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CHARACTERIZATION BY INTERSECTION GRAPH OF SOME FAMILIES OF FINITE NONSIMPLE GROUPS

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Abstract. For a finite group G, $\Gamma(G)$, the intersection graph of G, is a simple graph whose vertices are all nontrivial proper subgroups of G and two distinct vertices H and K are adjacent when $H \cap K \neq 1$. In this paper, we classify all finite nonsimple groups whose intersection graphs have a leaf and also we discuss the characterizability of them using their intersection graphs.

Keywords: intersection graph; leaf; nonsimple group; characterization

MSC 2020: 05C25, 20D99

1. Introduction

Throughout this paper all graphs are finite, undirected, with no loops and no multiple edges. The vertex set and the edge set of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A *complete graph* is a graph in which all vertices are adjacent and a *null graph* is a graph with no edges. For a vertex v, $\deg(v)$ is the number of vertices adjacent to v and is called the *degree* of v. A vertex of degree 1 is called a *leaf*. By $\mathrm{nl}(\Gamma)$, we denote the number of leaves of a graph Γ. A *path* between two distinct vertices $u, v \in V(\Gamma)$, is defined as a sequence of distinct vertices $u = v_0, v_1, \ldots, v_n = v$ such that $\{v_i, v_{i+1}\} \in E(\Gamma)$ for $0 \leqslant i \leqslant n-1$, and n is called the *length* of this path. The length of a shortest path between two distinct vertices u and v is called the *distance* between them and is denoted by $d(u, v)$. In case there is no path connecting u and v, we define $d(u, v)$ to be infinite. A *connected* graph is a graph in which there exists a path between each two distinct vertices. For a vertex $v \in V(\Gamma)$, the *neighbourhood* of v is denoted by $N(v)$ and is defined as $N(v) = \{u \in V(\Gamma)\colon \{u, v\} \in E(\Gamma)\}$. If S is a nonempty subset of $V(\Gamma)$, then the neighbourhood of S is defined as $N(S) = \bigcup\limits_{v \in S} N(v)$. We say two vertices u and v

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are *coneighbour* when $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. By this definition, two nonadjacent leaves are coneighbour when they are both adjacent to a vertex v_0.

For a finite group G, the intersection graph of G, denoted by $\Gamma(G)$, is an undirected graph whose vertex set consists of all nontrivial proper subgroups of G and two distinct vertices H and K are adjacent when $H \cap K \neq 1$. Csákány and Pollák in [2] introduced the intersection graph of a finite group. Later, some authors determined finite groups with disconnected intersection graphs, see [9], $K_{3,3}$-free intersection graphs, see [4], planar intersection graphs, see [6] and triangle-free intersection graphs, see [1]. Also the intersection graphs of abelian groups are discussed in [5] and [11]. The authors in [7] classified all finite groups with regular intersection graphs. Recently the authors in [8] classified all finite simple groups whose intersection graphs have a leaf and as a consequence, it is proved that these groups are uniquely determined by their intersection graphs. We say a finite group G is characterizable by its intersection graph when $\Gamma(G) \cong \Gamma(H)$ implies that $G \cong H$ for a finite group H.

In this paper, we first classify all finite nonsimple groups with at least a leaf in their intersection graphs. Then for a finite nonsimple group G with a leaf in $\Gamma(G)$, we show that in some cases G is characterizable by intersection graph and also in other cases we discuss the structure of G.

Throughout this paper, for a finite group G and a nontrivial proper subgroup H of G, by v_H we mean the corresponding vertex in $\Gamma(G)$. For two finite groups H and K by $H \ltimes K$ we mean a semidirect product of H and K, where H acts nontrivially on K. For a prime p, by $\mathbb{Z}_p^n$ we denote the elementary abelian group of order p^n. Also by D_n we mean the dihedral group of order n for an even number n. For a group G and two nontrivial proper subgroups L and M of G, we say M is an *overgroup* for L when $L < M$. Finally for two coprime integers $a > b > 0$ and a natural number n, a prime p is called a *primitive prime divisor* or briefly a PPD of $a^n - b^n$ when $p \,|\, (a^n - b^n)$ and $p \nmid (a^k - b^k)$ for each $0 < k < n$. In the case $b = 1$, we sometimes write $\mathrm{ord}_p(a) = n$.

2. Preliminary results

In this section, we state some lemmas which are necessary in the proofs of our main results.

Lemma 2.1 ([9], Theorem). *A finite group with a disconnected intersection graph is $\mathbb{Z}_p \times \mathbb{Z}_q$, where both p, q are primes, or a Frobenius group whose complement is a prime order group and the kernel is a minimal normal subgroup.*

Lemma 2.2. *Let G be a Frobenius group isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_q^n$, where a subgroup of order p acts irreducibly on the kernel. Then p is a PPD of $q^n - 1$.*

Proof. Let X be a generator of $\mathbb{Z}_p$, acting as $X \in \mathrm{Aut}(\mathbb{Z}_q^n) = \mathrm{GL}(n, q)$. Then saying that X acts irreducibly is the same as saying the characteristic polynomial $g_X(t) = \det(tI - X)$ is an irreducible factor of $t^p - 1 \in \mathbb{F}_q[t]$. Let K be the splitting field of $g_X(t)$ over $\mathbb{F}_q$. Thus $K = \mathbb{F}_q(\zeta)$, where ζ is a primitive root, and $\zeta^p = 1$. Then the Galois group of K over $\mathbb{F}_q$ is cyclic, generated by the Frobenius automorphism $\sigma(a) = a^q$, where $a \in \mathbb{F}_q$. If p divides $q^r - 1$, then $\sigma^r(\zeta) = \zeta$. Also

$$h(t) = \prod_{i=1}^{r} \left(t - \sigma^i(\zeta)\right)$$

is a member of $\mathbb{F}_q[t]$, because σ permutes the terms, and is a factor of

$$g_X(t) = \prod_{i=1}^{n} \left(t - \sigma^i(\zeta)\right).$$

This contradicts the irreducibility of $g_X(t)$. $\square$

Using the above results, we investigate the structure of finite groups with disconnected intersection graphs:

Corollary 2.3. *Let G be a finite group. Then the intersection graph of G is disconnected if and only if G is isomorphic to one of the following groups:*
(1) *$\mathbb{Z}_p \times \mathbb{Z}_q$ for some primes p and q (not necessarily distinct),*
(2) *$\mathbb{Z}_p \ltimes \mathbb{Z}_q^n$ for some distinct primes p and q, where p is a PPD of $q^n - 1$.*

Proof. It is obvious that the groups in (1) and (2) have disconnected intersection graphs. Conversely suppose that $\Gamma(G)$ is disconnected. If G is an abelian group, then by Lemma 2.1 we get the result. So suppose that G is a nonabelian group. Then by Lemma 2.1, $G \cong HN$, where N is a minimal normal subgroup of G and H is the complement of N of some prime order, say p. Let q be a prime such that $q \mid |N|$. Obviously $q \neq p$. We know that every minimal normal subgroup of G is an elementary abelian group or a direct product of some isomorphic nonabelian simple groups. Since N has a fixed-point-free automorphism of prime order, N is a nilpotent group and so N is an elementary abelian q-subgroup of G. Now by Lemma 2.2, we get the result. $\square$

Corollary 2.4. *Let G be a finite group whose intersection graph is disconnected. Then every connected component of $\Gamma(G)$ is an isolated vertex except possibly a connected component with at least four vertices.*

Corollary 2.5. *Let G be a finite group and $\Gamma(G)$ its intersection graph. Then $\Gamma(G)$ contains an isolated vertex if and only if G is one of the following groups:*
(1) $\mathbb{Z}_{p^2}$ *for some prime p,*
(2) $\mathbb{Z}_p \times \mathbb{Z}_q$ *for some primes p and q (not necessarily distinct),*
(3) $\mathbb{Z}_p \ltimes \mathbb{Z}_q^n$ *for some distinct primes p and q, where p is a PPD of $q^n - 1$.*

P r o o f. Suppose that $\Gamma(G)$ contains an isolated vertex, say v_L, for some nontrivial proper subgroup L. If $|V(\Gamma(G))| = 1$, then L is the only nontrivial proper subgroup of G and so $G \cong \mathbb{Z}_{p^2}$ for some prime p. If $|V(\Gamma(G))| > 1$, then $\Gamma(G)$ is disconnected and by Corollary 2.3, we get the result. The converse is obvious. $\square$

Lemma 2.6. *Let G be a finite group.*
(a) *If L is a nontrivial proper subgroup of G such that in the intersection graph of G we have $(N(v_L) \setminus \{v_H\}) \subsetneqq (N(v_H) \setminus \{v_L\})$ for each vertex v_H adjacent to v_L, then L is a minimal subgroup of G.*
(b) *If L is a minimal subgroup of G, then $(N(v_L) \setminus \{v_H\}) \subseteq (N(v_H) \setminus \{v_L\})$ for each subgroup H of G, where $H \cap L \neq 1$.*

P r o o f. (a) Let $K \neq 1$ be a subgroup of L. Then $K \cap S \neq 1$ implies that $L \cap S \neq 1$ for each subgroup S of G and hence $(N(v_K) \setminus \{v_L\}) \subseteq (N(v_L) \setminus \{v_K\})$, which implies that $K = L$ and so L is a minimal subgroup of G.

(b) If $v_T \in N(v_L) \setminus \{v_H\}$, then $L \subseteq T$ and hence $T \cap H \neq 1$. So we get the result. $\square$

Example 2.7. Consider the intersection graph of $\mathbb{Z}_{18} = \langle a \,|\, a^{18} = 1\rangle$ (see Figure 1).

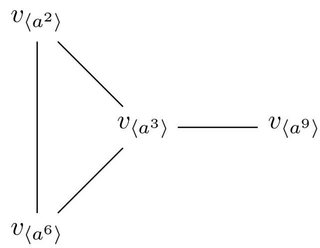

Figure 1. $\Gamma(\mathbb{Z}_{18})$.

Although $\langle a^6\rangle$ is a minimal subgroup of $\mathbb{Z}_{18}$, obviously $N(v_{\langle a^6\rangle}) \setminus \{v_{\langle a^2\rangle}\} = N(v_{\langle a^2\rangle}) \setminus \{v_{\langle a^6\rangle}\}$ and so the converse of (a) in the above lemma is not true in general. Also note that $N(v_{\langle a^2\rangle}) \setminus \{v_{\langle a^6\rangle}\} = N(v_{\langle a^6\rangle}) \setminus \{v_{\langle a^2\rangle}\}$ and $(N(v_{\langle a^2\rangle}) \setminus \{v_{\langle a^3\rangle}\}) \subseteq (N(v_{\langle a^3\rangle}) \setminus \{v_{\langle a^2\rangle}\})$, while $\langle a^2\rangle$ is not a minimal subgroup of $\mathbb{Z}_{18}$. Therefore the converse of (b) is not valid in general.

Lemma 2.8. *Let G be a finite group. Then $\Gamma(G)$ contains a leaf if and only if one of the following cases occurs:*

(i) *if G has only two nontrivial subgroups, then $G \cong \mathbb{Z}_{p^3}$ for some prime p,*

(ii) *if G has more than two nontrivial proper subgroups, then G contains a minimal subgroup, say L, such that L is contained properly in exactly one proper subgroup of G.*

Proof. Suppose that $\Gamma(G)$ contains a leaf, say v_L, which is adjacent to a unique vertex, say v_{M_L}, for some subgroups L and M_L of G, respectively. If v_{M_L} is a leaf too, then $\{v_L, v_{M_L}\}$ is a connected component of $\Gamma(G)$. So by Corollary 2.4, $\Gamma(G)$ is connected and hence L and M_L are the only nontrivial proper subgroups of G, which implies that $G \cong \mathbb{Z}_{p^3}$ for some prime p. If v_{M_L} is adjacent to some vertex other than v_L, then $(N(v_L) \setminus \{v_{M_L}\}) \subsetneqq (N(v_{M_L}) \setminus \{v_L\})$. Now using Lemma 2.6 (a), we see that L is a minimal subgroup of G. Moreover, M_L is the only overgroup of L. So we get the result. The converse is obvious by Lemma 2.6 (b). $\square$

Remark 2.9. Following the notation we used in the proof of Lemma 2.8, in the sequel of this paper when the intersection graph of a finite group G contains a leaf, we use v_L and v_{M_L} for a leaf of $\Gamma(G)$ and the unique vertex adjacent to this leaf, respectively. Then $L \cong \mathbb{Z}_p$ is a minimal subgroup of G, M_L is a maximal subgroup of G and M_L is the only overgroup of L.

Lemma 2.10. *Let G be a finite group with at least three nontrivial proper subgroups and let v_L be a leaf of $\Gamma(G)$, where $L \cong \mathbb{Z}_p$ for some prime p. Then M_L is one of the following groups:*

(1) *If M_L is a p-group, then $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_p$.*

(2) *If M_L is an abelian non-p-group, then $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_q$ for some prime q, where $q \neq p$.*

(3) *If M_L is a nonabelian group and $L \ntrianglelefteq M_L$, then $M_L \cong \mathbb{Z}_p \ltimes \mathbb{Z}_q^n$ for some prime q, where p is a PPD of $q^n - 1$.*

(4) *If M_L is a nonabelian group and $L \trianglelefteq M_L$, then $M_L \cong \mathbb{Z}_q \ltimes \mathbb{Z}_p$ for some prime q, where $q \,|(p-1)$.*

Proof. By Remark 2.9, M_L is the only overgroup of L. So v_L is an isolated vertex of $\Gamma(M_L)$. We claim that $\Gamma(M_L)$ is disconnected. If $\Gamma(M_L)$ is connected, then $|V(\Gamma(M_L))| = 1$. So by Corollary 2.5, $M_L \cong \mathbb{Z}_{p^2}$. If v_{M_L} is a leaf of $\Gamma(G)$, then by the proof of Lemma 2.8, $G \cong \mathbb{Z}_{p^3}$, which is a contradiction. So v_{M_L} is adjacent to at least one vertex other than v_L, say v_H, for some nontrivial proper subgroup H of G. Since $M_L \cap H \neq 1$, $M_L \cap H$ contains a minimal subgroup of G. But the only minimal subgroup of $M_L \cong \mathbb{Z}_{p^2}$ is L and therefore $L \leqslant H$, a contradiction. Thus $\Gamma(M_L)$ is disconnected. Now by Corollary 2.3, we get the result. $\square$

3. Main results

In the following theorem, we determine all finite groups whose intersection graphs have at least one leaf.

Theorem 3.1. *Let G be a finite group with at least three nontrivial proper subgroups such that $\Gamma(G)$ has a leaf. Then one of the following cases occurs:*

(a) *If G is a p-group, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, D_8 or $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$ for some odd prime p.*

(b) *If G is not a p-group, then there exist distinct prime divisors p and q of $|G|$ such that G is isomorphic to one of the following groups:*

(1) $\mathbb{Z}_p \times \mathbb{Z}_{q^2}$,

(2) $\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$, *where* $q \,|\, (p-1)$,

(3) $\mathbb{Z}_q \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, *where q is a PPD of $p^2 - 1$,*

(4) $\mathbb{Z}_p \ltimes N$, *where N is an extra special q-group with $\Phi(N) \cong \mathbb{Z}_q$ and a subgroup of order p acts irreducibly on $N/\Phi(N)$,*

(5) $\mathbb{Z}_p \ltimes N$, *where N is a q-group of order q^{2n} with a unique nontrivial proper characteristic subgroup $\Phi(N) \cong \mathbb{Z}_q^n$ and a subgroup of order p acts irreducibly on $N/\Phi(N)$,*

(6) *a finite nonabelian simple group in which there exist a minimal subgroup, say L, and a maximal subgroup, say M_L, such that M_L is the only overgroup of L in G.*

Proof. Let v_L be a leaf of $\Gamma(G)$. We consider two cases:

Case (a): Let G be a p-group. Clearly we get that $|G| = p^3$.

(i) *First suppose that G is an abelian p-group.* Since $\Gamma(G)$ has at least three vertices, so $G \ncong \mathbb{Z}_{p^3}$. Also it can be easily verified that the elementary abelian group of order p^3 has no leaf. Let $G = \langle a, b \colon a^{p^2} = b^p = 1, ab = ba\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. Obviously $\Phi(G) = \langle a^p \rangle$ and it can be easily verified that the structures of nontrivial proper subgroups of G and the number of subgroups isomorphic to each structure are as follows:

- ▷ p maximal subgroups $H_i = \langle b^i a\rangle \cong \mathbb{Z}_{p^2}$, where $1 \leqslant i \leqslant p$,
- ▷ one maximal subgroup $M = \langle a^p, b\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$,
- ▷ $p+1$ minimal subgroups $\Phi(G) = \langle a^p\rangle$ and $L_i = \langle ba^{ip}\rangle \cong \mathbb{Z}_p$, where $1 \leqslant i \leqslant p$.

Note that for each $i \neq j$, we have $H_i \cap H_j = \Phi(G)$. Also M is the only overgroup of L_i, where $1 \leqslant i \leqslant p$. So each v_{L_i} is a leaf of $\Gamma(G)$ for $1 \leqslant i \leqslant p$ and therefore $\mathrm{nl}(\Gamma(G)) = p$. In this case all leaves are coneighbour (see Figure 2).

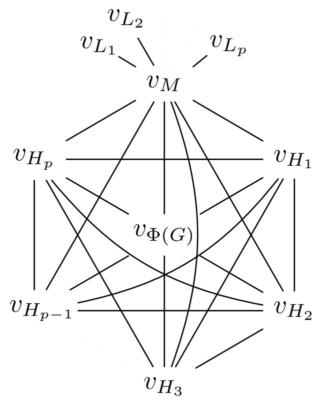

Figure 2. $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

(ii) *Now suppose that G is a nonabelian p-group.* If $p = 2$, then $G \cong D_8$, since $\Gamma(Q_8)$ has no leaf. So $G = \langle a, b\colon\ a^4 = b^2 = 1, bab = a^{-1}\rangle$. In this case, $\Phi(D_8) = \langle a^2\rangle$ and obviously $v_{\Phi(D_8)}$ is not a leaf. So each vertex correspondent to any minimal subgroup other than $\Phi(D_8)$ is a leaf of $\Gamma(D_8)$. Thus $M_L \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathrm{nl}(\Gamma(D_8)) = 4$ (see Figure 3).

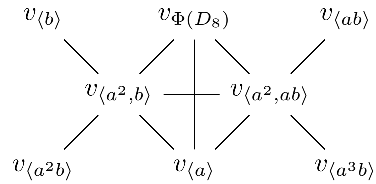

Figure 3. $\Gamma(D_8)$.

If $p > 2$, then there are two nonabelian groups of order p^3 which are isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$.

Let $G = \langle a, b\colon\ a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1}\rangle$, which is isomorphic to $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$. Then $\Phi(G) = Z(G) = \langle a^p\rangle$ and it can be easily verified that $(b^i a^j)^m = b^{mi} a^{(m(m-1)ip/2+m)j}$ for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant p^2$ and $m \geqslant 1$. So $o(b^i a) = p^2$ and $o(b^i a^p) = p$ for $1 \leqslant i \leqslant p$. Then nontrivial proper subgroups of G are as follows:

- p maximal subgroups $H_i = \langle b^i a\rangle \cong \mathbb{Z}_{p^2}$, where $1 \leqslant i \leqslant p$,
- one maximal subgroup $M = \langle a^p, b\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$,
- $p + 1$ minimal subgroups $\Phi(G) = \langle a^p\rangle$ and $L_i = \langle b^i a^p\rangle \cong \mathbb{Z}_p$, where $1 \leqslant i \leqslant p$.

It can be easily verified that for each $1 \leqslant i \leqslant p$, M is the only overgroup of L_i in G. Also $H_i \cap H_j = \Phi(G)$ for each $i \neq j$ and so $\mathrm{nl}(\Gamma(G)) = p$. By the above discussion, we see that $\Gamma(G)$ is isomorphic to $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ (see Figure 2).

Now let $G = \langle a,b,c\colon a^p = b^p = c^p = 1, ab = ba, ac = ca, c^{-1}bc = ab\rangle \cong \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$. Then $Z(G) = \langle a\rangle$ and it can be easily verified that $(b^ic^j)^n = a^{(-n(n-1)ij/2)}b^{ni}c^{nj}$. Hence every element $x \in G$ can be uniquely presented in the form $x = a^mb^nc^t$, where $1 \leqslant m,n,t \leqslant p$. Let $H_1 = \langle a,b\rangle$, $H_2 = \langle a,bc\rangle$, $H_3 = \langle a,b^2c\rangle,\ldots,H_p = \langle a,b^{p-1}c\rangle$ and $H_{p+1} = \langle a,c\rangle$. We claim that $H_i \cap H_j = \langle a\rangle$ for each $i \neq j$. If $x \in H_1 \cap H_j$, where $2 \leqslant j \leqslant p+1$, then $x = a^mb^n = a^s(b^{j-1}c)^t$ for some integers $0 \leqslant m,n,s,t \leqslant p-1$. So $a^mb^n = a^sa^{(-t(t-1)(j-1)/2)}b^{t(j-1)}c^t$ and hence $a^{s-t(t-1)(j-1)/2-m}b^{t(j-1)-n}c^t = 1$. Therefore we get that $t = n = 0$ and $m = s$. This shows that $x \in \langle a\rangle$. If $x \in H_i \cap H_j$, where $2 \leqslant i < j \leqslant p+1$, then similarly we get the result. So the maximal subgroups of G are $H_1,\ldots,H_{p+1}$ and all of them are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Thus every minimal subgroup of G, other than $Z(G)$, is contained properly in only one maximal subgroup of G. We know that each H_i, where $1 \leqslant i \leqslant p+1$, has $p+1$ minimal subgroups. Let L_{ij}, where $1 \leqslant j \leqslant p$, be the minimal subgroups of H_i other than $Z(G)$. Then $v_{L_{ij}}$ is a leaf of $\Gamma(G)$ for each i, j. By the above discussion, nontrivial proper subgroups of G are as follows:

▷ $p+1$ maximal subgroups $H_1,\ldots,H_{p+1}$, isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$,
▷ the minimal subgroup $Z(G) \cong \mathbb{Z}_p$,
▷ $p(p+1)$ minimal subgroups $L_{ij} \cong \mathbb{Z}_p$, where $1 \leqslant i \leqslant p+1$ and $1 \leqslant j \leqslant p$.

So $\mathrm{nl}(\Gamma(G)) = p(p+1)$ and each leaf belongs to a set of p coneighbour vertices (see Figure 4).

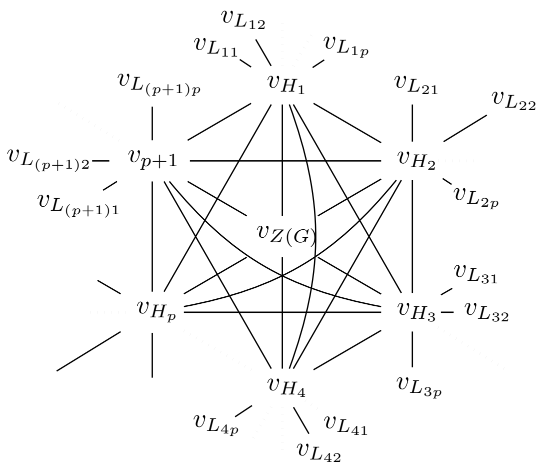

Figure 4. $\Gamma(\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p))$.

Case (b): Suppose that G is not a p-group. We consider two subcases:

(i) *Let L be a normal subgroup of G.* By assumption M_L is the only overgroup of L in G. So the quotient group G/L contains a unique nontrivial proper subgroup M_L/L, which implies that $G/L \cong \mathbb{Z}_{q^2}$ for some prime q. Since G is not a p-group, we get that $q \neq p$. Thus G is a split extension of $L \cong \mathbb{Z}_p$ by a Sylow q-subgroup isomorphic to $\mathbb{Z}_{q^2}$.

If G is an abelian group, then $G \cong \mathbb{Z}_p \times \mathbb{Z}_{q^2}$ and hence $G = \langle a, b \colon\ a^p = b^{q^2} = 1, ab = ba\rangle$. So nontrivial proper subgroups of G are $L = \langle a\rangle$, $M_L = \langle a, b^q\rangle$, $H = \langle b\rangle$ and $K = \langle b^q\rangle$. Thus in this case we have $\mathrm{nl}(\Gamma(G)) = 1$ (see Figure 5).

$$\begin{tikzcd}
& v_L \arrow[d, dash] & \\
& v_{M_L} \arrow[dl, dash] \arrow[dr, dash] & \\
v_H \arrow[rr, dash] & & v_K
\end{tikzcd}$$

Figure 5. $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{q^2})$.

If G is a nonabelian group, then $G \cong \mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$, where $q \,|\, (p-1)$. We note that if $q^2 \nmid (p-1)$, then there exists only one finite group of the form $\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$, while for $q^2 \,|\, (p-1)$, there exist two nonisomorphic nonabelian groups of the form $\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$.

Let $q^2 \nmid (p-1)$. Then

$$G_1 = \langle a, b \colon\ a^p = b^{q^2} = 1,\ b^{-1}ab = a^r,\ \mathrm{ord}_p(r) = q\rangle.$$

It can be easily verified that $ab^q = b^q a$ and $a^{-i}b^{-1}a^i = a^{i(r-1)}b^{-1}$ for $1 \leqslant i \leqslant p$. So $Z(G_1) = \langle b^q\rangle$ and nontrivial proper subgroups of G_1 are as follows:

- ▷ p maximal subgroups $H_i = \langle a^{i(r-1)}b^{-1}\rangle \cong \mathbb{Z}_{q^2}$, where $1 \leqslant i \leqslant p$,
- ▷ the minimal subgroup $Z(G_1) = \langle b^q\rangle \cong \mathbb{Z}_q$,
- ▷ the minimal subgroup $L = \langle a\rangle \cong \mathbb{Z}_p$,
- ▷ the maximal subgroup $M_L = \langle a, b^q\rangle \cong \mathbb{Z}_{pq}$.

Note that the maximal subgroup M_L is the only overgroup of L in G_1. So v_L is a leaf of $\Gamma(G_1)$ adjacent to v_{M_L}. Also for each $i \neq j$ we have $H_i \cap H_j = Z(G_1)$. Therefore $\mathrm{nl}(\Gamma(G_1)) = 1$ (see Figure 6). As an example for this case, we can give the dicyclic group of order 12 isomorphic to $\mathbb{Z}_4 \ltimes \mathbb{Z}_3$.

Now let $q^2 \,|\, (p-1)$. In this case, in addition to the above structure, there exists another finite group of the form $\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$, which is presented as

$$G_2 = \langle a, b \colon\ a^p = b^{q^2} = 1,\ b^{-1}ab = a^r,\ \mathrm{ord}_p(r) = q^2\rangle.$$

Note that in this case the subgroup of order pq is not isomorphic to $\mathbb{Z}_{pq}$, since $ab^q \neq b^q a$. Also we have $a^{-i}b^{-q}a^i = a^{i(r^q-1)}b^{-q}$ for $1 \leqslant i \leqslant p$. Thus nontrivial proper subgroups of G_2 are as follows:

- ▷ p maximal subgroups $H_i = \langle a^{i(r-1)}b^{-1}\rangle \cong \mathbb{Z}_{q^2}$, where $1 \leqslant i \leqslant p$,
- ▷ p minimal subgroups $K_i = \langle a^{i(r^q-1)}b^{-q}\rangle \cong \mathbb{Z}_q$, where $1 \leqslant i \leqslant p$,

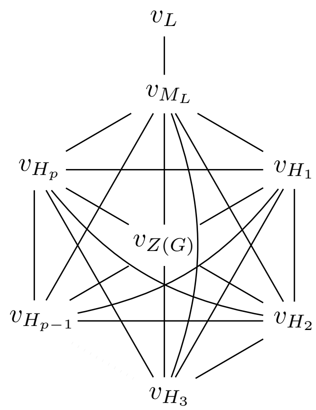

Figure 6. $\Gamma(G_1)$.

▷ one minimal subgroup $L = \langle a \rangle \cong \mathbb{Z}_p$,

▷ one maximal subgroup $M_L = \langle a, b^q \rangle \cong \mathbb{Z}_q \ltimes \mathbb{Z}_p$.

Similarly to the above M_L is the only overgroup of L in G_2. Also $H_i \cap M_L = K_i$ for $1 \leqslant i \leqslant p$. Therefore, $\mathrm{nl}(\Gamma(G_2)) = 1$ (see Figure 7). As an example for this case we can give $\mathrm{GA}_1(5) = \mathrm{Sz}(2) \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5$.

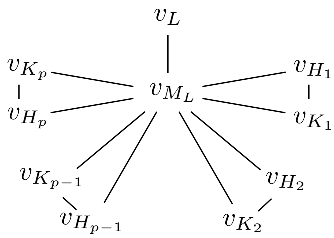

Figure 7. $\Gamma(G_2)$.

(ii) *Suppose that L is not a normal subgroup of G.* We consider two subcases:

Suppose that M_L is a normal subgroup of G. Then $[G : M_L] = r$ for some prime r. Using the possible structures of M_L which are recorded in Lemma 2.10, we consider four cases to determine the structure of G:

(1) If $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then obviously $r \neq p$, since G is not a p-group. In this case we have $G \cong \mathbb{Z}_r \times (\mathbb{Z}_p \times \mathbb{Z}_p)$ or $G \cong \mathbb{Z}_r \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$. If $G \cong \mathbb{Z}_r \times (\mathbb{Z}_p \times \mathbb{Z}_p)$, then every subgroup of order p lies in a subgroup of G isomorphic to $\mathbb{Z}_{pr}$, too. Hence v_L is not a leaf of $\Gamma(G)$, which is a contradiction. So $G \cong \mathbb{Z}_r \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, where $r \,|\, (p^2 - 1)$ and by Lemma 2.2, $r \nmid (p-1)$, since there exists no subgroup of order pr. Therefore $r \neq 2$ and $r \,|\, (p+1)$. Then nontrivial proper subgroups of G are as follows:

▷ one maximal subgroup $M \cong \mathbb{Z}_p \times \mathbb{Z}_p$,

▷ $p+1$ minimal subgroups $L_i \cong \mathbb{Z}_p$, where $1 \leqslant i \leqslant p+1$,
▷ p^2 minimal subgroups $K_j \cong \mathbb{Z}_r$, where $1 \leqslant j \leqslant p^2$.

In this case all leaves are coneighbours, $\Gamma(G)$ is disconnected and $\mathrm{nl}(\Gamma(G)) = p+1$ (see Figure 8). Note that A_4 is an example for this case.

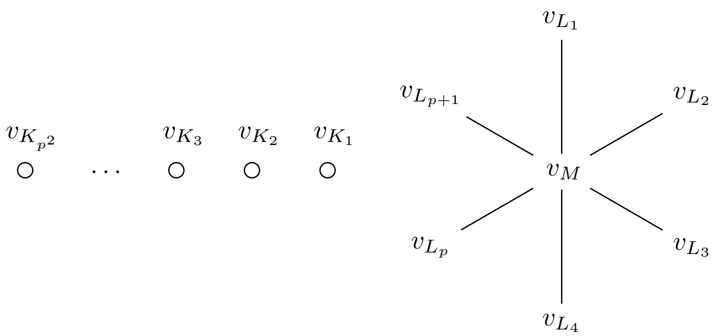

Figure 8. $\Gamma(\mathbb{Z}_r \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p))$, where r is a PPD of $p^2 - 1$.

(2) If $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_q$, then $L \cong \mathbb{Z}_p$ is a normal subgroup of G, a contradiction.

(3) If $M_L \cong \mathbb{Z}_p \ltimes \mathbb{Z}_q^n$, then we claim that this case implies a contradiction. If $r = p$, then L lies in a Sylow p-subgroup of G of order p^2, a contradiction. Thus $r \neq p$. Note that in this case $L \ntrianglelefteq M_L$ and so $N_G(L) = L$. Also $M_L \trianglelefteq G$ and M_L contains $L \cong \mathbb{Z}_p$. So every Sylow p-subgroup of G is contained in M_L. But the number of Sylow p-subgroups of M_L is $[M_L : L] = q^n$, while the number of Sylow p-subgroups of G is $[G : L] = rq^n$, which is clearly a contradiction.

(4) If $M_L \cong \mathbb{Z}_q \ltimes \mathbb{Z}_p$, then $L \cong \mathbb{Z}_p$ is a normal subgroup of G, a contradiction.

Suppose that M_L is not a normal subgroup of G. Then $N_G(M_L) = M_L$. Again we check each possible structure for M_L, by Lemma 2.10.

(1) If $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then M_L is a Sylow p-subgroup of G. Also $M_L = Z(N_G(M_L))$. Now by Burnside's normal p-complement theorem we see that there exists a proper normal subgroup of G, say N, such that $(|N|, p) = 1$ and hence $|G| = p^2|N|$. But this implies that NL is a proper subgroup of G and obviously $NL \neq M_L$, a contradiction.

(2) If $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_q$, then $L \cong \mathbb{Z}_p$ is a Sylow p-subgroup of G. Since $L \leqslant Z(N_G(L)) = M_L$, G has a normal p-complement subgroup, say N. Obviously $q \mid |N|$. We claim that N is a q-group. On the contrary suppose that for some prime r other than q we have $r \mid |N|$. Let R be a Sylow r-subgroup of G. If $R \trianglelefteq G$, then RL is a proper subgroup of G which contains properly L, a contradiction. So $R \ntrianglelefteq G$. Since $R \leqslant N$ and $N \trianglelefteq G$, so every Sylow r-subgroup of G is contained in N and hence $[G : N_G(R)] = [N : N_N(R)]$. Obviously $p \nmid [N : N_N(R)]$ and therfore $p \nmid [G : N_G(R)]$. But this implies that $p \mid |N_G(R)|$ and so there exists a supgroup $L_2 \cong \mathbb{Z}_p$ such that $L_2 \leqslant N_G(R)$. Thus RL_2 is a proper subgroup of G. Since Sylow p-subgroups of G

are conjugate in G, so for some element g we have $L \leqslant g^{-1}RL_2g$, a contradiction. So we get that $|N| = q^n$ and N is the Sylow q-subgroup of G. We claim that N is not a minimal normal subgroup of G, since otherwise L acts irreducibly on N and hence there is no proper subgroup of G containing properly L, which is a contradiction. So N has at least one nontrivial proper characteristic subgroup. It can be easily verified that $\Phi(N)$ is the only nontrivial proper characteristic subgroup of N, otherwise the subgroup L is contained properly in more than one subgroup of G, which is a contradiction. Moreover $|\Phi(N)| = q$, since $M_L \cong \mathbb{Z}_p \times \mathbb{Z}_q$. So N is the cyclic group of order q^2 or an extra special q-group. If $N \cong \mathbb{Z}_{q^2}$, then let a and b be some generators of N and L, respectively. Then $b^{-1}ab = a^r$, where $(r, q) = 1$. Since M_L is an abelian subgroup, we have $a^q b = ba^q$ and hence $q^2 \,|(rq - q)$ or equivalently $q\,|(r-1)$. So $r = tq+1$ for some $t > 0$. Note that if $t \equiv 0 \pmod q$, then G is an abelian group and hence $M_L \trianglelefteq G$, a contradiction. Moreover $q^2\,|(r^p - 1)$, since $a = b^{-p}ab^p = a^{r^p}$. But $r^p = (tq)^p + p(tq)^{p-1} + \ldots + ptq + 1$, which implies that $q^2 \,|\, ptq$, a contradiction. So N is an extra special q-group. Moreover L acts irreducibly on $N/\Phi(N)$ and hence p is a PPD of $q^{n-1} - 1$, by Lemma 2.2. In this case $\mathrm{nl}(\Gamma(G)) = [G : M_L] = q^{n-1}$. As an example for this case, we consider $SL_2(3) \cong \mathbb{Z}_3 \ltimes Q_8$, where $L \cong \mathbb{Z}_3$, $N \cong Q_8$ and $M_L \cong \mathbb{Z}_6$. Let a be an element of $SL_2(3)$ of order 3 and let $\{i, j, k\}$ be the generators of $N \cong Q_8$. Then nontrivial proper subgroups of $SL_2(3)$ are as follows:

- ▷ the maximal subgroup $N \cong Q_8$,
- ▷ the maximal subg three cyclic subgroups $\langle i \rangle$, $\langle j \rangle$, $\langle k \rangle$, isomorphic to $\mathbb{Z}_4$,
- ▷ the minimal subgroup $\Phi(N) = Z(SL_2(3)) \cong \mathbb{Z}_2$,
- ▷ four cyclic subgroups $L_1 = \langle a \rangle$, $L_2 = \langle a^i \rangle$, $L_3 = \langle a^j \rangle$, $L_4 = \langle a^k \rangle$, isomorphic to $\mathbb{Z}_3$,
- ▷ four maximal subgroups $M_{L_1} = \langle a, \Phi(N) \rangle$, $M_{L_2} = \langle a^i, \Phi(N) \rangle$, $M_{L_3} = \langle a^j, \Phi(N) \rangle$, $M_{L_4} = \langle a^k, \Phi(N) \rangle$, isomorphic to $\mathbb{Z}_6$.

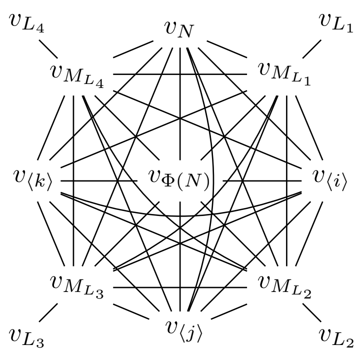

Figure 9. $\Gamma(SL_2(3))$.

(3) If $M_L \cong \mathbb{Z}_p \ltimes \mathbb{Z}_q^n$, then $L \cong \mathbb{Z}_p$ is a Sylow p-subgroup of G and similarly to Case (2), we get that $G \cong LN$, where N is a q-group with a unique nontrivial proper

characteristic subgroup $\Phi(N) \cong \mathbb{Z}_q^n$ and $|N| = q^m$, where $m > n \geqslant 1$. Hence N is an UCS q-group (groups with a unique nontrivial characteristic subgroup are called *UCS groups* and are well-studied in [3], [10]). By Lemma 2.2, p is a PPD of $q^n - 1$. Also we get that L acts irreducibly on $N/\Phi(N)$ and hence p is a PPD of $q^{m-n} - 1$. So $n = m - n$ or equivalently $2n = m$. It can be easily verified that v_{M_L} is adjacent to q^n leaves. Also each leaf is in correspondence with a subgroup of order p. In this case $\mathrm{nl}(\Gamma(G)) = [G : L] = q^{2n}$ and each leaf belongs to a set of q^n coneighbour vertices. As an examples for this case, we consider D_{18} (see Figure 10). It can be easily verified that nontrivial proper subgroups of D_{18} are as follows:

- ▷ one maximal subgroup $N \cong \mathbb{Z}_9$,
- ▷ one minimal subgroup $\Phi(N) = \Phi(G) \cong \mathbb{Z}_3$,
- ▷ nine minimal subgroups $L_{ij} \cong \mathbb{Z}_2$, where $1 \leqslant i, j \leqslant 3$,
- ▷ three maximal subgroups $M_{L_i} \cong \mathbb{S}_3$, where $1 \leqslant i \leqslant 3$.

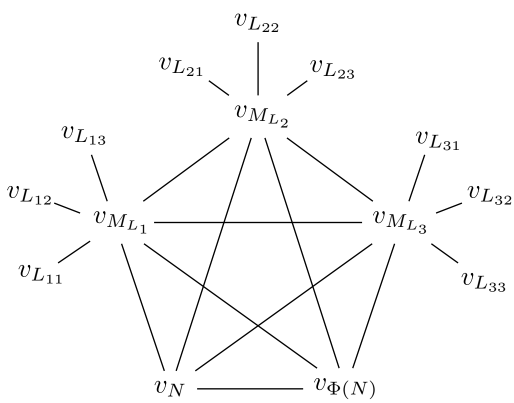

Figure 10. $\Gamma(D_{18})$.

The above example shows that N is not necessarily a special q-group.

(4) Finally suppose that $M_L \cong \mathbb{Z}_q \ltimes \mathbb{Z}_p$. Then $L \cong \mathbb{Z}_p$ is a Sylow p-subgroup of G. We claim that G is a simple group. On the contrary suppose that G is not simple and let N be a minimal normal subgroup of G. Obviously LN is a subgroup of G containing properly L. Thus $LN = M_L$ or $LN = G$. If $LN = M_L$, then $N \cong \mathbb{Z}_q$ and hence $M_L \cong \mathbb{Z}_{pq}$, a contradiction. Thus $LN = G$. Obviously $p \nmid |N|$. If a prime r other than q divides $|N|$, then N contains all Sylow r-subgroups of G. Let R be a Sylow r-subgroup of G. Then $[G : N_G(R)] = [N : N_N(R)]$, which implies that $p \,|\, |N_G(R)|$ and therefore there exits a subgroup of G of order $p|R|$, a contradiction. So N is an elementary abelian q-group and therefore L acts irreducibly on N, which implies that there is no proper subgroup of G containing properly L, a contradiction.

So we get the result. We note that in this case $\mathrm{nl}(\Gamma(G)) \geqslant [G : M_L] \geqslant 3$ (A_5 is an example for this case with $L \cong \mathbb{Z}_5$ and $M_L \cong D_{10}$). $\square$

Before passing to further results, it is very useful to summarize the results in the proof of Theorem 3.1. Let G be a finite nonsimple group such that $\Gamma(G)$ has more than two vertices and $\Gamma(G)$ has a leaf. Then the structures of G and $\mathrm{nl}(\Gamma(G))$ are as recorded in the following table:

#	the structure of G	$\mathrm{nl}(\Gamma(G))$	number of leaves in each set of coneighbour leaves	connectivity of $\Gamma(G)$
1	D_8	4	2	connected
2	$\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, p prime	p	p	connected
3	$\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, p odd prime	p	p	connected
4	$\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, p odd prime	$p(p+1)$	p	connected
5	$\mathbb{Z}_p \times \mathbb{Z}_{q^2}$, p and q distinct primes	1	1	connected
6	$\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$, p and q primes, $q \,\vert (p-1)$ and the subgroup of order pq is cyclic	1	1	connected
7	$\mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$, p and q primes, $q^2 \,\vert (p-1)$ and the subgroup of order pq is not cyclic	1	1	connected
8	$\mathbb{Z}_q \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, p prime and q is a PPD of p^2-1	$p+1$	$p+1$	disconnected
9	$\mathbb{Z}_p \ltimes N$, N is an extra special q-group of order q^n and a subgroup of order p acts irreducibly on $N/\Phi(N)$	q^{n-1}	1	connected
10	$\mathbb{Z}_p \ltimes N$, N is a UCS q-group of order q^{2n} with $\Phi(N) \cong \mathbb{Z}_q^n$ and a subgroup of order p acts irreducibly on $N/\Phi(N)$	q^{2n}	q^n	connected

Table 1. Finite nonsimple groups G whose $\Gamma(G)$ has a leaf.

In the sequel we state some definitions and lemmas which give a criterion for the simplicity of a finite group, when its intersection graph has a leaf.

Definition 3.2. In a simple graph Γ, a vertex $v \in V(\Gamma)$ is called an *ω-vertex* of Γ when $(N(v) \setminus \{v'\}) \subseteq (N(v') \setminus \{v\})$ for each $v' \in N(v)$.

Remark 3.3. If L is a minimal subgroup of G, then by Lemma 2.6 (b), v_L is an ω-vertex of $\Gamma(G)$. We note that the converse is not true in general.

Definition 3.4. Let Γ be a simple graph and S be a clique of Γ which contains an ω-vertex. Then S is called a *maximal ω-set*, when:
(1) for each pair of distinct vertices $u, v \in S$, u and v are coneighbour,
(2) if $v \in N(S) \setminus S$, then $N(v)$ contains at least one vertex, say z, such that $z \notin S$.

Obviously each vertex in a maximal ω-set is an ω-vertex. Also if $v \in N(S) \setminus S$, then v is adjacent to all vertices in S. It can be easily verified that each ω-vertex belongs to a unique maximal ω-set.

There are many graphs which have some maximal ω-sets. For example in every null graph Γ, $S = \{v\}$ is a maximal ω-set of Γ for each $v \in V(\Gamma)$. Also for every complete graph Γ, $S = V(\Gamma)$ is the only maximal ω-set of Γ. As an example in the intersection graphs of finite groups, see Figure 7, where $\{v_L\}$ and $\{v_{H_i}, v_{K_i}\}$, $1 \leqslant i \leqslant p$, are the maximal ω-sets of $\Gamma(G_2)$.

In the next lemma we get that maximal ω-sets play a key role in the study of intersection graphs of finite groups.

Lemma 3.5. *Let G be a finite group and $\Gamma(G)$ be its intersection graph.*
(a) *For each minimal subgroup L of G,*

$$S_L = \{v_K \colon L \text{ is the only minimal subgroup of } K\}$$

is the maximal ω-set of $\Gamma(G)$ containing v_L.
(b) *In every maximal ω-set of $\Gamma(G)$ there exists exactly one vertex v_L such that L is a minimal subgroup of G.*
(c) *v_H is an ω-vertex if and only if H has a unique minimal subgroup.*

Proof. (a) By Remark 3.3, v_L is an ω-vertex of $\Gamma(G)$ and obviously $v_L \in S_L$. Since L is the only minimal subgroup of K for each $v_K \in S_L$, we get that S_L is a clique of $\Gamma(G)$ and all vertices in S_L are coneighbour. Finally if $v_U \in N(S_L) \setminus S_L$, then U contains at least two minimal subgroups and hence $N(v_K) \setminus \{v_U\} \subsetneqq N(v_U) \setminus \{v_K\}$ for each vertex $v_K \in S_L$. Therefore S_L is a maximal ω-set of $\Gamma(G)$.

(b) Suppose that S is a maximal ω-set of $\Gamma(G)$ and let $v_H \in S$. If L is a minimal subgroup of H, then $(N(v_L) \setminus \{v_H\}) \subseteq (N(v_H) \setminus \{v_L\})$ and hence $v_L \in S$. But by (a), S_L is the unique ω-set of $\Gamma(G)$ containing v_L, which implies that $S = S_L$. Therefore L is the only minimal subgroup of H.

(c) Let v_H be an ω-vertex of $\Gamma(G)$ and L a minimal subgroup of H. If H has a minimal subgroup, say L_1, other than L, then $N(v_{L_1})\setminus\{v_H\} \subsetneqq N(v_H)\setminus\{v_{L_1}\}$, which is obviously a contradiction. So L is the only minimal subgroup of H. Conversely, if H is a subgroup of G containing a unique minimal subgroup L, then $v_H \in S_L$ and hence v_H is an ω-vertex of $\Gamma(G)$. □

Corollary 3.6. *The number of minimal subgroups of a finite group G is equal to the number of maximal ω-sets of $\Gamma(G)$.*

Theorem 3.7. *Let $\Gamma(G)$, the intersection graph of G, be connected and have a leaf. Then G is a nonabelian simple group if and only if for every ω-vertex of $\Gamma(G)$, say v_H, there exists an ω-vertex, say v_K, such that $d(v_H, v_K) \geqslant 3$.*

Proof. Let G be a finite nonabelian simple group and v_H an ω-vertex of $\Gamma(G)$. Then by Lemma 3.5 (c), H has a unique minimal subgroup L and hence v_H and v_L are coneighbour. So for each nontrivial proper subgroup K of G, where $K \neq \{L, H\}$, we have $d(v_H, v_K) = d(v_L, v_K)$.

Let v_{L_1} be a leaf of $\Gamma(G)$, which is adjacent to $v_{M_{L_1}}$. Since G is simple, $\mathrm{Core}_G(M_{L_1}) = 1$. Therefore there exists $g \in G$ such that v_L is not adjacent to $v_{(M_{L_1})^g}$. Hence $d(v_L, v_{L_1^g}) \geqslant 3$.

For the converse, on the contrary suppose that G is not a simple group. Then G is one of the groups listed in Table 1, where its intersection graph is connected. Then we can easily verify that in each case there exists at least one minimal subgroup, say L, such that $d(v_L, v_K) = 2$ for each minimal subgroup K of G, a contradiction. So G is a simple group and we get the result. □

Corollary 3.8. *Let G_1 be a finite group whose intersection graph has a leaf and let G_2 be a finite group such that $\Gamma(G_1) \cong \Gamma(G_2)$. Then G_1 is a simple group if and only if G_2 is simple.*

Proof. By Lemma 2.1, the simplicity of a group implies that its intersection graph is connected. Now let φ be a graph isomorphism from $\Gamma(G_1)$ to $\Gamma(G_2)$. It can be easily verified that v_H is an ω-vertex of $\Gamma(G_1)$ if and only if $\varphi(v_H)$ is an ω-vertex of $\Gamma(G_2)$ for each subgroup H of G_1. Also for each pair of distinct subgroups H and K of G_1, obviously $d(v_H, v_K) = d(\varphi(v_H), \varphi(v_K))$. So we get the result. □

In the following, we discuss the characterizability of a finite nonsimple group G by its intersection graph, when $\Gamma(G)$ has a leaf. We remark that in the following theorems, we use the notation from Theorem 3.1 (and so Table 1).

Theorem 3.9.

(a) *The dihedral group of order 8, D_8, is characterizable by its intersection graph.*

(b) *The group $\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, where p is an odd prime, is characterizable by its intersection graph.*

P r o o f. (a) Let G be a finite group such that $\Gamma(G) \cong \Gamma(D_8)$. Then using Corollary 3.8, we get that G is a nonsimple group and hence G is one of the groups listed in Table 1. Since $\mathrm{nl}(\Gamma(G)) = \mathrm{nl}(\Gamma(D_8)) = 4$, we see that G is isomorphic to D_8 or to a group in the last row of Table 1. If $G \neq D_8$, then $q = 2$ and $n = 1$, which implies that p is a PPD of $q^n - 1 = 1$, a contradiction. So we get the result.

(b) Let $G = \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$ and let H be a finite group such that $\Gamma(H) \cong \Gamma(G)$. Then by Corollary 3.8, H is a nonsimple group and by Table 1, we can easily verify that $H \cong \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$. □

Theorem 3.10. *Let G be a finite group whose intersection graph is isomorphic to $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for some prime p. Then:*

(a) *if $p = 2$, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$,*

(b) *if $p \neq 2$, then $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$.*

P r o o f. Similarly to the above theorem, we get that G is a nonsimple group and using Table 1, it can be easily verified that if $p = 2$, then the only possibility is $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Now let $p \neq 2$. Then by Theorem 3.1 (a) we see that $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, since $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2}) \cong \Gamma(\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2})$. □

Theorem 3.11. *Let $G \cong \mathbb{Z}_p \times \mathbb{Z}_{q^2}$ for some distinct primes p and q and let H be a finite group for which $\Gamma(H) \cong \Gamma(G)$. Then $H \cong \mathbb{Z}_r \times \mathbb{Z}_{s^2}$ for arbitrary distinct primes r and s.*

P r o o f. We note that $|V(\Gamma(H))| = 4$ and hence H is not isomorphic to any group in row 6 or 7 in Table 1. So the only possibility is $H \cong \mathbb{Z}_r \times \mathbb{Z}_{s^2}$ for some distinct primes r and s. It is clear that the primes r and s are arbitrary. □

Theorem 3.12. *Let $G \cong \mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$ for some primes p and q, where $q \,|(p-1)$ and the subgroup of order pq is cyclic. Let H be a finite group whose intersection graph is isomorphic to $\Gamma(G)$. Then $H \cong \mathbb{Z}_{r^2} \ltimes \mathbb{Z}_p$ for some prime r, where $r \,|(p-1)$ and the subgroup of order pr is cyclic. Particularly, if q is the only prime divisor of $p-1$, then $H \cong G$.*

P r o o f. Obviously H is isomorphic to a group in row 6 or 7 of Table 1. But comparing the degrees of the vertices in the graphs in Figure 6 and Figure 7, we get that H is isomorphic to a group in row 6. So $H \cong \mathbb{Z}_{r^2} \ltimes \mathbb{Z}_s$. Also we can easily verify that $s = p$ while r is not necessarily equal to q unless $p-1 = 2^m$ for some $m > 0$. □

Theorem 3.13. *Let $G \cong \mathbb{Z}_{q^2} \ltimes \mathbb{Z}_p$ for some distinct primes p and q, where $q^2 \,|(p-1)$ and the subgroup of order pq is not cyclic. If H is a finite group with $\Gamma(H) \cong \Gamma(G)$, then $H \cong \mathbb{Z}_{r^2} \ltimes \mathbb{Z}_p$ for some prime r, where $r^2 \,|(p-1)$ and the subgroup of order pr is not cyclic. Particularly, if $p - 1/q^2$ is square free, then $H \cong G$.*

P r o o f. Similarly to the above, we get that H is isomorphic to a group in row 7 of Table 1. Also it can be easily verified that $H \cong \mathbb{Z}_{r^2} \ltimes \mathbb{Z}_p$ for some prime r. Obviously if $p-1/q^2$ is square free, then $r = q$ and hence $H \cong G$. $\square$

Theorem 3.14. *Let $G \cong \mathbb{Z}_q \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$ for some distinct primes p and q, where q is a PPD of p^2-1 and let H be a finite group such that $\Gamma(H) \cong \Gamma(G)$. Then $H \cong \mathbb{Z}_r \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, where r is a PPD of p^2-1. Particularly, if q is the only PPD of p^2-1, then $H \cong G$.*

P r o o f. Since $\Gamma(H)$ is disconnected, the only possibility is $H \cong \mathbb{Z}_r \ltimes (\mathbb{Z}_s \times \mathbb{Z}_s)$ for some distinct primes r and s, where r is a PPD of s^2-1. Since $\mathrm{nl}(\Gamma(H)) = \mathrm{nl}(\Gamma(G))$, $s = p$ and so r is a PPD of p^2-1. $\square$

Theorem 3.15. *Let $G \cong \mathbb{Z}_p \ltimes N$, where N is an extra special q-group of order q^n and a subgroup of order p acts irreducibly on $N/\Phi(N)$. Let H be a finite group such that $\Gamma(H) \cong \Gamma(G)$. Then $H \cong \mathbb{Z}_r \ltimes N$, where r is a PPD of $q^{n-1}-1$. Particularly, if p is the only PPD of $q^{n-1}-1$, then $H \cong G$.*

P r o o f. Considering the number of leaves in $\Gamma(H)$ and also the number of leaves in each set of coneighbour leaves in $\Gamma(H)$, we get that $H \cong \mathbb{Z}_r \ltimes S$, where S is an extra special q-group of order q^n and r is a PPD of $q^{n-1}-1$. Note that if x is an element of G of order q^2, then $v_{\langle x\rangle}$ and $v_{\langle x^q\rangle}$ are coneighbour in $\Gamma(G)$. Thus we get that N and S have the same exponent and hence $N \cong S$. $\square$

Theorem 3.16. *Let $G \cong \mathbb{Z}_p \ltimes N$, where N is a UCS q-group of order q^{2n} with $\Phi(N) \cong \mathbb{Z}_q^n$ and a subgroup of order p acts irreducibly on $N/\Phi(N)$. Let H be a finite group such that $\Gamma(H) \cong \Gamma(G)$. Then $H \cong \mathbb{Z}_r \ltimes S$, where S is a UCS q-group of order q^{2n} with $\Phi(S) \cong \mathbb{Z}_q^n$ and r is a PPD of $q^{n-1}-1$.*

P r o o f. Similarly to the above we get that $H \cong \mathbb{Z}_r \ltimes S$, where S is a UCS q-group of order q^{2n} with $\Phi(S) \cong \mathbb{Z}_q^n$ and r is a PPD of $q^{n-1}-1$. $\square$

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