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REPDIGITS IN THE BASE b AS SUMS OF FOUR BALANCING NUMBERS

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Abstract. The sequence of balancing numbers (B_n) is defined by the recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ for $n \geqslant 2$ with initial conditions $B_0 = 0$ and $B_1 = 1$. B_n is called the nth balancing number. In this paper, we find all repdigits in the base b, which are sums of four balancing numbers. As a result of our theorem, we state that if B_n is repdigit in the base b and has at least two digits, then $(n, b) = (2, 5), (3, 6)$. Namely, $B_2 = 6 = (11)_5$ and $B_3 = 35 = (55)_6$.

Keywords: balancing number; repdigit; Diophantine equations; linear form in logarithms

MSC 2020: 11B39, 11J86, 11D61

1. Introduction

The sequence of balancing numbers (B_n) is defined by the recurrence relation $B_n = 6B_{n-1} - B_{n-2}$ for $n \geqslant 2$ with initial conditions $B_0 = 0$, $B_1 = 1$. B_n is called the nth balancing number. We have the Binet formula

$$B_n = \frac{\lambda^n - \delta^n}{4\sqrt{2}}, \tag{1.1}$$

where $\lambda = 3+2\sqrt{2}$ and $\delta = 3-2\sqrt{2}$, which are the roots of the characteristic equation $x^2 - 6x + 1 = 0$. It can be seen that $5 < \lambda < 6$, $0 < \delta < 1$, $\lambda\delta = 1$, and

$$B_n < \frac{\lambda^n}{4\sqrt{2}}. \tag{1.2}$$

For more information about the sequence of balancing numbers, see [11], [10], and [7]. A repdigit is a non-negative integer whose digits are all equal. Investigation of the

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repdigits in the second-order linear recurrence sequences has been of interest to mathematicians. In [4], the authors have found all Fibonacci and Lucas numbers, which are repdigits. The largest repdigits in Fibonacci and Lucas sequences are $F_5 = 55$ and $L_5 = 11$. After that, in [2], the authors showed that the largest Fibonacci number which is a sum of two repdigits is $F_{20} = 6765 = 6666 + 99$. In [3], the authors have found all Pell and Pell-Lucas numbers which are repdigits. The largest repdigits in Pell and Pell Lucas sequences are $P_3 = 5$ and $Q_2 = 6$. Later, Luca (see [5]) found all repdigits which are sums of three Fibonacci numbers. In [9], the authors have found all repdigits which are sums of three Pell numbers. In the subsequent work [6], the authors tackled the same problem by taking four Pell numbers instead of three Pell numbers. In this study, we determine all repdigits which are sums of four balancing numbers. Briefly, we solve the equation

$$N = B_{m_1} + B_{m_2} + B_{m_3} + B_{m_4} = \frac{d(b^n - 1)}{b - 1} \tag{1.3}$$

for $2 \leqslant b \leqslant 10$, $1 \leqslant d \leqslant 9$, $m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4 \geqslant 0$, and $n \geqslant 2$. If N is a solution of the equation (1.3), then $(m_1, m_2, m_3, m_4, b, d, n, N)$ is an element of the set

$$\begin{gathered}
\{(1,1,1,0,2,1,2,3), (1,1,1,1,3,1,2,4), (2,0,0,0,5,1,2,6), (2,1,0,0,2,1,3,7), \\
(2,1,0,0,6,1,2,7), (2,1,1,0,3,2,2,8), (2,1,1,0,7,1,2,8), (2,1,1,1,8,1,2,9), \\
(2,2,0,0,5,2,2,12), (2,2,1,0,3,1,3,13), (2,2,1,1,6,2,2,14), (2,2,2,0,5,3,2,18), \\
(2,2,2,0,8,2,2,18), (2,2,2,2,5,4,2,24), (2,2,2,2,7,3,2,24), (3,0,0,0,6,5,2,35), \\
(3,1,0,0,8,4,2,36), (3,2,1,0,4,2,3,42), (3,2,1,1,6,1,3,43), (3,2,2,1,7,6,2,48), \\
(3,3,0,0,9,7,2,70), (3,3,2,1,10,7,2,77), (3,3,3,2,10,1,3,111), \\
(4,2,2,2,10,2,3,222), (4,4,3,1,10,4,3,444)\}.
\end{gathered}$$

Furthermore, we conclude that if B_n is repdigit in the base b and has at least two digits, then $(n, b) = (2, 5), (3, 6)$. Namely, $B_2 = 6 = (11)_5$ and $B_3 = 35 = (55)_6$.

Our study can be viewed as a continuation of the previous works on this subject. We follow the approach and the method presented in [6]. In Section 2, we introduce necessary lemmas and theorems. Then, we prove our main theorem in Section 3.

2. Auxiliary results

In order to solve Diophantine equations of the exponential forms, the authors have used Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving

Diophantine equations of the similar form, we start with recalling some basic notions from the algebraic number theory.

Let η be an algebraic number of degree d with the minimal polynomial

$$a_0x^d + a_1x^{d-1} + \ldots + a_d = a_0\prod_{i=1}^{d}(x - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log(\max\{|\eta^{(i)}|, 1\})\right) \tag{2.1}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of the logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \leqslant h(\eta) + h(\gamma) + \log 2, \tag{2.2}$$

$$h(\eta\gamma^{\pm 1}) \leqslant h(\eta) + h(\gamma), \tag{2.3}$$

$$h(\eta^m) = |m|h(\eta). \tag{2.4}$$

The following lemma is deduced from Corollary 2.3 of Matveev (see [8]).

Lemma 2.1. *Assume that $\gamma_1, \gamma_2, \ldots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} \cdot t^{9/2} D^2 (1 + \log D)(1 + \log B) A_1 A_2 \ldots A_t),$$

where

$$B \geqslant \max\{|b_1|, \ldots, |b_t|\},$$

and $A_i \geqslant \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, \ldots, t$.

In the following lemma, $\|x\|$ denotes the distance from x to the nearest integer. That is, $\|x\| = \min\{|x-n|\colon\ n \in \mathbb{Z}\}$ for any real number x.

Lemma 2.2 ([1], Lemma 3.3). *Let $\nu_1, \nu_2, \beta \in \mathbb{R}$ be such that $\nu_1\nu_2\beta \neq 0$ and $x_1, x_2 \in \mathbb{Z}$. Put $\Lambda = \beta + x_1\nu_1 + x_2\nu_2$. Let c, δ be positive constants. Let X_0 be a (large) positive constant such that $\max\{|x_1|, |x_2|\} \leqslant X_0$. Put $\nu = -\nu_1/\nu_2$ and $\psi = \beta/\nu_2$. Let p/q be a convergent of ν with $q > X_0$. Suppose that $\|q\psi\| > 2X_0/q$ and $|\Lambda| < c\exp(-\delta X)$. Then*

$$X < \frac{1}{\delta}\log\frac{q^2c}{|\nu_2|X_0}.$$

3. Main theorem

Theorem 3.1. *Let $m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4 \geqslant 0$, $2 \leqslant b \leqslant 10$ and $N = B_{m_1} + B_{m_2} + B_{m_3} + B_{m_4}$. If N is a repdigit in the base b and has at least two digits, then (N, b) are elements of the set*

$$\big\{(3,2),(4,3),(6,5),(7,2),(7,6),(8,3),(8,7),(9,8),(12,5),(13,3),\\ (14,6),(18,5),(18,8),(24,5),(24,7),(35,6),(36,8),(42,4),\\ (43,6),(48,7),(70,9),(77,10),(111,10),(222,10),(444,10)\big\}.$$

Namely,

$$3=(11)_2,\ 4=(11)_3,\ 6=(11)_5,\ 7=(111)_2,\ 7=(11)_6,\ 8=(22)_3,\ 8=(11)_7,\\ 9=(11)_8,\ 12=(22)_5,\ 13=(111)_3,\ 14=(22)_6,\ 18=(33)_5,\ 18=(22)_8,\\ 24=(33)_7,\ 35=(55)_6,\ 36=(44)_8,\ 42=(222)_4,\ 43=(111)_6,\ 48=(66)_7,\\ 24=(44)_5,\ 70=(77)_9,\ 77=(77)_{10},\ 111=(111)_{10},\ 222=(222)_{10},\ 444=(444)_{10}.$$

Proof. Assume that $m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4 \geqslant 0$ and $N = B_{m_1} + B_{m_2} + B_{m_3} + B_{m_4}$. Assume that the equation (1.3) holds. A search in Mathematica in the range $0 \leqslant m_4 \leqslant m_3 \leqslant m_2 \leqslant m_1 \leqslant 299$ gives only the solutions in the statement of Theorem 3.1. Assume that $m_1 \geqslant 300$. Then

$$B_{300} \leqslant B_{m_1} + B_{m_2} + B_{m_3} + B_{m_4} = \frac{d(b^n-1)}{b-1} \leqslant b^n - 1 \leqslant 10^n - 1,$$

which gives us

$$228 \leqslant \frac{\log(1+B_{300})}{\log 10} \leqslant n.$$

That is, $n \geqslant 228$. Since

$$2^{n-1} \leqslant b^{n-1} \leqslant b^{n-1} + b^{n-2} + \ldots + 1 \leqslant \frac{d(b^n - 1)}{b-1}$$
$$= B_{m_1} + B_{m_2} + B_{m_3} + B_{m_4} \leqslant 4B_{m_1} \leqslant 4\frac{\lambda^{m_1}}{4\sqrt{2}} < \lambda^{m_1} < 2^{3m_1},$$

by (1.2), we get $3m_1 + 1 > n \geqslant 228$. Equation (1.3) can be rewritten as

$$\frac{1}{4\sqrt{2}}(\lambda^{m_1} - \delta^{m_1} + \lambda^{m_2} - \delta^{m_2} + \lambda^{m_3} - \delta^{m_3} + \lambda^{m_4} - \delta^{m_4}) = \frac{db^n}{b-1} - \frac{d}{b-1}. \tag{3.1}$$

We examine (3.1) in four different steps in the following way.

Step 1: Equation (3.1) can be reorganized as

$$\frac{\lambda^{m_1}}{4\sqrt{2}}(1 + \lambda^{m_2-m_1} + \lambda^{m_3-m_1} + \lambda^{m_4-m_1}) - \frac{db^n}{b-1} = -\frac{d}{b-1} + \frac{1}{4\sqrt{2}}(\delta^{m_1} + \delta^{m_2} + \delta^{m_3} + \delta^{m_4}). \tag{3.2}$$

This implies that

$$\left|\frac{\lambda^{m_1}}{4\sqrt{2}}(1 + \lambda^{m_2-m_1} + \lambda^{m_3-m_1} + \lambda^{m_4-m_1}) - \frac{db^n}{b-1}\right| \leqslant \frac{d}{b-1} + \frac{4}{4\sqrt{2}} < \frac{\lambda^2}{4\sqrt{2}}.$$

Dividing both sides of the above inequality by $\frac{1}{4\sqrt{2}}\lambda^{m_1}(1 + \lambda^{m_2-m_1} + \lambda^{m_3-m_1} + \lambda^{m_4-m_1})$, we get

$$|\Gamma_1| < \lambda^{2-m_1}, \tag{3.3}$$

where

$$\Gamma_1 = 1 - \lambda^{-m_4} b^n \frac{4d\sqrt{2}}{(b-1)(1 + \lambda^{m_1-m_4} + \lambda^{m_2-m_4} + \lambda^{m_3-m_4})}. \tag{3.4}$$

Suppose that $\Gamma_1 = 0$. Then

$$\lambda^{m_4} + \lambda^{m_1} + \lambda^{m_2} + \lambda^{m_3} = \frac{4d\sqrt{2}b^n}{b-1}.$$

Conjugating in $\mathbb{Q}(\sqrt{2})$ gives us

$$\delta^{m_4} + \delta^{m_1} + \delta^{m_2} + \delta^{m_3} = -\frac{4d\sqrt{2}b^n}{b-1}.$$

Then

$$\frac{4d\sqrt{2}b^n}{b-1} = |\delta^{m_4} + \delta^{m_1} + \delta^{m_2} + \delta^{m_3}| = \delta^{m_4} + \delta^{m_1} + \delta^{m_2} + \delta^{m_3} < 4,$$

which is impossible. Therefore $\Gamma_1 \neq 0$. Now we apply Lemma 2.1 to (3.4). Let

$$\gamma_1 := \lambda, \quad \gamma_2 := b, \quad \gamma_3 := \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4})}$$

and $b_1 := -m_4$, $b_2 := n$, $b_3 := 1$, where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$ and $b_1, b_2, b_3 \in \mathbb{Z}$. We can take $D = 2$. As $m_1 \geqslant m_4$ and $3m_1 + 1 \geqslant n$, we can also take $B := 3m_1 + 1 \geqslant \max\{|-m_4|, |n|, 1|\}$. It is clear that $h(\gamma_1) = h(\lambda) = \frac{1}{2}\log\lambda$ and $h(\gamma_2) = h(b) < h(10) = \log 10$ and so we can take $A_1 := 1.8$, $A_2 := 4.7$. Since

$$\gamma_3 = \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4})} < 4\sqrt{2}$$

and

$$\gamma_3^{-1} = \frac{(b-1)(1+\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4})}{4d\sqrt{2}} < \frac{b-1}{\sqrt{2}}\lambda^{m_1-m_4},$$

it follows that $|\log\gamma_3| < 2 + (m_1 - m_4)\log\lambda$. On the other hand,

$$\begin{aligned}
h(\gamma_3) &\leqslant h(4d\sqrt{2}) + h(b-1) + h(\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4}+1)\\
&\leqslant h(36\sqrt{2}) + h(b-1) + \log 2 + h(\lambda^{m_3-m_4}(\lambda^{m_1-m_3}+\lambda^{m_2-m_3}+1))\\
&\leqslant h(36) + h(\sqrt{2}) + h(b-1) + 2\log 2 + h(\lambda^{m_3-m_4}) + h(\lambda^{m_2-m_3}(\lambda^{m_1-m_2}+1))\\
&\leqslant h(36) + h(\sqrt{2}) + h(b-1) + 3\log 2 + h(\lambda^{m_3-m_4}) + h(\lambda^{m_2-m_3}) + h(\lambda^{m_1-m_2})\\
&\leqslant \log 36 + \frac{\log 2}{2} + \log(b-1) + 3\log 2 + (m_3-m_4)h(\lambda)\\
&\quad + (m_2-m_3)h(\lambda) + (m_1-m_2)h(\lambda)\\
&\leqslant 9 + \frac{1}{2}(m_1-m_4)\log\lambda.
\end{aligned}$$

Thus we can take $A_3 := 18 + (m_1 - m_4)\log\lambda$. By applying Lemma 2.1 to Γ_1 given by (3.4) and using (3.3), we get

$$\lambda^{2-m_1} > |\Gamma_1| > \exp(C(1+\log(3m_1+1)) \cdot 1.8 \cdot 4.7(18 + (m_1-m_4)\log\lambda)),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{9/2} \cdot 2^2(1+\log 2)$. Therefore we get

$$m_1\log\lambda - 2\log\lambda < 8.3 \cdot 10^{12}(1+\log(3m_1+1))(18+(m_1-m_4)\log\lambda). \tag{3.5}$$

Step 2: Equation (3.1) can be written as

$$\frac{\lambda^{m_1}}{4\sqrt{2}}(1+\lambda^{m_2-m_1}+\lambda^{m_3-m_1})-\frac{db^n}{b-1} = -\frac{d}{b-1}-\frac{\lambda^{m_4}}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}(\delta^{m_1}+\delta^{m_2}+\delta^{m_3}+\delta^{m_4}). \tag{3.6}$$

This gives

$$\left|\frac{\lambda^{m_1}}{4\sqrt{2}}(1+\lambda^{m_2-m_1}+\lambda^{m_3-m_1})-\frac{db^n}{b-1}\right| < \frac{\lambda^{m_4+2}}{4\sqrt{2}}. \tag{3.7}$$

Dividing both sides of (3.7) by $\frac{1}{4\sqrt{2}}\lambda^{m_1}(1+\lambda^{m_2-m_1}+\lambda^{m_3-m_1})$, we get

$$|\Gamma_2| < \frac{\lambda^{m_4-m_1+2}}{1+\lambda^{m_2-m_1}+\lambda^{m_3-m_1}} < \lambda^{2-(m_1-m_4)}, \tag{3.8}$$

where

$$\Gamma_2 = 1-\lambda^{-m_3}b^n\frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_3}+\lambda^{m_2-m_3})}. \tag{3.9}$$

It can be seen that $\Gamma_2 \neq 0$. Now we apply Lemma 2.1 to (3.9). Let

$$\gamma_1 := \lambda, \quad \gamma_2 := b, \quad \gamma_3 := \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_3}+\lambda^{m_2-m_3})}$$

and $b_1 := -m_3$, $b_2 := n$, $b_3 := 1$, where $\gamma_1, \gamma_2, \gamma_3 \in Q(\sqrt{2})$ and $b_1, b_2, b_3 \in Z$. We can take $D = 2$. As $m_1 \geqslant m_3$ and $3m_1+1 \geqslant n$, we can also take $B := 3m_1+1 \geqslant \max\{|-m_3|, |n|, 1\}$. It is clear that $h(\gamma_1) = h(\lambda) = \frac{1}{2}\log\lambda$ and $h(\gamma_2) = h(b) < h(10) = \log 10$. Therefore, we can take $A_1 := 1.8$, $A_2 := 4.7$. Since

$$\gamma_3 = \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_3}+\lambda^{m_2-m_3})} < 4\sqrt{2}$$

and

$$\gamma_3^{-1} = \frac{(b-1)(1+\lambda^{m_1-m_3}+\lambda^{m_2-m_3})}{4d\sqrt{2}} < \frac{27}{4\sqrt{2}}\lambda^{m_1-m_3},$$

it follows that $|\log\gamma_3| < 2+(m_1-m_3)\log\lambda$. On the other hand,

$$\begin{aligned}
h(\gamma_3) &\leqslant h(4d\sqrt{2})+h(b-1)+h(\lambda^{m_1-m_3}+\lambda^{m_2-m_3}+1)\\
&\leqslant h(36\sqrt{2})+h(b-1)+\log 2+h(\lambda^{m_2-m_3}(\lambda^{m_1-m_2}+1))\\
&\leqslant h(36)+h(\sqrt{2})+h(b-1)+2\log 2+h(\lambda^{m_2-m_3})+h(\lambda^{m_1-m_2})\\
&\leqslant \log 36+\frac{\log 2}{2}+\log(b-1)+2\log 2+(m_2-m_3)h(\lambda)+(m_1-m_2)h(\lambda)\\
&\leqslant 8+\frac{1}{2}(m_1-m_3)\log\lambda.
\end{aligned}$$

Thus we can take $A_3 := 16 + (m_1 - m_3)\log\lambda$. By applying Lemma 2.1 to Γ_2 given by (3.9) and using (3.8), we get

$$\lambda^{2-(m_1-m_4)} > |\Gamma_2| > \exp(C(1+\log(3m_1+1)) \cdot 1.8 \cdot 4.7(16 + (m_1 - m_3)\log\lambda)),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{9/2} \cdot 2^2(1 + \log 2)$. Thus we get

$$(m_1 - m_4)\log\lambda - 2\log\lambda < 8.3 \cdot 10^{12}(1+\log(3m_1+1))(16+(m_1-m_3)\log\lambda). \tag{3.10}$$

Step 3: Now, we write equation (3.1) as

$$\frac{\lambda^{m_1}}{4\sqrt{2}}(1+\lambda^{m_2-m_1}) - \frac{db^n}{b-1} = -\frac{d}{b-1} - \frac{\lambda^{m_3}+\lambda^{m_4}}{4\sqrt{2}} + \frac{\delta^{m_1}+\delta^{m_2}+\delta^{m_3}+\delta^{m_4}}{4\sqrt{2}}. \tag{3.11}$$

Thus

$$\left|\frac{\lambda^{m_1}}{4\sqrt{2}}(1+\lambda^{m_2-m_1}) - \frac{db^n}{b-1}\right| \leqslant \frac{d}{b-1} + \frac{1}{4\sqrt{2}}(\lambda^{m_3}+\lambda^{m_4}) + \frac{4}{4\sqrt{2}} < \frac{\lambda^{m_3+2}}{4\sqrt{2}}. \tag{3.12}$$

Dividing both sides of (3.12) by $\frac{1}{4\sqrt{2}}\lambda^{m_1}(1+\lambda^{m_2-m_1})$, we get

$$|\Gamma_3| < \frac{\lambda^{m_3-m_1+2}}{(1+\lambda^{m_2-m_1})} < \lambda^{2-(m_1-m_3)}, \tag{3.13}$$

where

$$\Gamma_3 = 1 - \lambda^{-m_2} b^n \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_2})}. \tag{3.14}$$

It can be seen that $\Gamma_3 \neq 0$. Now we apply Lemma 2.1 to (3.14). Let

$$\gamma_1 := \lambda, \quad \gamma_2 := b, \quad \gamma_3 := \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_2})}$$

and

$$b_1 := -m_2, \quad b_2 := n, \quad b_3 := 1,$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$ and $b_1, b_2, b_3 \in \mathbb{Z}$. We can take $D = 2$. As $m_1 \geqslant m_2$ and $3m_1 + 1 \geqslant n$, we can also take $B := 3m_1 + 1 \geqslant \max\{|-m_2|, |n|, 1\}$. It is clear that

$$h(\gamma_1) = h(\lambda) = \frac{\log\lambda}{2} \quad \text{and} \quad h(\gamma_2) = h(b) < h(10) = \log 10$$

and so we can take $A_1 := 1.8$, $A_2 := 4.7$. Since

$$\gamma_3 = \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_2})} < 4\sqrt{2}$$

and

$$\gamma_3^{-1} = \frac{(b-1)(1+\lambda^{m_1-m_2})}{4d\sqrt{2}} < \frac{9}{2\sqrt{2}}\lambda^{m_1-m_2},$$

it follows that $|\log\gamma_3| < 2 + (m_1-m_2)\log\lambda$. On the other hand,

$$\begin{aligned} h(\gamma_3) &\leqslant h(4d\sqrt{2}) + h(b-1) + \log 2 + h(\lambda^{m_1-m_2}) \\ &\leqslant h(36) + h(\sqrt{2}) + h(b-1) + \log 2 + (m_1-m_2)h(\lambda) \\ &= \log 36 + \frac{\log 2}{2} + \log(b-1) + \log 2 + \frac{1}{2}(m_1-m_2)\log\lambda \\ &\leqslant 7 + \frac{1}{2}(m_1-m_2)\log\lambda. \end{aligned}$$

Thus we can take $A_3 := 14 + (m_1-m_2)\log\lambda$. By applying Lemma 2.1 to Γ_3 given by (3.14) and using (3.13), we get

$$\lambda^{2-(m_1-m_3)} > |\Gamma_3| > \exp(C(1+\log(3m_1+1))\cdot 1.8\cdot 4.7(14+(m_1-m_2)\log\lambda)),$$

where $C = -1.4\cdot 30^6\cdot 3^{9/2}\cdot 2^2(1+\log 2)$. Then we get

$$\begin{aligned} &(m_1-m_3)\log\lambda - 2\log\lambda \\ &\qquad < 8.3\cdot 10^{12}(1+\log(3m_1+1))(14+(m_1-m_2)\log\lambda). \end{aligned} \tag{3.15}$$

Step 4: Equation (3.1) can be written as

$$\begin{aligned} \frac{\lambda^{m_1}}{4\sqrt{2}} - \frac{db^n}{b-1} = &-\frac{d}{b-1} - \frac{1}{4\sqrt{2}}(\lambda^{m_2}+\lambda^{m_3}+\lambda^{m_4}) \\ &+ \frac{1}{4\sqrt{2}}(\delta^{m_1}+\delta^{m_2}+\delta^{m_3}+\delta^{m_4}). \end{aligned} \tag{3.16}$$

This gives us

$$\left|\frac{\lambda^{m_1}}{4\sqrt{2}} - \frac{db^n}{b-1}\right| \leqslant \frac{d}{b-1} + \frac{1}{4\sqrt{2}}(\lambda^{m_2}+\lambda^{m_3}+\lambda^{m_4}) + \frac{4}{4\sqrt{2}} < \frac{\lambda^{m_2+2}}{4\sqrt{2}}. \tag{3.17}$$

Dividing both sides of (3.17) by $\frac{1}{4\sqrt{2}}\lambda^{m_1}$, we get

$$|\Gamma_4| < \lambda^{2-(m_1-m_2)}, \tag{3.18}$$

where

$$\Gamma_4 = 1 - \lambda^{-m_1} b^n \frac{4d\sqrt{2}}{b-1}. \tag{3.19}$$

It can be seen that $\Gamma_4 \neq 0$. Now we apply Lemma 2.1 to (3.19). Let

$$\gamma_1 := \lambda, \quad \gamma_2 := b, \quad \gamma_3 := \frac{4d\sqrt{2}}{b-1}$$

and $b_1 := -m_1$, $b_2 := n$, $b_3 := 1$, where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$ and $b_1, b_2, b_3 \in \mathbb{Z}$. We can take $D = 2$. As $3m_1 + 1 \geqslant n$, we can also take $B := 3m_1 + 1 \geqslant \max\{|-m_1|, |n|, 1\}$. It is clear that $h(\gamma_1) = h(\lambda) = \frac{1}{2}\log\lambda$ and $h(\gamma_2) = h(b) < h(10) = \log 10$. Therefore, we can take $A_1 := 1.8$, $A_2 := 4.7$. Since

$$\gamma_3 = \frac{4d\sqrt{2}}{b-1} \leqslant 4\sqrt{2} \quad \text{and} \quad \gamma_3^{-1} = \frac{b-1}{4d\sqrt{2}} \leqslant \frac{9}{4\sqrt{2}},$$

it follows that $|\log\gamma_3| < 1.8$. On the other hand,

$$\begin{aligned} h(\gamma_3) &\leqslant h(4d\sqrt{2}) + h(b-1) \leqslant h(36) + h(\sqrt{2}) + h(9) \\ &= \log 36 + \frac{\log 2}{2} + \log 9 < 6.2. \end{aligned}$$

Thus we can take $A_3 := 12.4$. By applying Lemma 2.1 to Γ_4 given by (3.19) and using (3.18), we get

$$\lambda^{2-(m_1-m_2)} > |\Gamma_4| > \exp(C(1 + \log(3m_1+1)) \cdot 1.8 \cdot 4.7 \cdot 12.4),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{9/2} \cdot 2^2(1 + \log 2)$. Therefore

$$(m_1 - m_2)\log\lambda - 2\log\lambda < 1.02 \cdot 10^{14}(1 + \log(3m_1+1)). \tag{3.20}$$

From (3.20), (3.15), (3.10), and (3.5), we get $m_1 < 1.38 \cdot 10^{61}$.

Let

$$\Lambda_1 = -m_1 \log\lambda + n\log b + \log\frac{4d\sqrt{2}}{b-1}. \tag{3.21}$$

From (3.16), we can see that

$$\begin{aligned} \frac{\lambda^{m_1}}{4\sqrt{2}} - \frac{db^n}{b-1} &= \frac{\lambda^{m_1}}{4\sqrt{2}}\Big(1 - \lambda^{-m_1} b^n \frac{4d\sqrt{2}}{b-1}\Big) = \frac{\lambda^{m_1}}{4\sqrt{2}}(1 - \exp\Lambda_1) \\ &= -\frac{d}{b-1} + \frac{\delta^{m_1}}{4\sqrt{2}} - B_{m_2} - B_{m_3} - B_{m_4} < -\frac{1}{9} + \frac{\delta^{300}}{4\sqrt{2}} < 0 \end{aligned}$$

as $m_1 \geqslant 300$. Thus $\Lambda_1 > 0$ and therefore from (3.18) we obtain

$$0 < \Lambda_1 < \exp\Lambda_1 - 1 = \left|1 - \lambda^{-m_1} b^n \frac{4d\sqrt{2}}{b-1}\right| < \lambda^{2+m_2-m_1}.$$

This means that

$$|\Lambda_1| < \lambda^2 \exp(-1.76(m_1 - m_2)) \tag{3.22}$$

with $m_1 - m_2 \leqslant m_1 \leqslant 1.38 \cdot 10^{61}$. In order to apply Lemma 2.2 to (3.21), we take $X_0 = 4.2 \cdot 10^{61} \geqslant 3m_1 + 1 \geqslant \max\{m_1, n\}$ and

$$c = \lambda^2, \quad \delta = 1.76, \quad \psi = \frac{1}{\log b}\log\frac{4d\sqrt{2}}{b-1},$$
$$v = \frac{\log\lambda}{\log b}, \quad v_1 = -\log\lambda, \quad v_2 = \log b, \quad \beta = \log\frac{4d\sqrt{2}}{b-1}.$$

We find that $q = q_{135}$ satisfies the hypothesis of Lemma 2.2 for $2 \leqslant b \leqslant 10$ and $1 \leqslant d \leqslant 9$. By Lemma 2.2, we get $m_1 - m_2 \leqslant 122$ for $2 \leqslant b \leqslant 10$ and so $m_2 \geqslant m_1 - 122 \geqslant 300 - 122 = 178$.

Let

$$\Lambda_2 = -m_2\log\lambda + n\log b + \log\frac{4d\sqrt{2}}{(b-1)(\lambda^{m_1-m_2}+1)}. \tag{3.23}$$

From (3.11) we can see that

$$\begin{aligned}
&\frac{\lambda^{m_1}}{4\sqrt{2}}(\lambda^{m_2-m_1}+1) - \frac{db^n}{b-1}\\
&\qquad = \frac{\lambda^{m_1}}{4\sqrt{2}}(1+\lambda^{m_2-m_1})(1-\lambda^{-m_2}b^n)\frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_2})}\\
&\qquad = \frac{\lambda^{m_1}}{4\sqrt{2}}(1+\lambda^{m_2-m_1})(1-\exp\Lambda_2)\\
&\qquad = -\frac{d}{b-1} + \frac{\delta^{m_1}}{4\sqrt{2}} + \frac{\delta^{m_2}}{4\sqrt{2}} - B_{m_3} - B_{m_4}\\
&\qquad \leqslant -\frac{1}{9} + \frac{\delta^{300}}{4\sqrt{2}} + \frac{\delta^{178}}{4\sqrt{2}} < 0
\end{aligned}$$

as $m_1 \geqslant 300$ and $m_2 \geqslant 178$. Therefore $\Lambda_2 > 0$ and so from (3.13), we obtain

$$0 < \Lambda_2 < \exp\Lambda_2 - 1 = \left|1 - \lambda^{-m_2} b^n \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_2})}\right| < \lambda^{2+m_3-m_1}.$$

This shows that

$$|\Lambda_2| < \lambda^2 \exp(-1.76(m_1 - m_3)) \tag{3.24}$$

with $m_1 - m_2 \leqslant m_1 \leqslant 1.38 \times 10^{61}$. In order to apply Lemma 2.2 to (3.23), we can take

$$c = \lambda^2, \quad \delta = 1.76, \quad X_0 = 4.2 \cdot 10^{61}, \quad \psi = \frac{1}{\log b} \log \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1 - m_2})},$$
$$v_1 = -\log \lambda, \quad v_2 = \log b, \quad v = \frac{\log \lambda}{\log b}, \quad \beta = \log \frac{4d\sqrt{2}}{(b-1)(1+\lambda^{m_1-m_2})}.$$

We find that $q = q_{174}$ satisfies the hypothesis of Lemma 2.2 for $2 \leqslant b \leqslant 10$ and $1 \leqslant d \leqslant 9$. By Lemma 2.2, we get $m_1 - m_3 \leqslant 180$ and so $m_3 \geqslant 120$.

Let

$$\Lambda_3 = -m_3 \log \lambda + n \log b + \log \frac{4d\sqrt{2}}{(b-1)(\lambda^{m_1-m_3} + \lambda^{m_2-m_3} + 1)}. \tag{3.25}$$

From (3.6), we can see that

$$\begin{aligned} &\frac{\lambda^{m_1}}{4\sqrt{2}}(\lambda^{m_3-m_1} + \lambda^{m_2-m_1} + 1)(1 - \exp \Lambda_3) \\ &\qquad = -\frac{d}{b-1} + \frac{1}{4\sqrt{2}}(\delta^{m_1} + \delta^{m_2} + \delta^{m_3}) - B_{m_4} < 0 \end{aligned}$$

as $m_1 \geqslant 300$, $m_2 \geqslant 178$, $m_3 \geqslant 120$. Thus $\Lambda_3 > 0$ and so from (3.8), we get

$$0 < \Lambda_3 < \exp \Lambda_3 - 1 = \left| 1 - \lambda^{-m_3} b^n \frac{4d\sqrt{2}}{(b-1)(1 + \lambda^{m_1-m_3} + \lambda^{m_2-m_3})} \right| < \lambda^{2+m_4-m_1}.$$

This implies that

$$|\Lambda_3| < \lambda^2 \exp(-1.76(m_1 - m_4)) \tag{3.26}$$

with $m_1 - m_4 \leqslant m_1 \leqslant 1.38 \cdot 10^{61}$. Again, in order to apply Lemma 2.2 to (3.25), we can take

$$c = \lambda^2, \quad \delta = 1.76, \quad X_0 = 4.2 \cdot 10^{61},$$
$$\psi = \frac{\log(4d\sqrt{2}) - \log(9(1 + \lambda^{m_1-m_3} + \lambda^{m_2-m_3}))}{\log b}, \quad v = \frac{\log \lambda}{\log b}, \quad v_1 = -\log \lambda,$$
$$v_2 = \log b, \quad \beta = \log \frac{4d\sqrt{2}}{(b-1)(1 + \lambda^{m_1-m_3} + \lambda^{m_2-m_3})}.$$

We find that $q = q_{146}$ satisfies the hypothesis of Lemma 2.2 for $2 \leqslant b \leqslant 10$ and $1 \leqslant d \leqslant 9$. Thus, by Lemma 2.2, we get $m_1 - m_4 \leqslant 137$ and so $m_4 \geqslant 163$.

Let

$$\Lambda_4 = -m_4 \log\lambda + n\log b + \log\frac{4d\sqrt{2}(b-1)^{-1}}{\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4}+1}. \tag{3.27}$$

From (3.2), we can see that

$$\begin{aligned}&\frac{\lambda^{m_1}}{4\sqrt{2}}(\lambda^{m_4-m_1}+\lambda^{m_3-m_1}+\lambda^{m_2-m_1}+1)(1-\exp\Lambda_4)\\ &\qquad = -\frac{d}{b-1}+\frac{1}{4\sqrt{2}}(\delta^{m_1}+\delta^{m_2}+\delta^{m_3}+\delta^{m_4})<0\end{aligned}$$

as $m_1 \geqslant 300$, $m_2 \geqslant 178 m_3 \geqslant 120$, $m_4 \geqslant 163$. Thus $\Lambda_4 > 0$ and so from (3.3) we obtain

$$\begin{aligned}0<\Lambda_4<\exp\Lambda_4-1 &= \left|1-\lambda^{-m_4}b^n\frac{4d\sqrt{2}}{(b-1)(\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4}+1)}\right|\\ &< \lambda^{2-m_1}.\end{aligned}$$

That is,

$$|\Lambda_4| < \lambda^2\exp(-1.76 m_1)$$

with $m_1 \leqslant 1.38\cdot 10^{61}$. Finally, in order to apply Lemma 2.2 to (3.27), we take

$$\begin{aligned}&c=\lambda^2,\quad \delta=1.76,\quad X_0=1.38\cdot 10^{61},\\ &\psi=\frac{1}{\log b}\log\frac{4d\sqrt{2}}{(b-1)(\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4}+1)},\\ &v=\frac{\log\lambda}{\log b},\quad v_1=-\log\lambda,\quad v_2=\log b,\\ &\beta=\log\frac{4d\sqrt{2}}{(b-1)(\lambda^{m_1-m_4}+\lambda^{m_2-m_4}+\lambda^{m_3-m_4}+1)}.\end{aligned}$$

We find that $q = q_{146}$ satisfies the hypothesis of Lemma 2.2 for $2 \leqslant b \leqslant 10$ and $1 \leqslant d \leqslant 9$. By Lemma 2.2, we get $m_1 \leqslant 138$, which contradicts our assumption that $m_1 \geqslant 300$. This completes the proof. □

Corollary 3.1. *If B_n is a repdigit in the base b and has at least two digits, then $n = 2, 3$. Namely, $B_2 = 6 = (11)_5$ and $B_3 = 35 = (55)_6$.*

Corollary 3.2. *Let b be an integer such that $2 \leqslant b \leqslant 10$. If $n \geqslant 4$, then the equation $B_n + 1 = b^k$ has no solution k in positive integers.*

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