Czechoslovak Mathematical Journal

Van An Le

Carleson measures and Toeplitz operators on small Bergman spaces on the ball

Czechoslovak Mathematical Journal, Vol. 71 (2021), No. 1, 211–229

Persistent URL: `http://dml.cz/dmlcz/148736`

Czechoslovak Mathematical Journal, 71 (146) (2021), 211–229

CARLESON MEASURES AND TOEPLITZ OPERATORS ON SMALL BERGMAN SPACES ON THE BALL

Van An Le, Marseille

Received June 10, 2019. Published online September 16, 2020.

Abstract. We study Carleson measures and Toeplitz operators on the class of so-called small weighted Bergman spaces, introduced recently by Seip. A characterization of Carleson measures is obtained which extends Seip's results from the unit disk of $\mathbb{C}$ to the unit ball of $\mathbb{C}^n$. We use this characterization to give necessary and sufficient conditions for the boundedness and compactness of Toeplitz operators. Finally, we study the Schatten p classes membership of Toeplitz operators for $1 < p < \infty$.

Keywords: Bergman space; Carleson measure; Toeplitz operator; Schatten classes

MSC 2020: 30H20, 47B35

1. Introduction

Let $\mathbb{C}^n$ denote the n-dimensional complex Euclidean space, $\mathbb{B}_n = \{z \in \mathbb{C}^n\colon\ |z| < 1\}$ be the unit ball and $\mathbb{S}_n = \{z \in \mathbb{C}^n\colon\ |z| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Denote by $H(\mathbb{B}_n)$ the space of all holomorphic functions on the unit ball $\mathbb{B}_n$. Let $\mathrm{d}v$ be the normalized volume measure on $\mathbb{B}_n$. The normalized surface measure on $\mathbb{S}_n$ is denoted by $\mathrm{d}\sigma$.

Let ϱ be a positive continuous and integrable function on $[0, 1)$. We extend it to $\mathbb{B}_n$ by $\varrho(z) = \varrho(|z|)$ and call such ϱ a weight function. The weighted Bergman space A^2_ϱ is the space of functions f in $H(\mathbb{B}_n)$ such that

$$\|f\|^2_\varrho = \int_{\mathbb{B}_n} |f(z)|^2 \varrho(z)\,\mathrm{d}v(z) < \infty.$$

Note that A^2_ϱ is a closed subspace of $L^2(\mathbb{B}_n, \varrho\,\mathrm{d}v)$ and hence it is a Hilbert space endowed with the inner product

$$\langle f, g\rangle_\varrho = \int_{\mathbb{B}_n} f(z)\overline{g(z)}\varrho(z)\,\mathrm{d}v(z), \quad f, g \in A^2_\varrho.$$

DOI: 10.21136/CMJ.2020.0265-19

When $\varrho(r) = (1-r^2)^\alpha$, $\alpha > -1$, we obtain the standard Bergman spaces A^2_α.

We impose a normalization condition on ϱ:

$$\int_0^1 x^{2n-1}\varrho(x)\,\mathrm{d}x = 1.$$

Consider the points $r_k \in [0,1)$ determined by the relation

$$\int_{r_k}^1 \varrho(x)\,\mathrm{d}x = 2^{-k}.$$

Denote by S the class of weights ϱ such that

$$\inf_k \frac{1-r_k}{1-r_{k+1}} > 1. \tag{1.1}$$

Since the function

$$\Phi_f(r) = \int_{\mathbb{S}_n} |f(r\xi)|^2\,\mathrm{d}\sigma(\xi)$$

is non-decreasing, we also have the equivalent norm

$$\|f\|^2_\varrho \asymp \sum_{k=1}^\infty 2^{-k}\int_{\mathbb{S}_n} |f(r_k\xi)|^2\,\mathrm{d}\sigma(\xi), \quad f \in A^2_\varrho. \tag{1.2}$$

The class S was introduced by Kristian Seip in [13]. It is easy to see that the functions

$$\varrho(x) = (1-x)^{-\beta}, \quad 0 < \beta < 1,$$

and

$$\varrho(x) = (1-x)^{-1}\Big(\log\frac{1}{1-x}\Big)^{-\alpha}, \quad 1 < \alpha < \infty,$$

belong to S.

In this paper we prove a characterization of the Carleson measure for weighted Bergman spaces A^2_ϱ with $\varrho \in S$. This result is then used to study spectral properties of Toeplitz operators on these spaces.

Let μ be a finite positive Borel measure on $\mathbb{B}_n$. We say that μ is a Carleson measure for a Hilbert space X of analytic functions in $\mathbb{B}_n$ if there exists a positive constant C such that

$$\int_{\mathbb{B}_n} |f(z)|^2\,\mathrm{d}\mu(z) \leqslant C\|f\|^2_X, \quad f \in X.$$

It is clear that μ is a Carleson measure for A^2_ϱ if and only if $A^2_\varrho \subset L^2(\mathbb{B}_n, \mathrm{d}\mu)$ and the identity operator $\mathrm{Id}\colon A^2_\varrho \to L^2(\mathbb{B}_n, \mathrm{d}\mu)$ is bounded. The Carleson constant of μ, denoted by $\mathcal{C}_\mu(A^2_\varrho)$, is the norm of this identity operator Id. Suppose that μ is a Carleson measure for A^2_ϱ. We say that μ is a vanishing Carleson measure for A^2_ϱ if the above identity operator Id is compact. That is,

$$\lim_{k\to\infty}\int_{\mathbb{B}_n}|f_k(z)|^2\,\mathrm{d}\mu(z)=0$$

whenever $\{f_k\}$ is a bounded sequence in A^2_ϱ which converges to 0 uniformly on compact subsets of $\mathbb{B}_n$.

The concept of Carleson measure was first introduced by Carleson (see [2], [3]) in order to study interpolating sequences and the corona problem on the algebra H^∞ of all bounded analytic functions on the unit disk. It has quickly become a powerful tool for the study of function spaces and operators acting on them. Carleson measures on Bergman spaces were studied by Hastings (see [4]), and later on by Luecking (see [6]) and many others. Recently, Pau and Zhao in [8] gave a characterization for Carleson measures and vanishing Carleson measures on the unit ball by using the products of functions in weighted Bergman spaces. In [9], Peláez and Rättyä gave a description of Carleson measures for A^2_ϱ on the unit disk when ϱ is such that

$$\frac{1}{(1-r)\varrho(r)}\int_r^1\varrho(t)\,\mathrm{d}t$$

is either equivalent to 1 or tends to ∞, and in [10] they then got a criterion for A^2_ϱ on the unit disk when $\varrho\in\widehat{\mathcal{D}}$, which means

$$\int_r^1\varrho(s)\,\mathrm{d}s\lesssim\int_{(r+1)/2}^1\varrho(s)\,\mathrm{d}s.$$

In [13], Seip gave a characterization of Carleson measures for A^2_ϱ with $\varrho\in S$ in the case $n=1$. One of our main results, Theorem 2.1, extends this result to the case $n>1$.

Given a function $\varphi\in L^\infty(\mathbb{B}_n)$, the Toeplitz operator T_φ on A^2_ϱ with symbol φ is defined by

$$T_\varphi f=P(\varphi f),\quad f\in A^2_\varrho,$$

where $P\colon\ L^2(\mathbb{B}_n,\varrho\,\mathrm{d}v)\to A^2_\varrho$ is the orthogonal projection onto A^2_ϱ. Using the integral representation of P, we can write T_φ as

$$T_\varphi f(z)=\int_{\mathbb{B}_n}K_\varrho(z,w)f(w)\varphi(w)\varrho(w)\,\mathrm{d}v(w),\quad z\in\mathbb{B}_n,$$

where $K_\varrho(z,w)$ is the reproducing kernel for A^2_ϱ. The Toeplitz operators can also be defined for unbounded symbols or for finite measures on $\mathbb{B}_n$. In fact, given a finite positive Borel measure μ on $\mathbb{B}_n$, the Toeplitz operator $T_\mu\colon A^2_\varrho \to A^2_\varrho$ is defined as

$$T_\mu f(z) = \int_{\mathbb{B}_n} K_\varrho(z,w) f(w)\,\mathrm{d}\mu(w), \quad z \in \mathbb{B}_n.$$

Note that

$$\langle T_\mu f, g\rangle_\varrho = \int_{\mathbb{B}_n} f(z)\overline{g(z)}\,\mathrm{d}\mu(z), \quad f,g \in A^2_\varrho.$$

The Toeplitz operators acting on various spaces of holomorphic functions have been extensively studied by many authors, and the theory is especially well understood in the case of Hardy spaces or standard Bergman spaces (see [14], [15] and the references therein). Luecking in [7] was the first to study Toeplitz operators on Bergman spaces with measures as symbols and some interesting results about Toeplitz operators acting on large Bergman spaces were obtained by Lin and Rochberg, see [5]. In this paper, we study the boundedness and compactness of T_μ on A^2_ϱ with $\varrho \in S$.

Next we study when our Toeplitz operators belong to the Schatten class. We refer to [15], Chapter 1 for a brief account on the Schatten classes. A description of the standard Bergman spaces on the unit disk was given (see [15], Chapter 7), and a description for the case of large Bergman spaces on the disk was obtained by Arroussi, Park, and Pau in 2015, see [1]. In 2016, Peláez and Rättyä in [11] gave an interesting characterization for the case of small Bergman spaces on the unit disk, where the weight $\varrho \in \widehat{\mathcal{D}}$. Note that $S \subsetneqq \widehat{\mathcal{D}}$, but $\{A^2_\varrho\colon \varrho \in S\} = \{A^2_\varrho\colon \varrho \in \widehat{\mathcal{D}}\}$. In fact, for $\varrho \in S \cup \widehat{\mathcal{D}}$, we can find $\widetilde{\varrho} \in S \cap \widehat{\mathcal{D}}$ such that $A^2_\varrho = A^2_{\tilde{\varrho}}$. Indeed, by the monotonicity of the functions Φ_f, we obtain that if $h_{\varrho_1} \gtrsim h_{\varrho_2}$, then $A^2_{\varrho_1} \subset A^2_{\varrho_2}$, where $h_\varrho(x) = \int_{1-x}^1 \varrho(t)\,\mathrm{d}t$. Correspondingly, if $h_{\varrho_1} \asymp h_{\varrho_2}$, then $A^2_{\varrho_1} = A^2_{\varrho_2}$. Now, if $\varrho \in S$, then we can interpolate h_ϱ linearly between the points $1-r_k$, $k \geqslant 1$, to get $h_{\tilde{\varrho}}$ such that $A^2_\varrho = A^2_{\tilde{\varrho}}$ and $h_{\tilde{\varrho}}(cx) \leqslant 2h_{\tilde{\varrho}}(x)$ for some $c > 1$. Hence, $h_{\tilde{\varrho}}(2x) \leqslant d h_{\tilde{\varrho}}(x)$ for some $d > 1$ and thus $\widetilde{\varrho} \in \widehat{\mathcal{D}}$. On the other hand, if $\varrho \in \widehat{\mathcal{D}}$, then we can interpolate $\log h_\varrho$ linearly between the points 2^{-k}, $k \geqslant 1$, to get $h_{\tilde{\varrho}}$ such that $A^2_\varrho = A^2_{\tilde{\varrho}}$ and $h_{\tilde{\varrho}}(dx) \leqslant 2h_{\tilde{\varrho}}(x)$ for some $d > 1$. Hence, $\tilde{\varrho} \in S$.

We introduce a subclass S^* of weights in S determined by the condition that $\varrho^*(r) \lesssim \varrho(r)$ for $r \in (0,1)$, where

$$\varrho^*(r) = \frac{1}{1-r}\int_r^1 \varrho(t)\,\mathrm{d}t.$$

For example, the weights

$$\varrho(x) = (1-x)^{-\beta}\Big(\log\frac{1}{1-x}\Big)^\alpha, \quad 0 < \beta < 1,\ \alpha \in \mathbb{R},$$

belong to S^*, but the weights

$$\varrho(x) = (1-x)^{-1}\Big(\log\frac{1}{1-x}\Big)^{\alpha}, \quad \alpha < -1,$$
$$\varrho(x) = (1-x)^{-1}\Big(\log\frac{1}{1-x}\Big)^{-1}\Big(\log\log\frac{1}{1-x}\Big)^{\alpha}, \quad \alpha < -1,$$

do not belong to S^*.

For weights ϱ in S^*, we obtain a characterization of the symbols of the Toeplitz operators in the Schatten classes $\mathcal{S}_p$. In [12], Peláez, Rättyä and Sierra gave a characterization for the case of dimension $n = 1$ when the weight is regular, that is $\varrho^*(r) \asymp \varrho(r)$. As an easy observation, our result is equivalent to their result when $n = 1$. We point out that our approach is completely different from that of [12], which does not seem to work in higher dimensions. On the other hand, for regular weights ϱ in $S \setminus S^*$, this characterization fails. A counterexample was given in [12].

In this paper, we restrict ourselves to the case $1 < p < \infty$. For the case $0 < p \leqslant 1$, the techniques we use should be modified.

The paper is organized as follows: The main results are stated in Section 2 and their proofs are given in Sections 3–5.

2. Main results

Throughout this text, we use the following notation. For every nonnegative integer k, set

$$\Omega_k = \{z \in \mathbb{B}_n \colon\ r_k \leqslant |z| < r_{k+1}\}$$

and let μ_k be the measure defined by $\mu_k = \chi_{\Omega_k}\mu$ whenever a nonnegative Borel measure μ on $\mathbb{B}_n$ is given. The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a positive constant C such that $U(z) \leqslant CV(z)$ holds for all z in the set in question, which may be a space of functions or a set of numbers. If both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$, then we write $U(z) \asymp V(z)$.

Our results are following:

Theorem 2.1. *Let $\varrho \in S$ and let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then:*

(i) *μ is a Carleson measure for A^2_ϱ if and only if each μ_k is a Carleson measure for the Hardy space H^2 with the Carleson constant $\mathcal{C}_{\mu_k}(H^2) \lesssim 2^{-k}$, $k \geqslant 0$.*

(ii) *μ is a vanishing Carleson measure for A^2_ϱ if and only if*

$$\lim_{k\to\infty} 2^k \mathcal{C}_{\mu_k}(H^2) = 0.$$

Theorem 2.1 (i) for the case $n = 1$ was obtained by Seip in [13].

Theorem 2.2. *Let $\varrho \in S$ and let μ be a finite positive Borel measure on $\mathbb{B}_n$. Then:*

(i) *The Toeplitz operator T_μ is bounded on A^2_ϱ if and only if μ is a Carleson measure for A^2_ϱ.*

(ii) *The Toeplitz operator T_μ is compact on A^2_ϱ if and only if μ is a vanishing Carleson measure for A^2_ϱ.*

Given $z \in \mathbb{B}_n$ and $0 < \alpha < 1$, we consider the Bergman metric ball

$$E(z,\alpha) = \{w \in \mathbb{B}_n \colon \beta(z,w) < \alpha\},$$

where $\beta(z,w)$ is the Bergman metric given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z,w \in \mathbb{B}_n.$$

Here, φ_z is the Möbius transformation on $\mathbb{B}_n$ that interchanges 0 and z.

We know that $E(0,\alpha)$ is actually a Euclidean ball of radius $R = \tanh\alpha$, centered at the origin, and

$$E(z,\alpha) = \varphi_z(E(0,\alpha)).$$

Moreover, for fixed α, $v(E(z,\alpha)) \asymp (1-|z|)^{n+1}$. See [14], Chapter 1 for more details.

For a measure μ on $\mathbb{B}_n$ and $\alpha > 0$, we define the function $\widehat{\mu}_\alpha$ by

$$\widehat{\mu}_\alpha(z) = \frac{2^k \mu(E(z,\alpha))}{(1-|z|)^n}, \quad z \in \Omega_k.$$

Let $\widetilde{T}_\mu$ be the Berezin transform of T_μ, defined by

$$\widetilde{T}_\mu(z) = \langle T_\mu k_z, k_z\rangle_\varrho, \quad z \in \mathbb{B}_n,$$

where k_z is the normalized reproducing kernel of A^2_ϱ. Set

$$\mathrm{d}\lambda_\varrho(z) = \frac{2^k \varrho(z)\,\mathrm{d}v(z)}{(1-|z|)^n}, \quad z \in \Omega_k.$$

Theorem 2.3. *Let ϱ be in S^*, μ be a finite positive Borel measure and $1 < p < \infty$. The following conditions are equivalent:*

(a) *The Toeplitz operator T_μ is in the Schatten class $\mathcal{S}_p$.*

(b) *The function $\widetilde{T}_\mu$ is in $L^p(\mathbb{B}_n, \mathrm{d}\lambda_\varrho)$.*

(c) *The function $\widehat{\mu}_\alpha$ is in $L^p(\mathbb{B}_n, \mathrm{d}\lambda_\varrho)$ for a sufficiently small $\alpha > 0$.*

3. Proof of Theorem 2.1

Given $a \in \mathbb{B}_n \setminus \{0\}$ and $r > 0$. Let $\delta(a) = \sqrt{2(1-|a|)}$. Define $Q(a,r) \subset \mathbb{B}_n$ and $O(a,r) \subset \mathbb{S}_n$ as follows:

$$Q(a,r) = \Big\{z \in \mathbb{B}_n\colon\ \sqrt{|1 - \langle a/|a|, z\rangle|} < r\Big\},$$
$$O(a,r) = \Big\{\zeta \in \mathbb{S}_n\colon\ \sqrt{|1 - \langle a/|a|, \zeta\rangle|} < r\Big\}.$$

For simplicity of notation, we write Q_a instead of $Q(a, \delta(a))$, O_a instead of $O(a, \delta(a))$.

We recall a well known characterization of Carleson measures for the Hardy space (see [14]): *A positive Borel measure μ on $\mathbb{B}_n$ is a Carleson measure for H^2 if and only if $\mu(Q_a) \lesssim (1-|a|)^n$ for all $a \in \mathbb{B}_n \setminus \{0\}$. Furthermore,*

$$\mathcal{C}_\mu(H^2) \asymp \sup_{a \in \mathbb{B}_n \setminus \{0\}} \mu(Q_a)(1-|a|)^{-n}.$$

We use the following covering lemma from [14], Lemma 4.7.

Lemma 3.1. *Suppose N is a natural number, $a_l \in \mathbb{B}_n \setminus \{0\}$, $1 \leqslant l \leqslant N$,*

$$E = \bigcup_{l=1}^{N} O_{a_l}.$$

There exists a subsequence $\{l_i\}$, $1 \leqslant i \leqslant M$, such that
(a) *$O_{a_{l_i}}$, $1 \leqslant i \leqslant M$, are disjoint.*
(b) *$O(a_{l_i}, 3\delta(a_{l_i}))$, $1 \leqslant i \leqslant M$, cover E.*

Lemma 3.2. *Let μ be a finite positive measure on $\mathbb{B}_n$. Then μ_k is a Carleson measure for H^2 if and only if $\mu_k(Q_a) \lesssim (1-|a|)^n$ for all $a \in \Omega_k$. Furthermore, $\mathcal{C}_{\mu_k}(H^2) \asymp \sup_{a \in \Omega_k} (1-|a|)^{-n} \mu_k(Q_a)$.*

Proof. Let $a \in \mathbb{B}_n \setminus \{0\}$. Then $a \in \Omega_l$ for some $l \geqslant 1$. If $l > k$, then $\mu_k(Q_a) = 0$ and there is nothing to prove. When $a \in \Omega_l$, $l \leqslant k$, we can cover $Q_a \setminus r_k\mathbb{B}_n$ by a finite family $\{Q_{a_l}\colon\ l \in \Lambda\}$ with $a_l \in \Omega_{k-1}$, where Λ is a finite index set. Applying Lemma 3.1 to the set $\{O_{a_l}\colon\ l \in \Lambda\}$, we get a subset Λ_0 of Λ such that O_{a_l}, $l \in \Lambda_0$, are disjoint and $O(a_l, 3\delta(a_l))$, $l \in \Lambda_0$, cover O_a. Moreover, it is easy to see that

$$Q_a \setminus r_k\mathbb{B}_n \subset \bigcup_{l \in \Lambda_0} Q(a_l, 3\delta(a_l)).$$

Then

$$\mu_k(Q_a) = \mu_k(Q_a \setminus r_k\mathbb{B}_n) \leqslant \sum_{l \in \Lambda_0} \mu_k(Q(a_l, 3\delta(a_l))).$$

Since $a_l \in \Omega_{k-1}$, we have $\mu_k(Q(a_l, 3\delta(a_l))) \lesssim (1-|a_l|)^n \asymp \sigma(O_{a_l})$. Hence

$$\mu_k(Q_a) \lesssim \sum_{l\in\Lambda_0} \sigma(O_{a_l}) = \sigma\Big(\bigcup_{l\in\Lambda_0} O_{a_l}\Big).$$

Finally,

$$\sigma\Big(\bigcup_{l\in\Lambda_0} O_{a_l}\Big) \lesssim \sigma(O_a) \asymp (1-|a|)^n.$$

Therefore $\mu_k(Q_a) \lesssim (1-|a|)^n$. This completes the proof. $\square$

Proof of part (i). ($\Leftarrow$) Since μ_k are Carleson measures for H^2 with Carleson constants $\lesssim 2^{-k}$, the same holds for H^2 on the smaller ball $r_{k+2}\mathbb{B}_n$. Indeed, we just use the characterization of Carleson measures and the fact that if $Q(a,\delta(a)) \cap r_{k+2}^{-1}\Omega_k \neq \emptyset$, then $1-|a| \gtrsim 1-r_{k+2}$ and, hence, $r_{k+2}Q(a,\delta(a)) \subset Q(a, M\delta(a))$ for some $M<\infty$ independent of a and k.

Therefore,

$$\int_{\Omega_k} |f(z)|^2 \,\mathrm{d}\mu(z) \lesssim 2^{-k} \int_{\mathbb{S}_n} |f(r_{k+2}\xi)|^2 \,\mathrm{d}\sigma(\xi)$$

for an arbitrary function f in A^2_ϱ and for all k. Summing this estimate over all $k \geqslant 1$, we get

$$\int_{\mathbb{B}_n} |f(z)|^2 \,\mathrm{d}\mu(z) \lesssim \sum_{k=1}^{\infty} 2^{-k} \int_{\mathbb{S}_n} |f(r_{k+2}\xi)|^2 \,\mathrm{d}\sigma(\xi) \asymp \|f\|^2_\varrho.$$

($\Rightarrow$) We just need to check that $\mu_k(Q_a) \lesssim 2^{-k}(1-|a|)^n$ when a is in Ω_k, $k \geqslant 0$. We use the test function

$$f_a(z) = (1-\langle a,z\rangle)^{-\gamma} \tag{3.1}$$

with large γ. By (1.2), we have

$$\|f_a\|^2_\varrho \asymp \sum_{j=1}^{\infty} 2^{-j} \int_{\mathbb{S}_n} \frac{1}{|1-\langle a, r_j\xi\rangle|^{2\gamma}} \,\mathrm{d}\sigma(\xi) \asymp \sum_{j=1}^{\infty} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}}.$$

Since $a \in \Omega_k$, relation (1.1) yields that

$$\|f_a\|^2_\varrho \asymp 2^{-k}(1-|a|)^{-2\gamma+n}. \tag{3.2}$$

Indeed,

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}} &= \sum_{j\leqslant k} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}} + \sum_{j>k} \frac{2^{-j}}{(1-r_j|a|)^{2\gamma-n}} \\ &\asymp \sum_{j\leqslant k} \frac{2^{-j}}{(1-r_j)^{2\gamma-n}} + \sum_{j>k} \frac{2^{-j}}{(1-|a|)^{2\gamma-n}} \asymp \frac{2^{-k}}{(1-r_k)^{2\gamma-n}} + \frac{2^{-k}}{(1-|a|)^{2\gamma-n}} \\ &\asymp 2^{-k}(1-|a|)^{-2\gamma+n}.\end{aligned}$$

On the other hand, for every z in Q_a we have

$$\begin{aligned}|1-\langle a,z\rangle| &= |(1-|a|)+|a|(1-\langle a/|a|,z\rangle)| \leqslant (1-|a|)+|a||1-\langle a/|a|,z\rangle| \\ &< (1-|a|)+2|a|(1-|a|) \leqslant 3(1-|a|).\end{aligned}$$

Hence,

$$|f_a(z)| \gtrsim (1-|a|)^{-\gamma}, \quad z \in Q_a. \tag{3.3}$$

Thus,

$$\int_{\mathbb{B}_n} |f_a(z)|^2 \,\mathrm{d}\mu(z) \gtrsim (1-|a|)^{-2\gamma}\mu(Q_a \cap \Omega_k).$$

Since μ is a Carleson measure for A^2_ϱ, we get

$$\mu(Q_a \cap \Omega_k) \lesssim 2^{-k}(1-|a|)^n.$$

This implies that μ_k is a Carleson measure for the Hardy space H^2 with the Carleson constant $\mathcal{C}_{\mu_k}(H^2) \lesssim 2^{-k}$. □

P r o o f of part (ii). Suppose that μ is a vanishing Carleson measure for A^2_ϱ. Given a in Ω_k, consider the function f_a defined by (3.1). By (3.2),

$$\|f_a\|^2_\varrho \asymp 2^{-k}(1-|a|)^{-2\gamma+n}.$$

Set

$$h_a(z) = \frac{(1-\langle a,z\rangle)^{-\gamma}}{2^{-k/2}(1-|a|)^{-\gamma+n/2}}. \tag{3.4}$$

Then $\|h_a\|^2_\varrho \asymp 1$ and, by (3.3),

$$|h_a(z)|^2 \gtrsim \frac{2^k}{(1-|a|)^n}, \quad z \in Q_a.$$

Since μ is a vanishing Carleson measure for A^2_ϱ and h_a tends to 0 uniformly on compact subsets of the unit ball as $|a| \to 1$, we have

$$\lim_{|a|\to 1} \int_{\mathbb{B}_n} |h_a(z)|^2 \,\mathrm{d}\mu(z) = 0.$$

Thus,

$$\sup_{a\in\Omega_k} \frac{2^k \mu_k(Q_a \cap \Omega_k)}{(1-|a|)^n} \to 0$$

as $k \to \infty$. Hence, $\lim\limits_{k\to\infty} 2^k \mathcal{C}_{\mu_k}(H^2) = 0$.

Conversely, let $\mu^r = \mu|_{\mathbb{B}_n \setminus \overline{r\mathbb{B}_n}}$, where $r\mathbb{B}_n = \{z \in \mathbb{B}_n \colon |z| < r\}$. Then $(\mu^r)_k \leqslant \mu_k$, $k \geqslant 1$, and $(\mu^r)_k = 0$ if $r_{k+1} \leqslant r$. Therefore, part (i) of Theorem 2.1 implies that

$$\int_{\mathbb{B}_n} |h(z)|^2 \, \mathrm{d}\mu^r(z) \leqslant C_r \|h\|_\varrho^2, \quad h \in A_\varrho^2,$$

where

$$C_r = \sup_{k \colon r_{k+1} > r} 2^k \mathcal{C}_{\mu_k}(H^2) \quad \text{and} \quad \lim_{r \to 1} C_r = 0. \tag{3.5}$$

Let $\{f_k\}$ be a bounded sequence in A_ϱ^2 converging uniformly to 0 on compact subsets of $\mathbb{B}_n$. Let $\varepsilon > 0$. By (3.5), there exists a $r_0 \in (0,1)$ such that $C_r < \varepsilon$ for all $r \geqslant r_0$. Moreover, by the uniform convergence on compact subsets, we may choose $k_0 \in \mathbb{N}$ such that $|f_k(z)|^2 < \varepsilon$ for all $k \geqslant k_0$ and $z \in \overline{r_0\mathbb{B}_n}$. It follows that

$$\begin{aligned}
\int_{\mathbb{B}_n} |f_k(z)|^2 \, \mathrm{d}\mu(z) &= \int_{\overline{r_0\mathbb{B}_n}} |f_k(z)|^2 \, \mathrm{d}\mu(z) + \int_{\mathbb{B}_n \setminus \overline{r_0\mathbb{B}_n}} |f_k(z)|^2 \, \mathrm{d}\mu(z) \\
&< \varepsilon\mu(\overline{r_0\mathbb{B}_n}) + \int_{\mathbb{B}_n} |f_k(z)|^2 \, \mathrm{d}\mu^{r_0}(z) \\
&\leqslant \varepsilon\mu(\overline{r_0\mathbb{B}_n}) + C_{r_0}\|f_k\|_\varrho^2 \leqslant \varepsilon C, \quad k \geqslant k_0,
\end{aligned}$$

for some positive constant C. Hence, μ is a vanishing Carleson measure for A_ϱ^2. □

4. Proof of Theorem 2.2

Proof of part (i). (⇒) Given a in Ω_k, we define h_a by (3.4). Then

$$\|h_a\|_\varrho^2 \asymp 1 \quad \text{and} \quad |h_a(z)|^2 \gtrsim 2^k(1-|a|)^{-n}, \quad z \in Q_a.$$

Consider the function

$$T_\mu^{\#}(a) = \langle T_\mu h_a, h_a \rangle_\varrho = \int_{\mathbb{B}_n} |h_a|^2 \, \mathrm{d}\mu(z). \tag{4.1}$$

Since T_μ is bounded, $A := \sup_{a \in \mathbb{B}_n} T_\mu^{\#}(a) < \infty$. Then

$$\begin{aligned}
A &\geqslant \int_{\mathbb{B}_n} |h_a(z)|^2 \, \mathrm{d}\mu(z) \geqslant \int_{\mathbb{B}_n} |h_a(z)|^2 \, \mathrm{d}\mu_k(z) \\
&\geqslant \int_{Q_a} |h_a(z)|^2 \, \mathrm{d}\mu_k(z) \gtrsim 2^k(1-|a|)^{-n}\mu_k(Q_a).
\end{aligned} \tag{4.2}$$

Hence, $\mu_k(Q_a) \lesssim 2^{-k}(1-|a|)^n$ for every $a \in \Omega_k$. By Theorem 2.1 and Lemma 3.2, μ is a Carleson measure for A^2_ϱ.

($\Leftarrow$) For every $f, g \in A^2_\varrho$ we have

$$\langle T_\mu f, g\rangle_\varrho = \int_{\mathbb{B}_n} f(z)\overline{g(z)}\,\mathrm{d}\mu(z).$$

Then by Cauchy-Schwarz inequality, we get

$$|\langle T_\mu f, g\rangle_\varrho| \leqslant \int_{\mathbb{B}_n} |f(z)||g(z)|\,\mathrm{d}\mu(z) \leqslant \Big(\int_{\mathbb{B}_n} |f(z)|^2\,\mathrm{d}\mu(z)\Big)^{1/2}\Big(\int_{\mathbb{B}_n} |g(z)|^2\,\mathrm{d}\mu(z)\Big)^{1/2}.$$

Since μ is a Carleson measure for A^2_ϱ, there exists a positive constant C such that

$$\int_{\mathbb{B}_n} |f(z)|^2\,\mathrm{d}\mu(z) \leqslant C\|f\|^2_\varrho \quad \text{and} \quad \int_{\mathbb{B}_n} |g(z)|^2\,\mathrm{d}\mu(z) \leqslant C\|g\|^2_\varrho.$$

Hence,

$$|\langle T_\mu f, g\rangle_\varrho| \leqslant C\|f\|_\varrho\|g\|_\varrho \quad \forall\, f, g \in A^2_\varrho.$$

Thus, T_μ is bounded on A^2_ϱ. □

Proof of part (ii). We need the following auxiliary results. □

Proposition 4.1. *Suppose that $f \in A^2_\varrho$ with $\varrho \in S$. Then*

$$|f(z)|^2 \leqslant \frac{C2^k}{(1-|z|)^n}\|f\|^2_\varrho, \quad z \in \Omega_k,\ k \geqslant 0, \tag{4.3}$$

where C is a positive constant independent of k and z.

Proof. Let $z \in \Omega_k$. Applying [14], Corollary 4.5 to the function $g(z) = f(r_{k+2}z)$ at the point $z/(r_{k+2})$, we obtain

$$|f(z)|^2 \leqslant \int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2 \frac{(1-|z/r_{k+2}|^2)^n}{|1-\langle z/r_{k+2}, \zeta\rangle|^{2n}}\,\mathrm{d}\sigma(\zeta).$$

By (1.1), $|1-\langle z/r_{k+2}, \zeta\rangle| \geqslant 1-|\langle z/r_{k+2}, \zeta\rangle| \geqslant 1-|z||\zeta|/r_{k+2} = 1-|z|/r_{k+2} \gtrsim 1-|z|$ for $z \in \Omega_k$, $\zeta \in \mathbb{S}_n$. Thus,

$$\begin{aligned}|f(z)|^2 &\lesssim \int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2 \frac{(1-|z|^2)^n}{(1-|z|)^{2n}}\,\mathrm{d}\sigma(\zeta) \leqslant \frac{(1+|z|)^n}{(1-|z|)^n}\int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2\,\mathrm{d}\sigma(\zeta)\\ &\lesssim \frac{2^k}{(1-|z|)^n}2^{-k}\int_{\mathbb{S}_n} |f(r_{k+2}\zeta)|^2\,\mathrm{d}\sigma(\zeta)\\ &\leqslant \frac{2^k}{(1-|z|)^n}\sum_{j=1}^{\infty} 2^{-j}\int_{\mathbb{S}_n} |f(r_{j+2}\zeta)|^2\,\mathrm{d}\sigma(\zeta) \lesssim \frac{2^k}{(1-|z|)^n}\|f\|^2_\varrho\end{aligned}$$

with constants independent of k and z. □

Corollary 4.2. *A sequence of functions $\{f_k\} \subset A^2_\varrho$ converges to 0 weakly in A^2_ϱ if and only if it is bounded in A^2_ϱ and converges to 0 uniformly on each compact subset of $\mathbb{B}_n$.*

Proof of part (ii) of Theorem 2.2. Suppose that T_μ is compact on A^2_ϱ. We define h_a, $a \in \mathbb{B}_n$ by (3.4) and $T^{\#}_\mu$ by (4.1). Then $\|h_a\|^2_\varrho \asymp 1$ and h_a converges uniformly to 0 on compact subsets of $\mathbb{B}_n$ as $|a| \to 1$. Since T_μ is compact, $T^{\#}_\mu(a) \to 0$ as $|a| \to 1$. By (4.2) this implies that

$$\sup_{a\in\Omega_k} \frac{2^k \mu_k(Q_a)}{(1-|a|)^n} \to 0 \quad \text{as } k \to \infty.$$

Hence,

$$\lim_{k\to\infty} 2^k \mathcal{C}_{\mu_k}(H^2) = 0.$$

By part (ii) of Theorem 2.1, μ is a vanishing Carleson measure for A^2_ϱ.

Conversely, assume that μ is a vanishing Carleson measure for A^2_ϱ. For every $h \in A^2_\varrho$ we have

$$\|T_\mu h\|_\varrho = \sup_{\substack{g\in A^2_\varrho \\ \|g\|_\varrho \leqslant 1}} |\langle T_\mu h, g\rangle_\varrho|.$$

Furthermore,

$$\begin{aligned} |\langle T_\mu h, g\rangle_\varrho| &= \left| \int_{\mathbb{B}_n} h(z)\overline{g(z)}\, \mathrm{d}\mu(z) \right| \leqslant \int_{\mathbb{B}_n} |h(z)||g(z)|\, \mathrm{d}\mu(z) \\ &\leqslant \left(\int_{\mathbb{B}_n} |h(z)|^2\, \mathrm{d}\mu(z) \right)^{1/2} \left(\int_{\mathbb{B}_n} |g(z)|^2\, \mathrm{d}\mu(z) \right)^{1/2} \\ &\lesssim \left(\int_{\mathbb{B}_n} |h(z)|^2\, \mathrm{d}\mu(z) \right)^{1/2} \|g\|_\varrho. \end{aligned}$$

The last inequality follows from the fact that μ is a Carleson measure for A^2_ϱ. Therefore,

$$\|T_\mu h\|_\varrho \lesssim \left(\int_{\mathbb{B}_n} |h(z)|^2\, \mathrm{d}\mu(z) \right)^{1/2}, \quad h \in A^2_\varrho.$$

Now, let $\{f_k\} \subset A^2_\varrho$ be bounded and converge uniformly to 0 on compact subsets of $\mathbb{B}_n$. Since μ is a vanishing Carleson measure for A^2_ϱ,

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k(z)|^2\, \mathrm{d}\mu(z) = 0.$$

It follows that $\|T_\mu f_k\|_\varrho \to 0$ and hence T_μ is compact. $\square$

5. Proof of Theorem 2.3

Proposition 5.1. *Let $K_\varrho(z,w)$ be the reproducing kernel of A^2_ϱ.*

(a) *Let $k \geqslant 1$, $z \in \Omega_k$. Then*

$$K_\varrho(z,z) \asymp \frac{2^k}{(1-|z|)^n}. \tag{5.1}$$

(b) *There exists $\alpha = \alpha(\varrho) > 0$ such that for every $z \in \mathbb{B}_n$,*

$$|K_\varrho(z,w)|^2 \asymp K_\varrho(z,z)K_\varrho(w,w) \tag{5.2}$$

whenever $w \in E(z,\alpha)$.

Proof. (a) Fix $k \geqslant 1$. Given $z \in \Omega_k$, let L_z be the point evaluation at z on A^2_ϱ. It is well known that

$$K_\varrho(z,z) = \|L_z\|^2.$$

By Proposition 4.1,

$$\|L_z\|^2 \lesssim \frac{2^k}{(1-|z|)^n}.$$

Furthermore, choosing h_z by (3.4), we have $\|h_z\|_\varrho \asymp 1$ and

$$|h_z(z)|^2 \gtrsim \frac{2^k}{(1-|z|)^n}.$$

Hence,

$$\|L_z\|^2 \gtrsim \frac{2^k}{(1-|z|)^n}.$$

Thus

$$K_\varrho(z,z) \asymp \frac{2^k}{(1-|z|)^n}, \quad z \in \Omega_k.$$

(b) In this proof, we use an argument of Lin and Rochberg, see [5]. It is well known that

$$|K_\varrho(z,w)|^2 \leqslant K_\varrho(z,z)K_\varrho(w,w)$$

for all $z,w \in \mathbb{B}_n$. For any fixed $z_0 \in \Omega_k$, consider the subspace $A^2_\varrho(z_0)$ defined as

$$A^2_\varrho(z_0) = \{f \in A^2_\varrho\colon\ f(z_0) = 0\}.$$

Denote by $\mathcal{L}_{z_0}$ the one-dimensional subspace spanned by the function

$$k_{\varrho,z_0}(z) = \frac{K_\varrho(z,z_0)}{\sqrt{K_\varrho(z_0,z_0)}}.$$

Then we have the orthogonal decomposition

$$A^2_\varrho = A^2_\varrho(z_0) \oplus \mathcal{L}_{z_0}.$$

Hence $K_\varrho(z,w) = K_{\varrho,z_0}(z,w) + \overline{k_{\varrho,z_0}(w)} k_{\varrho,z_0}(z)$, where K_{ϱ,z_0} is the reproducing kernel of $A^2_\varrho(z_0)$. Therefore,

$$K_\varrho(z_0,w) = \overline{k_{\varrho,z_0}(w)} k_{\varrho,z_0}(z_0)$$

and

$$K_\varrho(w,w) = K_{\varrho,z_0}(w,w) + |k_{\varrho,z_0}(w)|^2. \tag{5.3}$$

We are going to prove that there exists $\alpha > 0$ such that

$$K_{\varrho,z_0}(w,w) < \frac{1}{2} K_\varrho(w,w), \quad w \in E(z_0,\alpha). \tag{5.4}$$

By (1.1), there exists $\alpha_1 > 0$ such that $E(z_0,\alpha) \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}$, $0 < \alpha < \alpha_1$. Hence, for every $f \in A^2_\varrho(z_0)$ such that $\|f\|_\varrho = 1$, by Proposition 4.1 we have

$$|f(w)|^2 \lesssim \frac{2^k}{(1-|w|)^n} \asymp \frac{2^k}{(1-|z_0|)^n} \tag{5.5}$$

whenever $w \in E(z_0,\alpha)$. Since $E(z_0,\alpha) = \varphi_{z_0}(E(0,\alpha))$, we can rewrite (5.5) as

$$|f(\varphi_{z_0}(\eta))|^2 \lesssim \frac{2^k}{(1-|z_0|)^n} \tag{5.6}$$

whenever $\eta \in E(0,\alpha)$. Note that $f(z_0) = f(\varphi_{z_0}(0)) = 0$. Therefore, by the Schwarz lemma, we get

$$|f(\varphi_{z_0}(\eta))|^2 \lesssim |\eta|^2 \frac{2^k}{(1-|z_0|)^n} \asymp |\eta|^2 \frac{2^k}{(1-|\varphi_{z_0}(\eta)|)^n}$$

whenever $\eta \in E(0,\alpha)$. This implies that there is a constant $C > 0$ such that

$$|f(\varphi_{z_0}(\eta))|^2 \leqslant C|\eta|^2 \frac{2^k}{(1-|\varphi_{z_0}(\eta)|)^n}, \quad \eta \in E(0,\alpha).$$

Therefore, we can choose α so small that

$$|f(\varphi_{z_0}(\eta))|^2 < \frac{1}{2} K_\varrho(\varphi_{z_0}(\eta), \varphi_{z_0}(\eta)), \quad \eta \in E(0,\alpha).$$

This proves (5.4).

Now, from (5.3) and (5.4), we obtain that $|k_{\varrho,z_0}(w)|^2 > \frac{1}{2}K_\varrho(w,w)$ whenever $w \in E(z_0,\alpha)$. This means that

$$|K_\varrho(w,z_0)|^2 > \frac{1}{2}K_\varrho(z_0,z_0)K_\varrho(w,w)$$

whenever $w \in E(z_0,\alpha)$, which completes the proof. □

Lemma 5.2. *Let T be a positive operator on A^2_ϱ and let $\widetilde{T}$ be the Berezin transform of T, defined by*

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle_\varrho, \quad z \in \mathbb{B}_n.$$

(a) *Let $0 < p \leqslant 1$. If $\widetilde{T} \in L^p(\mathbb{B}_n, \mathrm{d}\lambda_\varrho)$, then T is in $\mathcal{S}_p$.*
(b) *Let $p \geqslant 1$. If T is in $\mathcal{S}_p$, then $\widetilde{T} \in L^p(\mathbb{B}_n, \mathrm{d}\lambda_\varrho)$. Here,*

$$\mathrm{d}\lambda_\varrho(z) = \frac{2^k\varrho(z)\,\mathrm{d}v(z)}{(1-|z|)^n}$$

if $z \in \Omega_k$.

Proof. Note that $\mathrm{d}\lambda_\varrho(z) \asymp K(z,z)\varrho(z)\,\mathrm{d}v(z) = \|K_z\|^2\varrho(z)\,\mathrm{d}v(z)$.

The proof is similar to the proof of [1], Lemma 4.2. The positive operator T is in $\mathcal{S}_p$ if and only if T^p is in the trace class $\mathcal{S}_1$. Fix an orthonormal basis $\{e_k\}$ of A^2_ϱ. Since T^p is positive, it is in S_1 if and only if $\sum\limits_k\langle T^pe_k, e_k\rangle_\varrho < \infty$. Let $U = \sqrt{T^p}$. By Fubini's theorem, the reproducing property of K_z, and Parseval's identity, we have

$$\begin{aligned}\sum_k \langle T^pe_k, e_k\rangle_\varrho &= \sum_k \|Ue_k\|^2_\varrho = \sum_k \int_{\mathbb{B}_n} |Ue_k(z)|^2\varrho(z)\,\mathrm{d}v(z)\\ &= \int_{\mathbb{B}_n}\Big(\sum_k |Ue_k(z)|^2\Big)\varrho(z)\,\mathrm{d}v(z) = \int_{\mathbb{B}_n}\Big(\sum_k |\langle Ue_k, K_z\rangle_\varrho|^2\Big)\varrho(z)\,\mathrm{d}v(z)\\ &= \int_{\mathbb{B}_n}\Big(\sum_k |\langle e_k, UK_z\rangle_\varrho|^2\Big)\varrho(z)\,\mathrm{d}v(z) = \int_{\mathbb{B}_n} \|UK_z\|^2_\varrho\varrho(z)\,\mathrm{d}v(z)\\ &= \int_{\mathbb{B}_n} \langle T^pK_z, K_z\rangle_\varrho\varrho(z)\,\mathrm{d}v(z) = \int_{\mathbb{B}_n} \langle T^pk_z, k_z\rangle_\varrho\|K_z\|^2_\varrho\varrho(z)\,\mathrm{d}v(z)\\ &\asymp \int_{\mathbb{B}_n} \langle T^pk_z, k_z\rangle_\varrho\,\mathrm{d}\lambda_\varrho(z).\end{aligned}$$

Hence, both (a) and (b) are the consequences of the well known inequalities (see [15], Proposition 1.31)

$$\begin{aligned}\langle T^pk_z, k_z\rangle_\varrho &\leqslant \langle Tk_z, k_z\rangle^p_\varrho = (\widetilde{T}(z))^p, \quad 0 < p \leqslant 1,\\ \langle T^pk_z, k_z\rangle_\varrho &\geqslant \langle Tk_z, k_z\rangle^p_\varrho = (\widetilde{T}(z))^p, \quad p \geqslant 1.\end{aligned}$$

□

Lemma 5.3. *Let $\varrho \in S^*$ and $z \in \Omega_k$. Then there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0)$ we have*

$$|f(z)|^2 \lesssim \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \varrho(w) \, \mathrm{d}v(w)$$

for all $f \in H(\mathbb{B}_n)$.

P r o o f. Let $z \in \Omega_k$. For each $f \in H(\mathbb{B}_n)$, by the subharmonicity of the function $w \mapsto |f(w)|^2$ and the estimate $v(E(z,\alpha)) \asymp (1-|z|)^{n+1}$, we have

$$|f(z)|^2 \lesssim \frac{1}{(1-|z|)^{n+1}} \int_{E(z,\alpha)} |f(w)|^2 \, \mathrm{d}v(w).$$

It is easy to see that $1-|z| \asymp 1-|w|$ for $w \in E(z,\alpha)$. Hence,

$$\begin{aligned} |f(z)|^2 &\lesssim \frac{1}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \frac{1}{1-|w|} \, \mathrm{d}v(w) \\ &= \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \frac{2^{-k}}{1-|w|} \, \mathrm{d}v(w). \end{aligned} \tag{5.7}$$

By (1.1), for small α_0 we have $E(z,\alpha_0) \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}$. Therefore, for every $\alpha \in (0,\alpha_0)$, we have $r_{k-1} < |w| < r_{k+2}$ for $w \in E(z,\alpha)$. Since $\int_{r_{k+2}}^1 \varrho(t) \, \mathrm{d}t = 2^{-k-2}$, we obtain $2^{-k} \lesssim \int_{|w|}^1 \varrho(t) \, \mathrm{d}t$ for every $w \in E(z,\alpha)$, $\alpha \in (0,\alpha_0)$. Plugging this into (5.7) and using that $\varrho^*(w) \lesssim \varrho(w)$, we get

$$\begin{aligned} |f(z)|^2 &\lesssim \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \varrho^*(w) \, \mathrm{d}v(w) \\ &\lesssim \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |f(w)|^2 \varrho(w) \, \mathrm{d}v(w). \end{aligned}$$

This completes the proof. □

P r o o f of Theorem 2.3. (a) $\Rightarrow$ (b). This follows from Lemma 5.2 (b).

(b) $\Rightarrow$ (c). By Proposition 5.1 (b), for sufficiently small $\alpha > 0$, we have

$$|K_z(w)|^2 \asymp \|K_z\|_\varrho^2 \|K_w\|_\varrho^2, \quad w \in E(z,\alpha), \ z \in \mathbb{B}_n.$$

Then by Proposition 5.1 (a), we get

$$\begin{aligned} \widetilde{T}_\mu(z) &= \int_{\mathbb{B}_n} |k_z(w)|^2 \, \mathrm{d}\mu(w) = \|K_z\|_\varrho^{-2} \int_{\mathbb{B}_n} |K_z(w)|^2 \, \mathrm{d}\mu(w) \\ &\geqslant \|K_z\|_\varrho^{-2} \int_{E(z,\alpha)} |K_z(w)|^2 \, \mathrm{d}\mu(w) \asymp \int_{E(z,\alpha)} \|K_w\|_\varrho^2 \, \mathrm{d}\mu(w) \asymp \widehat{\mu}_\alpha(z). \end{aligned}$$

Since $\widetilde{T}_\mu$ is in $L^p(\mathbb{B}_n, \mathrm{d}\lambda_\varrho)$, $\widehat{\mu}_\alpha$ is also in $L^p(\mathbb{B}_n, \mathrm{d}\lambda_\varrho)$.

(c) $\Rightarrow$ (a). For every orthonormal basis $\{e_l\}$ of A^2_ϱ, we have

$$\sum_l \langle T_\mu e_l, e_l\rangle_\varrho^p = \sum_l \Big(\int_{\mathbb{B}_n} |e_l(z)|^2 \, \mathrm{d}\mu(z)\Big)^p. \tag{5.8}$$

By Lemma 5.3,

$$|e_l(z)|^2 \lesssim \frac{2^k}{(1-|z|)^n} \int_{E(z,\alpha)} |e_l(w)|^2 \varrho(w) \, \mathrm{d}v(w), \quad z \in \Omega_k.$$

By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{B}_n} |e_l(z)|^2 \, \mathrm{d}\mu(z) &\lesssim \int_{\mathbb{B}_n} |e_l(w)|^2 \widehat{\mu}_\alpha(w) \varrho(w) \, \mathrm{d}v(w) \\
&\leqslant \Big(\int_{\mathbb{B}_n} |e_l(w)|^2 \widehat{\mu}_\alpha(w)^p \varrho(w) \, \mathrm{d}v(w)\Big)^{1/p} \Big(\int_{\mathbb{B}_n} |e_l(w)|^2 \varrho(w) \, \mathrm{d}v(w)\Big)^{1/q} \\
&= \Big(\int_{\mathbb{B}_n} |e_l(w)|^2 \widehat{\mu}_\alpha(w)^p \varrho(w) \, \mathrm{d}v(w)\Big)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$. Thus, (5.8) implies that

$$\begin{aligned}
\sum_l \langle T_\mu e_l, e_l\rangle_\varrho^p &\lesssim \int_{\mathbb{B}_n} \Big(\sum_l |e_l(w)|^2\Big) \widehat{\mu}_\alpha(w)^p \varrho(w) \, \mathrm{d}v(w) \\
&= \int_{\mathbb{B}_n} \|K_w\|_\varrho^2 \widehat{\mu}_\alpha(w)^p \varrho(w) \, \mathrm{d}v(w) \asymp \int_{\mathbb{B}_n} \widehat{\mu}_\alpha(w)^p \, \mathrm{d}\lambda_\varrho(w) < \infty.
\end{aligned}$$

This proves (a). □

Remark 5.4. Let $1 < p < \infty$. In the case of large weighted Bergman spaces, Arroussi, Park and Pau proved in [1], Theorem 4.6 that

$$T_\mu \in \mathcal{S}_p \Leftrightarrow \widetilde{\mu}_\varepsilon(z) = \frac{\mu(B(z,\varepsilon))}{(1-|z|)^{2n}} \text{ is in the corresponding weighted } L^p,$$

where $B(z,\varepsilon)$ is the Euclidean ball with the center z and radius $\varepsilon(1-|z|)$. When the dimension $n = 1$, we can see that $\widetilde{\mu}_\varepsilon$ is in L^p if and only if $\widehat{\mu}_\varepsilon$ is in L^p. However, for $n > 1$, this equivalence is not true anymore.

Let us verify this. Choose $z_k \in \mathbb{B}_n$ such that $|z_k|$ tend to 1 sufficiently rapidly as $k \to \infty$. Consider

$$\mu = \sum_{k=1}^\infty c_k \chi_{B(z_k,\varepsilon)} \quad \text{and} \quad \mu^* = \sum_{k=1}^\infty c_k \chi_{B(z_k,3\varepsilon)},$$

where $c_k > 0$ will be chosen later. We have

$$\mu \lesssim \widetilde{\mu}_\varepsilon \lesssim \mu^*$$

and

$$\sum_{k=1}^{\infty} c_k \frac{v(B(z_k,\varepsilon))}{v(E(z_k,\varepsilon))} \chi_{E(z_k,\varepsilon)} \lesssim \widehat{\mu}_\varepsilon \lesssim \sum_{k=1}^{\infty} c_k \frac{v(B(z_k,\varepsilon))}{v(E(z_k,\varepsilon))} \chi_{E(z_k,3\varepsilon)}.$$

Hence

$$\widetilde{\mu}_\varepsilon \in L^p \Leftrightarrow \sum_{k=1}^{\infty} c_k^p v(B(z_k,\varepsilon)) < \infty$$

and

$$\widehat{\mu}_\varepsilon \in L^p \Leftrightarrow \sum_{k=1}^{\infty} c_k^p \frac{(v(B(z_k,\varepsilon)))^p}{(v(E(z_k,\varepsilon)))^{p-1}} < \infty.$$

Since

$$\frac{c_k^p (v(B(z_k,\varepsilon)))^p (v(E(z_k,\varepsilon)))^{1-p}}{c_k^p v(B(z_k,\varepsilon))} = \Big(\frac{v(B(z_k,\varepsilon))}{v(E(z_k,\varepsilon))}\Big)^{p-1} \asymp (1-|z_k|)^{(n-1)(p-1)} \to 0$$

as $k \to \infty$, we can choose c_k such that $\widehat{\mu}_\varepsilon \in L^p$ but $\widetilde{\mu}_\varepsilon \notin L^p$. On the other hand, one can easily see that $\widetilde{\mu}_\varepsilon \in L^p$ implies $\widehat{\mu}_\varepsilon \in L^p$.

Acknowledgments. I am deeply grateful to my advisors Professors Alexander Borichev and El Hassan Youssfi for their help and many suggestions during the preparation of this paper.

References

[1] *H. Arroussi, I. Park, J. Pau*: Schatten class Toeplitz operators acting on large weighted Bergman spaces. Stud. Math. *229* (2015), 203–221. zbl MR doi
[2] *L. Carleson*: An interpolation problem for bounded analytic functions. Am. J. Math. *80* (1958), 921–930. zbl MR doi
[3] *L. Carleson*: Interpolations by bounded analytic functions and the corona problem. Ann. Math. *76* (1962), 547–559. zbl MR doi
[4] *W. W. Hastings*: A Carleson measure theorem for Bergman spaces. Proc. Am. Math. Soc. *52* (1975), 237–241. zbl MR doi
[5] *P. Lin, R. Rochberg*: Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights. Pac. J. Math. *173* (1996), 127–146. zbl MR doi
[6] *D. Luecking*: A technique for characterizing Carleson measures on Bergman spaces. Proc. Am. Math. Soc. *87* (1983), 656–660. zbl MR doi
[7] *D. Luecking*: Trace ideal criteria for Toeplitz operators. J. Func. Anal. *73* (1987), 345–368. zbl MR doi
[8] *J. Pau, R. Zhao*: Carleson measures and Toeplitz operators for weighted Bergman spaces on the unit ball. Mich. Math. J. *64* (2015), 759–796. zbl MR doi

[9] *J. Á. Peláez, J. Rättyä*: Weighted Bergman spaces induced by rapidly increasing weights. Mem. Am. Math. Soc. *227* (2014), 124 pages. zbl MR doi
[10] *J. Á. Peláez, J. Rättyä*: Embedding theorems for Bergman spaces via harmonic analysis. Math. Ann. *362* (2015), 205–239. zbl MR doi
[11] *J. Á. Peláez, J. Rättyä*: Trace class criteria for Toeplitz and composition operators on small Bergman spaces. Adv. Math. *293* (2016), 606–643. zbl MR doi
[12] *J. Á. Peláez, J. Rättyä, K. Sierra*: Berezin transform and Toeplitz operators on Bergman spaces induced by regular weights. J. Geom. Anal. *28* (2018), 656–687. zbl MR doi
[13] *K. Seip*: Interpolation and sampling in small Bergman spaces. Collect. Math. *64* (2013), 61–72. zbl MR doi
[14] *K. Zhu*: Spaces of Holomorphic Functions in the Unit Ball. Graduate Texts in Mathematics 226, Springer, New York, 2005. zbl MR doi
[15] *K. Zhu*: Operator Theory in Function Spaces. Mathematical Surveys and Monographs 138, American Mathematical Society, Providence, 2007. zbl MR doi

Author's address: Van An Le, Aix-Marseille University, CNRS, Centrale Marseille, I2M, Marseille, France; University of Quynhon, Department of Mathematics and Statistics, 170 An Duong Vuong, Quy Nhon, Vietnam, e-mail: `vanandkkh@gmail.com`, `levanan@qnu.edu.vn`.