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# A NOTE ON SKOLEM-NOETHER ALGEBRAS

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*Abstract.* The paper was motivated by Kovacs' paper (1973), Isaacs' paper (1980) and a recent paper, due to Brešar et al. (2018), concerning Skolem-Noether algebras. Let $K$ be a unital commutative ring, not necessarily a field. Given a unital $K$-algebra $S$, where $K$ is contained in the center of $S$, $n \in \mathbb{N}$, the goal of this paper is to study the question: when can a homomorphism $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ be extended to an inner automorphism of $\mathrm{M}_n(S)$? As an application of main results presented in the paper, it is proved that if $S$ is a semilocal algebra with a central separable subalgebra $R$, then any homomorphism from $R$ into $S$ can be extended to an inner automorphism of $S$.

*Keywords*: Skolem-Noether algebra; (inner) automorphism; matrix algebra; central simple algebra; central separable algebra; semilocal ring; unique factorization domain (UFD); stably finite ring; Dedekind-finite ring

*MSC 2020*: 16K20, 16W20, 16S50

## 1. Introduction

Throughout, rings or algebras are always associative and unital. All algebras are assumed to be algebras over a commutative ring $K$ (not necessarily a field), any subalgebra has the unit of the whole algebra, and all homomorphisms are linear and send 1 to 1. Given a ring $S$, let $Z(S)$ denote the center of $S$ and, given $n \in \mathbb{N}$, let $\mathrm{M}_n(S)$ denote the ring of $n \times n$ matrices over the ring $S$. For $x, y \in S$, let $[x, y] := xy - yx$, the additive commutator of $x$ and $y$. Given additive subgroups $A$, $B$ of $S$, let $[A, B]$ denote the additive subgroup of $S$ generated by all $[a, b]$ for $a \in A$ and $b \in B$.

Our study of this paper was motivated by Kovacs' paper (see [6]), Isaacs' paper (see [4]) and a recent paper (see [1]) concerning Skolem-Noether algebras due to

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Brešar et al. Let $K$ be a commutative ring. In [4] Isaacs studied automorphisms of the $K$-algebra $\mathrm{M}_n(K)$. Although an automorphism of $\mathrm{M}_n(K)$ is not in general inner (see [4], Example 6), its $n$th power and the commutator of two automorphisms are always inner (see [2], [4], Theorems 5 and 11, and [10]). Another viewpoint was given in [6], Kovacs proved that, given any endomorphism $\varphi$ of $\mathrm{M}_n(K)$, there exists a commutative ring $U$ containing $K$ such that $\varphi$ can be extended to an inner automorphism of $\mathrm{M}_n(U)$. Therefore, it seems natural to study the following:

**Question I.** Given a $K$-algebra $S$, where $K \subseteq Z(S)$, $n \in \mathbb{N}$, when can a homomorphism $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ be extended to an inner automorphism of $\mathrm{M}_n(S)$?

Question I is also rather related to a recent paper due to Brešar et al., see [1]. Motivated by the celebrated Skolem-Noether theorem (see [3], Theorem 4.3.1), in [1] Brešar et al. initiated the study of Skolem-Noether algebras (SN algebras for short).

An algebra $S$ over a field $F$ is called an SN algebra if, given any finite-dimensional central simple $F$-algebra $R$, every homomorphism $\varphi\colon R \to R \otimes_F S$ can be extended to an inner automorphism of $R \otimes_F S$. SN algebras can be also characterized by homomorphisms of matrix algebras. To be precise, an algebra $S$ over a field $F$ is an SN algebra if and only if, for every $n \in \mathbb{N}$ and a homomorphism $\varphi\colon \mathrm{M}_n(F) \to \mathrm{M}_n(S)$, there exists a unit $c \in \mathrm{M}_n(S)$ such that $\varphi(x) = cxc^{-1}$ for every $x \in \mathrm{M}_n(F)$ (see [1], Proposition 2.1). As pointed out by Brešar et al., although the characterization seems to facilitate the process of showing that a certain algebra $S$ is an SN algebra, it does not simplify their proofs of results in [1]. However, the viewpoint from matrix algebras over a field is indeed more concrete and basic properties concerning matrix algebras over a field are also more familiar to the reader. Another point of view is that any ring can be considered as an algebra over its center, but its center maybe contains no subfield. Therefore, following both [1], Proposition 2.1 and Kovacs' viewpoint [6] to study Question I we define SN algebras over a commutative ring, not necessarily a field, as follows.

**Definition.** An algebra $S$ over a commutative ring $K$, where $K$ is a subring of $Z(S)$, is called an SN algebra if, for every $n \in \mathbb{N}$, any homomorphism $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ can be extended to an inner automorphism of $\mathrm{M}_n(S)$.

As observed, a $K$-algebra algebra $S$ being SN does not really depend on $K \subseteq Z(S)$ since the homomorphisms are considered to be linear. For a concrete algebra $S$ it suffices to test the SN property on the smallest possible $K$, i.e., the subring generated by the identity. It is also clear that an algebra $S$ over a commutative ring $K$, where $K$ is contained in $Z(S)$, is an SN algebra if and only if $S$ is an SN-algebra over its center $Z(S)$. Therefore, it will be no confusion when we only say an algebra $S$ to be an SN algebra without mentioning its coefficient ring. Also, a commutative ring $K$

is an SN algebra if and only if, given any $n \in \mathbb{N}$, every endomorphism of $\mathrm{M}_n(K)$ is an inner automorphism. Moreover, if $S$ is an algebra over a field $K$, then in view of [1], Proposition 2.1 $S$ is an SN algebra if and only if $S$ is an SN algebra in the sense of [1].

The celebrated Skolem-Noether theorem asserts that any field is an SN algebra. A ring is called subdirectly irreducible if the intersection of its proper ideals is nonzero (see [8]). Kovacs proved that any commutative Noetherian subdirectly irreducible ring is an SN algebra (see [6], Theorem 2). However, $\mathbb{Z}[\sqrt{-5}]$ is not an SN algebra (see [4], Example 6). In this paper we want to study Question I, which means also to study SN algebras from the viewpoints of [1], Proposition 2.1, Kovacs' paper [6] and Isaacs's paper [4].

## 2. Results

Let $B$ be a ring with a subring $A$. A map $\varphi\colon A \to B$ is said to be defined by a linear generalized polynomial (LGP for short) if there exist finitely many $a_i, b_i \in B$ such that $\varphi(x) = \sum\limits_i a_i x b_i$ for all $x \in A$. In this case, we say that $\varphi$ is defined by the LGP $\sum\limits_i a_i X b_i$. Since all main results of [1] are derived from the technical basic lemma (see [1], Lemma 4.1), this its key part is worth mentioning. It is proved in [1], Lemma 4.1, that if $R$ is a finite-dimensional central simple $F$-algebra and if $S$ is an arbitrary $F$-algebra, then every homomorphism $\varphi\colon R \to R \otimes_F S$ is defined by an LGP. The result depends on the fact that $R$ is a finite-dimensional central simple $F$-algebra and $S$ can be considered as a vector space over the field $F$. Therefore, to study SN algebras over a commutative ring, we need the following main proposition (see Proposition 2.1 below), which is parallel to [1], Lemma 4.1.

A ring $R$ is called Dedekind-finite if $xy = 1$ implies $yx = 1$ in $R$, and $R$ is called stably finite if $\mathrm{M}_n(R)$ is Dedekind-finite for any $n \in \mathbb{N}$. Dedekind-finite rings include the following: reversible rings (that is, rings satisfying $ab = 0$ if and only if $ba = 0$), reduced rings, left or right Noetherian rings, semiprime Artinian rings, PI-rings (in particular, finite-dimensional algebras over a field), algebraic algebras over a field, rings with only finitely many nilpotents, group algebras $K[G]$ of an arbitrary group $G$ over any field $K$ of characteristic 0 (see [5]).

We list the set $\{E_{ij}\colon 1 \leqslant i, j \leqslant n\}$ of the standard matrix units of a given matrix algebra $\mathrm{M}_n(S)$ as $E_1, E_2, \ldots, E_{n^2}$. Let $I_n := \sum\limits_{i=1}^{n} E_{ii}$.

**Proposition 2.1.** *Let $S$ be an algebra over $K$, where $K$ is a subring of $Z(S)$, $n \in \mathbb{N}$. Suppose that $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ is a homomorphism. Then there exist*

$A_{ks}, B_{sk}, C_j \in \mathrm{M}_n(S)$, $1 \leqslant k, s \leqslant n$ *and* $1 \leqslant j \leqslant n^2$, *such that the following statements hold:*

(i) $xA_{ks} = A_{ks}\varphi(x)$ *for all* $x \in \mathrm{M}_n(K)$.

(ii) $A_{ks}B_{sk} = w_{s,kk}I_n$, *where* $\varphi(E_{ss}) := \sum\limits_{1\leqslant i,j\leqslant n} w_{s,ij}E_{ij}$ *with* $w_{s,ij} \in S$ *for all* $i$, $j$.

(iii) $\varphi(x) = \sum\limits_{j=1}^{n^2} E_j x C_j$ *for all* $x \in \mathrm{M}_n(K)$.

(iv) $xC_j = C_j\varphi(x)$ *for all* $x \in \mathrm{M}_n(K)$.

(v) $\sum\limits_{k=1}^{n^2} E_j C_j = I_n$.

(vi) *Write* $C_j = \sum\limits_{i=1}^{n^2} E_i s_{ji}$, *where* $s_{ji} \in S$ *for* $1 \leqslant i, j \leqslant n^2$. *Then, given* $j$, $i$, *there exists* $D_{ji} \in \mathrm{M}_n(S)$ *such that* $C_j D_{ji} = s_{ji} I_n$.

*Consequently, if* $S$ *is a stably finite ring and if there exist* $j$, $i$ *such that* $s_{ji}$ *is invertible in* $S$, *then* $C_j$ *is invertible in* $\mathrm{M}_n(S)$ *and* $\varphi(x) = C_j^{-1} x C_j$ *for all* $x \in \mathrm{M}_n(K)$.

We first say a few words about the proof. One of the most important things is to prove that $\varphi$ is defined by an LGP with some specific properties. The corresponding LGP for $\varphi$ in our situation is proved by some arguments from [4] and [6]. Essentially, our proof below follows the arguments from [1], Lemma 4.1, [4], Theorem 2 and [6], Theorem 2.

P r o o f of Proposition 2.1. Let $F_{ij} := \varphi(E_{ij})$ for $1 \leqslant i, j \leqslant n$ and, given $1 \leqslant s, k \leqslant n$, let

$$(1)\qquad A_{ks} := \sum_{\nu=1}^{n} E_{\nu k} F_{s\nu}, \quad B_{sk} := \sum_{\nu=1}^{n} F_{\nu s} E_{k\nu}, \quad \text{and} \quad \varphi(E_{ss}) := \sum_{1\leqslant i,\, j\leqslant n} w_{s,ij} E_{ij}$$

where $w_{s,ij} \in S$, $1 \leqslant s, i, j \leqslant n$. Then $F_{ij}F_{jt} = F_{it}$ and $F_{ij}F_{lt} = 0$ if $j \neq l$ and so

$$E_{ij}A_{ks} = \sum_{\nu=1}^{n} E_{ij}E_{\nu k}F_{s\nu} = E_{ik}F_{sj} \quad \text{and} \quad A_{ks}F_{ij} = \sum_{\nu=1}^{n} E_{\nu k}F_{s\nu}F_{ij} = E_{ik}F_{sj},$$

implying that

$$(2)\qquad E_{ij}A_{ks} = A_{ks}F_{ij} = A_{ks}\varphi(E_{ij})$$

for all $1 \leqslant i, j \leqslant n$. Therefore, it follows from (2) that

$$(3)\qquad xA_{ks} = A_{ks}\varphi(x)$$

for all $x \in \mathrm{M}_n(K)$ and $1 \leqslant k, s \leqslant n$. This proves (i). Moreover,

$$(4)\qquad A_{ks}B_{sk} = \sum_{1\leqslant \nu,\, \mu\leqslant n} E_{\nu k}F_{s\nu}F_{\mu s}E_{k\mu} = \sum_{\nu=1}^{n} E_{\nu k}F_{ss}E_{k\nu} = w_{s,kk}I_n.$$

This proves (ii). Furthermore,

$$(5)\quad \sum_{k=1}^{n} B_{sk}A_{ks} = \sum_{k=1}^{n} \sum_{1\leqslant \nu,\, \mu\leqslant n} F_{\mu s}E_{k\mu}E_{\nu k}F_{s\nu} = \sum_{k=1}^{n}\sum_{\nu=1}^{n} F_{\nu s}E_{kk}F_{s\nu} = \varphi(I_n) = I_n.$$

Given $x \in \mathrm{M}_n(K)$, it follows from (5) and (3) that

$$\varphi(x) = \sum_{k=1}^{n} B_{sk}A_{ks}\varphi(x) = \sum_{k=1}^{n} B_{sk}xA_{ks}.$$

Recall that the set $\{\mathrm{E}_{ij}\colon 1 \leqslant i, j \leqslant n\}$ is listed as $E_1, E_2, \ldots, E_{n^2}$. Write $B_{sk} = \sum_{j=1}^{n^2} E_j z_{j,sk}$, where $z_{j,sk} \in S$. Then, by the fact that $[S, \mathrm{M}_n(K)] = 0$, we have

$$\varphi(x) = \sum_{k=1}^{n} B_{sk}xA_{ks} = \sum_{k=1}^{n}\sum_{j=1}^{n^2} E_j z_{j,sk}xA_{ks} = \sum_{j=1}^{n^2} E_j x\left(\sum_{k=1}^{n} z_{j,sk}A_{ks}\right) = \sum_{j=1}^{n^2} E_j x C_j$$

for all $x \in \mathrm{M}_n(K)$, where $C_j := \sum_{k=1}^{n} z_{j,sk}A_{ks}$. This proves (iii). By (3) and the fact that $[S, \mathrm{M}_n(K)] = 0$, we get

$$(6)\qquad xC_j = C_j\varphi(x)$$

for all $x \in \mathrm{M}_n(K)$ and $j = 1, \ldots, n^2$. This proves (iv). Moreover, by (5), we have $\sum_{j=1}^{n^2} E_jC_j = I_n$, implying (v). Write

$$(7)\qquad C_j = \sum_{i=1}^{n^2} E_i s_{ji},$$

where $s_{ji} \in S$ for $i, j = 1, \ldots, n^2$. For $s_{ji}$, we claim that there exists $D_{ji} \in \mathrm{M}_n(S)$ such that

$$(8)\qquad C_jD_{ji} = s_{ji}I_n.$$

Given $i \in \{1, 2, \ldots, n^2\}$, there exist $k$, $l$ such that $E_i = E_{kl}$. By (7) we have

$$E_{sk}C_jE_{ls} = E_{sk}\left(\sum_{t=1}^{n^2} E_t s_{jt}\right)E_{ls} = E_{ss}s_{ji}$$

for $s = 1, \ldots, n$. This implies

$$(9)\qquad \sum_{s=1}^{n} E_{sk}C_jE_{ls} = \sum_{s=1}^{n} E_{ss}s_{ji} = s_{ji}I_n.$$

However, by (6) we have

$$\sum_{s=1}^{n} E_{sk}C_jE_{ls} = \sum_{s=1}^{n} \big(E_{sk}C_j\big)E_{ls} = \sum_{s=1}^{n} C_j\varphi(E_{sk})E_{ls} = C_jD_{ji}, \tag{10}$$

where $D_{ji} := \sum\limits_{s=1}^{n} \varphi(E_{sk})E_{ls} \in \mathrm{M}_n(S)$. It follows from (9) and (10) that $C_jD_{ji} = s_{ji}I_n$, implying (vi).

Finally, assume that $S$ is a stably finite ring and that there exist $j$, $i$ such that $s_{ji}$ is invertible in $S$. Therefore, $\mathrm{M}_n(S)$ is Dedekind-finite, implying $C_jD_{ji}s_{ji}^{-1} = I_n$ and $D_{ji}s_{ji}^{-1}C_j = I_n$. Hence, $C_j$ is invertible in $\mathrm{M}_n(S)$ and $\varphi(x) = C_j^{-1}xC_j$ for all $x \in \mathrm{M}_n(K)$, as asserted. □

Let $R$ be a ring with idempotents $e$, $f$. It is known that $eR \cong fR$ as right $R$-modules if and only if there exist $u, v \in R$ such that $uv = e$ and $vu = f$.

**Theorem 2.2.** *Let $K$ be a commutative ring, $n \in \mathbb{N}$, and let $\mathcal{A} := \mathrm{M}_n(K)$. Suppose that $\varphi$ is an endomorphism of the $K$-algebra $\mathcal{A}$. If $E_{11}\mathcal{A} \cong \varphi(E_{11})\mathcal{A}$ as right $\mathcal{A}$-modules, then $\varphi$ is an inner automorphism of $\mathcal{A}$.*

Proof. Given $1 \leqslant i \leqslant n$, $E_{1i}E_{i1} = E_{11}$ and $E_{i1}E_{1i} = E_{ii}$, implying $E_{11}\mathcal{A} \cong E_{ii}\mathcal{A}$ as right $\mathcal{A}$-modules. Similarly, $\varphi(E_{11})\mathcal{A} \cong \varphi(E_{ii})\mathcal{A}$ as right $\mathcal{A}$-modules. Therefore, $E_{ii}\mathcal{A} \cong \varphi(E_{ii})\mathcal{A}$ as right $\mathcal{A}$-modules for $i = 1, \ldots, n$. In view of [7], Exercises 15, page 334, there exists a unit $w \in \mathcal{A}$ such that $w\varphi(E_{ii})w^{-1} = E_{ii}$ for $i = 1, \ldots, n$. Since $\varphi$ is inner if and only if the homomorphism $x \mapsto w\varphi(x)w^{-1}$ for $x \in \mathcal{A}$ is, we may assume from the start that $\varphi(E_{ii}) = E_{ii}$ for $i = 1, \ldots, n$. Assume that $\varphi$ is of the form given in Proposition 2.1. Then, by Proposition 2.1 (i) and (ii), we have

$$w_{1,11} = 1, \quad A_{11}B_{11} = I_n, \quad \text{and} \quad xA_{11} = A_{11}\varphi(x) \quad \text{for all } x \in \mathcal{A}.$$

Note that $\mathcal{A}$ is Dedekind-finite since it is a PI-algebra. Therefore, $B_{11}A_{11} = I_n$, implying that $A_{11}$ is invertible in $\mathcal{A}$. Hence, $\varphi(x) = A_{11}^{-1}xA_{11}$ for all $x \in \mathcal{A}$. It means that $\varphi$ is inner, as desired. □

It is known that, in a simple Artinian ring $R$, any two minimal right ideals of $R$ are isomorphic as right $R$-modules and, moreover, any automorphism of $R$ sends any minimal right ideal to a minimal right ideal. Therefore, the following is an immediate consequence of Theorem 2.2.

**Corollary 2.3** (Skolem-Noether)**.** *Let $K$ be a field, $n \in \mathbb{N}$. Then every automorphism of the $K$-algebra $\mathrm{M}_n(K)$ is inner. That is, any field is an SN algebra.*

Since Proposition 2.1 in this paper is parallel to the key lemma (see [1], Lemma 4.1), many main results are indeed derived from it. We will illustrate Proposition 2.1 in studying SN algebras (over a commutative ring) by focusing only on proving the following: UFDs and semilocal algebras are SN algebras. We also give an application. It is proved that if $S$ is a semilocal algebra over a field $F$, then every homomorphism from a finite-dimensional central simple subalgebra of $S$ into $S$ can be extended to an inner automorphism of $S$ (see Corollary 2.9 below). This gives a generalization of [1], Theorem 6.1, where $S$ is required to be a finite-dimensional algebra. In fact, we get a more generalized result proved in Theorem 2.8 below.

We begin with the following (see [1], Proposition 7.1).

**Theorem 2.4.** *Let $S$ be an algebra over $K$, where $K$ is a subring of $Z(S)$, $n \in \mathbb{N}$, and let $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ be a homomorphism. If $S$ is embedded in a division ring $\Delta$ with $K \subseteq Z(\Delta)$, then there exists $C \in \mathrm{M}_n(S)$, which is invertible in $\mathrm{M}_n(\Delta)$, such that*

$$C^{-1} = Ds^{-1} \quad \text{for some } s \in S,\ D \in \mathrm{M}_n(S),$$

*and $\varphi(x) = C^{-1}xC$ for all $x \in \mathrm{M}_n(K)$. Moreover, if $D = \sum\limits_{k=1}^{n^2} E_k s_k$ and $C = \sum\limits_{l=1}^{n^2} E_l t_l$, where $s_k, t_l \in S$, then $s_k s^{-1} t_l \in S$ for all $l$, $k$.*

P r o o f. Since $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ is a homomorphism, we may assume that $\varphi$ is of the form as given in Proposition 2.1. There exist $C := C_k \neq 0$ in $\mathrm{M}_n(S)$ and $s := s_{kl} \neq 0$ in $S$ such that $CD = sI_n$, where $D := D_{kl} \in \mathrm{M}_n(S)$, and $xC = C\varphi(x)$ for all $x \in \mathrm{M}_n(K)$. Note that $s$ is invertible in $\Delta$ and that $\mathrm{M}_n(\Delta)$ is Dedekind-finite. Therefore, $C^{-1} = Ds^{-1}$ in $\mathrm{M}_n(\Delta)$ and so $\varphi(x) = C^{-1}xC$ for all $x \in \mathrm{M}_n(K)$.

Since $C^{-1} = Ds^{-1} = \sum\limits_{k=1}^{n^2} E_k s_k s^{-1}$ and $C = \sum\limits_{l=1}^{n^2} E_l t_l$, we get

$$\varphi(x) = \sum_{1 \leqslant k,\, l \leqslant n^2} E_k s_k s^{-1} x E_l t_l = \sum_{1 \leqslant k,\, l \leqslant n^2} E_k x E_l s_k s^{-1} t_l \tag{11}$$

for all $x \in \mathrm{M}_n(K)$, where we have used the fact that $[S, \mathrm{M}_n(K)] = 0$. Then, for $1 \leqslant m, p \leqslant n$, given $x \in \mathrm{M}_n(K)$ we have $\sum\limits_{q=1}^{n} E_{qm}\varphi(E_{pq}x) \in \mathrm{M}_n(S)$ and, by (11),

$$\begin{aligned}
\sum_{q=1}^{n} E_{qm}\varphi(E_{pq}x) &= \sum_{q=1}^{n} E_{qm} \sum_{1 \leqslant k,\, l \leqslant n^2} E_k (E_{pq}x) E_l s_k s^{-1} t_l = \sum_{q=1}^{n} \sum_{1 \leqslant l \leqslant n^2} E_{qq} x E_l s_j s^{-1} t_l \\
&= \sum_{1 \leqslant l \leqslant n^2} x E_l s_j s^{-1} t_l \in \mathrm{M}_n(S),
\end{aligned}$$

where $E_j = E_{mp}$. This implies $s_j s^{-1} t_l \in S$. $\square$

**Corollary 2.5.** *Every UFD is an SN algebra.*

Proof. Let $K$ be a UFD, $n \in \mathbb{N}$, and let $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(K)$ be a homomorphism. We claim that $\varphi$ is an inner automorphism. We set $S = K$ in Theorem 2.4 and $\varphi$ satisfies the conclusion of Theorem 2.4. Let $t_0$ be the greatest common divisor of $t_1, \ldots, t_{n^2}$. Replacing $C$ by $Ct_0^{-1}$ we may assume that $t_1, \ldots, t_{n^2}$ are coprime.

It suffices to claim that $C^{-1} = Ds^{-1} \in \mathrm{M}_n(K)$, that is, $s$ is a divisor of $s_k$ for all $k$. Since $s_k s^{-1} t_l \in K$ for all $l$, $k$, this means that $s$ is a divisor of $s_k t_l$ for all $k$, $l$. Note that $K$ is a UFD and $t_1, \ldots, t_{n^2}$ are coprime. Therefore, $s$ is a divisor of $s_k$ for all $k$, as desired. □

We remark that Corollary 2.5 is not new. Indeed, by [2], Proposition 6.1, every endomorphism of a central separable algebra is an automorphism. Therefore, it follows from [4], Corollary 15 that every endomorphism of a matrix algebra over a UFD is an inner automorphism. This proves that every UFD is an SN algebra.

We next turn to prove the theorem: every semilocal algebra is an SN algebra. Recall that a ring $S$ is called local (or semilocal) if $S/J(S)$ is a division ring (or semisimple Artinian ring), where $J(S)$ is the Jacobson radical of $S$. It is easy to check that every local ring is stably finite and the set of its nonunits forms an ideal.

**Proposition 2.6.** *Every local algebra is an SN algebra.*

Proof. Let $S$ be a local algebra. Given $n \in \mathbb{N}$, let $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ be a homomorphism, where $K$ is a commutative ring contained in $Z(S)$. We claim that $\varphi$ can be extended to an inner automorphism of $\mathrm{M}_n(S)$. We keep all notations of the proof of Proposition 2.1. Write $\varphi(E_{ss}) = \sum\limits_{1 \leqslant i,j \leqslant n} w_{s,ij} E_{ij}$, where $w_{s,ij} \in S$ for $1 \leqslant s, i, j \leqslant n$. Since $\sum\limits_{s=1}^{n} \varphi(E_{ss}) = I_n$, we get

$$\sum_{s=1}^{n} w_{s,ii} = 1$$

for $i = 1, \ldots, n$. Fix an $s$, $1 \leqslant s \leqslant n$. Since the set of all nonunits of $S$ is an additive subgroup, $w_{s,rr}$ is a unit in $S$ for some $r$. In view of Proposition 2.1 (ii), $A_{rs} B_{sr} w_{s,rr}^{-1} = I_n$ and so $B_{sr} w_{s,rr}^{-1} A_{rs} = I_n$, implying that $A_{rs}$ is invertible in $\mathrm{M}_n(S)$. In view of Proposition 2.1 (i), $\varphi(x) = A_{rs}^{-1} x A_{rs}$ for all $x \in \mathrm{M}_n(K)$, as asserted. □

In Proposition 2.6, where we consider a commutative local algebra over itself, it is known that any endomorphism of matrix algebra over a commutative local ring is inner (see [1] and [6], page 163 for local algebras over a field).

Let $B$ be a $K$-algebra with a subalgebra $A$. Given a homomorphism $\varphi\colon A \to B$, which is defined by the LGP $\sum\limits_{i} a_i X b_i$, and any ideal $I$ of $B$, $\varphi$ induces a canonical homomorphism $\bar{\varphi}\colon A/A\cap I \to B/I$, which is defined by $\bar{\varphi}(\bar{x}) = \sum_i \bar{a}_i\bar{x}\bar{b}_i$ for $\bar{x} \in A/A\cap I$. In view of Proposition 2.1, given an algebra $S$ over $K$ with $K \subseteq Z(S)$, any homomorphism from $\mathrm{M}_n(K)$ to $\mathrm{M}_n(S)$ is defined by an LGP. We are now ready to prove the following.

**Theorem 2.7.** *Every semilocal algebra is an SN algebra.*

P r o o f. We separate the proof into several steps.

*Step 1*: If $S$ and $T$ are SN algebras, then $S\oplus T$ is an SN algebra. Given $n \in \mathbb{N}$, let

$$\varphi\colon \mathrm{M}_n(Z(S)\oplus Z(T)) \to \mathrm{M}_n(S\oplus T)$$

be a homomorphism over $Z(S)\oplus Z(T)$. Clearly, we have

$$\varphi(\mathrm{M}_n(Z(S)) \subseteq \mathrm{M}_n(S) \quad \text{and} \quad \varphi(\mathrm{M}_n(Z(T)) \subseteq \mathrm{M}_n(T).$$

Since $S$ and $T$ are SN algebras, there exist units $u \in \mathrm{M}_n(S)$ and $v \in \mathrm{M}_n(T)$ such that $\varphi(x) = uxu^{-1}$ for all $x \in \mathrm{M}_n(Z(S))$ and $\varphi(y) = vyv^{-1}$ for all $y \in \mathrm{M}_n(Z(T))$. These imply that

$$\varphi((x,y)) = (u,v)(x,y)(u^{-1},v^{-1}) = (u,v)(x,y)(u,v)^{-1}$$

for all $(x,y) \in \mathrm{M}_n(S)\oplus\mathrm{M}_n(T) = \mathrm{M}_n(S\oplus T)$.

*Step 2*: Every semisimple Artinian algebra is an SN algebra. Let $S$ be a semisimple Artinian algebra. By the Wedderburn-Artin theorem and Step 1, it suffices to assume that $S$ is a simple Artinian algebra with center $K$. Given $n \in \mathbb{N}$, let $\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$ be a $K$-algebra homomorphism. Since $\mathrm{M}_n(S)$ is still a simple Artinian algebra with the center $K$, it follows from the Skolem-Noether theorem (see [3], Theorem 4.3.1) that there exists a unit $u \in \mathrm{M}_n(S)$ such that $\varphi(x) = uxu^{-1}$ for $x \in \mathrm{M}_n(K)$, as asserted.

*Step 3*: If an algebra $S$ is stably finite and $S/J(S)$ is an SN algebra, then $S$ is also an SN algebra. Let $K$ denote the center of $S$, $n \in \mathbb{N}$, and let

$$\varphi\colon \mathrm{M}_n(K) \to \mathrm{M}_n(S)$$

be a homomorphism as $K$-algebras. In view of Proposition 2.1, $\varphi$ is defined by an LGP. Therefore, $\varphi$ induces a canonical homomorphism

$$\bar{\varphi}\colon \mathrm{M}_n(\overline{K}) \to \mathrm{M}_n(S/J(S)),$$

which is defined by $\bar{\varphi}(\bar{x}) = \overline{\varphi(x)}$ for $\bar{x} \in \mathrm{M}_n(\overline{K})$, where $\overline{K} := K + J(S)/J(S)$. Note that $\overline{K} \subseteq Z(S/J(S))$. Since $S/J(S)$ is an SN algebra, there exists a unit

$\bar{u} \in \mathrm{M}_n(\overline{S}) = \mathrm{M}_n(S)/J(\mathrm{M}_n(S))$, where $u \in \mathrm{M}_n(S)$, such that $\bar{\varphi}(\bar{x}) = \bar{u}\bar{x}\bar{u}^{-1}$ for $\bar{x} \in \mathrm{M}_n(\overline{K})$. Note that $u$ is also a unit of $\mathrm{M}_n(S)$. Therefore,

$$u^{-1}\varphi(x)u - x \in J(\mathrm{M}_n(S))$$

for all $x \in \mathrm{M}_n(K)$. Replacing $\varphi$ by the homomorphism $x \mapsto u^{-1}\varphi(x)u$ for $x \in \mathrm{M}_n(K)$, we may assume from the start that

$$\varphi(x) - x \in J(\mathrm{M}_n(S))$$

for all $x \in \mathrm{M}_n(K)$. Since $\varphi$ is a homomorphism, we may assume that $\varphi$ is of the form as given in Proposition 2.1. We now follow the argument in the proof of [1], Theorem 5.4. By Proposition 2.1 (iii) and (vi),

$$\varphi(x) = \sum_{j=1}^{n^2} E_j x C_j = \sum_{1 \leqslant j,i \leqslant n^2} E_j x E_i s_{ji}$$

for all $x \in \mathrm{M}_n(K)$. The goal is to show that at least one $s_{ji}$ is invertible in $S$ (see Proposition 2.1 (vi) and (iv)). Write $1 = \sum_{k=1}^{n^2} \lambda_k E_k$, where $\lambda_k = 1$ if $E_k$ is one of $E_{11}, \ldots, E_{nn}$ and is zero otherwise. Note that

$$x = I_n x I_n = \sum_{1 \leqslant j,i \leqslant n^2} \lambda_j \lambda_i E_j x E_i$$

for all $x \in \mathrm{M}_n(K)$. Therefore,

$$\varphi(x) - x = \sum_{1 \leqslant j,i \leqslant n^2} E_j x E_i (s_{ji} - \lambda_j \lambda_i) \in J(\mathrm{M}_n(S)) \tag{12}$$

for all $x \in \mathrm{M}_n(K)$. We may list $E_j = E_{jj}$ for $j = 1, \ldots, n$. Therefore, for $x \in \mathrm{M}_n(K)$, by (12) we have

$$\sum_{k=1}^{n} E_{k1} \sum_{1 \leqslant j,i \leqslant n^2} E_j (E_{1k} x) E_i (s_{ji} - \lambda_j \lambda_i) \in J(\mathrm{M}_n(S)).$$

That is,

$$\sum_{1 \leqslant i \leqslant n^2} x E_i (s_{1i} - \lambda_1 \lambda_i) \in J(\mathrm{M}_n(S)) = \mathrm{M}_n(J(S))$$

for all $x \in \mathrm{M}_n(K)$. Since $\lambda_1 = 1$, this implies that $s_{11} - 1 \in J(S)$ and so $s_{11}$ is invertible in $S$, as asserted.

Since $S$ is stably finite [7], Theorem 2.13 and the algebra $S/J(S)$ is semisimple Artinian, the theorem follows from Steps 2 and 3. $\square$

Therefore, every right Artinian algebra is an SN algebra since it is semilocal. In particular, any finite ring is an SN algebra even if its center contains no subfield. Applying the fact that every finite-dimensional algebra is an SN algebra, Brešar et al. proved the theorem: Let $A$ be a finite-dimensional algebra and let $R$ be a finite-dimensional central simple subalgebra of $A$. Then every homomorphism from $R$ into $A$ can be extended to an inner automorphism of $A$. Our next aim is to prove a generalization of the theorem above.

**Theorem 2.8.** *Let $S$ be a semilocal algebra over $K$, where $K \subseteq Z(S)$, and let $R$ be a central separable $K$-subalgebra of $S$. Then any homomorphism from $R$ into $S$ can be extended to an inner automorphism of $S$.*

Since every finite-dimensional central simple algebra is a central separable algebra, the following is an immediate consequence of Theorem 2.8.

**Corollary 2.9.** *Let $S$ be a semilocal algebra over a field $F$. Then every homomorphism from a finite-dimensional central simple subalgebra of $S$ into $S$ can be extended to an inner automorphism of $S$.*

We now turn to the proof of Theorem 2.8. It is known that, given a commutative ring $K$, $\mathrm{M}_n(K)$ is a central separable $K$-algebra free of rank $n^2$ over $K$. Given a ring $T$ and $n \in \mathbb{N}$, $T$ is semisimple (or right Artinian) if and only if $\mathrm{M}_n(T)$ is semisimple (or right Artinian). Thus, $T$ is semilocal if and only if $\mathrm{M}_n(T)$ is semilocal. We begin with the following preliminary result.

**Lemma 2.10.** *Let $S$ be a semilocal algebra over $K$, where $K$ is a subring of $Z(S)$. If $\mathrm{M}_n(K) \subseteq S$, then any homomorphism from $\mathrm{M}_n(K)$ into $S$ can be extended to an inner automorphism of $S$.*

P r o o f. Let $\varphi\colon \mathrm{M}_n(K) \to S$ be a homomorphism as $K$-algebras. Since we have $\mathrm{M}_n(K) \subseteq S$, the semilocal algebra $S$ contains the usual matrix units $E_{ij}$'s, $1 \leqslant i, j \leqslant n$. Let $T := \{x \in S\colon [x, E_{ij}] = 0\, \forall i, j\}$. Then $K \subseteq Z(T)$ and $S = \mathrm{M}_n(T)$. Since $S$ is semilocal, so is $T$. In view of Theorem 2.7, $T$ is an SN algebra. Thus, $\varphi$ can be extended to an inner automorphism of $S$. $\square$

We need the following result in our proof.

**Theorem 2.11** (Srivastava and Shah, [12], Theorem 1)**.** *Let $S$ be a semilocal ring and let $R$ be a unital subring of $S$. Suppose that $R$ is a direct summand of $S$ as a left $R$-module. Then $R$ is a semilocal ring.*

**Proposition 2.12.** *Let $S$ be an algebra over a commutative ring $K$ and let $R$ be a central separable $K$-algebra. If $S \otimes_K R$ is a semilocal algebra, then so is $S$.*

P r o o f. In view of [2], Lemma 3.1, $K$ is a direct summand of $R$, that is, $R = B \oplus K$ for some $K$-submodule $B$ of $R$. Therefore,

$$S \otimes_K R = S \otimes_K (B \oplus K) \cong (S \otimes_K B) \oplus (S \otimes_K K) \cong (S \otimes_K B) \oplus S.$$

This implies that $S$ is a direct summand of $S \otimes_K R$ as a left $S$-module. It follows from Theorem 2.11 that $S$ is itself a semilocal algebra. $\square$

**Lemma 2.13.** *Let $S$ be a semilocal algebra over a commutative ring $K$ and let $R$ be a central separable $K$-algebra, which is a free module over $K$. Then $S \otimes_K R$ is also a semilocal algebra.*

P r o o f. It is known that $R$ is finitely generated $K$-module. Therefore, $R$ has a finite free rank, say $s$, over $K$ and so $R \otimes_K R^{\mathrm{op}} \cong \mathrm{M}_s(K)$. Then

$$(S \otimes_K R) \otimes_K R^{\mathrm{op}} \cong S \otimes_K (R \otimes_K R^{\mathrm{op}}) \cong S \otimes_K \mathrm{M}_s(K) \cong \mathrm{M}_s(S),$$

which is a semilocal algebra since $S$ is. Note that $R^{\mathrm{op}}$ is also a central separable $K$-algebra. In view of Proposition 2.12, $S \otimes_K R$ is a semilocal algebra. $\square$

We need two more basic facts.

**Fact 1.** *Let $R$ be a central separable $K$-algebra. It is known that there exists a canonical isomorphism $\eta\colon R \otimes_K R^{\mathrm{op}} \to \mathrm{Hom}_K(R, R)$, which is defined by*

$$\eta\Big(\sum_i a_i \otimes b_i\Big)(x) = \sum_i a_i x b_i$$

*for all $x \in R$ and $\sum_i a_i \otimes b_i \in R \otimes_K R^{\mathrm{op}}$. In addition, suppose that $R$ is a free module over $K$ of rank $q$. Therefore, if $r_1, \ldots, r_q$ form a basis of $R$ over $K$, given $z \in R$, there exist finitely many $a_i, b_i \in R$ such that*

$$\sum_i a_i r_j b_i = 0 \quad \text{for } j = 2, \ldots, q \quad \text{and} \quad \sum_i a_i r_1 b_i = z.$$

**Fact 2.** *Let $S$ be a $K$-algebra with a subalgebra $R$. If $\sum_i a_i \otimes b_i = 0$ in $S \otimes_K R$, then $\sum_i a_i x b_i = 0$ for all $x \in S$.*

**Proposition 2.14.** *In Theorem 2.8, if $R$ is a free $K$-module, then every homomorphism from $R$ into $S$ can be extended to an inner automorphism of $S$.*

P r o o f. Since $R$ is a central separable $K$-subalgebra of $S$, $R$ is a finitely generated module over its center $K$. By the fact that $R$ is a free $K$-module, $R$ has a finite free rank, say $q$. Let $\varphi\colon R \to S$ be a homomorphism. Let

$$\varphi \otimes 1\colon R \otimes_K R^{\mathrm{op}} \to S \otimes_K R^{\mathrm{op}}$$

be the homomorphism defined by $(\varphi \otimes 1)(x \otimes y) = \varphi(x) \otimes y$ for $x \in R$ and $y \in R^{\mathrm{op}}$. Note that $R \otimes_K R^{\mathrm{op}} \cong \mathrm{M}_q(K)$. In view of Lemma 2.13, $S \otimes_K R^{\mathrm{op}}$ is also semilocal. By Lemma 2.10, there exists a unit $u \in S \otimes_K R^{\mathrm{op}}$ such that

$$\varphi(x) \otimes y = (\varphi \otimes 1)(x \otimes y) = u(x \otimes y)u^{-1} \tag{13}$$

for all $x \in R$ and $y \in R^{\mathrm{op}}$. Choose $r_1, \ldots, r_q$ to be a basis of the free $K$-module $R$. Write $u = \sum\limits_{i=1}^{q} s_i \otimes r_i$, where $s_i \in S$, $1 \leqslant i \leqslant q$. Let $x \in R$ and $y \in R^{\mathrm{op}}$. By (13) we get

$$(\varphi(x) \otimes y) \sum_{i=1}^{q} s_i \otimes r_i = \left(\sum_{i=1}^{q} s_i \otimes r_i\right)(x \otimes y),$$

that is,

$$\sum_{i=1}^{q} \varphi(x) s_i \otimes r_i y = \sum_{i=1}^{q} s_i x \otimes y r_i. \tag{14}$$

Replacing $y$ by 1 in (14), we get

$$\sum_{i=1}^{q} \bigl(\varphi(x) s_i - s_i x\bigr) \otimes r_i = 0 \tag{15}$$

for all $x \in R$.

To apply Fact 2, we need to notice that $\sum\limits_i y_i \otimes z_i = 0$ in $S \otimes_F R^{\mathrm{op}}$ if and only if $\sum\limits_i y_i \otimes z_i = 0$ in $S \otimes_F R$. Applying Fact 2 to (15) with the remark above, we have

$$\sum_{i=1}^{q} (\varphi(x) s_i - s_i x) y r_i = 0 \tag{16}$$

for all $x, y \in R$. By Fact 1, there exist finitely many $a_j, b_j \in R$ such that

$$\sum_j a_j r_k b_i = 0 \quad \text{for } k = 2, \ldots, q, \quad \text{but} \quad \sum_j a_j r_1 b_j = 1.$$

By (16) we have

$$0=\sum_{j}\Big(\sum_{i=1}^{q}(\varphi(x)s_i-s_ix)a_jr_i\Big)b_j=\sum_{i=1}^{q}\Big((\varphi(x)s_i-s_ix)\sum_{j}a_jr_ib_j\Big)=\varphi(x)s_1-s_1x$$

for all $x\in R$. That is, $\varphi(x)s_1=s_1x$ for all $x\in R$. Analogously, $\varphi(x)s_i=s_ix$ for all $x\in R$, where $i=2,\dots,q$. We now rewrite (14) as

$$\sum_{i=1}^{q}s_ix\otimes r_iy=\sum_{i=1}^{q}s_ix\otimes yr_i \tag{17}$$

for all $x,y\in R$. By Fact 2, we get

$$w_xy=\sum_{i=1}^{q}(s_ix)yr_i, \tag{18}$$

where $w_x:=\sum\limits_{i=1}^{q}s_ixr_i$ for all $x,y\in R$. Write $1=\beta_1r_1+\ldots+\beta_qr_q$, where $\beta_i\in K$. It follows from (18) that

$$\sum_{j=1}^{q}\big(\beta_jw_x-s_jx\big)yr_j=0. \tag{19}$$

Applying the same argument as above, we get $\beta_jw_x-s_jx=0$ for all $x\in R$ and all $j$. That is,

$$\sum_{i=1}^{q}\beta_js_ixr_i=\beta_jw_x=s_jx=\sum_{i=1}^{q}\beta_is_jxr_i \tag{20}$$

for all $x\in R$ and all $j$. Applying the same argument to (20), we get $\beta_js_i=\beta_is_j$ for $1\leqslant i,j\leqslant q$. Therefore, for $j\in\{1,\dots,q\}$, we have

$$\begin{aligned}u(r_j\beta_j\otimes 1)&=\sum_{i=1}^{q}s_ir_j\beta_j\otimes r_i=\sum_{i=1}^{q}s_i\beta_jr_j\otimes r_i=\sum_{i=1}^{q}s_j\beta_ir_j\otimes r_i\\&=\sum_{i=1}^{q}s_jr_j\otimes\beta_ir_i=s_jr_j\otimes 1.\end{aligned}$$

Hence,

$$u=u\Big(\sum_{j=1}^{q}r_j\beta_j\otimes 1\Big)=\sum_{j=1}^{q}u(r_j\beta_j\otimes 1)=\sum_{j=1}^{q}(s_jr_j\otimes 1)=\Big(\sum_{j=1}^{q}s_jr_j\Big)\otimes 1,$$

implying $u \in S$. The last step is to show $u^{-1} \in S$. Write $u^{-1} = \sum\limits_{j=1}^{q} t_j \otimes r_j$, where $t_j \in S$ for $j = 1, \ldots, q$. Then

$$1 = uu^{-1} = \sum_{j=1}^{q} (ut_j) \otimes r_j = \sum_{j=1}^{q} \beta_j \otimes r_j,$$

implying that $ut_j = \beta_j$ for $j = 1, \ldots, q$. Therefore,

$$u \sum_{j=1}^{q} t_j r_j = \sum_{j=1}^{q} (ut_j) r_j = \sum_{j=1}^{q} \beta_j r_j = 1.$$

This implies that $u^{-1} = \sum\limits_{j=1}^{q} t_j r_j \in S$, as desired. □

**Lemma 2.15.** *Let $R$ be a central separable $K$-algebra and let $S$ be an algebra over $K$, where $K \subseteq Z(S)$. If $\varphi\colon R \to S$ is a homomorphism, then there exist finitely many $a_i \in R$, $s_i \in S$, $1 \leqslant i \leqslant m$, such that $\varphi(x) = \sum\limits_{i=1}^{m} a_i x s_i$ for all $x \in R$.*

Proof. Since $R$ is a finitely generated projective module over $K$, there exist finitely many $m_i \in R$ and $K$-module maps $f_i\colon R \to K$ such that $x = \sum\limits_{i} f_i(x) m_i$ for all $x \in R$. Since $f_i \in \mathrm{Hom}_K(R, R)$, by Fact 1 there exist finitely many $a_{ij}, b_{ij} \in R$ such that

$$f_i(x) = \sum_{j} a_{ij} x b_{ij}$$

for all $x \in R$. Therefore,

$$\varphi(x) = \varphi\biggl(\sum_{i} f_i(x) m_i\biggr) = \sum_{i} f_i(x) \varphi(m_i) = \sum_{i} \sum_{j} a_{ij} x b_{ij} \varphi(m_i)$$

for all $x \in R$. Note that $a_{ij} \in R$ and $b_{ij}\varphi(m_i) \in S$ for all $i$, $j$. □

We are now ready to give the proof of Theorem 2.8.

Proof of Theorem 2.8. Suppose first that $S$ is a simple Artinian algebra. Since $Z(S)$ is a field and $K \subseteq Z(S)$, $K$ is an integral domain. Denote by $\widetilde{K}$ the quotient field of $K$. Therefore, $\widetilde{K} \subseteq Z(S)$. By the fact that $K$ is an integral domain, $R$ is a prime ring (see [2], Corollary 3.2). Moreover, $R$ is a PI-ring since $R$ is a finitely generated $K$-module. In view of the Posner-Rowen theorem (see [11]), $R\widetilde{K}$ is a finite-dimensional central simple $\widetilde{K}$-algebra. Clearly, $\varphi$ can be extended to a $\widetilde{K}$-algebra homomorphism from $R\widetilde{K}$ into $S$. In view of Proposition 2.14, $\varphi$ can be extended to an inner automorphism of $S$.

We now consider the general case. Since $S$ is a semilocal algebra, $S/J(S)$ is a semisimple Artinian algebra. It follows from the Wedderburn-Artin theorem that

$$S/J(S) = T_1 \oplus \ldots \oplus T_n,$$

where all $T_i$ are simple Artinian algebras. Note that $T_i \cong S/M_i$ for some maximal ideal of $S$ with $J(S) \subseteq M_i$, $i = 1, \ldots, n$. In view of Lemma 2.15, $\varphi(R \cap M_i) \subseteq M_i$ for all $i$. Consider the map

$$\varphi_i\colon\ R/M_i \cap R \to S/M_i,$$

which is defined by $\varphi_i(\bar{x}) = \overline{\varphi(x)} := \varphi(x) + M_i$ for $x \in R$. In view of [2], Proposition 1.11, $R/R \cap M_i$ is a central separable algebra over $K + (R \cap M_i)/R \cap M_i$. We identify $R/M_i \cap R$ with the subalgebra $R + M_i/M_i$ of $S/M_i$. In view of the case in the first paragraph, $\varphi_i$ can be extended to an inner automorphism of $S/M_i$, say

$$\bar{x} \mapsto \bar{u}_i \bar{x} \bar{u}_i^{-1} \quad \text{for } \bar{x} \in R/R \cap M_i,$$

where $\bar{u}_i$ is a unit of $S/M_i$. Note that $\varphi(R \cap J(S)) \subseteq J(S)$ (see Lemma 2.15). This implies that the following canonical $\overline{K}$-algebra homomorphism

$$\bar{\varphi}\colon\ R/R \cap J(S) \to S/J(S)$$

can be extended to an inner automorphism of $S/J(S)$. Note that an element $u \in S$ is invertible if and only if $u + J(S)$ is invertible in $S/J(S)$. Hence there exists a unit $w \in S$ such that

$$\varphi(x) - wxw^{-1} \in J(S)$$

for all $x \in R$. Equivalently,

$$w^{-1}\varphi(x)w - x \in J(S) \tag{21}$$

for all $x \in R$. Since the map $x \mapsto w^{-1}\varphi(x)w$ for $x \in R$ is a $K$-algebra homomorphism, it follows from Lemma 2.15 that there are finitely many $r_i \in R$, $s_i \in S$, $i = 1, \ldots, q$, such that

$$w^{-1}\varphi(x)w = \sum_{i=1}^{q} r_i x s_i \tag{22}$$

for all $x \in R$. By [2], Lemma 3.1, there exists a $K$-submodule $B$ of $R$ such that $R = K \oplus B$ as $K$-modules. Therefore, we may assume further that $r_1 = 1$ and $r_i \in B$ for $i = 2, \ldots, q$. We claim that $s_1$ is a unit of $S$.

Consider the $K$-module map $g\colon R \to R$ defined by $g(\beta) = \beta$ for $\beta \in K$ and $g(b) = 0$ for $b \in B$. By Fact 1, there exist finitely many $c_j, d_j \in R$ such that $g(x) = \sum_j c_j x d_j$ for all $x \in R$. Therefore, $\sum_j c_j r_1 d_j = 1$ but $\sum_j c_j r_k d_j = 0$ for $k = 2, \ldots, q$. By (21) and (22), we have $\sum_{i=1}^{q} r_i x s_i - x \in J(S)$ for all $x \in R$ and so

$$\sum_j c_j \left( \sum_{i=1}^{q} r_i (d_j x) s_i - (d_j x) \right) = \sum_{i=1}^{q} \left( \sum_j c_j r_i d_j \right) x s_i - \left( \sum_j c_j d_j \right) x = x(s_1 - 1) \in J(S)$$

for all $x \in R$. In particular, $s_1 - 1 \in J(S)$, implying that $s_1$ is a unit of $S$.

Set $\widetilde{\varphi}(x) := w^{-1}\varphi(x)w$ for $x \in R$. Let $x, y \in R$. We have

$$\widetilde{\varphi}(xy) = \widetilde{\varphi}(x)\widetilde{\varphi}(y) = \sum_{i=1}^{q} r_i x s_i \widetilde{\varphi}(y).$$

On the other hand, $\widetilde{\varphi}(xy) = \sum_{i=1}^{q} r_i x y s_i$. Comparing the two equalities, we get

$$\sum_{i=1}^{q} r_i x (y s_i - s_i \widetilde{\varphi}(y)) = 0.$$

Therefore,

$$0 = \sum_j c_j \sum_{i=1}^{q} r_i (d_j x)(y s_1 - s_1 \widetilde{\varphi}(y)) = x(y s_i - s_i \widetilde{\varphi}(y))$$

for all $x, y \in R$. That is,

$$y s_1 = s_1 \widetilde{\varphi}(y)$$

for all $y \in R$. Since $s_1$ is a unit of $S$, we get

$$\varphi(y) = w\widetilde{\varphi}(y)w^{-1} = w s_1^{-1} \varphi(y) s_1 w^{-1} = (w s_1^{-1}) x (w s_1^{-1})^{-1}$$

for all $y \in R$. $\square$

We end this paper with a generalization of Theorem 2.8.

**Theorem 2.16.** *Let $R$ be a central separable algebra over $K$ and let $S$ be a semilocal algebra over $K$, where $K \subseteq Z(S)$. If $f, g\colon R \to S$ are $K$-algebra homomorphisms, then there exists a unit $u$ of $S$ such that $g(x) = u f(x) u^{-1}$ for all $x \in R$.*

P r o o f. Since $f\colon R \to S$ is a $K$-algebra homomorphism and $R$ is a central separable algebra over $K$, $f$ is a monomorphism (see [2], Corollary 3.7). Therefore, $R \cong f(R)$ as $K$-algebras. This means that $f(R)$ is a central separable $K$-subalgebra of $S$. Let $\varphi\colon f(R) \to S$ be the map defined by $\varphi(f(x)) = g(x)$ for all $x \in R$. Clearly, $\varphi$ is well-defined and is a homomorphism. In view of Theorem 2.8, $\varphi$ can be extended to an inner automorphism of $S$; that is, there exists a unit $u$ of $S$ such that $g(x) = uf(x)u^{-1}$ for all $x \in R$, as desired. $\square$

We remark that the theorem above gives a generalization of [9], Theorem 1.

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