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ON GENERALIZED DOUGLAS-WEYL RANDERS METRICS

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Abstract. We characterize generalized Douglas-Weyl Randers metrics in terms of their Zermelo navigation data. Then, we study the Randers metrics induced by some important classes of almost contact metrics. Furthermore, we construct a family of generalized Douglas-Weyl Randers metrics which are not R-quadratic. We show that the Randers metric induced by a Kenmotsu manifold is a Douglas metric which is not of isotropic S-curvature. We show that the Randers metric induced by a Kenmotsu or Sasakian manifold is not Einsteinian. By using D-homothetic deformation of a Kenmotsu or Sasakian manifold, we construct a family of generalized Douglas-Weyl Randers metrics and show that the Lie group of projective transformations does not act transitively on the set of generalized Douglas-Weyl Randers metrics.

Keywords: generalized Douglas-Weyl metric; Randers metric; Kenmotsu manifold; Sasakian manifold

MSC 2020: 53B40, 53C60

1. Introduction

Projective invariants are important in geometry. In Riemannian geometry, the Weyl tensor is a natural projective invariant. The Weyl tensor can be defined in Finsler geometry, too. There are some non-Riemannian projective invariants in Finsler geometry. Douglas metrics and generalized Douglas-Weyl metrics (for simplicity GDW-metrics) are non-Riemannian projective invariant classes of Finsler metrics. Indeed, if F_1 and F_2 are two projectively related Finsler metrics on a manifold M, then F_1 is a Douglas metric (or GDW-metric) if and only if F_2 is a Douglas metric (or GDW-metric). It is worth mentioning that every Riemannian manifold is a Douglas metric. For a manifold M, let $\mathcal{GDW}(M)$ denote the class of all Finsler metrics on M satisfying

$$D^i_{jkl;m}y^m = T_{jkl}y^i,$$

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where $D^i_{jkl;m}$ denotes the horizontal covariant derivatives of the Douglas tensor D^i_{jkl} with respect to the Berwald connection of the Finsler metric F. It is known that Douglas and Weyl's metrics are in $\mathcal{GDW}(M)$. It has been stated in [1] that $\mathcal{GDW}(M)$ is closed under projective changes. More precisely, if F is projectively equivalent to a Finsler metric in $\mathcal{GDW}(M)$, then F is in $\mathcal{GDW}(M)$. As mentioned in [11], $\mathcal{GDW}(M)$ contains R-quadratic metrics as a special case, but the class of R-quadratic metrics is not closed under projective transformations. For more recent progress on generalized Douglas-Weyl metrics, see [5], [20], [21], [22].

Almost contact geometry is a very fruitful branch of differential geometry. Similar to [7], we use some important classes of almost contact metrics, namely, cosymplectic, Sasakian and Kenmotsu manifolds, to make Randers metrics with special curvature properties. Although the trans-Sasakian structure contains these three classes, we prefer to restrict our study to each structure separately.

A Finsler metric is of Randers type if and only if it is a solution of the navigation problem on a Riemannian manifold, see [4]. In this paper, we characterize the GDW-Randers metric $F = \alpha + \beta$ with $\|\beta\|_\alpha = \text{const.}$ in terms of its Zermelo navigation data (h, W). We construct a family of Randers metrics which are in $\mathcal{GDW}(M)$ and are not R-quadratic. Then, we prove that the Randers metric induced by a Kenmotsu manifold is a Douglas metric while it is not of isotropic S-curvature.

The study of Einstein Finsler metrics is an important problem in Finsler geometry. As is well known, a Riemann-Einstein metric has constant Ricci curvature for $n \geqslant 3$. In this paper, we prove that the Randers metric induced by Kenmotsu or Sasakian manifold is not Einsteinian. Since the induced Randers metric of a cosymplectic manifold is Berwaldian and obviously is Douglas, we omit them.

Two Finsler metrics on a manifold M are said to be pointwise projectively related if they have the same geodesics as point sets. Two Finsler metrics are said to be projectively related if there exists a diffeomorphism between them such that the pull-back metric is pointwise projectively related to the other one. It is a good idea to study projectively related Finsler metrics. To this aim, we use D-homothetic deformation. A D-homothetic deformation of a Sasakian (Kenmotsu) structure is also a Sasakian (Kenmotsu) structure. Therefore, we can construct a new Randers metric as $F_t = \alpha_t + \varepsilon\eta_t$ associated to $(M, \eta, \varphi, \xi, \alpha)$ by its D-homothetic deformation. If $F = \alpha + \varepsilon\eta$ is the induced Randers metric of a Sasakian (Kenmotsu) structure, we show that F is not projectively related to F_t. Hence, we show that the Lie group of projective transformations of a Finsler metrics does not act transitively on the set of GDW-metrics.

2. Preliminaries

Let M be a differentiable manifold of dimension $2n+1$. Suppose η, ξ and φ are a 1-form, a vector field, and a $(1,1)$-tensor, respectively. The triple (η, ξ, φ) is called an *almost contact structure on* M if it satisfies the following conditions:

$$\varphi(\xi) = 0, \quad \eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi.$$

A differentiable manifold of odd dimension $2n+1$ with an almost contact structure is called an *almost contact manifold*. On an almost contact manifold, we have the following:

$$\operatorname{rank} \varphi = 2\, n.$$

Let us suppose that a manifold M with the (η, ξ, φ) structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M is called an *almost contact metric structure* and g is called a *compatible metric*. An almost contact structure is normal (see [3]) if the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ, vanishes identically. An almost contact metric structure becomes a contact metric structure if $\Phi = d\eta$, where Φ is the fundamental 2-form defined as

$$\Phi(X, Y) = g(X, \varphi Y).$$

The following is true for every contact metric:

$$\nabla_i \varphi^i_j = -2n\eta_j, \tag{2.1}$$

where ∇ stands for the Levi-Civita connection of g. An almost contact metric structure (η, ξ, φ, g) on M is called a *trans-Sasakian structure* (see [14]) if it satisfies

$$(\nabla_X \varphi)Y = k_1\{g(X, Y)\xi - \eta(Y)X\} + k_2\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} \tag{2.2}$$

for some scalar functions k_1 and k_2. Moreover, (2.2) is equivalent to

$$(\nabla_X \eta)(Y) = -\, k_1 g(\varphi X, Y) + k_2\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.3}$$

$$\begin{aligned}(\nabla_X \Phi)(Y, Z) = {} & k_1\{g(X, Z)\eta(Y) - g(X, Y)\eta(Z)\} \\ & - k_2\{g(X, \varphi Z)\eta(Y) - g(X, \varphi Y)\eta(Z)\}\end{aligned} \tag{2.4}$$

In local coordinates, (2.2) and (2.3) can be written as follows:

$$\varphi^j_{k|i} = k_1(a_{ki}\xi^j - \delta^j_i\eta_k) + k_2(\varphi_{ki}\xi^j - \varphi^j_{\ i}\eta_k), \tag{2.5}$$

$$\eta_{i|j} = k_1\varphi_{ji} + k_2(a_{ij} - \eta_i\eta_j), \tag{2.6}$$

where $|$ is the covariant derivative with respect to ∇, a_{ij} are the local components of g and $\varphi_{ij} = a_{is}\varphi^s_j$ are the local components of the fundamental 2-form Φ. Thus, equation (2.5) can be written as

$$\varphi_{jk|i} = k_1(a_{ki}\eta_j - a_{ij}\eta_k) + k_2(\varphi_{ki}\eta_j - \varphi_{ji}\eta_k). \tag{2.7}$$

It is easy to see that trans-Sasakian manifolds are normal, see [14]. A trans-Sasakian structure is reduced to a Sasakian (Kenmotsu) structure if $k_1 = 1$ and $k_2 = 0$ ($k_1 = 0$ and $k_2 = 1$) and cosymplectic if $k_1 = k_2 = 0$.

A D-homothetic deformation of $(M, \eta, \varphi, \xi, \alpha)$ is a change of structure tensors in the following form:

$$\eta_t = t\eta, \quad \xi_t = t^{-1}\xi, \quad \varphi_t = \varphi, \quad \alpha_t = t\alpha + t(1-t)\eta \otimes \eta, \quad t > 0. \tag{2.8}$$

It is easy to see that $(\eta_t, \varphi_t, \xi_t, \alpha_t)$ is also an almost contact metric structur, see [17].

Suppose $(M, \eta, \varphi, \xi, g)$ is a Sasakian (Kenmotsu) manifold. We define $\alpha\colon TM \to \mathbb{R}$ by $\alpha(x, y) := \sqrt{g_x(y, y)}$ for every tangent vector $y \in T_xM$. Indeed, α is the norm induced by the Riemannian metric g. In Finsler geometry, we refer to α as a Riemannian metric on M (for example, see [4]). Let $F = \alpha + \varepsilon\eta$ be a Randers metric associated with $(M, \eta, \varphi, \xi, g)$, where $0 < \varepsilon < 1$. It can be seen that $(M, \eta_t, \varphi_t, \xi_t, \alpha_t)$ is also a Sasakian (Kenmotsu) manifold, where α_t is the norm induced by g_t, see [17] and [23].

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F\colon TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of the tangent bundle TM,
(iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite:

$$g_y(u, v) := \tfrac{1}{2}[F^2(y + su + tv)]\big|_{s,t=0}, \quad u, v \in T_xM. \tag{2.9}$$

Randers metrices are an important class of Finsler metrics since they are computable and have diverse applications in many branches. They are defined by a Riemannian metric α and a 1-form β on the manifold as $F = \alpha + \beta$. It is easy to see that a Randers metric is strongly convex if and only if $b := \|\beta\|_\alpha < 1$, see [9], [13], [18], [19].

Similarly to [7], let $(M, \alpha, \eta, \xi, \varphi)$ be an almost contact metric manifold and put $\beta := \varepsilon\eta$, where $0 < \varepsilon < 1$ is a constant. Then $F = \alpha + \beta$ is a Randers metric, since $\|\beta\|_\alpha = \varepsilon < 1$.

Define

$$b_{i|j} := \frac{\partial b_i}{\partial x^j} - b_k \widetilde{\Gamma}^k_{ij},$$

where $\widetilde{\Gamma}^k_{ij}$ denote the Christoffel symbols of α. For a Randers metric $F = \alpha + \beta$, let us put

$$\begin{aligned} r_{ij} &:= \tfrac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \tfrac{1}{2}(b_{i|j} - b_{j|i}), \\ s_i &:= b^m s_{mi}, \quad r_j := b^m r_{mj}, \quad t_{ij} := s_{im} s^m_j. \end{aligned} \tag{2.10}$$

We will denote $r_{00} = r_{ij} y^i y^j$, $s^i_j = a^{im} s_{mj}$ and $s^i_0 := s^i_j y^j$, etc.

A spray G on M is a smooth vector field on $TM_0 := TM - \{0\}$, locally expressed in the following form:

$$\mathrm{G} = y^i \frac{\partial}{\partial x^i} - 2\, G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions on TM_0 which are homogeneous of degree 2 with respect to y. Let F be a Finsler metric. The associated spray to F is given by

$$G^i(x, y) := \tfrac{1}{4} g^{il} \{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}.$$

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics. For a vector $y \in T_x M_0$, the Riemann curvature operator $\mathrm{R}_y \colon T_x M \to T_x M$ is defined by

$$\mathrm{R}_y(u) := R^i_k(y) u^k \frac{\partial}{\partial x^i},$$

where

$$R^i_k(y) = 2\, \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2\, G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

The family $\mathrm{R} := \{\mathrm{R}_y\}_{y \in TM_0}$ is called the *Riemann curvature*. We define the Ricci curvature as the trace of R_y, i.e., $\mathrm{Ric}(x, y) := \mathrm{trace}(\mathrm{R}_y)$. A Finsler metric is said to be R-quadratic if its Riemann curvature coefficients R^i_k are quadratic in $y \in T_x M$ at every point $x \in M$.

Let us recall some notions about the curvatures of a Finsler manifold (M, F). Let $P \subset T_x M$ be a tangent plane and $y \in P - \{0\}$. The pair $\{P, y\}$ is called a *flag* in $T_x M$. Then $P = \mathrm{span}\{y, u\}$, where $u \in P$ is an arbitrary vector linearly independent of y. Define

$$\mathrm{K}(P, x, y) = \frac{\mathrm{g}_y(\mathrm{R}_y(u), u)}{\mathrm{g}_y(y, y)\mathrm{g}_y(u, u) - \mathrm{g}_y(y, u)\mathrm{g}_y(u, u)}.$$

The quantity $\mathrm{K}(P,x,y)$ is called the *flag curvature of the flag* $\{P,y\}$. A Finsler metric is of scalar curvature $\mathrm{K}(x,y)$ if and only if the flag curvature is independent of the tangent planes P containing $y \in T_xM$. A Finsler manifold (M,F) is of constant flag curvature K if and only if $H^h_{ijk} = \mathrm{K}(g_{ij}\delta^h_k - g_{ik}\delta^h_j)$, in the natural coordinate system on TM_0, where H^h_{ijk} are the local components of the hh-curvature of the Berwald connection of F. Contrary to the Riemannian case, the classification of Finsler metrics of constant (scalar) flag curvature is an open problem.

There is a notion of distortion $\tau = \tau(x,y)$ on TM associated with the Busemann-Hausdorff volume form on M, i.e., $dV_{BH} = \sigma(x)dx^1 \wedge dx^2 \ldots \wedge dx^n$, which is defined by

$$\tau(x,y) = \ln \frac{\sqrt{\det(g_{ij}(x,y))}}{\sigma(x)}, \quad \sigma(x) = \frac{\mathrm{Vol}(B^n(1))}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \colon\ F(x, y^i(\partial/\partial x^i)|_x) < 1\}}.$$

Then the S-curvature is defined by

$$S(x,y) = \frac{\mathrm{d}}{\mathrm{dt}}[\tau(c(t),\dot{c}(t))]\big|_{t=0},$$

where $c = c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$, see [15]. From the definition, we see that the S-curvature $S(x,y)$ measures the rate of change in the distortion on (T_xM, F_x) in the direction $y \in T_xM$. In the local coordinates, the S-curvature is given by

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m}(\ln \sigma).$$

Let (M,F) be an n-dimensional Finsler manifold, $S(x,y)$ its S-curvature. Suppose $c = c(x)$ is a scalar function on M. If $S = (n+1)cF$, then F is said to be of isotropic S-curvature, and F is said to be of constant S-curvature if $c = \text{const}$.

Any Randers metric $F = \alpha + \beta$ on the manifold M is a solution of the following Zermelo navigation problem:

$$h\Big(x, \frac{y}{F} - W_x\Big) = 1,$$

where $h = \sqrt{h_{ij}y^iy^j}$ is a Riemannian metric and $W = W^i\partial/\partial x^i$ is a vector field such that $\|W\|^2_h = h_{ij}W^iW^j < 1$. Note that here we follow the notations of [4] and consider the induced norm of the Reimannian metric h to be the same as h. In fact, α and β are given by

$$\alpha = \frac{\sqrt{\lambda h^2 + W_0}}{\lambda}, \quad \beta = -\frac{W_0}{\lambda},$$

respectively, and moreover, $\lambda = 1 - \|W\|^2_h$ and $W_0 = h_{ij}W^iy^j$, see [4]. Now, F can be written as follows:

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}. \tag{2.11}$$

Given a Randers metric $F = \alpha + \beta$, the pair (α, β) and its navigation data (h, W) are related to each other as follows:

$$h^2 = \lambda(\alpha^2 - \beta^2), \quad W_0 = -\lambda\beta. \tag{2.12}$$

We use the following traditional conventions:

$$\mathcal{R}_{ij} := \tfrac{1}{2}\{\overline{\nabla}_j W_i + \overline{\nabla}_i W_j\},\ \ \mathcal{S}_{ij} := \tfrac{1}{2}\{\overline{\nabla}_j W_i - \overline{\nabla}_i W_j\},\ \ \mathcal{S}^i{}_j := h^{ih}\mathcal{S}_{hj},\ \ \mathcal{S}_j := \mathcal{S}_{ij}W^i,$$
$$\mathcal{R}_j := \mathcal{R}_{ij}W^i,\ \ \mathcal{R} := \mathcal{R}_j W^j,\ \ \mathcal{R}_{i0} := \mathcal{R}_{ij}y^j,\ \ \mathcal{R}_{00} := \mathcal{R}_j y^j,\ \ \mathcal{S}_{i0} := \mathcal{S}_{ij}y^j,$$

where $\overline{\nabla}$ denotes the covariant derivative with respect to h.

A Randers metric with isotropic S-curvature can be expressed in terms of h and W as follows, see [24].

Theorem 2.1 ([24])**.** *Let $F = \alpha + \beta$ be a Randers metric on a manifold M, which is expressed in terms of a Riemann metric h and a vector field W by (2.12). Then F is of isotropic S-curvature, $S = (n+1)cF$, if and only if W satisfies*

$$\mathcal{R}_{00} = -2\,ch^2. \tag{2.13}$$

A Finsler metric F on an n-dimensional manifold M is called an *Einstein metric* if its Ricci curvature satisfies $\mathrm{Ric} = (n-1)K(x)F^2$. Therefore, if an n-dimensional Finsler metric has constant flag curvature K, then its Ricci curvature is $\mathrm{Ric} = (n-1)\mathrm{K}F^2$, which implies that it is Einsteinian. Moreover, it is said to have Einstein constant σ if $\mathrm{K}(x) = \sigma = \mathrm{const}$. Finsler metrics with isotropic flag curvature are Einstein metrics. Riemannian Einstein metrics with dimension $n > 3$ must be of constant Ricci curvature. However, the analogous proposition in the Finsler setting is still open.

Theorem 2.2 ([2])**.** *Suppose that the Randers metric $F = \alpha + \beta$ is the solution of Zermelo's navigation problem on a Riemann space (M, h) under the influence of a vector field W with $h(x, W) < 1$. Then (M, F) is Einstein with Einstein scalar $K = \sigma(x)$ if and only if there exists a constant c such that h and W satisfy the following conditions:*

(1) *h is Einstein with Einstein scalar $\mu = \sigma(x) + c^2$, that is,*

$$\mathrm{Ric}_{ik} = (n-1)\mu h_{ik}. \tag{2.14}$$

(2) *W is an infinitesimal homothety of h, namely,*

$$\mathcal{R}_{ij} = -2\,ch_{ij}. \tag{2.15}$$

Furthermore, σ must vanish whenever h is not Ricci-flat.

Since studying the projectively related Finsler metrics is important, we use D-homothetic deformation to make new Finsler metrics F_t. It is natural to investigate whether F and F_t are projectively related or not. In [16], the authors prove the following:

Theorem 2.3 ([16]). *A Randers metric $F = \alpha+\beta$ is pointwise projectively related to another Randers metric $\overline{F} = \overline{\alpha} + \overline{\beta}$ if and only if one of the following cases holds:*
(1) *$\alpha \neq \lambda(x)\overline{\alpha}$, β and $\overline{\beta}$ are closed, and α is pointwise projectively related to $\overline{\alpha}$.*
(2) *$\alpha = \lambda\overline{\alpha}$ for some positive constant λ, and $\beta - \lambda\overline{\beta}$ is closed.*

3. GDW-Randers metrics via their navigation data

The characterization of GDW-Randers metrics on a manifold M is given in [12] as follows.

Theorem 3.1 ([12]). *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Then F is in $\mathcal{GDW}(M)$ if and only if*

$$s_{ij|k} = \frac{1}{n-1}\{a_{ik}s^m_{j|m} - a_{jk}s^m_{i|m}\}, \tag{3.1}$$

where $|$ denotes the covariant derivative with respect to α.

Now, suppose $F = \alpha + \beta$ is a Randers metric with navigation data (h, W). The aim of this section is expressing (3.1) in terms of h and W. Let $\overline{\mathcal{G}}^i$ and $\widetilde{\mathcal{G}}^i$ be the spray coefficients of h and α, respectively. Then by [4]

$$\widetilde{\mathcal{G}}^i = \overline{\mathcal{G}}^i + \zeta^i,$$

where

$$\begin{aligned}\zeta^i = {}& \frac{1}{\lambda}y^i(\mathcal{R}_0 + \mathcal{S}_0) + \frac{1}{2}W^i\mathcal{R}_{00} + \Big(\frac{1}{2\lambda}h^2 + \frac{1}{\lambda^2}W_0^2\Big)[W^i\mathcal{R} - (\mathcal{R}^i + \mathcal{S}^i)]\\ &+ \frac{1}{\lambda}W_0[W^i\mathcal{R} + \mathcal{S}^i_0].\end{aligned} \tag{3.2}$$

Then the Christoffel coefficients of h and α are related as follows:

$$\begin{aligned}\zeta^i_{jk} = {}& \frac{1}{\lambda}[\delta^i_j(\mathcal{R}_k + \mathcal{S}_k) + \delta^i_k(\mathcal{R}_j + \mathcal{S}_j)] + W^i\mathcal{R}_{jk} + \Big(\frac{1}{\lambda}h_{jk} + \frac{2}{\lambda^2}W_jW_k\Big)\\ &\times [W^i\mathcal{R} - (\mathcal{R}^i + \mathcal{S}^i)] + \frac{1}{\lambda}W_j(W^i\mathcal{R}_k + \mathcal{S}^i_k) + \frac{1}{\lambda}W_k(W^i\mathcal{R}_j + \mathcal{S}^i_j).\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
(3.3)\qquad r_{jk} =&\ -\mathcal{R}_{jk} - \Big(\frac{1}{\lambda}h_{jk} + \frac{2}{\lambda^2}W_jW_k\Big)\mathcal{R} + \frac{1}{\lambda^2}(\mathcal{S}_jW_k + \mathcal{S}_kW_j)\\
&+ \frac{1-\lambda}{\lambda^2}(\mathcal{R}_jW_k + \mathcal{R}_kW_j),\\
(3.4)\qquad s_{jk} =&\ -\frac{1}{\lambda}\mathcal{S}_{jk} + \frac{1}{\lambda^2}[(\mathcal{R}_j + \mathcal{S}_j)W_k - (\mathcal{R}_k + \mathcal{S}_k)W_j],\\
r_k =&\ \mathcal{R}_k + \frac{1}{\lambda}\mathcal{R}W_k - \frac{1-\lambda}{\lambda}(\mathcal{R}_k + \mathcal{S}_k),\\
(3.5)\qquad s_k =&\ \mathcal{S}_k - \frac{1}{\lambda}\mathcal{R}W_k + \frac{1-\lambda}{\lambda}(\mathcal{R}_k + \mathcal{S}_k).
\end{aligned}
$$

One can see that $\lambda_{;k} = -2(\mathcal{R}_k + \mathcal{S}_k)$, see [4].

Although characterizing general GDW-Randers metrics in terms of their Zermelo navigation data is interesting, the calculation is quite long and cumbersome. For this reason, and since for a Randers metric induced by a trans-Sasakian manifold we have $\|\beta\|_\alpha = \text{const.}$, in the sequel, we suppose that $r_j + s_j = 0$. In this case, we also have $\mathcal{R}_j + \mathcal{S}_j = 0$ and thus (3.3) and (3.4) reduce to the following:

$$
\begin{aligned}
(3.6)\qquad r_{jk} =&\ -\mathcal{R}_{jk} - \Big(\frac{1}{\lambda}h_{jk} + \frac{2}{\lambda^2}W_jW_k\Big)\mathcal{R} + \frac{1}{\lambda^2}(\mathcal{S}_jW_k + \mathcal{S}_kW_j)\\
&+ \frac{1-\lambda}{\lambda^2}(\mathcal{R}_jW_k + \mathcal{R}_kW_j),\\
s_{jk} =&\ -\frac{1}{\lambda}\mathcal{S}_{jk}, \quad r_k = \mathcal{R}_k + \frac{1}{\lambda}\mathcal{R}W_k,\\
(3.7)\qquad s_k =&\ S_k - \frac{1}{\lambda}\mathcal{R}W_k, \quad t_k = -\mathcal{T}_k + \mathcal{R}\mathcal{S}_k.
\end{aligned}
$$

By taking covariant derivatives of s_{ij} and using $\lambda_{;k} = -2(\mathcal{R}_k + \mathcal{S}_k) = 0$, we have

$$
(3.8)\qquad s_{ij|k} = \frac{1}{\lambda}(-\mathcal{S}_{ij;k} + \mathcal{S}_{im}\zeta^m_{jk} + \mathcal{S}_{mj}\zeta^m_{ik}),
$$

where ; and | are covariant derivatives with respect to h and α, respectively. A direct computation shows

$$
\begin{aligned}
\mathcal{S}_{im}\zeta^m_{jk} =&\ -\mathcal{S}_i\mathcal{R}_{jk} - \mathcal{S}_i\mathcal{R}\Big(\frac{1}{\lambda}h_{jk} + \frac{2}{\lambda^2}W_jW_k\Big)\\
&+ \frac{1}{\lambda}[W_j(-\mathcal{S}_i\mathcal{R}_k + \mathcal{T}_{ik}) + W_k(-\mathcal{S}_i\mathcal{R}_j + \mathcal{T}_{ij})],\\
\mathcal{S}_{mj}\zeta^m_{ik} =&\ \mathcal{S}_j\mathcal{R}_{ik} + \mathcal{S}_j\mathcal{R}\Big(\frac{1}{\lambda}h_{ik} + \frac{2}{\lambda^2}W_iW_k\Big)\\
&+ \frac{1}{\lambda}[W_i(\mathcal{S}_j\mathcal{R}_k - \mathcal{T}_{jk}) + W_k(\mathcal{S}_j\mathcal{R}_i - \mathcal{T}_{ij})],
\end{aligned}
$$

where $\mathcal{T}_{ij} = \mathcal{S}_{im}\mathcal{S}^m_j$. On the other hand, $s^t_{j|k} = a^{it}s_{ij|k}$, where $a^{ti} = \lambda(h^{ti} - W^iW^t)$. By raising the index i and contracting k and i in (3.8), we have

$$(3.9)\quad s^t_{j|t} = \mathcal{R}^t_t\mathcal{S}_j - \mathcal{S}^t\mathcal{R}_{jt} + S_{j;t}W^t - \mathcal{S}^t_{j;t} - \frac{1}{\lambda}[(\mathcal{S}^m\mathcal{R}_m - \mathcal{S}_m\mathcal{S}^m - \mathcal{T}^t_t)W_j - n\mathcal{R}S_j] - \mathcal{Q}_j,$$

where $\mathcal{Q}_i := \mathcal{R}_m\mathcal{S}^m_i$. If we substitute equations (3.8) and (3.9) into (3.1), we get the following:

Theorem 3.2. *Suppose that the Randers metric $F = \alpha + \beta$ is the solution of Zermelo's navigation problem on a Riemann space (M, h) under the influence of a vector field W with $h(x, W) = \varepsilon < 1$, where ε is constant. Then (M, F) is in $\mathcal{GDW}(M)$ if and only if*

$$(3.10)\quad \begin{aligned} h_{ik}Q_j\lambda^2 &+ \mathcal{S}^t_{j;t}h_{ik}\lambda^2 + \mathcal{S}_{i;t}W^tW_jW_k\lambda + \mathcal{R}^t_t\mathcal{S}_iW_jW_k\lambda + \mathcal{S}^tW_iW_k\mathcal{R}_{jt}\lambda \\ &+ \mathcal{S}^m\mathcal{R}_mh_{ik}W_j\lambda + \mathcal{S}^m\mathcal{S}_mW_ih_{jk}\lambda + \mathcal{S}_{i;t}W^th_{jk}\lambda^2 \\ &+ W_iQ_jW_k\lambda + W_i\mathcal{S}^t_{j;t}W_k\lambda + \mathcal{R}^t_t\mathcal{S}_ih_{jk}\lambda^2 + \mathcal{S}^th_{ik}\mathcal{R}_{jt}\lambda^2 \\ &+ (n-1)[\mathcal{R}_i\mathcal{S}_jW_k\lambda + W_i\mathcal{S}_j\mathcal{R}_k\lambda + \mathcal{R}W_i\mathcal{S}_jW_k + \mathcal{R}_{ik}\mathcal{S}_j\lambda^2 + \mathcal{T}_{ik}W_j\lambda] \\ &+ T^t_tW_ih_{jk}\lambda + \mathcal{R}\mathcal{S}_iW_jW_k + \mathcal{R}\mathcal{S}_ih_{jk}\lambda - (i \mid j) = (n-1)\mathcal{S}_{ij;k}\lambda^2, \end{aligned}$$

where $(i \mid j)$ is the cyclic permutation of i and j.

Corollary 3.1. *Suppose that the Randers metric $F = \alpha + \beta$ is the solution of Zermelo's navigation problem on a Riemann space (M, h) under the influence of a Killing vector field W with $h(x, W) = \varepsilon < 1$, where ε is constant. Then (M, F) is in $\mathcal{GDW}(M)$ if and only if*

$$(3.11)\quad \begin{aligned} T^t_t(W_ih_{jk} - W_jh_{ik}) + (\mathcal{S}^t_{j;t}W_i - \mathcal{S}^t_{i;t}W_j)W_k + \lambda(\mathcal{S}^t_{j;t}h_{ik} - \mathcal{S}^t_{i;t}h_{jk}) \\ = 2n(\lambda\mathcal{S}_{ij;k} + T_{ik}W_j - T_{jk}W_i). \end{aligned}$$

Proof. Let W be a Killing vector field, i.e., $\mathcal{R}_{ij} = 0$. Then $\mathcal{R}_i = 0$. Moreover, $\mathcal{R}_j + \mathcal{S}_j = 0$ yields $\mathcal{S}_j = 0$. By substituting these results in (3.10), we obtain the result. □

Here, we want to characterize R-quadratic Randers metrics in terms of their navigation datum. Similarly to the $\mathcal{GDW}(M)$ case, we also suppose $\|W\|_h = \text{const}$. First, we recall the following.

Theorem 3.3 ([8]). *A Randers metric $F = \alpha + \beta$ on a manifold is R-quadratic if and only if*

$$(3.12)\quad r_{00} + 2\,s_0\beta = 2\,c(\alpha^2 - \beta^2),$$

$$(3.13)\quad s_{ij|k} = a_{ik}(2\,cs_j + c^2b_j + t_j) - a_{jk}(2\,cs_i + c^2b_i + t_i),$$

where c is a constant.

By Theorem 2.1, equation (3.12) is equivalent to $\mathcal{R}_{00} = -2ch^2$. If we substitute equations (3.7) and (3.8) in (3.13), we have the following:

Theorem 3.4. *Suppose that the Randers metric $F = \alpha + \beta$ is the solution of Zermelo's navigation problem on a Riemann space (M, h) under the influence of W with $h(x, W) = \varepsilon < 1$, where ε is constant. Then (M, F) is R-quadratic if and only if $\mathcal{R}_{00} = -2ch^2$ and*

$$\begin{aligned}\mathcal{S}_{ij;k} = \frac{1}{\lambda^2}[&\mathcal{R}(\mathcal{S}_i W_j W_k \lambda + \mathcal{S}_i h_{jk}\lambda^2 - 2\,W_i c h_{jk}\lambda - 2\,\mathcal{S}_i W_j W_k - \mathcal{S}_i h_{jk}\lambda)\\ &+ [2\;\mathcal{S}_i W_j W_k c + 2\,\mathcal{S}_i c h_{jk}\lambda - W_i c^2 h_{jk} + \mathcal{R}_i \mathcal{S}_j W_k + \mathcal{R}_{ik}\mathcal{S}_j\lambda\\ &- \mathcal{R}_k \mathcal{S}_i W_j - \mathcal{T}_i(W_j W_k + h_{jk}\lambda) + \mathcal{T}_{ik} W_j]\lambda - (i \mid j)],\end{aligned} \tag{3.14}$$

where $(i \mid j)$ is the cyclic permutation of i and j.

Corollary 3.2. *Suppose that the Randers metric $F = \alpha + \beta$ is the solution of Zermelo's navigation problem on a Riemann space (M, h) under the influence of a Killing vector field W with $h(x, W) = \varepsilon < 1$, where ε is constant. Then (M, F) is R-quadratic if and only if*

$$\mathcal{S}_{ij;k} = \frac{1}{\lambda}[W_j \mathcal{T}_{ik} - W_i \mathcal{T}_{jk}]. \tag{3.15}$$

4. Randers metrics associated to a Sasakian structure

Let $(M, \eta, \varphi, \xi, \alpha)$ be a Sasakian manifold. In this section, we restrict our study to the Randers metric induced by the Sasakian manifold $(M, \alpha, \eta, \varphi, \xi)$. Putting $k_1 = 1$ and $k_2 = 0$ in (2.5), (2.6) and (2.7), we obtain directly

$$r_{ij} = 0, \quad s_{ij} = \varepsilon\varphi_{ji}, \quad t_{ij} = \varepsilon^2(\eta_i\eta_j - a_{ij}), \quad r_j = 0. \tag{4.1}$$

In this section, we express our findings in terms of (h, W). If we compare $\beta = \varepsilon\eta$ with $\beta = -W_0/\lambda$, we obtain $W_0 = -\varepsilon\lambda\eta$. By using (3.3), (3.4) and (4.1), we obtain the following:

$$\mathcal{R}_{ij} = 0, \quad \mathcal{S}_{ij} = -\lambda s_{ij} = \lambda\varepsilon\varphi_{ij}. \tag{4.2}$$

Corollary 4.1. *Suppose $F = \alpha + \beta$ is the Randers metric induced by the Sasakian structure α and $\beta = \varepsilon\eta$, where $0 < \varepsilon < 1$. Then F is of vanishing S-curvature.*

Theorem 4.1. *Let $(M, \eta, \varphi, \xi, \alpha)$ be a Sasakian metric. Suppose $F = \alpha + \beta$ is the Randers metric induced by the Sasakian structure α and $\beta = \varepsilon\eta$, where $0 < \varepsilon < 1$. Then F is a generalized Douglas-Weyl metric.*

P r o o f. Suppose $F = \alpha + \beta$ is the Randers metric induced by the Sasakian manifold $(M, \eta, \varphi, \xi, \alpha)$. Then by direct calculation of (4.2)

$$\mathcal{S}_{ij;k} = \mathcal{S}_{ij|k} + \frac{1}{\lambda}(T_{ik}W_j - T_{jk}W_i), \quad \mathcal{S}^t_{j;t} = \mathcal{S}^t_{j|t} + \frac{1}{\lambda}T^t_t W_j, \tag{4.3}$$

where ; and | are covariant derivatives with respect to h and α, respectively. By using (2.7) and (2.12), we have

$$\mathcal{S}_{ij|k} = \frac{1}{\lambda}(h_{ik}W_j - h_{jk}W_i), \quad \mathcal{S}^t_{i|t} = \frac{2n}{\lambda}W_j. \tag{4.4}$$

By using (4.3) and (4.4), one can see that (3.11) holds. This means F is in $\mathcal{GDW}(M)$. □

In [11] the authors prove that $\mathcal{GDW}(M)$ contains R-quadratic metrics as a special case, but the class of R-quadratic metrics is not closed under projective transformations. Here, we prove that the Randers metric induced by Sasakian metric is not actually R-quadratic.

Theorem 4.2. *Let $(M, \eta, \varphi, \xi, \alpha)$ be a Sasakian metric. Suppose $F = \alpha + \beta$ is a Randers metric induced by a Sasakian structure α and $\beta = \varepsilon\eta$, where $0 < \varepsilon < 1$. Then F is not R-quadratic.*

P r o o f. Suppose F is an induced Randers metric of a Sasakian metric, then by Corollary 3.2, we have

$$\mathcal{S}_{ij;k} = \frac{1}{\lambda}[W_j\mathcal{T}_{ik} - W_i\mathcal{T}_{jk}].$$

Comparing the above equation and equation (4.3) gives that $S_{ij|k}$ should be zero, which by (4.4) implies that $h_{ik}W_j - h_{jk}W_i = 0$. Contracting the previous relation with W^i yields that (h_{jk}) is of rank 1, which is impossible. Thus, F cannot be R-quadratic. □

For a Sasakian manifold, we have

$$\mathrm{Ric}(X, \xi) = 2n\eta(X) \quad (\text{or } Q\xi = 2n\xi). \tag{4.5}$$

Theorem 4.3. *Let $F = \alpha + \beta$ be the Randers metric induced by a Sasakian manifold $(M, \eta, \varphi, \xi, \alpha)$. Then F is not Einsteinian.*

P r o o f. In general, for the Randers metric induced by a Sasakian structure, we have $\mathcal{R}_{ij} = 0$ and $\mathcal{S}_{ij} = \lambda\varepsilon\varphi_{ij}$. Suppose F is Einsteinian, then by Theorem 2.2, $c = 0$. In this case, μ reduces to $\mu = \sigma(x)$. We also have

$$\mathrm{Ric}_{ik} = 2n\sigma h_{ik}. \tag{4.6}$$

By contracting (4.6) in $\xi^j = -(\lambda/\varepsilon)W^j$ and using $1 - \varepsilon^2 = \lambda$, we have

$$\mathrm{Ric}_{ij}\xi^j\xi^i = 2\, n\sigma\lambda^2. \tag{4.7}$$

On the other hand, by (4.5) we have

$$\mathrm{Ric}_{ij}\xi^j\xi^i = 2n, \tag{4.8}$$

and consequently, h is not Ricci-flat. Thus it follows from Theorem 2.2 that $\sigma = 0$, which is a contradiction. □

Let $(M, \eta, \varphi, \xi, \alpha)$ be a Sasakian manifold and $(M, \eta_t, \varphi_t, \xi_t, \alpha_t)$ be its D-homothetic deformation. Then, we prove that two Riemannian metrics α and α_t are not projectively related.

Proposition 4.1. *Let $(M, \eta, \varphi, \xi, \alpha)$ be a Sasakian structure and α_t be the D-homothetic deformation of α. In this case, α_t and α are not projectively related.*

P r o o f. Let

$$\overline{g}_{ij} := tg_{ij} + t(1-t)\eta_i\eta_j$$

be the D-homothetic deformation of g_{ij}. The inverse of $\overline{g}_{ij}$ is obtained as follows:

$$\overline{g}^{ij} = \frac{1}{t}g^{ij} + \lambda\xi^i\xi^j, \tag{4.9}$$

where $\lambda = (t-1)/t(2-t)$. By direct calculation we have

$$\begin{aligned}\bar{\Gamma}^k_{ij} &= \frac{1}{2}\overline{g}^{ks}\Big\{\frac{\partial\overline{g}_{is}}{\partial x^j} + \frac{\partial\overline{g}_{js}}{\partial x^i} - \frac{\partial\overline{g}_{ij}}{\partial x^s}\Big\} \\ &= \Gamma^k_{ij} + \frac{1-t}{2}g^{ks}\Big\{\eta_s\frac{\partial\eta_i}{\partial x^j} + \eta_s\frac{\partial\eta_j}{\partial x^i} + \eta_i\frac{\partial\eta_s}{\partial x^j} + \eta_j\frac{\partial\eta_s}{\partial x^i} - \frac{\partial(\eta_i\eta_j)}{\partial x^s}\Big\} \\ &\quad + \frac{1}{2}\lambda t\xi^k\xi^s\Big\{\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\Big\} \\ &\quad + \frac{1}{2}\lambda t(1-t)\xi^k\xi^s\Big\{\eta_s\frac{\partial\eta_i}{\partial x^j} + \eta_s\frac{\partial\eta_j}{\partial x^i} + \eta_i\frac{\partial\eta_s}{\partial x^j} + \eta_j\frac{\partial\eta_s}{\partial x^i} - \frac{\partial(\eta_i\eta_j)}{\partial x^s}\Big\}.\end{aligned} \tag{4.10}$$

By (4.1), we see that for a Sasakian metric

$$r_{ij} = \frac{\partial \eta_i}{\partial \eta_j} + \frac{\partial \eta_j}{\partial \eta_i} - 2\eta_l \Gamma^l_{ij} = 0 \quad \text{and} \quad \frac{\partial \eta_i}{\partial \eta_j} = \frac{\partial \eta_j}{\partial \eta_i} + \varphi_{ij}.$$

Thus (4.10) reduces to

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{t-1}{2}(\eta_j \varphi^k_i + \eta_i \varphi^k_j). \tag{4.11}$$

Suppose α and α_t are projectively related metrics. Then we have

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \pi_j + \delta^k_j \pi_i, \tag{4.12}$$

where π_i is homogeneous of degree zero. By (4.11) and (4.12) we have

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{t-1}{2}(\eta_j \varphi^k_i + \eta_i \varphi^k_j) = \Gamma^k_{ij} + \delta^k_i \pi_j + \delta^k_j \pi_i. \tag{4.13}$$

By contracting (4.13) with i and k we have $(n+1)\pi_j = 0$, which is contradiction. Thus, α and α_t are not projectively related metrics. $\square$

Now, suppose $F = \alpha + \varepsilon\eta$ is the induced Randers metric of a Sasakian structure. Suppose that $\alpha = \lambda\alpha_t$ for some scalar function λ on M. Fix a point x at M. Considering $y \in \ker \eta_x \subset T_xM$ and taking into account (2.8), one can see that $\lambda(x) = t$. Similarly, considering $y = \xi(x) \in T_xM$ implies that $\lambda(x) = t(1-t)$, which is a contradiction since t is a positive number and $t \neq 1$. Therefore, α cannot be a multiple of α_t. Proposition 4.1 infers that α_t and α are not projectively related. In this case, by Theorem 2.3, F cannot be projectively related to F_t. In fact, we prove the following.

Corollary 4.2. *Suppose $F = \alpha + \varepsilon\eta$ is the induced Randers metric of a Sasakian structure. Let $F_t = \alpha_t + \varepsilon\eta_t$ be the Randers metric associated to $(M, \eta, \varphi, \xi, \alpha)$ by its D-homothetic deformation. In this case, F is not projectively related to F_t.*

5. Randers metric associated to a Kenmotsu structure

Let $F = \alpha + \beta$ be a Randers metric induced by a Kenmotsu structure. Then we have

$$r_{ij} = \varepsilon(a_{ij} - \eta_i\eta_j), \quad s_{ij} = 0, \quad t_{ij} = 0. \tag{5.1}$$

We can express the above equation in terms of (h, W). For the induced Randers metric, we have

$$\lambda_{;k} = -2(\mathcal{R}_k + \mathcal{S}_k) = 0.$$

Then, by using (5.1) and (3.4) we have

$$\mathcal{R}_{ij} = -\frac{\varepsilon}{\lambda}\Big(\frac{W_iW_j}{1-\lambda} - h_{ij}\Big), \quad \mathcal{S}_{ij} = 0, \quad \mathcal{S}_j = 0. \tag{5.2}$$

Since in this case $\mathcal{R}_j = 0$, we have $\mathcal{R} = 0$.

Proposition 5.1 ([4]). *Let $F = \alpha+\beta$ be a Randers metric expressed by navigation data (h, W). Then β is closed if and only if*

$$\mathcal{S}_{jk} = \frac{1}{1-\lambda}(W_j\mathcal{S}_k - W_k\mathcal{S}_j), \quad \mathcal{S}_k = \mathcal{R}W_k - (1-\lambda)\mathcal{R}_k.$$

Corollary 5.1. *Let $F = \alpha + \beta$ be a Randers metric induced by a Kenmotsu metric. Then F is a Douglas metric.*

Proof. For the induced Randers metric we have $\mathcal{S}_{ij} = 0$. By Proposition 5.1, β is closed, i.e., F is Douglas metric. □

Corollary 5.2. *Let $F = \alpha + \beta$ be a Randers metric induced by a Kenmotsu metric. Then F is not of isotropic S-curvature.*

Proof. For the induced Randers metric, we have

$$\mathcal{R}_{ij} = \frac{\varepsilon}{\lambda}\Big(\frac{W_iW_j}{1-\lambda} - h_{ij}\Big).$$

By Theorem 2.1, F is of isotropic S-curvature if and only if $\mathcal{R}_{00} = \hat{a}2ch^2$. Thus F is not of isotropic S-curvature. □

Note that in the previous corollary, we have $\|\beta\|_\alpha = \varepsilon$, which is a positive constant. Thus, F cannot be a Riemannain metric.

Theorem 5.1. *Let $F = \alpha + \beta$ be an associated Randers metric of a Kenmotsu structure. Then F is not Einsteinian.*

Proof. For a Randers metric associated with a Kenmotsu structure, $\mathcal{R}_{ij}$ does not satisfy (2.15) for any constant c, which means that F is not Einsteinian. □

Let $(M, \eta, \varphi, \xi, \alpha)$ be a Kenmotsu metric, $(\eta_t, \varphi_t, \xi_t, \alpha_t)$ be its D-homothetic deformation and $F_t = \alpha_t + \varepsilon\eta_t$. Since α_t is also Kenmotsu, by a direct calculation we obtain

$$\bar{s}_{ij} = 0, \quad \bar{r}_{ij} = t^{-1}r_{ij}. \tag{5.3}$$

Thus, a deformed Randers metric F_t is also of constant S-curvature $c = \frac{1}{2}t^{-1}$. For a deformed Kenmotsu structure (see [10]), we have the following:

$$\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y + \frac{t-1}{t} g(\varphi X, \varphi Y)\xi, \\
\overline{R}(X,Y)Z &= R(X,Y)Z + \frac{t-1}{t}\{g(\varphi Y, \varphi Z)X - g(\varphi X, \varphi Z)Y\}, \\
\overline{S}(Y,Z) &= S(Y,Z) + \frac{2n(t-1)}{t} g(\varphi Y, \varphi Z),
\end{aligned} \tag{5.4}$$

where S is the Ricci tensor of α. On the other hand, the Weyl tensor of a Riemannian metric is as follows:

$$W(X,Y)Z = K(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}. \tag{5.5}$$

By substituting (5.4) in (5.5), one can conclude that α and α_t have the same Weyl curvature tensor.

Proposition 5.2. *Let $(M, \eta, \varphi, \xi, \alpha)$ be a Kenmotsu structure and α_t be the D-homothetic deformation of α. In this case, α_t and α are not projectively related.*

Proof. Let $\widehat{g}_{ij} = tg_{ij} + t(t-1)\eta_i\eta_j$ be the D-homothetic deformation of g_{ij}. The inverse of $\widehat{g}_{ij}$ is obtained as follows:

$$\widehat{g}^{ij} = \frac{1}{t} g^{ij} + \frac{t-1}{t^2}\xi^i\xi^j. \tag{5.6}$$

By direct calculation we have

$$\begin{aligned}
\widehat{\Gamma}^k_{ij} &= \frac{1}{2}\widehat{g}^{ks}\Big\{\frac{\partial \widehat{g}_{is}}{\partial x^j} + \frac{\partial \widehat{g}_{js}}{\partial x^i} - \frac{\partial \widehat{g}_{ij}}{\partial x^s}\Big\} \\
&= \Gamma^k_{ij} + \frac{t-1}{2} g^{ks}\Big\{\eta_s\frac{\partial \eta_i}{\partial x^j} + \eta_s\frac{\partial \eta_j}{\partial x^i} + \eta_i\frac{\partial \eta_s}{\partial x^j} + \eta_j\frac{\partial \eta_s}{\partial x^i} - \frac{\partial(\eta_i\eta_j)}{\partial x^s}\Big\} \\
&\quad - \frac{t-1}{2t}\xi^k\xi^s\Big\{\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s}\Big\} \\
&\quad - \frac{(t-1)^2}{2t}\xi^k\xi^s\Big\{\eta_s\frac{\partial \eta_i}{\partial x^j} + \eta_s\frac{\partial \eta_j}{\partial x^i} + \eta_i\frac{\partial \eta_s}{\partial x^j} + \eta_j\frac{\partial \eta_s}{\partial x^i} - \frac{\partial(\eta_i\eta_j)}{\partial x^s}\Big\}.
\end{aligned} \tag{5.7}$$

By (4.1), we see that for a Kenmotsu metric $d\eta = 0$, which means that $\partial\eta_s/\partial x^j = \partial\eta_j/\partial x^s$. Thus (5.7) reduces to

$$\widehat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{t-1}{2t}\xi^k\Big\{\frac{\partial \eta_i}{\partial x^j} + \frac{\partial \eta_j}{\partial x^i} - 2\eta_l\Gamma^l_{ij}\Big\}.$$

On the other hand, by (4.1) we have

$$r_{ij} = g_{ij} - \eta_i\eta_j,$$

which means that

$$\widehat{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{t-1}{2t}\xi^k(g_{ij} - \eta_i\eta_j). \tag{5.8}$$

Suppose α and α_t are projectively related metrics, then we have

$$\widehat{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k\pi_j + \delta_j^k\pi_i,$$

where π_i is homogeneous of degree zero. Thus we have

$$\widehat{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{t-1}{2t}\xi^k(g_{ij} - \eta_i\eta_j) = \Gamma_{ij}^k + \delta_i^k\pi_j + \delta_j^k\pi_i. \tag{5.9}$$

By contracting (5.9) with i and k we obtain $(n+1)\pi_j = 0$, which is a contradiction. Thus, α and α_t are not projectively related metrics. $\square$

Remark 5.1. Although α and α_t have the same Weyl curvature tensor, Proposition 5.2 proves that they are not projectively related. In fact, if two metrics are projectively related, they will have the same Weyl curvature tensor, but the converse is not true in general, see [6].

Let $F = \alpha + \varepsilon\eta$ be the induced Randers metric of a Kenmotsu structure. By Theorem 2.3 we prove the following:

Theorem 5.2. *Suppose $F = \alpha + \varepsilon\eta$ is the induced Randers metric of a Kenmotsu structure. Let $F_t = \alpha_t + \varepsilon\eta_t$ be the Randers metric associated to $(M, \eta, \varphi, \xi, \alpha)$ by its D-homothetic deformation. In this case, F is not projectively related to F_t.*

Proof. By the same argument, since α cannot be a multiple of α_t and for an induced Randers metric of a Kenmotsu structure, we have $s_{ij} = 0$. Since α and α_t are not projectively related, by Theorem 2.3 F is not projectively related to F_t. $\square$

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