DISTRIBUTIVE LATTICES HAVE THE INTERSECTION PROPERTY

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Abstract. Distributive lattices form an important, well-behaved class of lattices. They are instances of two larger classes of lattices: congruence-uniform and semidistributive lattices. Congruence-uniform lattices allow for a remarkable second order of their elements: the core label order; semidistributive lattices naturally possess an associated flag simplicial complex: the canonical join complex. In this article we present a characterization of finite distributive lattices in terms of the core label order and the canonical join complex, and we show that the core label order of a finite distributive lattice is always a meet-semilattice.

Keywords: distributive lattice; congruence-uniform lattice; canonical join complex; core label order; intersection property

MSC 2010: 06D05

1. Introduction

A finite lattice $\mathcal{L}$ is congruence uniform if for both $\mathcal{L}$ and its dual there is a bijection between the set of join-irreducible elements of $\mathcal{L}$ and the set of join-irreducible congruences of $\mathcal{L}$. Congruence-uniform lattices play an important role in the theory of free lattices, because they are precisely the finite lattices that can be realized as bounded-homomorphic images of free lattices (see [7], Theorem 5.1).

Motivated by his research on the characterization of congruence-uniform lattices of regions of simplicial hyperplane arrangements, Reading observed that there is a natural way to order the elements of a congruence-uniform lattice $\mathcal{L}$ in a second way. This order has been dubbed the *core label order* in [15], denoted by $\mathrm{CLO}(\mathcal{L})$, and it has interesting combinatorial properties. In certain special cases the core label order was investigated in [1], [5], [11], [12], [14], [16], [17]. A general study of the core label order of a congruence-uniform lattice was carried out in [15].

It follows from the results of Day that a finite lattice is congruence uniform if and only if it can be obtained from the singleton lattice by a finite sequence of interval

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doublings (see [7], Theorem 5.1). It was shown in [9] that finite distributive lattices can be obtained from the singleton lattice by the successive doubling of principal order ideals, which implies in particular that they are congruence uniform. In this article we investigate the core label order of finite distributive lattices.

In his solution of the word problem of free lattices, Whitman (see [20], [21]) showed that every element of a free lattice admits a canonical join and a canonical meet representation. It can be shown that lattices, in which every element admits these canonical forms, are semidistributive, which is a weaker form of distributivity. Moreover, a finite lattice is semidistributive if and only if every element admits a canonical join and a canonical meet representation. It is straightforward to show that every subset of a canonical representation is itself a canonical representation. This gives rise to the definition of the *canonical join complex* of a semidistributive lattice. In this complex, the faces are therefore indexed by the elements of the lattice. The canonical join complex was thoroughly studied in [2].

It turns out that we can use the core label order and the canonical join complex to characterize distributive lattices.

Theorem 1.1. *A finite congruence-uniform lattice $\mathcal{L}$ is distributive if and only if* $\mathrm{CLO}(\mathcal{L})$ *is the face poset of the canonical join complex of $\mathcal{L}$.*

We want to point out that we can also use the core label order to characterize finite Boolean lattices. They are precisely the congruence-uniform lattices that are isomorphic to their own core label order (see [15], Theorem 1.5). Consequently, the canonical join complex of a finite Boolean lattice is a simplex.

In [19], Problem 9.5, Reading asked under what conditions the core label order is again a lattice. In [15], Section 4.2 we found one such property, which we call the *intersection property*. This property can be used to characterize the congruence-uniform lattices whose core label orders are meet-semilattices (see [15], Theorem 4.8). We conclude this article with the observation that every distributive lattice has the intersection property.

Theorem 1.2. *Every finite distributive lattice $\mathcal{L}$ has the intersection property. Consequently,* $\mathrm{CLO}(\mathcal{L})$ *is a meet-semilattice, and it is a lattice if and only if $\mathcal{L}$ is isomorphic to a Boolean lattice.*

We first recall the necessary basic notions in Section 2. After that we define the core label order of a congruence-uniform lattice in Section 3.1 and the canonical join complex of a semidistributive lattice in Section 3.2, where we also prove Theorem 1.1. In Section 3.3 we define the intersection property for congruence-uniform lattices and prove Theorem 1.2.

2.1. Basic notions. Let $\mathcal{P} = (P, \leqslant)$ be a partially ordered set (*poset* for short). The *dual poset* of $\mathcal{P}$ is $\mathcal{P}^* \stackrel{\text{def}}{=} (P, \geqslant)$.

An element $x \in P$ is *minimal* in $\mathcal{P}$ if $y \leqslant x$ implies $y = x$ for all $y \in P$. Dually, $x \in P$ is *maximal* in $\mathcal{P}$ if it is minimal in $\mathcal{P}^*$.

An *order ideal* of $\mathcal{P}$ is a set $X \subseteq P$ that is downwards closed, i.e. if $x \in X$ and $y \leqslant x$, then $y \in X$. Dually, $X \subseteq P$ is an *order filter* of $\mathcal{P}$ if it is an order ideal of $\mathcal{P}^*$. Every subset $X \subseteq P$ generates the order ideal

$$P_{\leqslant X} \stackrel{\text{def}}{=} \{y \in P\colon\ y \leqslant x \text{ for some } x \in X\}.$$

If $|X| = 1$, then we call $P_{\leqslant X}$ a *principal order ideal*. We denote by $P^{\geqslant X}$ the order filter of $\mathcal{P}$ generated by X. Moreover, we denote the (po)set of all order ideals of $\mathcal{P}$ by $\mathcal{I}(\mathcal{P})$.

A *cover relation* of $\mathcal{P}$ is a pair (x, y) such that $x < y$ and there is no $z \in P$ such that $x < z < y$. We usually write $x \lessdot y$ for a cover relation and we denote the set of all cover relations of $\mathcal{P}$ by $\mathcal{E}(\mathcal{P})$. Moreover, if $x \lessdot y$, then we call x a *lower cover* of y, and y an *upper cover* of x.

A *chain* of $\mathcal{P}$ is a totally ordered subset of P and it is *saturated* if it can be written as a sequence of cover relations. A saturated chain is *maximal* if it contains a minimal and a maximal element of $\mathcal{P}$.

We say that $\mathcal{P}$ is a *lattice* if for every two elements $x, y \in P$ there exists a greatest lower bound $x \wedge y$ (the *meet*) and a least upper bound $x \vee y$ (the *join*). Observe that every finite lattice has a unique minimal element (denoted by $\hat{0}$) and a unique maximal element (denoted by $\hat{1}$).

2.2. Characterizations of finite distributive lattices. A lattice $\mathcal{L} = (L, \leqslant)$ is *distributive* if for every three elements $x, y, z \in L$ the following two identities hold:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z);$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Finite distributive lattices admit a nice representation as ordered families of sets which was first observed by Birkhoff (see [4]). To that end recall that an element $j \in L$ is *join irreducible* if for every $x, y \in L$ with $j = x \vee y$ we have $j \in \{x, y\}$. Let $\mathcal{J}(\mathcal{L})$ denote the (po)set of join-irreducible elements of $\mathcal{L}$. We remark that in a finite lattice every join-irreducible element j has a unique lower cover, which we denote by j_*.

Theorem 2.1 ([13], Theorem II.1.9)**.** *A finite lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L} \cong \mathcal{I}\bigl(\mathcal{J}(\mathcal{L})\bigr)$.*

Figure 1 (a) shows a distributive lattice with its set of join-irreducible elements highlighted. Figure 1 (b) shows the corresponding lattice of order ideals of the poset of join-irreducibles.

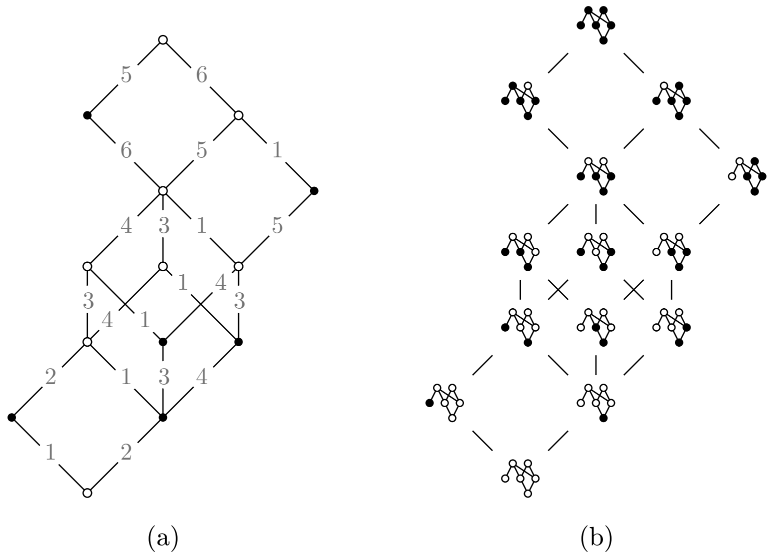

Figure 1. An illustration of Theorem 2.1: (a) A distributive lattice. The cover relations are labeled by the map (2.2); (b) The lattice of order ideals of the poset of join-irreducible elements of the lattice in Figure 1 (a).

As a consequence of Theorem 2.1 we may view a distributive lattice as a family of sets ordered by inclusion, where joins and meets are given by the set union and set intersection, respectively. If $\mathcal{L} = (L, \leqslant)$ is distributive, then we use the bijection

$$\iota\colon\ L \to \mathcal{I}\bigl(\mathcal{J}(\mathcal{L})\bigr), \quad x \mapsto L_{\leqslant\{x\}} \cap \mathcal{J}(\mathcal{L}), \tag{2.1}$$

to switch between elements of a distributive lattice and their representing order ideals of join-irreducible elements.

Another consequence of Theorem 2.1 is that distributive lattices are *graded*, i.e. every maximal chain has the same cardinality. This can be quickly seen as follows: let $x \in L \setminus \{\hat{1}\}$ and let $j \in \mathcal{J}(\mathcal{L})$ be a minimal element of $\mathcal{J}(\mathcal{L}) \setminus \iota(x)$. Then, $\iota(x) \cup \{j\}$ is an order ideal of $\mathcal{J}(\mathcal{L})$ which therefore represents an element $x' \in L$, and we have $x \lessdot x'$. We thus obtain a natural map

$$\lambda\colon\ \mathcal{E}(\mathcal{L}) \to \mathcal{J}(\mathcal{L}), \quad (x, y) \mapsto \iota(y) \setminus \iota(x). \tag{2.2}$$

(a) The lattice N_5. (b) The lattice M_3.

Figure 2. The two forbidden sublattices of a distributive lattice.

There is another characterization of finite distributive lattices due to Birkhoff, first described in [3], that will be useful later. Let N_5 denote the lattice shown in Figure 2 (a) and let M_3 denote the lattice shown in Figure 2 (b).

Theorem 2.2 ([13], Theorem II.1.1). *A finite lattice is distributive if and only if it does not have a sublattice isomorphic to N_5 or M_3.*

2.3. Canonical join representations. Let $\mathcal{L} = (L, \leqslant)$ be a lattice. A *join representation* of $x \in L$ is a set $X \subseteq L$ with $x = \bigvee X$. A join representation X of x is *irredundant* if no proper subset of X joins to x, and it is *canonical* if $L_{\leqslant X} \subseteq L_{\leqslant Y}$ for every join representation Y of x. We denote the canonical join representation of $x \in L$ by $\Gamma(x)$ (if it exists).

It turns out that the finite lattices in which every element admits a canonical join representation can be characterized algebraically. We say that $\mathcal{L}$ is *join semidistributive* if for every $x, y, z \in L$ with $x \vee y = x \vee z$ it holds that $x \vee y = x \vee (y \wedge z)$. The lattice $\mathcal{L}$ is *meet semidistributive* if $\mathcal{L}^*$ is join semidistributive. It is *semidistributive* if it is both join and meet semidistributive.

Theorem 2.3 ([10], Theorem 2.24). *A finite lattice is join semidistributive if and only if every element admits a canonical join representation.*

2.4. Interval doubling. Day introduced a way of constructing bigger lattices from smaller ones by the so-called doubling (see [7]). Let $\mathbf{2}$ denote the unique lattice with two elements 0 and 1. Moreover, let $\mathcal{P} = (P, \leqslant)$ be a poset and let $X \subseteq P$. The *doubling* of $\mathcal{P}$ by X is the subposet $\mathcal{P}[X]$ of the direct product $\mathcal{P} \times \mathbf{2}$ given by the ground set

$$(P_{\leqslant X} \times \{0\}) \uplus (((P \setminus P_{\leqslant X}) \cup X) \times \{1\}).$$

Here, $\uplus$ denotes the disjoint set union. We can use this doubling construction to characterize finite distributive lattices.

Theorem 2.4 ([9], Theorem 3)**.** *A finite lattice is distributive if and only if it can be obtained from the singleton lattice by a finite sequence of doublings by principal order filters.*

Recall from [7], Theorem 5.1 that a lattice is *congruence uniform* if and only if it can be constructed from the singleton lattice by a finite sequence of doublings by intervals. See also [8]. Since every principal order filter in a finite lattice is an interval, Theorem 2.4 implies that every finite distributive lattice is congruence uniform. But we have more than that.

Theorem 2.5 ([7], Theorem 4.2)**.** *Every congruence-uniform lattice is semidistributive.*

Let $\mathcal{L} = (L, \leqslant)$ be a finite congruence-uniform lattice. As a consequence of Theorems 2.3 and 2.5, every $x \in L$ admits a canonical join representation. This canonical join representation is determined completely by the lower covers of x. To see how that works, let us say that two cover relations $(x, y), (u, v) \in \mathcal{E}(\mathcal{L})$ are *perspective* if either $v \vee x = y$ and $v \wedge x = u$ or $u \vee y = v$ and $u \wedge y = x$. Let us consider the map

$$\gamma\colon\ \mathcal{E}(\mathcal{L}) \to \mathcal{J}(\mathcal{L}), \quad (x, y) \mapsto j, \tag{2.3}$$

where j is the unique join-irreducible element of $\mathcal{L}$ such that (x, y) and (j_*, j) are perspective. It follows from [10], Theorem 2.30 and [12], Lemma 2.6 that this map is well defined.

Proposition 2.6 ([12], Proposition 2.9)**.** *Let $\mathcal{L} = (L, \leqslant)$ be a finite congruence-uniform lattice. For $x \in L$ we have*

$$\Gamma(x) = \{\gamma(y, x)\colon\ y \lessdot x\}.$$

3. A new characterization of distributive lattices

3.1. The core label order of a distributive lattice. Motivated by the study of the poset of regions of real hyperplane arrangements, Reading introduced an alternate way to order the elements of a congruence-uniform lattice (see [19], Section 9, 7.4). For $x \in L$, let us define the *nucleus* of x by

$$x_{\downarrow} \overset{\text{def}}{=} \bigwedge_{y \in L\colon\ y \lessdot x} y.$$

We call the interval $[x_{\downarrow}, x]$ the *core* of x. This definition enables us to define the set of *core labels* of x by

$$\Psi(x) \stackrel{\text{def}}{=} \{\gamma(u,v)\colon\ x_{\downarrow} \leqslant u \lessdot v \leqslant x\}.$$

The *core label order* of $\mathcal{L}$ is defined by $x \sqsubseteq y$ if $\Psi(x) \subseteq \Psi(y)$, and we usually write $\mathrm{CLO}(\mathcal{L}) \stackrel{\text{def}}{=} (L, \sqsubseteq)$.

In this section we investigate this core label order of a distributive lattice. We start by observing that the two maps (2.2) and (2.3) coincide for distributive lattices.

Lemma 3.1. *In a finite distributive lattice the maps λ and γ coincide.*

Proof. Let $(x,y) \in \mathcal{E}(\mathcal{L})$ and let $j = \lambda(x,y)$. By definition we have $\iota(y) \setminus \iota(x) = j$, which implies that $\iota(x) \cup \iota(j) = \iota(y)$ and $\iota(x) \cap \iota(j) = \iota(j) \setminus \{j\} = \iota(j_*)$. This means that (j_*, j) and (x,y) are perspective, and by definition we obtain $\gamma(x,y) = j$. □

Proposition 3.2. *Let $\mathcal{L} = (L, \leqslant)$ be a finite distributive lattice. For every $x \in L$ we have*

$$\Gamma(x) = \iota(x) \setminus \iota(x_{\downarrow}).$$

Proof. In view of Proposition 2.6 and Lemma 3.1 we conclude for $x \in L$:

$$\begin{aligned}\Gamma(x) &= \bigcup_{y \in L\colon\, y \lessdot x} \gamma(y,x) = \bigcup_{y \in L\colon\, y \lessdot x} \lambda(y,x) = \bigcup_{y \in L\colon\, y \lessdot x} \big(\iota(x) \setminus \iota(y)\big)\\ &= \iota(x) \setminus \bigcap_{y \in L\colon\, y \lessdot x} \iota(y) = \iota(x) \setminus \iota\Big(\bigwedge_{y \in L\colon\, y \lessdot x} y\Big) = \iota(x) \setminus \iota(x_{\downarrow}).\end{aligned}$$

□

Recall that a finite *Boolean lattice* is a lattice, which is isomorphic to the power set of a finite set ordered by inclusion. We write $\mathrm{Bool}(k)$ for the Boolean lattice with 2^k elements. Let us recall the following result.

Proposition 3.3 ([15], Proposition 4.2)**.** *Let $\mathcal{L} = (L, \leqslant)$ be a finite congruence-uniform lattice and let $x \in L$. We have $\Gamma(x) = \Psi(x)$ if and only if $[x_{\downarrow}, x] \cong \mathrm{Bool}(k)$, where $k = |\Gamma(x)|$.*

Proposition 3.4. *Let $\mathcal{L} = (L, \leqslant)$ be a finite distributive lattice. For $x \in L$ we have $\Psi(x) = \Gamma(x)$.*

P r o o f. Let us first prove that the core of x is isomorphic to $\mathrm{Bool}(k)$, where k denotes the number of lower covers of x. This is trivially true if $k \leqslant 1$. Now suppose that $k = 2$. Let y_1, y_2 denote the lower covers of x and let $x' = y_1 \wedge y_2$. Pick $z \in L$ with $x_\downarrow < z \leqslant y_1$. We obtain

$$z = x' \vee z = (y_1 \wedge y_2) \vee (y_1 \wedge z) = y_1 \wedge (y_2 \vee z) = y_1 \wedge x = y_1$$

and we may thus conclude that $x_\downarrow$ is a lower cover of both y_1 and y_2, which establishes the claim.

Now suppose that x has k lower covers $y_1, y_2, \ldots, y_k$ and let $x' = y_1 \wedge y_2 \wedge \ldots \wedge y_{k-1}$. By induction, the interval $[x', x]$ is isomorphic to $\mathrm{Bool}(k-1)$. Analogously to the reasoning in the first paragraph we may show that $x_\downarrow \lessdot x'$, which implies that $y_k \wedge x' \in \{x', x_\downarrow\}$. We may then show inductively that

$$x = (y_1 \vee y_k) \wedge (y_2 \vee y_k) \wedge \ldots \wedge (y_{k-1} \vee y_k) = (y_1 \wedge y_2 \wedge \ldots \wedge y_{k-1}) \vee y_k = x' \vee y_k,$$

which implies $x' \not\leqslant y_k$, since $x' < x$. Therefore, $x' \wedge y_k = x_\downarrow$ must hold. Analogously we see that for every $z \in [x', x]$ the element $z \wedge y_k$ is a lower cover of z. Therefore, the interval $[x_\downarrow, x]$ is isomorphic to $\mathrm{Bool}(k)$. Proposition 3.3 implies that $\Psi(x) = \Gamma(x)$. $\square$

Proposition 3.5. *Let $\mathcal{L} = (L, \leqslant)$ be a finite congruence-uniform lattice. If for every $x \in L$ we have $\Gamma(x) = \Psi(x)$, then $\mathcal{L}$ is distributive.*

P r o o f. We proceed by contraposition and assume that $\mathcal{L}$ is not distributive. By Theorem 2.2 we conclude that it contains a sublattice isomorphic to N_5 or M_3. We know from [6] that M_3 is not semidistributive, and in view of Theorem 2.5 it cannot appear as a sublattice of a congruence-uniform lattice.

We conclude that $\mathcal{L}$ contains a sublattice $\mathcal{K}$ isomorphic to N_5. Let x and y denote the least and greatest element of $\mathcal{K}$. Define the *length* of a lattice to be the maximal size of a maximal chain. We choose $\mathcal{K}$ minimal in such a way that every sublattice of the interval $[x, y]$ in $\mathcal{L}$, whose length is strictly smaller than the length of $\mathcal{K}$, is distributive. We say that a set $X \subseteq L$ *contradicts the choice of* $\mathcal{K}$ if X induces a proper sublattice of the interval $[x, y]$ that is isomorphic to N_5 and has smaller length than $\mathcal{K}$.

We will show that $\Gamma(x) \subsetneq \Psi(x)$ in $\mathcal{L}$. Since intervals of congruence-uniform lattices are congruence uniform again (see [7], Theorem 4.3), we may assume without loss of generality that $x = \hat{0}$ and $y = \hat{1}$. (Here $\hat{0}$ and $\hat{1}$ denote the least and greatest elements of $\mathcal{L}$.)

In other words, there are three elements $b, c, d \in L$ such that $b < c$ and $b \wedge d = \hat{0} = c \wedge d$ and $b \vee d = \hat{1} = c \vee d$.

We may choose b and c such that they form a cover relation in $\mathcal{L}$. (Observe that for every $z \in L$ with $b \leqslant z \leqslant c$ we have $\hat{1} = b \vee d \leqslant z \vee d$ and $\hat{0} = c \wedge d \geqslant z \wedge d$, which implies $z \wedge d = \hat{0}$ and $z \vee d = \hat{1}$.)

We may as well choose b in such a way that it covers $\hat{0}$. (Observe that if there is some $z \in L$ with $\hat{0} < z < b$ such that $z \vee d < \hat{1}$, then the set $\{z, b, c, z \vee d, \hat{1}\}$ contradicts the choice of $\mathcal{K}$.)

Since $\mathcal{K}$ is finite we can find elements y_1, y_2 such that $c \leqslant y_1 \lessdot \hat{1}$ and $d \leqslant y_2 \lessdot \hat{1}$. If $y_1 = y_2$, then the set $\{\hat{0}, b, c, d, y_1\}$ contradicts the choice of $\mathcal{K}$. We thus have $y_1 \neq y_2$. Since $b \vee d = \hat{1}$, we conclude that $b \nleqslant y_2$ and the same is true for c.

Let $z = y_1 \wedge y_2$. Suppose that $\hat{0} < z$. Since $\hat{0} \lessdot b$, we conclude that $b \wedge z = \hat{0}$. We also have $c \wedge z = \hat{0}$, since otherwise $\{c \wedge z, c, y_1, y_2, \hat{1}\}$ contradicts the choice of $\mathcal{K}$.

Moreover, we have $b \vee z = y_1$, since otherwise $\{z, b \vee z, y_1, y_2, \hat{1}\}$ contradicts the choice of $\mathcal{K}$. The analogous argument shows that $c \vee z = y_1$. Then, however, $\{\hat{0}, b, c, z, y_1\}$ contradicts the choice of $\mathcal{K}$.

We thus conclude that $y_1 \wedge y_2 = \hat{0}$. The dual of Proposition 2.9 in [15] implies that y_1 and y_2 are the only lower covers of $\hat{1}$. However, $\mathcal{L}$ has cardinality $\geqslant 5$, which implies that it is not isomorphic to Bool(2). Proposition 3.3 implies that $\Psi(\hat{1}) \neq \Gamma(\hat{1})$ and we are done. $\square$

Recall from [15] that the *Boolean defect* of a congruence-uniform lattice $\mathcal{L} = (L, \leqslant)$ is

$$\mathrm{bdef}(\mathcal{L}) \stackrel{\mathrm{def}}{=} \sum_{x \in L} |\Psi(x) \setminus \Gamma(x)|.$$

We obtain the following result, which strengthens Proposition 5.2 of [15].

Proposition 3.6. *A finite congruence-uniform lattice* $\mathcal{L}$ *satisfies* $\mathrm{bdef}(\mathcal{L}) = 0$ *if and only if* $\mathcal{L}$ *is distributive.*

P r o o f. This follows from Propositions 3.4 and 3.5. $\square$

3.2. The canonical join complex of a distributive lattice. Given a finite set M, a *simplicial complex* on M is a family $\Delta(M)$ of subsets of M such that for every $F \in \Delta(M)$ and every $F' \subseteq F$ we have $F' \in \Delta(M)$. The members of $\Delta(M)$ are *faces*. The *face poset* of $\Delta(M)$ is the poset $(\Delta(M), \subseteq)$.

Reading observed in [18], Proposition 2.2 that the set of canonical join representations of a lattice is closed under taking subsets. In other words, it forms a simplicial complex; the *canonical join complex* of $\mathcal{L}$, denoted by $\mathrm{Can}(\mathcal{L})$.

We are now ready to prove Theorem 1.1.

P r o o f of Theorem 1.1. Let $\mathcal{L} = (L, \leqslant)$ be a finite congruence-uniform lattice. By definition, the face poset of $\operatorname{Can}(\mathcal{L})$ is precisely $(\{\Gamma(x)\colon x \in L\}, \subseteq)$ and the core label order of $\mathcal{L}$ is $(\{\Psi(x)\colon x \in L\}, \subseteq)$. Propositions 3.4 and 3.5 now imply that the sets $\{\Gamma(x)\colon x \in L\}$ and $\{\Psi(x)\colon x \in L\}$ are equal if and only if $\mathcal{L}$ is distributive. $\square$

Figure 3 (a) shows the core label order of the lattice from Figure 1 (a) and Figure 3 (b) shows the canonical join complex of the lattice in Figure 1 (a). It is quickly verified that Theorem 1.1 holds.

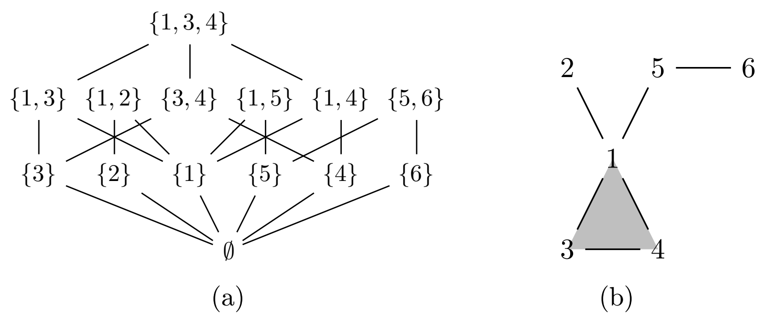

Figure 3. An illustration of Theorem 1.1: (a) The core label order of the lattice in Figure 1 (a); (b) The canonical join complex of the lattice in Figure 1 (a). The highlighted region indicates a two-dimensional face.

3.3. The intersection property. Reading asked in [19], Problem 9.5 for conditions on a congruence-uniform lattice $\mathcal{L}$ which would imply that $\operatorname{CLO}(\mathcal{L})$ is a lattice, too. We gave one such property in [15], Section 4.2: a finite congruence-uniform lattice $\mathcal{L} = (L, \leqslant)$ *has the intersection property* if for all $x, y \in L$ there exists $z \in L$ such that $\Psi(x) \cap \Psi(y) = \Psi(z)$.

Theorem 3.7 ([15], Theorems 1.3 and 4.7)**.** *Let $\mathcal{L}$ be a finite congruence-uniform lattice. The core label order $\operatorname{CLO}(\mathcal{L})$ is a meet-semilattice if and only if $\mathcal{L}$ has the intersection property. It is a lattice if and only if $\hat{1}_{\downarrow} = \hat{0}$.*

We conclude this article with the proof of Theorem 1.2.

P r o o f of Theorem 1.2. Let $\mathcal{L} = (L, \leqslant)$ be a finite distributive lattice. For $x, y \in L$ we conclude from Proposition 3.4 that $\Psi(x) = \Gamma(x)$ and $\Psi(y) = \Gamma(y)$. It follows that $Z = \Gamma(x) \cap \Gamma(y)$ is a face of $\operatorname{Can}(\mathcal{L})$, which means that there exists $z \in L$ with $Z = \Gamma(z) = \Psi(z)$. We have thus established that $\mathcal{L}$ has the intersection property.

Lemma 3.9 of [15] states that $\operatorname{CLO}(\mathcal{L})$ has a greatest element if and only if $\hat{1}_{\downarrow} = \hat{0}$, which in view of Proposition 3.3 is the case precisely when $\mathcal{L}$ is isomorphic to a Boolean lattice.

The claims then follow from Theorem 3.7. $\square$

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