \dot{H}^2) \cap L^2(0,T_*; \dot{H}^3), \\ \partial_t u \in L^2(0,T_*; \dot{H}^1), \quad \text{and} \quad \sqrt{\varrho}\,\partial_t u \in L^\infty(0,T_*; L^2). \end{gathered}$$

Kim [14] gave the following regularity criterion:

$$(1.10)\qquad u \in L^{2p/(p-3)}(0,T; L^p_w) \text{ with } 3 < p \leqslant \infty.$$

Here L^p_w denotes the weak-L^p space and $L^\infty_w \equiv L^\infty$. Then Fan and Ozawa [7] refined it as

$$(1.11)\qquad u \in L^{2/(1-\alpha)}(0,T; \dot{B}^{-\alpha}_{\infty,\infty}) \text{ with } 0 < \alpha < 1.$$

Very recently, Hou, Xu and Ye [13] improved (1.11) as

$$(1.12)\qquad \int_0^T \frac{\|u(t)\|^{2/(1-\alpha)}_{\dot{B}^{-\alpha}_{\infty,\infty}}}{\log(\mathrm{e} + \|u(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}})}\, \mathrm{d}t < \infty \text{ with } 0 < \alpha < 1.$$

On the other hand, when $u = 0$, system (1.4), (1.5), and (1.6) reduces to the time-dependent Ginzburg-Landau, which has received many studies, e.g. [1], [2], [4], [5], [8], [9], [10], [11], [12], [16], [17]. Paper [5] showed the existence of global weak solutions. Paper [8], [4] proved the uniqueness of weak solutions.

The aim of this note is to prove (1.12) as a regularity criterion for (1.1)–(1.7) under the assumption that $|\psi_0| \leqslant 1$ on $\mathbb{R}^3$. We will prove:

Theorem 1.1. *Let* (1.8) *hold true. Let* $\psi_0, A_0 \in H^1$ *and* $h \in H^2$ *with* $\operatorname{div} u_0 = \operatorname{div} A_0 = 0$ *and* $|\psi_0| \leqslant 1$ *in* $\mathbb{R}^3$. *Let* $(\varrho, u, \pi, \psi, A, \varphi)$ *be a local strong solution to the problem* (1.1)–(1.7). *If* (1.12) *holds true with some* $0 < T < \infty$, *then the solution* $(\varrho, u, \pi, \psi, A, \varphi)$ *can be extended beyond* $T > 0$.

R e m a r k 1.1. We can prove similar results under the Lorentz gauge.

R e m a r k 1.2. Note that the system (1.1)–(1.5) holds its form under the scaling $(\varrho, u, \pi, \psi, A, \varphi, h) \to (\varrho_\lambda, u_\lambda, \pi_\lambda, \psi_\lambda, A_\lambda, \varphi_\lambda, h_\lambda) := (\varrho, \lambda u, \lambda^2\pi, \lambda\psi, \lambda A, \lambda^2\varphi, h)$ $(\lambda^2 t, \lambda x)$ for any $\lambda > 0$ when neglecting the linear lower order term ψ in (1.4). Thus (1.10), (1.11), and (1.12) are optimal in this sense.

R e m a r k 1.3. The regularity criterion includes only one unknown velocity, which plays an important role. In fact, if $u = 0$, then the GL system has a unique global strong (smooth) solution.

Applying div to (1.5) and using (1.6), we have

$$-\Delta\varphi = \operatorname{div}\operatorname{Re}\Big\{\Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}\Big\}. \tag{1.13}$$

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to use the classical Banach fixed-point theorem to show the local well-posedness of strong solutions, we only need to establish a priori estimates.

First, it follows from (1.1) and (1.2) that

$$\|\varrho(t)\|_{L^p} = \|\varrho_0\|_{L^p} \text{ with } \frac{3}{2} \leqslant p \leqslant \infty. \tag{2.1}$$

Similarly to the method in [2], [5], it is standard to prove that

$$|\psi| \leqslant 1. \tag{2.2}$$

Testing (1.4) by $\overline{\psi}$, taking the real parts and using (1.1), we get

$$\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\,\mathrm{d}x + \int\Big|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x + \int|\psi|^4\,\mathrm{d}x = \int|\psi|^2\,\mathrm{d}x,$$

which leads to

$$\int|\psi|^2\,\mathrm{d}x + \int_0^T\int\Big|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.3}$$

Testing (1.5) by A, using (1.6), (2.2), and (2.3), we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|A|^2\,\mathrm{d}x + \int|\nabla A|^2\,\mathrm{d}x &= -\operatorname{Re}\int\Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}A\,\mathrm{d}x\\ &\leqslant \Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2}\|\psi\|_{L^\infty}\|A\|_{L^2} \leqslant \Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2}\|A\|_{L^2},\end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;L^2)} + \|A\|_{L^2(0,T;H^1)} \leqslant C. \tag{2.4}$$

It follows from (2.2), (2.3), and (2.4) that

$$\int_0^T \int |\psi A|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \|\psi\|_{L^\infty(0,T;L^\infty)} \int_0^T \int |A|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C,$$

whence

$$\|\psi\|_{L^2(0,T;H^1)} \leqslant C. \tag{2.5}$$

Testing (1.13) by φ, and using (2.2) and (2.3), we compute

$$\begin{aligned}\|\varphi\|_{L^2(0,T;L^2)} &\leqslant C\|\nabla\varphi\|_{L^2(0,T;L^{6/5})} \\ &\leqslant C\Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2(0,T;L^2)} \|\psi\|_{L^\infty(0,T;L^3)} \leqslant C\end{aligned} \tag{2.6}$$

and

$$\|\nabla\varphi\|_{L^2(0,T;L^2)} \leqslant C\Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2(0,T;L^2)} \|\psi\|_{L^\infty(0,T;L^\infty)} \leqslant C. \tag{2.7}$$

Testing (1.3) by u and using (1.1), (1.2), (2.1), and (2.2), we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho|u|^2\,\mathrm{d}x + \int |\nabla u|^2\,\mathrm{d}x &= \int |\psi|^2 \nabla h\cdot u\,\mathrm{d}x \leqslant \|\psi\|_{L^6}^2 \|\nabla h\|_{L^2}\|u\|_{L^6} \\ &\leqslant C\|\nabla u\|_{L^2} \leqslant \frac{1}{2}\int |\nabla u|^2\,\mathrm{d}x + C,\end{aligned}$$

which gives

$$\|\sqrt{\varrho}u\|_{L^\infty(0,T;L^2)} + \|\nabla u\|_{L^2(0,T;L^2)} \leqslant C. \tag{2.8}$$

Testing (1.5) by $-\Delta A$, using (1.6), (2.3), and (2.2), we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla A|^2\,\mathrm{d}x + \int |\Delta A|^2\,\mathrm{d}x &= \mathrm{Re}\int \Big(\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big)\overline{\psi}\cdot\Delta A\,\mathrm{d}x \\ &\leqslant \Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2} \|\psi\|_{L^\infty}\|\Delta A\|_{L^2} \leqslant \frac{1}{2}\|\Delta A\|_{L^2}^2 + C\Big\|\frac{\mathrm{i}}{k}\nabla\psi + \psi A\Big\|_{L^2}^2,\end{aligned}$$

which yields

$$\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} \leqslant C. \tag{2.9}$$

Equation (1.4) can be written as

$$\eta\partial_t\psi - \frac{1}{k^2}\Delta\psi = -\mathrm{i}\eta k\varphi\psi - u\cdot\nabla\psi - \frac{2\mathrm{i}}{k}A\cdot\nabla\psi - |A|^2\psi - (|\psi|^2-1)\psi. \tag{2.10}$$

Testing (2.10) by $-\Delta\overline{\psi}$, taking the real parts, using (1.1), (2.6), (2.9), and (2.2), we compute

$$\begin{aligned}
&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\psi|^2\,\mathrm{d}x+\frac{1}{k^2}\int|\Delta\psi|^2\,\mathrm{d}x\\
&\quad=\mathrm{Re}\int \mathrm{i}\eta k\varphi\psi\Delta\overline{\psi}\,\mathrm{d}x-\sum_j\mathrm{Re}\int\nabla u_j\partial_j\psi\nabla\overline{\psi}\,\mathrm{d}x+\mathrm{Re}\int\frac{2\mathrm{i}}{k}A\cdot\nabla\psi\cdot\Delta\overline{\psi}\,\mathrm{d}x\\
&\qquad+\mathrm{Re}\int|A|^2\psi\Delta\overline{\psi}\,\mathrm{d}x+\mathrm{Re}\int(|\psi|^2-1)\psi\Delta\overline{\psi}\,\mathrm{d}x\\
&\quad\leqslant C\|\varphi\|_{L^2}\|\Delta\psi\|_{L^2}+C\|\nabla u\|_{L^2}\|\nabla\psi\|_{L^4}^2+C\|A\|_{L^4}\|\nabla\psi\|_{L^4}\|\Delta\psi\|_{L^2}\\
&\qquad+C\|A\|_{L^4}^2\|\Delta\psi\|_{L^2}+C\|\psi\|_{L^2}\|\Delta\psi\|_{L^2}\\
&\quad\leqslant C\|\varphi\|_{L^2}\|\Delta\psi\|_{L^2}+C\|\nabla u\|_{L^2}\cdot\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}\\
&\qquad+C\|\psi\|_{L^\infty}^{1/2}\|\Delta\psi\|_{L^2}^{3/2}+C\|\Delta\psi\|_{L^2}\\
&\quad\leqslant\frac{1}{2k^2}\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C,
\end{aligned}$$

which implies

$$\|\psi\|_{L^\infty(0,T;H^1)}+\|\psi\|_{L^2(0,T;H^2)}\leqslant C. \tag{2.11}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|_{L^4}^2\leqslant C\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}. \tag{2.12}$$

Testing (1.3) by $\partial_t u$, using (1.1), (1.2), (2.1), and (2.2), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla u|^2\,\mathrm{d}x+\int\varrho|\partial_t u|^2\,\mathrm{d}x=-\int\varrho u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x+\int|\psi|^2\nabla h\cdot\partial_t u\,\mathrm{d}x\\
&\quad\leqslant\|\sqrt{\varrho}\|_{L^\infty}\|\sqrt{\varrho}\partial_t u\|_{L^2}\|u\cdot\nabla u\|_{L^2}+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x-\int\nabla h\cdot u\partial_t|\psi|^2\,\mathrm{d}x\\
&\quad\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^{1+\alpha}}+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x\\
&\qquad+C\|u\|_{L^6}\|\nabla h\|_{L^3}(\|\Delta\psi\|_{L^2}+\|\varphi\|_{L^2}+\|A\|_{L^3}\|\nabla\psi\|_{L^6}+\|A\|_{L^4}^2+1)\\
&\qquad+C\|u\|_{L^6}^2\|\nabla\psi\|_{L^2}\|\nabla h\|_{L^6}\\
&\quad\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\alpha}\|\Delta u\|_{L^2}^{\alpha}\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C\\
&\quad\leqslant\delta\|\sqrt{\varrho}\partial_t u\|_{L^2}^2+\delta\|\Delta u\|_{L^2}^2+C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{2/(1-\alpha)}\|\nabla u\|_{L^2}^2\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\nabla h\cdot u\,\mathrm{d}x+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C
\end{aligned} \tag{2.13}$$

for any $0<\delta<1$.

Here we have used the inequality [15]:

$$\|u\cdot\nabla u\|_{L^2}\leqslant C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|u\|_{\dot H^{1+\alpha}} \text{ with } 0<\alpha<1. \tag{2.14}$$

On the other hand, thanks to the H^2-theory of the Stokes system, it follows from (1.3), (2.1), (2.2), and (2.14) that

$$\begin{aligned}\|\Delta u\|_{L^2}&\leqslant C\|\nabla\pi-\Delta u\|_{L^2}\leqslant C\|\varrho\partial_t u+\varrho u\cdot\nabla u-|\psi|^2\nabla h\|_{L^2}\\&\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}+C\|u\cdot\nabla u\|_{L^2}+C\\&\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}+C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{L^2}^{1-\alpha}\|\Delta u\|_{L^2}^{\alpha}+C,\end{aligned}$$

which leads to

$$\|\Delta u\|_{L^2}\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2}+C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{1/(1-\alpha)}\|\nabla u\|_{L^2}+C. \tag{2.15}$$

Inserting (2.15) into (2.13), taking δ small enough and summing with (2.15), we have

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\int|\nabla u|^2\,\mathrm dx+\int\varrho|\partial_t u|^2\,\mathrm dx+\int|\Delta u|^2\,\mathrm dx\\&\leqslant C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{2/(1-\alpha)}\|\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2\\&\quad+C\|\varphi\|_{L^2}^2+C+\frac{\mathrm d}{\mathrm dt}\int|\psi|^2\nabla h\cdot u\,\mathrm dx\\&=\frac{C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{2/(1-\alpha)}}{\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})}\|\nabla u\|_{L^2}^2\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})\\&\quad+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C+\frac{\mathrm d}{\mathrm dt}\int|\psi|^2\nabla h\cdot u\,\mathrm dx\\&\leqslant\frac{C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{2/(1-\alpha)}}{\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})}\|\nabla u\|_{L^2}^2\log(\mathrm e+y)\\&\quad+C\|\nabla u\|_{L^2}^2+C\|\Delta\psi\|_{L^2}^2+C\|\varphi\|_{L^2}^2+C+\frac{\mathrm d}{\mathrm dt}\int|\psi|^2\nabla h\cdot u\,\mathrm dx,\end{aligned}$$

which implies

$$\|\nabla u\|_{L^\infty(t_0,t;L^2)}^2+\int_{t_0}^t\int(\varrho|\partial_t u|^2+|\Delta u|^2)\,\mathrm dx\,\mathrm ds\leqslant C(\mathrm e+y)^{C_0\varepsilon} \tag{2.16}$$

with

$$y(t):=\sup_{[t_0,t]}\|D^{3/2-\alpha}u(s)\|_{L^2}$$

for any $0<t_0\leqslant t\leqslant T$ where C_0 is an absolute constant, provided that

$$\int_{t_0}^T\frac{\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{2/(1-\alpha)}}{\log(\mathrm e+\|u\|_{\dot B^{-\alpha}_{\infty,\infty}})}\,\mathrm dt\leqslant\varepsilon<<1. \tag{2.17}$$

Applying ∂_t to (1.3), testing by $\partial_t u$, and using (1.1) and (1.2), we get

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho|\partial_t u|^2\,\mathrm{d}x + \int |\nabla\partial_t u|^2\,\mathrm{d}x
&= -\int \partial_t\varrho|\partial_t u|^2\,\mathrm{d}x - \int \partial_t\varrho u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x \\
&\quad - \int \varrho\partial_t u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x - \int \partial_t|\psi|^2\nabla h\cdot\partial_t u\,\mathrm{d}x \\
&= -\int \varrho u\cdot\nabla|\partial_t u|^2\,\mathrm{d}x - \int \varrho u\cdot\nabla(u\cdot\nabla u\cdot\partial_t u)\,\mathrm{d}x \\
&\quad - \int \varrho\partial_t u\cdot\nabla u\cdot\partial_t u\,\mathrm{d}x - \int \partial_t|\psi|^2\nabla h\cdot\partial_t u\,\mathrm{d}x =: \sum_{j=1}^{4} I_j.
\end{aligned}
\tag{2.18}
$$

We use the Hölder inequality, (2.1), and (2.2) to bound I_j $(j=1,\dots,4)$ as follows.

$$
\begin{aligned}
|I_1| &\leqslant C\|u\|_{L^6}\|\sqrt{\varrho}\partial_t u\|_{L^3}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\sqrt{\varrho}\partial_t u\|_{L^6}^{1/2}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\nabla\partial_t u\|_{L^2}^{3/2} \\
&\leqslant \delta\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\sqrt{\varrho}\partial_t u\|_{L^2}^2
\end{aligned}
$$

for any $0<\delta<1$;

$$
\begin{aligned}
|I_2| &\leqslant C\|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla u\|_{L^2}\|\partial_t u\|_{L^6} + C\|u\|_{L^6}^2\|\Delta u\|_{L^2}\|\partial_t u\|_{L^6} \\
&\quad + C\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}\|\nabla\partial_t u\|_{L^2} \\
&\leqslant \delta\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2
\end{aligned}
$$

for any $0<\delta<1$;

$$
\begin{aligned}
|I_3| &\leqslant \|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^4}^2 \leqslant \|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\sqrt{\varrho}\partial_t u\|_{L^6}^{3/2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\varrho}\partial_t u\|_{L^2}^{1/2}\|\nabla\partial_t u\|_{L^2}^{3/2} \\
&\leqslant \delta\|\nabla\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\sqrt{\varrho}\partial_t u\|_{L^2}^2
\end{aligned}
$$

for any $0<\delta<1$;

$$
|I_4| \leqslant C\|\partial_t\psi\|_{L^2}\|\nabla h\|_{L^3}\|\partial_t u\|_{L^6} \leqslant C\|\partial_t\psi\|_{L^2}^2 + \delta\|\nabla\partial_t u\|_{L^2}^2
$$

for any $0<\delta<1$.

Inserting the above estimates into (2.18) and taking δ small enough, we have

$$\begin{aligned}(2.19)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho|\partial_t u|^2\,\mathrm{d}x + \frac{1}{2}\int |\nabla\partial_t u|^2\,\mathrm{d}x \\ &\quad \leqslant C\|\nabla u\|_{L^2}^4\|\sqrt{\varrho}\partial_t u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\Delta u\|_{L^2}^2 + C\|\partial_t\psi\|_{L^2}^2.\end{aligned}$$

Testing (2.10) by $\partial_t\overline{\psi}$, taking the real parts, using (2.2), (2.9), and (2.16), we have

$$\begin{aligned}&\frac{1}{2k^2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\,\mathrm{d}x + \int \eta|\partial_t\psi|^2\,\mathrm{d}x \\ &\quad \leqslant C(\|\varphi\|_{L^2} + \|u\|_{L^6}\|\nabla\psi\|_{L^3} + \|A\|_{L^4}\|\nabla\psi\|_{L^4} + \|A\|_{L^4}^2 + \|\psi\|_{L^2})\|\partial_t\psi\|_{L^2} \\ &\quad \leqslant \frac{1}{2}\eta\|\partial_t\psi\|_{L^2}^2 + C\|\varphi\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\psi\|_{L^3}^2 + C\|\nabla\psi\|_{L^4}^2 + C,\end{aligned}$$

which gives

$$(2.20)\qquad \int_{t_0}^t\int |\partial_t\psi|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}.$$

Integrating (2.19) over (t_0,t) and using (2.20), we arrive at

$$(2.21)\qquad \int \varrho|\partial_t u|^2\,\mathrm{d}x + \int_{t_0}^t\int |\nabla\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}.$$

Similarly to (2.15), we find that

$$\begin{aligned}\|\Delta u\|_{L^2} &\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|u\cdot\nabla u\|_{L^2} + C \leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C \\ &\leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}\cdot\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2} + C,\end{aligned}$$

which yields

$$(2.22)\qquad \|\Delta u\|_{L^2} \leqslant C\|\sqrt{\varrho}\partial_t u\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C.$$

Here we have used the Gagliardo-Nirenberg inequality

$$(2.23)\qquad \|\nabla u\|_{L^3}^2 \leqslant C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}.$$

Similarly to (2.15) again, we have

$$\begin{aligned}(2.24)\quad \|\Delta u\|_{L^{3/(2+\alpha)}} &\leqslant C\|\nabla\pi - \Delta u\|_{L^{3/(2+\alpha)}} \leqslant C\|\varrho\partial_t u + \varrho u\cdot\nabla u - |\psi|^2\nabla h\|_{L^{3/(2+\alpha)}} \\ &\leqslant C(\|\sqrt{\varrho}\|_{L^{6/(1+2\alpha)}}\|\sqrt{\varrho}\partial_t u\|_{L^2} \\ &\qquad + \|\varrho\|_{L^{3/\alpha}}\|u\|_{L^6}\|\nabla u\|_{L^2} + \|\psi\|_{L^{12/(3+2\alpha)}}^2\|\nabla h\|_{L^6}) \\ &\leqslant C(\|\sqrt{\varrho}\partial_t u\|_{L^2} + \|\nabla u\|_{L^2}^2 + 1).\end{aligned}$$

Noting the imbedding inequality

$$\|D^{3/2-\alpha}u\|_{L^2} \leqslant C\|\Delta u\|_{L^{3/(2+\alpha)}}, \tag{2.25}$$

we have

$$y \leqslant C(\mathrm{e}+y)^{C_0\varepsilon}, \tag{2.26}$$

which gives

$$\|\Delta u\|_{L^2} + \|\Delta u\|_{L^{3/(2+\alpha)}} \leqslant C. \tag{2.27}$$

Thus

$$\|\partial_t u\|_{L^2(0,T;H^1)} \leqslant C. \tag{2.28}$$

Equation (1.3) can be rewritten as

$$-\Delta u + \nabla\pi = f := |\psi|^2\nabla h - \varrho\partial_t u - \varrho u\cdot\nabla u \in L^2(0,T;L^2\cap L^6). \tag{2.29}$$

We have

$$\|\nabla u\|_{W^{1,q}} \leqslant C\|f\|_{L^q} + C\|\nabla u\|_{L^2} \text{ with } 3 < q \leqslant 6$$

and therefore,

$$\|\nabla u\|_{L^2(0,T;W^{1,q})} \leqslant C. \tag{2.30}$$

Now it is standard to deduce that

$$\|\varrho\|_{C([0,T];L^{3/2}\cap H^2)} \leqslant C. \tag{2.31}$$

This completes the proof. □

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