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Czechoslovak Mathematical Journal, Vol. 71 (2021), No. 1, 253–267

Persistent URL: http://dml.cz/dmlcz/148738

Czechoslovak Mathematical Journal, 71 (146) (2021), 253–267

DRINFELD DOUBLES VIA DERIVED HALL ALGEBRAS AND BRIDGELAND'S HALL ALGEBRAS

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Received July 16, 2019. Published online October 20, 2020.

Abstract. Let $\mathcal{A}$ be a finitary hereditary abelian category. We give a Hall algebra presentation of Kashaev's theorem on the relation between Drinfeld double and Heisenberg double. As applications, we obtain realizations of the Drinfeld double Hall algebra of $\mathcal{A}$ via its derived Hall algebra and Bridgeland's Hall algebra of m-cyclic complexes.

Keywords: Heisenberg double; Drinfeld double; derived Hall algebra; Bridgeland's Hall algebra

MSC 2020: 16G20, 17B20, 17B37

1. Introduction

The Hall algebra $\mathfrak{H}(A)$ of a finite dimensional algebra A over a finite field was introduced by Ringel in 1990, see [9]. Ringel in [8] and [9] proved that if A is a hereditary algebra of finite type, the twisted Hall algebra $\mathfrak{H}_v(A)$, called the *Ringel-Hall algebra*, is isomorphic to the positive part of the corresponding quantized enveloping algebra. In 1995, Green in [3] generalized Ringel's work to any hereditary algebra A and showed that the composition subalgebra of $\mathfrak{H}_v(A)$ generated by simple A-modules gives a realization of the positive part of the quantized enveloping algebra associated with A. Moreover, he introduced a bialgebra structure on $\mathfrak{H}_v(A)$ via a significant formula called *Green's formula*. In 1997, Xiao [13] provided the antipode on $\mathfrak{H}_v(A)$ and proved that the extended Ringel-Hall algebra is a Hopf algebra. Furthermore,

The authors were supported partially by the National Natural Science Foundation of China (No.s 11471177, 11801273), Natural Science Foundation of Jiangsu Province of China (No. BK20180722) and Natural Science Foundation of Jiangsu Higher Education Institutions of China (No. 18KJB110017).

DOI: 10.21136/CMJ.2020.0313-19

he considered the Drinfeld double of the extended Ringel-Hall algebras and obtained a realization of the full quantized enveloping algebra.

In order to give an intrinsic realization of the entire quantized enveloping algebra via Hall algebra approach, one tried to define the Hall algebra of a triangulated category (see for example, [5], [12], [14]). Kapranov in [5] considered the Heisenberg double of the extended Ringel-Hall algebras and defined an associative algebra, called the *lattice algebra*, for the bounded derived category of a hereditary algebra A. By using the fibre products of model categories, Toën in [12] defined an associative algebra, called the *derived Hall algebra*, for a DG-enhanced triangulated category. Later on, Xiao and Xu in [14] generalized the definition of the derived Hall algebra to any triangulated category with some homological finiteness conditions. In particular, the derived Hall algebra $\mathcal{DH}(A)$ of the bounded derived category of a hereditary algebra A can be defined and it is proved in [12] that there exist certain Heisenberg double structures in $\mathcal{DH}(A)$.

Recently, for any hereditary algebra A, Bridgeland in [1] defined an associative algebra, called the *Bridgeland's Hall algebra*, which is the Ringel-Hall algebra of 2-cyclic complexes over projective A-modules with some localization and reduction. He proved that the quantized enveloping algebra associated to A can be embedded into its Bridgeland's Hall algebra. This provides a beautiful realization of the full quantized enveloping algebra by Hall algebras. Afterwards, Yanagida in [15] (see also [16]) showed that Bridgeland's Hall algebra of 2-cyclic complexes of a hereditary algebra is isomorphic to the Drinfeld double of its extended Ringel-Hall algebras. Inspired by the work of Bridgeland, Chen and Deng in [2] introduced Bridgeland's Hall algebra $\mathcal{DH}_m(A)$ of m-cyclic complexes of a hereditary algebra A for each non-negative integer $m \neq 1$. If $m = 0$ or $m > 2$, the algebra structure of $\mathcal{DH}_m(A)$ has a characterization in [17], in particular, it is proved that there exist Heisenberg double structures in $\mathcal{DH}_m(A)$.

Kashaev in [6] established a relation between the Drinfeld double and Heisenberg double of a Hopf algebra. Explicitly, he showed that the Drinfeld double is representable as a subalgebra in the tensor square of the Heisenberg double.

In this paper, let $\mathcal{A}$ be a finitary hereditary abelian category. We first give a Hall algebra presentation of Kashaev's theorem on the relation between the Drinfeld double and Heisenberg double. Then we apply this presentation to Bridgeland's Hall algebra and the derived Hall algebra of $\mathcal{A}$.

Throughout the paper, all tensor products are assumed to be over the complex number field $\mathbb{C}$. Let k be a fixed finite field with q elements and set $v = \sqrt{q} \in \mathbb{C}$. Let $\mathcal{A}$ be a finitary hereditary abelian k-category. We denote by $\mathrm{Iso}(\mathcal{A})$ and $K(\mathcal{A})$ the set of isoclasses of objects in $\mathcal{A}$ and the Grothendieck group of $\mathcal{A}$, respectively. For each object M in $\mathcal{A}$, the class of M in $K(\mathcal{A})$ is denoted by $\widehat{M}$, and the automorphism group of M is denoted by $\mathrm{Aut}(M)$. For a finite set S, we denote by $|S|$ its cardinality, and we also write a_M for $|\mathrm{Aut}(M)|$. For a positive integer m, we denote the quotient ring $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. By convention, $\mathbb{Z}_0 = \mathbb{Z}$.

2. Preliminaries

In this section, we recall definitions of the Ringel-Hall algebra, Heisenberg double, and Drinfeld double (see [5], [10], [13]).

2.1. Hall algebras. For objects $M, N_1, \ldots, N_t \in \mathcal{A}$, let $g^M_{N_1 \ldots N_t}$ be the number of the filtrations

$$M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_{t-1} \supseteq M_t = 0$$

such that $M_{i-1}/M_i \cong N_i$ for all $1 \leqslant i \leqslant t$. In particular, if $t = 2$, $g^M_{N_1 N_2}$ is the number of subobjects X of M such that $X \cong N_2$ and $M/X \cong N_1$. One defines the *Hall algebra* $\mathfrak{H}(\mathcal{A})$ to be the vector space over $\mathbb{C}$ with the basis $[M] \in \mathrm{Iso}(\mathcal{A})$ and with the multiplication defined by

$$[M] \diamond [N] = \sum_{[L]} g^L_{MN} [L].$$

By definition, it is easy to see that for each $1 < i < t$,

$$g^M_{N_1 \ldots N_t} = \sum_{[X]} g^M_{N_1 \ldots N_{i-1} X} g^X_{N_i \ldots N_t} = \sum_{[Y]} g^Y_{N_1 \ldots N_i} g^M_{Y N_{i+1} \ldots N_t}.$$

For any $M, N \in \mathcal{A}$, define

$$\langle M, N \rangle := \dim_k \mathrm{Hom}_{\mathcal{A}}(M, N) - \dim_k \mathrm{Ext}^1_{\mathcal{A}}(M, N).$$

It induces a bilinear form

$$\langle \cdot, \cdot \rangle\colon\ K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z},$$

known as the *Euler form*. We also consider the *symmetric Euler form*

$$(\cdot, \cdot)\colon\ K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z},$$

defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in K(\mathcal{A})$.

The twisted Hall algebra $\mathfrak{H}_v(\mathcal{A})$, called the *Ringel-Hall algebra*, is the same vector space as $\mathfrak{H}(\mathcal{A})$ but with the twisted multiplication defined by

$$[M][N] = v^{\langle M, N \rangle} \cdot [M] \diamond [N].$$

We can form the *extended Ringel-Hall algebra* $\mathfrak{H}_v^e(\mathcal{A})$ by adjoining symbols K_α for all $\alpha \in K(\mathcal{A})$ and imposing the relations

$$K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_\alpha[M] = v^{(\alpha,\widehat{M})} \cdot [M]K_\alpha. \tag{2.1}$$

Green in [3] introduced a (topological) bialgebra structure on $\mathfrak{H}_v^e(\mathcal{A})$ by defining the comultiplication as

$$\Delta([L]K_\alpha) = \sum_{[M],[N]} v^{\langle M,N\rangle} \frac{a_M a_N}{a_L} g_{MN}^L [M]K_{\widehat{N}+\alpha} \otimes [N]K_\alpha \quad \text{for any } L \in \mathcal{A},\ \alpha \in K(\mathcal{A}).$$

The fact that Δ is a homomorphism of algebras amounts to the following crucial formula.

Theorem 2.1 (Green's formula)**.** *Given $M, N, M', N' \in \mathcal{A}$, we have the formula*

$$\begin{aligned} a_M a_N a_{M'} a_{N'} \sum_{[L]} g_{MN}^L g_{M'N'}^L \frac{1}{a_L} \\ = \sum_{[A],[A'],[B],[B']} \frac{|\mathrm{Ext}^1_{\mathcal{A}}(A,B')|}{|\mathrm{Hom}_{\mathcal{A}}(A,B')|} g_{AA'}^M g_{BB'}^N g_{AB}^{M'} g_{A'B'}^{N'} a_A a_{A'} a_B a_{B'}. \end{aligned} \tag{2.2}$$

2.2. Heisenberg doubles. Let A and B be Hopf algebras, and let $\varphi\colon A \times B \to \mathbb{C}$ be a Hopf pairing. The *Heisenberg double* $HD(A, B, \varphi)$ is defined to be the free product $A * B$ imposed by the relation (with $a \in A$ and $b \in B$)

$$b * a = \sum \varphi(a_2, b_1) a_1 * b_2,$$

where (and elsewhere) we use Sweedler's notation $\Delta(a) = \sum a_1 \otimes a_2$.

There exists a so-called *Green's pairing* $\varphi_0\colon \mathfrak{H}_v^e(\mathcal{A}) \times \mathfrak{H}_v^e(\mathcal{A}) \to \mathbb{C}$ defined by

$$\varphi_0([M]K_\alpha, [N]K_\beta) = \delta_{[M],[N]} \frac{v^{(\alpha,\beta)}}{a_M},$$

which is a Hopf pairing.

Now let us apply construction of the Heisenberg double to Ringel-Hall algebras. Let $H^+(\mathcal{A})$ (or $H^-(\mathcal{A})$) be the Ringel-Hall algebra $\mathfrak{H}_v^e(\mathcal{A})$ with any $[M]K_\alpha$ rewritten as $\mu_M^+ K_\alpha^+$ (or $\mu_M^- K_\alpha^-$). Thus, considering $A = H^-(\mathcal{A})$, $B = H^+(\mathcal{A})$ and $\varphi = \varphi_0$, we obtain the *Heisenberg double Hall algebra*, denoted by $HD(\mathcal{A})$. By direct calculations, we give the characterization of $HD(\mathcal{A})$ via generators and generating relations (with $\alpha, \beta \in K(\mathcal{A})$ and $[M], [N] \in \mathrm{Iso}(\mathcal{A})$) as follows (cf. [5]):

$$\mu_M^+ \mu_N^+ = \sum_{[L]} v^{\langle M,N\rangle} g_{MN}^L \mu_L^+, \tag{2.3}$$

$$\mu_M^- \mu_N^- = \sum_{[L]} v^{\langle M,N\rangle} g_{MN}^L \mu_L^-,$$

$$K_\alpha^+ \mu_M^+ = v^{(\alpha,\widehat{M})} \mu_M^+ K_\alpha^+, \tag{2.4}$$

$$K_\alpha^- \mu_M^- = v^{(\alpha,\widehat{M})} \mu_M^- K_\alpha^-,$$

$$K_\alpha^\pm K_\beta^\pm = K_{\alpha+\beta}^\pm, \tag{2.5}$$

$$K_\alpha^+ K_\beta^- = v^{(\alpha,\beta)} K_\beta^- K_\alpha^+,$$

$$K_\alpha^+ \mu_M^- = \mu_M^- K_\alpha^+, \tag{2.6}$$

$$K_\alpha^- \mu_M^+ = v^{-(\alpha,\widehat{M})} \mu_M^+ K_\alpha^-,$$

$$\mu_M^+ \mu_N^- = \sum_{[X],[Y]} v^{\langle \widehat{N}-\widehat{Y}, \widehat{X}-\widehat{Y}\rangle} \gamma_{MN}^{XY} K_{\widehat{N}-\widehat{Y}}^- \mu_Y^- \mu_X^+, \tag{2.7}$$

where (and elsewhere) $\gamma_{MN}^{XY} = a_X a_Y / a_M a_N \sum_{[L]} a_L g_{LX}^M g_{YL}^N$.

Similarly, one defines the *dual Heisenberg double Hall algebra* $\check{H}D(\mathcal{A})$, which is given by the generators and generating relations (with $\alpha, \beta \in K(\mathcal{A})$ and $[M], [N] \in \mathrm{Iso}(\mathcal{A})$) as follows:

$$\nu_M^+ \nu_N^+ = \sum_{[L]} v^{\langle M,N\rangle} g_{MN}^L \nu_L^+, \tag{2.8}$$

$$\nu_M^- \nu_N^- = \sum_{[L]} v^{\langle M,N\rangle} g_{MN}^L \nu_L^-,$$

$$\mathcal{K}_\alpha^+ \nu_M^+ = v^{(\alpha,\widehat{M})} \nu_M^+ \mathcal{K}_\alpha^+, \tag{2.9}$$

$$\mathcal{K}_\alpha^- \nu_M^- = v^{(\alpha,\widehat{M})} \nu_M^- \mathcal{K}_\alpha^-,$$

$$\mathcal{K}_\alpha^\pm \mathcal{K}_\beta^\pm = \mathcal{K}_{\alpha+\beta}^\pm, \tag{2.10}$$

$$\mathcal{K}_\alpha^+ \mathcal{K}_\beta^- = v^{-(\alpha,\beta)} \mathcal{K}_\beta^- \mathcal{K}_\alpha^+,$$

$$\mathcal{K}_\alpha^- \nu_M^+ = \nu_M^+ \mathcal{K}_\alpha^-, \tag{2.11}$$

$$\mathcal{K}_\alpha^+ \nu_M^- = v^{-(\alpha,\widehat{M})} \nu_M^- \mathcal{K}_\alpha^+,$$

$$\nu_N^- \nu_M^+ = \sum_{[X],[Y]} v^{\langle \widehat{N}-\widehat{Y}, \widehat{Y}-\widehat{X}\rangle} \gamma_{NM}^{YX} \mathcal{K}_{\widehat{N}-\widehat{Y}}^+ \nu_X^+ \nu_Y^-. \tag{2.12}$$

2.3. Drinfeld doubles. Let A and B be Hopf algebras, and let $\varphi\colon A \times B \to \mathbb{C}$ be a Hopf pairing. The *Drinfeld double* $D(A, B, \varphi)$ is defined to be the free product

$A * B$ imposed by the relations (with $a \in A$ and $b \in B$)

$$\sum \varphi(a_1, b_2) b_1 * a_2 = \sum \varphi(a_2, b_1) a_1 * b_2. \tag{2.13}$$

Applying construction of the Drinfeld double to the Ringel-Hall algebras $H^-(\mathcal{A})$ and $H^+(\mathcal{A})$, we obtain the *Drinfeld double Hall algebra* denoted by $D(\mathcal{A})$, which is defined by the generators and generating relations (with $\alpha, \beta \in K(\mathcal{A})$, $[M], [N] \in \mathrm{Iso}(\mathcal{A})$) as

$$\omega_M^+ \omega_N^+ = \sum_{[L]} v^{\langle M,N \rangle} g_{MN}^L \omega_L^+, \tag{2.14}$$

$$\omega_M^- \omega_N^- = \sum_{[L]} v^{\langle M,N \rangle} g_{MN}^L \omega_L^-,$$

$$\mathscr{K}_\alpha^+ \omega_M^+ = v^{(\alpha, \widehat{M})} \omega_M^+ \mathscr{K}_\alpha^+, \tag{2.15}$$

$$\mathscr{K}_\alpha^- \omega_M^- = v^{(\alpha, \widehat{M})} \omega_M^- \mathscr{K}_\alpha^-,$$

$$\mathscr{K}_\alpha^\pm \mathscr{K}_\beta^\pm = \mathscr{K}_{\alpha+\beta}^\pm, \tag{2.16}$$

$$\mathscr{K}_\alpha^+ \mathscr{K}_\beta^- = \mathscr{K}_\beta^- \mathscr{K}_\alpha^+,$$

$$\mathscr{K}_\alpha^+ \omega_M^- = v^{-(\alpha, \widehat{M})} \omega_M^- \mathscr{K}_\alpha^+, \tag{2.17}$$

$$\mathscr{K}_\alpha^- \omega_M^+ = v^{-(\alpha, \widehat{M})} \omega_M^+ \mathscr{K}_\alpha^-,$$

$$\sum_{[X],[Y]} v^{\langle \widehat{M} - \widehat{X}, \widehat{M} - \widehat{N} \rangle} \gamma_{MN}^{XY} \mathscr{K}_{\widehat{M} - \widehat{X}}^- \omega_Y^- \omega_X^+ \tag{2.18}$$

$$= \sum_{[X],[Y]} v^{\langle \widehat{M} - \widehat{X}, \widehat{N} - \widehat{M} \rangle} \gamma_{NM}^{YX} \mathscr{K}_{\widehat{M} - \widehat{X}}^+ \omega_X^+ \omega_Y^-.$$

3. Kashaev's theorem: Hall algebra presentation

In this section, we prove Kashaev's theorem (see [6], Theorem 2) in the form of Ringel-Hall algebras. There are some similar constructions in [4], but they are not so natural.

Theorem 3.1. *There exists an embedding of algebras $I\colon D(\mathcal{A}) \hookrightarrow HD(\mathcal{A}) \otimes \check{H}D(\mathcal{A})$ defined on generators by*

$$\mathscr{K}_\alpha^+ \mapsto K_\alpha^+ \otimes \mathcal{K}_\alpha^+, \quad \omega_M^+ \mapsto \sum_{[M_1],[M_2]} v^{\langle M_1, M_2 \rangle} \frac{a_{M_1} a_{M_2}}{a_M} g_{M_1 M_2}^M \mu_{M_1}^+ K_{\widehat{M_2}}^+ \otimes \nu_{M_2}^+,$$

$$\mathscr{K}_\alpha^- \mapsto K_\alpha^- \otimes \mathcal{K}_\alpha^-, \quad \omega_M^- \mapsto \sum_{[M_1],[M_2]} v^{\langle M_2, M_1 \rangle} \frac{a_{M_1} a_{M_2}}{a_M} g_{M_2 M_1}^M \mu_{M_1}^- \otimes \nu_{M_2}^- \mathcal{K}_{\widehat{M_1}}^-.$$

P r o o f. In order to prove that I is a homomorphism of algebras, it suffices to show that the relations from (2.14) to (2.18) are preserved under I. We prove only the relations (2.14) and (2.18), since the other relations can be easily proved.

For the first relation in (2.14),

$$\sum_{[L]} v^{\langle M,N\rangle} g^L_{MN} I(\omega^+_L) = \sum_{[L],[L_1],[L_2]} v^{\langle M,N\rangle+\langle L_1,L_2\rangle} \frac{a_{L_1}a_{L_2}}{a_L} g^L_{MN} g^L_{L_1L_2} \mu^+_{L_1} K^+_{\widehat{L}_2} \otimes \nu^+_{L_2},$$

$$(*) \qquad I(\omega^+_M I(\omega^+_N) = \sum_{[M_1],[M_2],[N_1],[N_2]} v^{\langle M_1,M_2\rangle+\langle N_1,N_2\rangle} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_M a_N}$$
$$\times g^M_{M_1M_2} g^N_{N_1N_2} \mu^+_{M_1} K^+_{\widehat{M}_2} \mu^+_{N_1} K^+_{\widehat{N}_2} \otimes \nu^+_{M_2}\nu^+_{N_2}$$
$$= \sum_{[M_1],[M_2],[N_1],[N_2]} v^{x_0} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_M a_N}$$
$$\times g^M_{M_1M_2} g^N_{N_1N_2} \mu^+_{M_1} \mu^+_{N_1} K^+_{\widehat{M}_2+\widehat{N}_2} \otimes \nu^+_{M_2}\nu^+_{N_2}$$
$$= \sum_{[M_1],[M_2],[N_1],[N_2],[L_1],[L_2]} v^{x_1} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_M a_N}$$
$$\times g^M_{M_1M_2} g^N_{N_1N_2} g^{L_1}_{M_1N_1} g^{L_2}_{M_2N_2} \mu^+_{L_1} K^+_{\widehat{L}_2} \otimes \nu^+_{L_2},$$

where $x_0 = \langle M_1, M_2\rangle + \langle N_1, N_2\rangle + (M_2, N_1)$ and $x_1 = \langle M_1, M_2\rangle + \langle N_1, N_2\rangle + (M_2, N_1) + \langle M_1, N_1\rangle + \langle M_2, N_2\rangle$. For any fixed L_1, L_2, noting that in $(*)$ $\widehat{M} = \widehat{M_1} + \widehat{M_2}$, $\widehat{N} = \widehat{N}_1 + \widehat{N}_2$, $\widehat{L}_i = \widehat{M}_i + \widehat{N}_i$ for $i = 1, 2$, we obtain that $x_1 = \langle M, N\rangle + \langle L_1, L_2\rangle - 2\langle M_1, N_2\rangle$. Thus, by Green's formula, we conclude that

$$\sum_{[M_i],[N_i],i=1,2} v^{x_1} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_M a_N} g^M_{M_1M_2} g^N_{N_1N_2} g^{L_1}_{M_1N_1} g^{L_2}_{M_2N_2}$$
$$= \sum_{[L]} v^{\langle M,N\rangle+\langle L_1,L_2\rangle} \frac{a_{L_1}a_{L_2}}{a_L} g^L_{MN} g^L_{L_1L_2}$$

and thus

$$I(\omega^+_M) I(\omega^+_N) = \sum_{[L]} v^{\langle M,N\rangle} g^L_{MN} I(\omega^+_L).$$

Similarly, we can prove that the second relation in (2.14) is also preserved under I.

Now, we come to prove that the relation in (2.18) is preserved under I. First of all, substituting $\gamma^{XY}_{MN} = (a_X a_Y / a_M a_N) \sum_{[L]} a_L g^M_{LX} g^N_{YL}$ into (2.18), we rewrite (2.18) as

$$\sum_{[X],[Y],[L]} v^{\langle \widehat{L}, \widehat{M}-\widehat{N}\rangle} a_X a_Y a_L g^M_{LX} g^N_{YL} \mathscr{K}^-_{\widehat{L}} \omega^-_Y \omega^+_X$$
$$= \sum_{[X],[Y],[L]} v^{\langle \widehat{L}, \widehat{N}-\widehat{M}\rangle} a_X a_Y a_L g^M_{XL} g^N_{LY} \mathscr{K}^+_{\widehat{L}} \omega^+_X \omega^-_Y.$$

On the one hand,

$$\begin{aligned}
\text{LHS} :=& \sum_{[X],[Y],[L]} v^{\langle \widehat{L}, \widehat{M}-\widehat{N}\rangle} a_X a_Y a_L g_{LX}^M g_{YL}^N I(\mathscr{K}_{\widehat{L}}^-) I(\omega_Y^-) I(\omega_X^+) \\
=& \sum_{\substack{[X],[Y],[L],\\ [Y_1],[Y_2],[X_1],[X_2]}} v^{y_0} a_{X_1} a_{X_2} a_{Y_1} a_{Y_2} a_L g_{LX}^M g_{X_1 X_2}^X g_{Y_2 Y_1}^Y g_{YL}^N K_{\widehat{L}}^- \\
& \times \mu_{Y_1}^- \mu_{X_1}^+ K_{\widehat{X}_2}^+ \otimes \mathcal{K}_{\widehat{L}}^- \nu_{Y_2}^- \mathcal{K}_{\widehat{Y}_1}^- \nu_{X_2}^+ \\
=& \sum_{[L],[X_1],[X_2],[Y_1],[Y_2]} v^{y_1} a_{X_1} a_{X_2} a_{Y_1} a_{Y_2} a_L g_{L X_1 X_2}^M g_{Y_2 Y_1 L}^N K_{\widehat{L}}^- \\
& \times \mu_{Y_1}^- \mu_{X_1}^+ K_{\widehat{X}_2}^+ \otimes \mathcal{K}_{\widehat{Y}_1+\widehat{L}}^- \nu_{Y_2}^- \nu_{X_2}^+,
\end{aligned}$$

where

$$y_0 = \langle \widehat{L}, \widehat{M}-\widehat{N}\rangle + \langle X_1, X_2\rangle + \langle Y_2, Y_1\rangle$$

and

$$y_1 = y_0 - (Y_1, Y_2) = \langle \widehat{L}, \widehat{M}-\widehat{N}\rangle + \langle X_1, X_2\rangle - \langle Y_1, Y_2\rangle.$$

By (2.12),

$$\begin{aligned}
\nu_{Y_2}^- \nu_{X_2}^+ =& \sum_{[A],[B]} v^{\langle \widehat{Y}_2-\widehat{B}, \widehat{B}-\widehat{A}\rangle} \gamma_{Y_2 X_2}^{BA} \mathcal{K}_{\widehat{Y}_2-\widehat{B}}^+ \nu_A^+ \nu_B^- \\
=& \sum_{[A],[B],[C]} v^{\langle \widehat{C}, \widehat{B}-\widehat{A}\rangle} \frac{a_A a_B a_C}{a_{X_2} a_{Y_2}} g_{CB}^{Y_2} g_{AC}^{X_2} \mathcal{K}_{\widehat{C}}^+ \nu_A^+ \nu_B^-.
\end{aligned}$$

Thus

$$\begin{aligned}
\text{LHS} =& \sum_{[L],[X_1],[Y_1],[A],[B],[C]} v^{y_2} a_L a_{X_1} a_A a_C a_B a_{Y_1} g_{L X_1 A C}^M g_{C B Y_1 L}^N \\
& \times K_{\widehat{L}}^- \mu_{Y_1}^- \mu_{X_1}^+ K_{\widehat{A}+\widehat{C}}^+ \otimes \mathcal{K}_{\widehat{Y}_1+\widehat{L}}^- \mathcal{K}_{\widehat{C}}^+ \nu_A^+ \nu_B^- \\
=& \sum_{[L],[X_1],[Y_1],[A],[B],[C]} v^{y_3} a_L a_{X_1} a_A a_C a_B a_{Y_1} g_{L X_1 A C}^M g_{C B Y_1 L}^N \\
& \times K_{\widehat{L}}^- \mu_{Y_1}^- \mu_{X_1}^+ K_{\widehat{A}+\widehat{C}}^+ \otimes \mathcal{K}_{\widehat{C}}^+ \nu_A^+ \nu_B^- \mathcal{K}_{\widehat{Y}_1+\widehat{L}}^-,
\end{aligned}$$

where

$$y_2 = y_1 + \langle \widehat{C}, \widehat{B}-\widehat{A}\rangle = \langle \widehat{L}, \widehat{M}-\widehat{N}\rangle + \langle \widehat{X}_1, \widehat{A}+\widehat{C}\rangle - \langle \widehat{Y}_1, \widehat{B}+\widehat{C}\rangle + \langle \widehat{C}, \widehat{B}-\widehat{A}\rangle$$

and

$$y_3 = \langle \widehat{L}, \widehat{M}-\widehat{N}\rangle + \langle \widehat{X}_1, \widehat{A}+\widehat{C}\rangle - \langle \widehat{Y}_1, \widehat{B}+\widehat{C}\rangle + \langle \widehat{C}, \widehat{B}-\widehat{A}\rangle + (\widehat{L}+\widehat{Y}_1, \widehat{B}+\widehat{C}).$$

On the other hand,

$$\begin{aligned}
\mathrm{RHS} :=& \sum_{[X],[Y],[L]} v^{\langle \widehat{L}, \widehat{N}-\widehat{M}\rangle} a_X a_Y a_L g_{XL}^M g_{LY}^N I(\mathscr{K}_{\widehat{L}}^+) I(\omega_X^+) I(\omega_Y^-) \\
=& \sum_{\substack{[X],[Y],[L],\\ [X_1],[X_2],[Y_1],[Y_2]}} v^{z_0} a_{X_1} a_{X_2} a_{Y_1} a_{Y_2} a_L g_{X_1X_2}^X g_{XL}^M g_{LY}^N g_{Y_2Y_1}^Y \\
& \times K_{\widehat{L}}^+ \mu_{X_1}^+ K_{\widehat{X}_2}^+ \mu_{Y_1}^- \otimes \mathcal{K}_{\widehat{L}}^+ \nu_{X_2}^+ \nu_{Y_2}^- \mathcal{K}_{\widehat{Y}_1}^- \\
=& \sum_{[L],[X_1],[X_2],[Y_1],[Y_2]} v^{z_1} a_{X_1} a_{X_2} a_{Y_1} a_{Y_2} a_L g_{X_1X_2L}^M g_{LY_2Y_1}^N \\
& \times K_{\widehat{X}_2+\widehat{L}}^+ \mu_{X_1}^+ \mu_{Y_1}^- \otimes \mathcal{K}_{\widehat{L}}^+ \nu_{X_2}^+ \nu_{Y_2}^- \mathcal{K}_{\widehat{Y}_1}^-,
\end{aligned}$$

where $z_0 = \langle \widehat{L}, \widehat{N}-\widehat{M}\rangle + \langle X_1, X_2\rangle + \langle Y_2, Y_1\rangle$ and $z_1 = z_0 - (X_1, X_2) = \langle \widehat{L}, \widehat{N}-\widehat{M}\rangle + \langle Y_2, Y_1\rangle - \langle X_2, X_1\rangle$. By (2.7),

$$\begin{aligned}
\mu_{X_1}^+ \mu_{Y_1}^- =& \sum_{[A],[B]} v^{\langle \widehat{Y}_1 - \widehat{B}, \widehat{A}-\widehat{B}\rangle} \gamma_{X_1Y_1}^{AB} K_{\widehat{Y}_1-\widehat{B}}^- \mu_B^- \mu_A^+ \\
=& \sum_{[A],[B],[C]} v^{\langle \widehat{C}, \widehat{A}-\widehat{B}\rangle} \frac{a_A a_B a_C}{a_{X_1} a_{Y_1}} g_{CA}^{X_1} g_{BC}^{Y_1} K_{\widehat{C}}^- \mu_B^- \mu_A^+.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{RHS} =& \sum_{[L],[X_2],[Y_2],[A],[B],[C]} v^{z_2} a_C a_A a_{X_2} a_L a_{Y_2} a_B g_{CAX_2L}^M g_{LY_2BC}^N \\
& \times K_{\widehat{X}_2+\widehat{L}}^+ K_{\widehat{C}}^- \mu_B^- \mu_A^+ \otimes \mathcal{K}_{\widehat{L}}^+ \nu_{X_2}^+ \nu_{Y_2}^- \mathcal{K}_{\widehat{B}+\widehat{C}}^- \\
=& \sum_{[L],[X_2],[Y_2],[A],[B],[C]} v^{z_3} a_C a_A a_{X_2} a_L a_{Y_2} a_B g_{CAX_2L}^M g_{LY_2BC}^N \\
& \times K_{\widehat{C}}^- \mu_B^- \mu_A^+ K_{\widehat{X}_2+\widehat{L}}^+ \otimes \mathcal{K}_{\widehat{L}}^+ \nu_{X_2}^+ \nu_{Y_2}^- \mathcal{K}_{\widehat{B}+\widehat{C}}^-,
\end{aligned}$$

where $z_2 = z_1 + \langle \widehat{C}, \widehat{A}-\widehat{B}\rangle = \langle \widehat{L}, \widehat{N}-\widehat{M}\rangle + \langle \widehat{Y}_2, \widehat{B}+\widehat{C}\rangle - \langle \widehat{X}_2, \widehat{A}+\widehat{C}\rangle + \langle \widehat{C}, \widehat{A}-\widehat{B}\rangle$ and $z_3 = \langle \widehat{L}, \widehat{N}-\widehat{M}\rangle + \langle \widehat{Y}_2, \widehat{B}+\widehat{C}\rangle - \langle \widehat{X}_2, \widehat{A}+\widehat{C}\rangle + \langle \widehat{C}, \widehat{A}-\widehat{B}\rangle + (\widehat{L}+\widehat{X}_2, \widehat{A}+\widehat{C}$. Identifying L, X_1, A, C, B, Y_1 in LHS with C, A, X_2, L, Y_2, B in RHS, respectively, we obtain that $y_3 = \langle \widehat{C}, \widehat{M}-\widehat{N}\rangle + \langle \widehat{A}, \widehat{X}_2+\widehat{L}\rangle - \langle \widehat{B}, \widehat{Y}_2+\widehat{L}\rangle + \langle \widehat{L}, \widehat{Y}_2-\widehat{X}_2\rangle + (\widehat{B}+\widehat{C}, \widehat{Y}_2+\widehat{L})$. Noting that in RHS $\widehat{M}-\widehat{N} = \widehat{X}-\widehat{Y} = (\widehat{X}_1-\widehat{Y}_1) + (\widehat{X}_2-\widehat{Y}_2) = (\widehat{A}-\widehat{B}) + (\widehat{X}_2-\widehat{Y}_2)$, we have that

$$\begin{aligned}
y_3 =& \langle \widehat{C}, \widehat{A}-\widehat{B}\rangle + \langle \widehat{C}, \widehat{X}_2\rangle - \langle \widehat{C}, \widehat{Y}_2\rangle + \langle \widehat{A}, \widehat{X}_2\rangle + \langle \widehat{A}, \widehat{L}\rangle - \langle \widehat{B}, \widehat{Y}_2+\widehat{L}\rangle \\
&+ \langle \widehat{L}, \widehat{Y}_2-\widehat{X}_2\rangle + (\widehat{C}, \widehat{L}) + \langle \widehat{C}, \widehat{Y}_2\rangle + \langle \widehat{Y}_2, \widehat{C}\rangle + \langle \widehat{B}, \widehat{Y}_2+\widehat{L}\rangle + \langle \widehat{Y}_2, \widehat{B}\rangle + \langle \widehat{L}, \widehat{B}\rangle \\
=& \langle \widehat{C}, \widehat{A}-\widehat{B}\rangle + \langle \widehat{A}+\widehat{C}, \widehat{X}_2\rangle + \langle \widehat{A}, \widehat{L}\rangle + \langle \widehat{L}, \widehat{Y}_2-\widehat{X}_2\rangle + (\widehat{C}, \widehat{L}) \\
&+ \langle \widehat{Y}_2, \widehat{B}+\widehat{C}\rangle + \langle \widehat{L}, \widehat{B}\rangle
\end{aligned}$$

and

$$\begin{aligned} z_3 &= \langle \widehat{L}, \widehat{B}\rangle - \langle \widehat{L}, \widehat{A}\rangle + \langle \widehat{L}, \widehat{Y}_2 - \widehat{X}_2\rangle + \langle \widehat{Y}_2, \widehat{B} + \widehat{C}\rangle - \langle \widehat{X}_2, \widehat{A} + \widehat{C}\rangle + \langle \widehat{C}, \widehat{A} - \widehat{B}\rangle \\ &\quad + (\widehat{C}, \widehat{L}) + \langle \widehat{L}, \widehat{A}\rangle + \langle \widehat{A}, \widehat{L}\rangle + \langle \widehat{X}_2, \widehat{A} + \widehat{C}\rangle + \langle \widehat{A} + \widehat{C}, \widehat{X}_2\rangle \\ &= \langle \widehat{L}, \widehat{B}\rangle + \langle \widehat{L}, \widehat{Y}_2 - \widehat{X}_2\rangle + \langle \widehat{Y}_2, \widehat{B} + \widehat{C}\rangle + \langle \widehat{C}, \widehat{A} - \widehat{B}\rangle + (\widehat{C}, \widehat{L}) + \langle \widehat{A}, \widehat{L}\rangle \\ &\quad + \langle \widehat{A} + \widehat{C}, \widehat{X}_2\rangle = y_3. \end{aligned}$$

Hence, LHS = RHS and we have proved that I is a homomorphism of algebras.

Since $D(\mathcal{A}) \cong H^+(\mathcal{A}) \otimes H^-(\mathcal{A})$ as a vector space and the restriction of I to the positive (negative) part is injective, we conclude that I is injective. Therefore, we have completed the proof. □

4. Applications

In this section, we apply Theorem 3.1 to Bridgeland's Hall algebras of m-cyclic complexes and derived Hall algebras.

4.1. Bridgeland's Hall algebras. Assume that $\mathcal{A}$ has enough projectives, Bridgeland's Hall algebra of 2-cyclic complexes of $\mathcal{A}$ was introduced in [1]. Inspired by the work of Bridgeland, for each nonnegative integer $m \neq 1$, Chen and Deng in [2] introduced Bridgeland's Hall algebra $\mathcal{DH}_m(\mathcal{A})$ of m-cyclic complexes. For $m = 0$ or $m > 2$, we recall the algebra structure of $\mathcal{DH}_m(\mathcal{A})$ by [17] as follows:

Proposition 4.1 ([17]). *Let $m = 0$ or $m > 2$. Then $\mathcal{DH}_m(\mathcal{A})$ is an associative and unital $\mathbb{C}$-algebra generated by the elements in $\{e_{M,i}\colon [M] \in \mathrm{Iso}(\mathcal{A}), i \in \mathbb{Z}_m\}$ and $\{K_{\alpha,i}\colon \alpha \in K(\mathcal{A}), i \in \mathbb{Z}_m\}$, and the relations*

$$K_{\alpha,i}K_{\beta,i} = K_{\alpha+\beta,i}, \quad K_{\alpha,i}K_{\beta,j} = \begin{cases} v^{(\alpha,\beta)}K_{\beta,j}K_{\alpha,i}, & i = j+1, \\ v^{-(\alpha,\beta)}K_{\beta,j}K_{\alpha,i}, & i = m-1+j, \\ K_{\beta,j}K_{\alpha,i}, & \text{otherwise}, \end{cases} \tag{4.1}$$

$$K_{\alpha,i}e_{M,j} = \begin{cases} v^{(\alpha,\widehat{M})}e_{M,j}K_{\alpha,i}, & i = j, \\ v^{-(\alpha,\widehat{M})}e_{M,j}K_{\alpha,i}, & i = m-1+j, \\ e_{M,j}K_{\alpha,i}, & \text{otherwise}, \end{cases} \tag{4.2}$$

$$e_{M,i}e_{N,i} = \sum_{[L]} v^{\langle M,N\rangle} g_{MN}^{L} e_{L,i}, \tag{4.3}$$

$$e_{M,i+1}e_{N,i} = \sum_{[X],[Y]} v^{\langle \widehat{M}-\widehat{X}, \widehat{X}-\widehat{Y}\rangle}\gamma_{MN}^{XY} K_{\widehat{M}-\widehat{X},i}e_{Y,i}e_{X,i+1}, \tag{4.4}$$

$$e_{M,i}e_{N,j} = e_{N,j}e_{M,i}, \quad i-j \neq 0, 1 \text{ or } m-1. \tag{4.5}$$

Corollary 4.2. *Let $m = 0$ or $m > 2$. Then for each $i \in \mathbb{Z}_m$,*

(1) *there exists an embedding of algebras $\kappa_i\colon\ HD(\mathcal{A}) \hookrightarrow \mathcal{DH}_m(\mathcal{A})$ defined on generators by*

$$K_\alpha^+ \mapsto K_{\alpha,i+1}, \quad K_\alpha^- \mapsto K_{\alpha,i}, \quad \mu_M^+ \mapsto e_{M,i+1}, \quad \mu_M^- \mapsto e_{M,i},$$

(2) *there exists an embedding of algebras $\check{\kappa}_i\colon\ \check{H}D(\mathcal{A}) \hookrightarrow \mathcal{DH}_m(\mathcal{A})$ defined on generators by*

$$\mathcal{K}_\alpha^+ \mapsto K_{\alpha,i}, \quad \mathcal{K}_\alpha^- \mapsto K_{\alpha,i+1}, \quad \nu_M^+ \mapsto e_{M,i}, \quad \nu_M^- \mapsto e_{M,i+1}.$$

P r o o f. By Proposition 4.1 the defining relations of $HD(\mathcal{A})$ and $\check{H}D(\mathcal{A})$ are preserved under κ_i and $\check{\kappa}_i$, respectively, and we obtain that κ_i and $\check{\kappa}_i$ are homomorphisms of algebras. According to [17], Proposition 2.7, we conclude that they are injective. □

As a first application of Theorem 3.1, we have the following:

Theorem 4.3. *Let $m = 0$ or $m > 2$. Then for each $i \in \mathbb{Z}_m$, there exists an embedding of algebras $\psi_i\colon\ D(\mathcal{A}) \hookrightarrow \mathcal{DH}_m(\mathcal{A}) \otimes \mathcal{DH}_m(\mathcal{A})$ defined on generators by*

$$\mathscr{K}_\alpha^+ \mapsto K_{\alpha,i+1} \otimes K_{\alpha,i}, \quad \omega_M^+ \mapsto \sum_{[M_1],[M_2]} v^{\langle M_1,M_2\rangle} \frac{a_{M_1}a_{M_2}}{a_M} g_{M_1M_2}^M e_{M_1,i+1} K_{\widehat{M_2},i+1} \otimes e_{M_2,i}$$

and

$$\mathscr{K}_\alpha^- \mapsto K_{\alpha,i} \otimes K_{\alpha,i+1}, \quad \omega_M^- \mapsto \sum_{[M_1],[M_2]} v^{\langle M_2,M_1\rangle} \frac{a_{M_1}a_{M_2}}{a_M} g_{M_2M_1}^M e_{M_1,i} \otimes e_{M_2,i+1} K_{\widehat{M_1},i+1}.$$

P r o o f. For each $i \in \mathbb{Z}_m$, by the commutative diagram

$$\begin{array}{ccc} D(\mathcal{A}) & \xhookrightarrow{\ I\ } & HD(\mathcal{A}) \otimes \check{H}D(\mathcal{A}) \\ & \searrow^{\psi_i} & \Big\downarrow{\scriptstyle \kappa_i \otimes \check{\kappa}_i} \\ & & \mathcal{DH}_m(\mathcal{A}) \otimes \mathcal{DH}_m(\mathcal{A}) \end{array}$$

we complete the proof. □

Remark 4.4. As mentioned in Introduction, there is an isomorphism $\varrho\colon D(\mathcal{A}) \to \mathcal{DH}_2(\mathcal{A})$, which is defined on generators by

$$\omega_M^+ \mapsto \frac{E_M}{a_M}, \quad \omega_M^- \mapsto \frac{F_M}{a_M}, \quad \mathscr{K}_\alpha^+ \mapsto K_\alpha, \quad \mathscr{K}_\alpha^- \mapsto K_\alpha^*,$$

where the notations E_M, F_M, K_α and K_α^* are the same as those in [16]. Hence, Theorem 4.3 establishes a relation between Bridgeland's Hall algebra of 2-cyclic complexes and that of m-cyclic complexes.

4.2. Derived Hall algebras. The derived Hall algebra $\mathcal{DH}(\mathcal{A})$ of the bounded derived category of $\mathcal{A}$ was introduced in [12] (see also [14]).

Proposition 4.5 ([12]). *The derived Hall algebra $\mathcal{DH}(\mathcal{A})$ is an associative and unital $\mathbb{C}$-algebra generated by the elements in $\{Z_M^{[i]}\colon [M] \in \mathrm{Iso}(\mathcal{A}),\ i \in \mathbb{Z}\}$ and the relations*

$$Z_M^{[i]} Z_N^{[i]} = \sum_{[L]} g_{MN}^L Z_L^{[i]}, \tag{4.6}$$

$$Z_M^{[i+1]} Z_N^{[i]} = \sum_{[X],[Y]} q^{-\langle Y,X\rangle} \gamma_{MN}^{XY} Z_Y^{[i]} Z_X^{[i+1]}, \tag{4.7}$$

$$Z_M^{[i]} Z_N^{[j]} = q^{(-1)^{i-j}\langle N,M\rangle} Z_N^{[j]} Z_M^{[i]}, \quad i-j>1. \tag{4.8}$$

According to [11], we twist the multiplication in $\mathcal{DH}(\mathcal{A})$ as

$$Z_M^{[i]} * Z_N^{[j]} = v^{(-1)^{i-j}\langle M,N\rangle} Z_M^{[i]} Z_N^{[j]}. \tag{4.9}$$

The *twisted derived Hall algebra* $\mathcal{DH}_{\mathrm{tw}}(\mathcal{A})$ is the same vector space as $\mathcal{DH}(\mathcal{A})$, but with the twisted multiplication. In order to relate the modified Ringel-Hall algebra, which is isomorphic to corresponding Bridgeland's Hall algebra if $\mathcal{A}$ has enough projectives, to the derived Hall algebra, Lin in [7] introduced the *completely extended twisted derived Hall algebra* $\mathcal{DH}_{\mathrm{tw}}^{\mathrm{ce}}(\mathcal{A})$.

Definition 4.6 ([7]). The completely extended twisted derived Hall algebra $\mathcal{DH}_{\mathrm{tw}}^{\mathrm{ce}}(\mathcal{A})$ is the associative and unital $\mathbb{C}$-algebra generated by the elements in $\{Z_M^{[i]}\colon [M] \in \mathrm{Iso}(\mathcal{A}),\ i \in \mathbb{Z}\}$ and $\{K_\alpha^{[i]}\colon\ \alpha \in K(\mathcal{A}),\ i \in \mathbb{Z}\}$, and the relations

$$K_\alpha^{[i]} K_\beta^{[i]} = K_{\alpha+\beta}^{[i]}, \quad K_\alpha^{[i]} Z_M^{[i]} = \begin{cases} v^{(\alpha,\widehat{M})} Z_M^{[i]} K_\alpha^{[i]}, & i=-1,0, \\ Z_M^{[i]} K_\alpha^{[i]}, & \text{otherwise}, \end{cases} \tag{4.10}$$

$$K_\alpha^{[i+1]} K_\beta^{[i]} = v^{(\alpha,\beta)} K_\beta^{[i]} K_\alpha^{[i+1]}, \quad K_\alpha^{[i]} K_\beta^{[j]} = K_\beta^{[j]} K_\alpha^{[i]}, \quad |i-j|>1, \tag{4.11}$$

$$K_\alpha^{[i]} Z_M^{[i+1]} = \begin{cases} v^{-(\alpha,\widehat{M})} Z_M^{[i+1]} K_\alpha^{[i]}, & i=-1,0, \\ Z_M^{[i+1]} K_\alpha^{[i]}, & \text{otherwise}, \end{cases} \tag{4.12}$$

$$K_\alpha^{[i]} Z_M^{[i-1]} = \begin{cases} v^{-(\alpha,\widehat{M})} Z_M^{[i-1]} K_\alpha^{[i]}, & i=-1,0, \\ Z_M^{[i-1]} K_\alpha^{[i]}, & \text{otherwise}, \end{cases} \tag{4.13}$$

For any $|i-j|>1$,

$$K_\alpha^{[i]} Z_M^{[j]} = \begin{cases} v^{(-1)^j(\alpha,\widehat{M})} Z_M^{[j]} K_\alpha^{[i]}, & i=0 \text{ and } |j|>1, \\ v^{(-1)^{j+1}(\alpha,\widehat{M})} Z_M^{[j]} K_\alpha^{[i]}, & i=-1 \text{ and } |j+1|>1, \\ Z_M^{[j]} K_\alpha^{[i]}, & \text{otherwise}, \end{cases} \tag{4.14}$$

$$Z_M^{[i]} Z_N^{[i]} = \sum_{[L]} v^{\langle M,N\rangle} g_{MN}^L Z_L^{[i]}, \tag{4.15}$$

$$Z_M^{[i+1]} Z_N^{[i]} = \sum_{[X],[Y]} v^{-\langle M,N\rangle - \langle Y,X\rangle} \gamma_{MN}^{XY} Z_Y^{[i]} Z_X^{[i+1]}, \tag{4.16}$$

$$Z_M^{[i]} Z_N^{[j]} = v^{(-1)^{i-j}(M,N)} Z_N^{[j]} Z_M^{[i]}, \quad i-j>1. \tag{4.17}$$

Remark 4.7. In Definition 4.6, we have employed the linear Euler form, not the multiplicative Euler form used in [7]; $K_\alpha^{[i]}$ and $Z_M^{[i]}$ here are equal to $K_\alpha^{[-i]}$ and $Z_M^{[-i]}$ in [7], respectively.

Now we reformulate Theorem 5.3 and Corollary 5.5 in [7] as follows:

Theorem 4.8. *Assume that $\mathcal{A}$ has enough projectives. Then there exists an isomorphism of algebras $\phi\colon \mathcal{DH}_{\mathrm{tw}}^{\mathrm{ce}}(\mathcal{A}) \to \mathcal{DH}_0(\mathcal{A})$ defined on generators (with $n>0$) by*

$$Z_M^{[0]} \mapsto e_{M,0}, \quad K_\alpha^{[n]} \mapsto K_{\alpha,n},$$

$$Z_M^{[n]} \mapsto v^{n\langle M,M\rangle} e_{M,n} \prod_{i=1}^{n} K_{(-1)^i\widehat{M},n-i}, \quad Z_M^{[-n]} \mapsto v^{-n\langle M,M\rangle} e_{M,-n} \prod_{i=0}^{n-1} K_{(-1)^{i+1}\widehat{M},i-n}.$$

Remark 4.9.

(1) The inverse of ϕ in Theorem 4.8 is the homomorphism $\phi^{-1}\colon \mathcal{DH}_0(\mathcal{A}) \to \mathcal{DH}_{\mathrm{tw}}^{\mathrm{ce}}(\mathcal{A})$ defined on generators (with $n>0$) by

$$e_{M,0} \mapsto Z_M^{[0]}, \quad K_{\alpha,n} \mapsto K_\alpha^{[n]},$$

$$e_{M,n} \mapsto v^{-n\langle M,M\rangle} Z_M^{[n]} \prod_{i=0}^{n-1} K_{(-1)^{n-i-1}\widehat{M},i}, \quad e_{M,-n} \mapsto v^{n\langle M,M\rangle} Z_M^{[-n]} \prod_{i=1}^{n} K_{(-1)^{n-i}\widehat{M},-i}.$$

(2) Theorem 4.8 establishes the relation between Bridgeland's Hall algebra of bounded complexes over projectives of $\mathcal{A}$ and the derived Hall algebra of the bounded derived category $D^b(\mathcal{A})$. In other words, one can realize the derived Hall algebra via Bridgeland's construction.

As a second application of Theorem 3.1, we have the following

Theorem 4.10. *For each $i \in \mathbb{Z}$, there exists an embedding of algebras φ_i: $D(\mathcal{A}) \hookrightarrow \mathcal{DH}_{\mathrm{tw}}^{\mathrm{ce}}(\mathcal{A}) \otimes \mathcal{DH}_{\mathrm{tw}}^{\mathrm{ce}}(\mathcal{A})$. Explicitly,*

(1) *if $i=-1$, φ_i is defined on generators by*

$$\mathscr{K}_\alpha^+ \mapsto K_\alpha^{[0]} \otimes K_\alpha^{[-1]}, \ \omega_M^+ \mapsto \sum_{[M_1],[M_2]} v^{\langle M,M_2\rangle} \frac{a_{M_1}a_{M_2}}{a_M} g_{M_1M_2}^M Z_{M_1}^{[0]} K_{\widehat{M_2}}^{[0]} \otimes Z_{M_2}^{[-1]} K_{\widehat{M_2}}^{[-1]},$$

$$\mathscr{K}_\alpha^- \mapsto K_\alpha^{[-1]} \otimes K_\alpha^{[0]}, \ \omega_M^- \mapsto \sum_{[M_1],[M_2]} v^{\langle M,M_1\rangle} \frac{a_{M_1}a_{M_2}}{a_M} g_{M_2M_1}^M Z_{M_1}^{[-1]} K_{\widehat{M_1}}^{[-1]} \otimes Z_{M_2}^{[0]} K_{\widehat{M_1}}^{[0]},$$

(2) *if* $i=0$, φ_i *is defined on generators by*

$$\mathscr{K}_\alpha^+ \mapsto K_\alpha^{[1]} \otimes K_\alpha^{[0]},\ \omega_M^+ \mapsto \sum_{[M_1],[M_2]} v^{-\langle \widehat{M},\widehat{M_1}\rangle} \frac{a_{M_1}a_{M_2}}{a_M} g^M_{M_1M_2} Z^{[1]}_{M_1} K^{[1]}_{\widehat{M_2}} K^{[0]}_{\widehat{M_1}} \otimes Z^{[0]}_{M_2},$$

$$\mathscr{K}_\alpha^- \mapsto K_\alpha^{[0]} \otimes K_\alpha^{[1]},\ \omega_M^- \mapsto \sum_{[M_1],[M_2]} v^{-\langle \widehat{M},\widehat{M_2}\rangle} \frac{a_{M_1}a_{M_2}}{a_M} g^M_{M_2M_1} Z^{[0]}_{M_1} \otimes Z^{[1]}_{M_2} K^{[1]}_{\widehat{M_1}} K^{[0]}_{\widehat{M_2}},$$

(3) *if* $i<-1$, φ_i *is defined on generators by*

$$\mathscr{K}_\alpha^+ \mapsto K_\alpha^{[i+1]} \otimes K_\alpha^{[i]}, \quad \mathscr{K}_\alpha^- \mapsto K_\alpha^{[i]} \otimes K_\alpha^{[i+1]},$$

$\omega_M^+ \mapsto$

$$\sum_{[M_1],[M_2]} v^{x} \frac{a_{M_1}a_{M_2}}{a_M} g^M_{M_1M_2} Z^{[i+1]}_{M_1} \prod_{j=1}^{-(i+1)} K^{[-j]}_{(-1)^{i+j+1}\widehat{M_1}} K^{[i+1]}_{\widehat{M_2}} \otimes Z^{[i]}_{M_2} \prod_{j=1}^{-i} K^{[-j]}_{(-1)^{i+j}\widehat{M_2}},$$

$\omega_M^- \mapsto$

$$\sum_{[M_1],[M_2]} v^{y} \frac{a_{M_1}a_{M_2}}{a_M} g^M_{M_2M_1} Z^{[i]}_{M_1} \prod_{j=1}^{-i} K^{[-j]}_{(-1)^{i+j}\widehat{M_1}} \otimes Z^{[i+1]}_{M_2} \prod_{j=1}^{-(i+1)} K^{[-j]}_{(-1)^{i+j+1}\widehat{M_2}} K^{[i+1]}_{\widehat{M_1}},$$

where $x=\langle \widehat{M_1}, \widehat{M_2}-\widehat{M_1}\rangle - i(\langle M_1,M_1\rangle + \langle M_2,M_2\rangle)$ *and* $y=\langle \widehat{M_2}, \widehat{M_1}-\widehat{M_2}\rangle - i(\langle M_1,M_1\rangle + \langle M_2,M_2\rangle)$.

(4) *if* $i>0$, φ_i *is defined on generators by*

$$\mathscr{K}_\alpha^+ \mapsto K_\alpha^{[i+1]} \otimes K_\alpha^{[i]}, \quad \mathscr{K}_\alpha^- \mapsto K_\alpha^{[i]} \otimes K_\alpha^{[i+1]},$$

$\omega_M^+ \mapsto$

$$\sum_{[M_1],[M_2]} v^{x} \frac{a_{M_1}a_{M_2}}{a_M} g^M_{M_1M_2} Z^{[i+1]}_{M_1} \prod_{j=0}^{i} K^{[j]}_{(-1)^{i-j}\widehat{M_1}} K^{[i+1]}_{\widehat{M_2}} \otimes Z^{[i]}_{M_2} \prod_{j=0}^{i-1} K^{[j]}_{(-1)^{i-j-1}\widehat{M_2}},$$

$\omega_M^- \mapsto$

$$\sum_{[M_1],[M_2]} v^{y} \frac{a_{M_1}a_{M_2}}{a_M} g^M_{M_2M_1} Z^{[i]}_{M_1} \prod_{j=0}^{i-1} K^{[j]}_{(-1)^{i-j-1}\widehat{M_1}} \otimes Z^{[i+1]}_{M_2} \prod_{j=0}^{i} K^{[j]}_{(-1)^{i-j}\widehat{M_2}} K^{[i+1]}_{\widehat{M_1}},$$

where $x=\langle \widehat{M_1}, \widehat{M_2}-\widehat{M_1}\rangle - i(\langle M_1,M_1\rangle + \langle M_2,M_2\rangle)$ *and* $y=\langle \widehat{M_2}, \widehat{M_1}-\widehat{M_2}\rangle - i(\langle M_1,M_1\rangle + \langle M_2,M_2\rangle)$.

P r o o f. By the commutative diagram

$$\begin{array}{ccc} D(\mathcal{A}) & \xhookrightarrow{\psi_i} & \mathcal{DH}_0(\mathcal{A}) \otimes \mathcal{DH}_0(\mathcal{A}) \\ & \searrow^{\varphi_i} & \cong \big\downarrow \phi^{-1}\otimes\phi^{-1} \\ & & \mathcal{DH}^{\rm ce}_{\rm tw}(\mathcal{A}) \otimes \mathcal{DH}^{\rm ce}_{\rm tw}(\mathcal{A}) \end{array}$$

we complete the proof. □

Acknowledgments. The authors are grateful to the anonymous referees for helpful comments.

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