Mohamed Tahar Kadaoui Abbassi; Noura Amri

On g-natural conformal vector fields on unit tangent bundles

Czechoslovak Mathematical Journal, Vol. 71 (2021), No. 1, 75–109

Czechoslovak Mathematical Journal, 71 (146) (2021), 75–109

ON g-NATURAL CONFORMAL VECTOR FIELDS ON UNIT TANGENT BUNDLES

Mohamed Tahar Kadaoui Abbassi, Noura Amri, Fez

Received April 24, 2019. Published online August 17, 2020.

Abstract. We study conformal and Killing vector fields on the unit tangent bundle, over a Riemannian manifold, equipped with an arbitrary pseudo-Riemannian g-natural metric. We characterize the conformal and Killing conditions for classical lifts of vector fields and we give a full classification of conformal fiber-preserving vector fields on the unit tangent bundle endowed with an arbitrary pseudo-Riemannian Kaluza-Klein type metric.

Keywords: conformal vector field; unit tangent bundle; g-natural metric

MSC 2020: 53C07, 53C24, 53C25

1. Introduction and main results

A smooth vector field ξ on a (pseudo-)Riemannian manifold (M, g) is said to be a *conformal* vector field if there exists a smooth function f on M, called the *potential function* of ξ, that satisfies $\mathcal{L}_\xi g = 2fg$, where $\mathcal{L}_\xi g$ is the Lie derivative of g with respect to ξ, that is, the flow of the vector field ξ consists of conformal transformations of the Riemannian manifold (M, g). When f is constant (in particular $f = 0$), the flow of ξ is given by homothetic (isometric) transformations of (M, g), and X is called a *homothetic* (*Killing*) *vector field*.

Conformal vector fields are considered by specialists as useful tools for understanding the geometry of a pseudo-Riemannian manifold. For instance, they have shown their efficiency to characterize some classical geometric spaces (see [15] and the references therein). Furthermore, like all symmetries, they have many interesting applications in physics (see [19]). In this context, we can find various studies focusing on conformal or Killing vector fields on some special pseudo-Riemannian manifolds. For example, in the framework of the Riemannian geometry of tangent bundles, Killing and conformal vector fields had been classified on tangent bundles of

DOI: 10.21136/CMJ.2020.0193-19

Riemannian manifolds, equipped with the Sasaki metric (see [28] and [29]) and the Cheeger-Gromoll metric (see [8] and [18]), respectively. When the tangent bundle is endowed with an arbitrary g-natural metric, it is not easy to find a full classification of conformal or Killing vector fields, but we can find some partial results on the subject (see [17] for Killing vector fields and [2], [20], [25] for conformal vector fields).

Concerning the unit tangent bundle T_1M of a Riemannian manifold (M, g), equipped with the Sasaki metric, Konno succeeded in giving a classification of fiber-preserving Killing vector fields (see [22]) and a full classification of Killing vector fields in the case when the base manifold is three-dimensional (see [23]). At the best of our knowledge, conformal vector fields on unit tangent bundles had not been studied yet, even in the case of the Sasaki metric.

In this paper, we are interested in the study of conformal/Killing vector fields on the unit tangent bundle equipped with an arbitrary pseudo-Riemannian g-natural metric, i.e. a metric determined by four fixed constants a, b, c, d, $a \neq 0$, $a(a+c) - b^2 \neq 0$, $a + c + d \neq 0$, as follows:

$$
\begin{aligned}
\widetilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,u)g(Y,u), \\
\widetilde{G}_{(x,u)}(X^h, Z^v) &= bg_x(X,Z), \\
\widetilde{G}_{(x,u)}(Z^v, W^v) &= ag_x(Z,W)
\end{aligned} \tag{1.1}
$$

for all $(x,u) \in T_1M$, $X, Y \in M_x$ and $Z, W \in \{u\}^\perp \subset M_x$, where X^h and Y^h (or Z^v and W^v) are the horizontal (or vertical) lifts to T_1M of X and Y (or Z and W). When $b = d = 0$, then $\widetilde{G}$ is said to be a *Kaluza-Klein metric*, and when $b = 0$ it is said to be a *Kaluza-Klein type metric*.

However, it turns out that it is difficult to give a full classification of conformal or Killing vector fields on the unit tangent bundle, endowed with an arbitrary pseudo-Riemannian g-natural metric. Hence, we will focus on three questions:

- ▷ to give necessary conditions for horizontal, tangential and complete lifts of a vector field to the unit tangent bundle to be conformal or Killing;
- ▷ to find a full classification of fiber-preserving conformal vector fields on the unit tangent bundle, endowed with a Kaluza-Klein type metric;
- ▷ to find some examples of non-fiber preserving conformal or Killing vector fields on the unit tangent bundle, endowed with a Kaluza-Klein type metric.

For tangential lifts of vector fields on M to T_1M, i.e. the tangential components of vertical lifts to T_1M of vector fields on M, we have the following result:

Theorem 1.1. *Let (M, g) be a Riemannian manifold, $\widetilde{G}$ a pseudo-Riemannian g-natural metric on T_1M, and ξ a nonzero vector field on M. Then the tangential lift ξ^t of ξ to T_1M is never conformal on $(T_1M, \widetilde{G})$.*

As concerns the horizontal lifts to T_1M of vector fields on M, we get:

Theorem 1.2. *Let (M,g) be a Riemannian manifold, $\widetilde{G}$ a pseudo-Riemannian g-natural metric on T_1M, ξ a vector field on M, and ξ^h its horizontal lift to T_1M. Then the following assertions are equivalent*

(i) *ξ^h is a conformal vector field on $(T_1M,\widetilde{G})$;*
(ii) *ξ^h is a Killing vector field on $(T_1M,\widetilde{G})$;*
(iii) *either: $\widetilde{G}$ is a Kaluza-Klein metric on T_1M, ξ is a Killing vector field on (M,g) and $R(\xi,.).=0$, R being the curvature tensor of (M,g),*
or: ξ is parallel.

Corollary 1.1. *Let (M,g) be a flat manifold. Then, for every pseudo-Riemannian Kaluza-Klein metric $\widetilde{G}$ on T_1M, a vector field ξ is a Killing vector field on (M,g) if and only if ξ^h is a Killing vector field on $(T_1M,\widetilde{G})$.*

When we take the tangential component of the complete lift ξ^c to T_1M of a vector field ξ on M, we obtain the complete lift $\xi^{\bar{c}}$ to T_1M of ξ. For this special kind of vector fields, we have:

Theorem 1.3. *Let (M,g) be a Riemannian manifold and $\widetilde{G}$ a pseudo-Riemannian g-natural metric on T_1M. Suppose that $\dim M>2$ and let ξ be a vector field on M and $\xi^{\bar{c}}$ its complete lift vector field to T_1M. Then the following assertions are equivalent*

(i) *$\xi^{\bar{c}}$ is a conformal vector field on $(T_1M,\widetilde{G})$;*
(ii) *$\xi^{\bar{c}}$ is a Killing vector field on $(T_1M,\widetilde{G})$;*
(iii) *ξ is a Killing vector field on (M,g).*

Concerning the geodesic flow vector field on T_1M, we can assert the following:

Theorem 1.4. *Let $\widetilde{G}$ be a pseudo-Riemannian g-natural metric on T_1M given by* (1.1) *and ζ the geodesic flow vector field on T_1M. The following assertions are equivalent*

(i) *ζ is conformal on $(T_1M,\widetilde{G})$;*
(ii) *ζ is a Killing vector field on $(T_1M,\widetilde{G})$;*
(iii) *$\widetilde{G}$ is of Kaluza-Klein type and the base manifold (M,g) has constant sectional curvature $(a+c)/a$.*

As a corollary, we have the following characterization of Riemannian manifolds of non zero constant sectional curvatures by means of the conformality of its geodesic flow vector field with respect to a pseudo-Riemannian Kaluza-Klein type metric on T_1M:

Corollary 1.2. *A Riemannian manifold (M, g) has a nonzero constant sectional curvature if and only if the geodesic flow vector field on T_1M is a conformal (or Killing) vector field with respect to a pseudo-Riemannian Kaluza-Klein type metric on T_1M.*

A vector field V on T_1M is called *fiber-preserving* if its flow consists of local fiber preserving transformations, i.e., local diffeomorphisms on T_1M preserving fibers. Horizontal, tangential and complete lifts to T_1M of vector fields on M are examples of fiber-preserving vector fields, while the geodesic flow vector field is not fiber-preserving. Another non classical example of fiber-preserving vector field on T_1M is the vector field $\tilde{\imath}P$, whose value at $(x, u) \in T_1M$ is the tangential lift at (x, u) of the vector $P(u) \in M_x$, where P is a $(1, 1)$-tensor field on M.

When we restrict ourselves to pseudo-Riemannian Kaluza-Klein type metrics, we can give a full classification of fiber-preserving conformal vector fields on the unit tangent bundles, namely:

Theorem 1.5. *Let $(T_1M, \widetilde{G})$ be the tangent bundle of a Riemannian manifold (M, g) endowed with a pseudo-Riemannian g-natural metric $\widetilde{G}$ of Kaluza-Klein type, i.e. of the form (3.5) with $b = 0$, and let V be a fiber-preserving vector field on $(T_1M, \widetilde{G})$. Then the following assertions are equivalent*

(i) *V is a conformal vector field on $(T_1M, \widetilde{G})$;*

(ii) *V is a Killing vector field on $(T_1M, \widetilde{G})$;*

(iii) *one of the following cases occurs:*

  (1) *$\widetilde{G}$ is a Kaluza-Klein metric and $V = \xi^{\bar{c}} + \tilde{\imath}P$, where ξ is a Killing vector field on (M, g) and P is a skew-symmetric parallel $(1, 1)$-tensor field on M;*

  (2) *$V = \xi^{\bar{c}}$, where ξ is a Killing vector field on (M, g).*

When we analyze the classifications of Killing vector fields on the tangent bundle endowed with the Sasaki metric or the Cheeger-Gromoll metric (see [8] and [29]), we realize that the class of such vector fields is generated by three types of lifted vector fields from the base manifold, two of them are fiber-preserving and the third is non-fiber-preserving. Since Theorem 1.5 gives the full classification of fiber-preserving conformal vector fields on T_1M, equipped with a pseudo-Riemannian g-natural metric of Kaluza-Klein type, it is worthwhile to consider a large class of (non-fiber-preserving) vector fields on T_1M and investigate their conformality.

For any real number λ and any vector field ξ on M, we define a vector field $\bar{*}\xi_\lambda$ on T_1M by

$$\bar{*}\xi_\lambda(x, u) := \lambda \xi^t(x, u) + h\{C(\xi)(u)\} \quad \text{for all } (x, u) \in T_1M,$$

where $C(\xi)$ is the $(1,1)$-tensor field on M defined by $g(C(\xi)Y, Z) = -g(Y, \nabla_Z \xi)$ for all vector fields Y and Z on M, ∇ being the Levi-Civita connection of (M, g), and $h\{C(\xi)(u)\}$ is expressed, in local coordinates, as $h\{C(\xi)(u)\} = \sum_i u^i [C(\xi)(\partial/\partial x^i)]^h$. Then we have:

Theorem 1.6. *Let (M, g) be a space of constant sectional curvature k of dimension $n > 2$, and $\widetilde{G}$ a pseudo-Riemannian g-natural metric on T_1M of Kaluza-Klein type, i.e. of the form (3.5) with $b = 0$, and $d \neq ak$. Let $\lambda \in \mathbb{R}$ and let ξ be a nonzero vector field on M. Then the following assertions are equivalent*

(i) *$\bar{*}\xi_\lambda$ is a nonzero conformal vector field on $(T_1M, \widetilde{G})$;*

(ii) *$\bar{*}\xi_\lambda$ is a nonzero Killing vector field on $(T_1M, \widetilde{G})$;*

(iii) *$\lambda = 0$, $k = (a + c)/a$ and $\bar{*}\xi_0$ is, up to a nonzero real factor, the geodesic flow vector field on T_1M.*

2. Preliminaries

Let (M, g) be a (pseudo-)Riemannian manifold with a Levi-Civita connection ∇ and a curvature tensor R. Recall that a smooth vector field ξ on (M, g) is *conformal* if there exists a smooth function f on M, called the *potential function* of ξ, that satisfies

$$\mathcal{L}_\xi g = 2fg, \tag{2.1}$$

where $\mathcal{L}_\xi g$ is the Lie derivative of g with respect to ξ, i.e.

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2fg(X, Y) \quad \text{for all } X, Y \in \mathfrak{X}(M), \tag{2.2}$$

or equivalently,

$$g(\nabla_X \xi, X) = fg(X, X) \quad \text{for all } X \in \mathfrak{X}(M). \tag{2.3}$$

We have the following classical result (see [15] for the Riemannian case whose generalization to the pseudo-Riemannian case is straightforward):

Lemma 2.1. *Let ξ be a conformal vector field on a (pseudo-)Riemannian manifold (M, g) with potential function f. Then we have*

$$R(\xi, X)Y + \nabla^2 \xi(Y, X) = X(f)Y + Y(f)X - \mathrm{grad} f g(X, Y) \tag{2.4}$$

for all $X, Y \in \mathfrak{X}(M)$*, where* $\operatorname{grad} f$ *denotes the gradient of* f *and* $\nabla^2\xi$ *is the second covariant derivative of* ξ *given by* $\nabla^2\xi(Y,X) = \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi$ *for all* $X, Y \in \mathfrak{X}(M)$*.*

In particular, if f *is constant, i.e.* ξ *is a homothetic or a Killing vector field, then we get*

$$R(\xi, X)Y + \nabla^2\xi(Y, X) = 0 \tag{2.5}$$

for all $X, Y \in \mathfrak{X}(M)$*.*

Now, we recall some basic facts and formulas and fix notation about tangent bundles. For more detail, we refer to [31], [32] for classical lifts of vector fields on tangent bundles and [16] for the geometry of tangent bundles of Riemannian manifolds.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of g. We will denote by M_x the tangent space of M at a point $x \in M$ and by $p\colon TM \to M$ the bundle projection. The tangent space of TM at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For $(x, u) \in TM$ and $X \in M_x$, there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_*X^h = X$, where $p\colon TM \to M$ is the natural projection. We call X^h the *horizontal lift* of X to the point $(x, u) \in TM$. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in V_{(x,u)}$ such that $X^v(df) = Xf$ for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e., $(df)(x, u) = uf$).

Observe that the map $X \to X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between the vector spaces M_x and $V_{(x,u)}$. Obviously, each tangent vector $\widetilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\widetilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

Horizontal and vertical lifts of vector fields on M are defined in the corresponding way. Each system of local coordinates $\{(U;\ x^i, i = 1, \ldots, n)\}$ in M induces on TM a system of local coordinates $\{(p^{-1}(U);\ x^i, u^i, i = 1, \ldots, n)\}$. Let $X = \sum_i X^i(\partial/\partial x^i)_x$ be the local expression in $\{(U;\ x^i, i = 1, \ldots, n)\}$ of a vector X in M_x, $x \in M$. Then the horizontal lift X^h and the vertical lift X^v of X are given, with respect to the induced coordinates, by

$$X^h = \sum_i X^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma^i_{jk} u^j X^k \frac{\partial}{\partial u^i}, \tag{2.6}$$

and

$$X^v = \sum_i X^i \frac{\partial}{\partial u^i}, \tag{2.7}$$

where (Γ^i_{jk}) denote the Christoffel's symbols of g.

Next, we introduce some notation which will be used describing vectors obtained from lifted vectors by basic operations on TM. Let T be a tensor field of type $(1, s)$ on M. If $X_1, X_2, \ldots, X_{s-1} \in M_x$, then $h\{T(X_1, \ldots, u, \ldots, X_{s-1})\}$ and $v\{T(X_1, \ldots, u, \ldots, X_{s-1})\})$ are horizontal and vertical vectors at (x, u) which are introduced by the formulas

$$h\{T(X_1, \ldots, u, \ldots, X_{s-1})\} = \sum u^\lambda \Big(T\Big(X_1, \ldots, \Big(\frac{\partial}{\partial x^\lambda}\Big)_x, \ldots, X_{s-1}\Big)\Big)^h,$$
$$v\{T(X_1, \ldots, u, \ldots, X_{s-1})\} = \sum u^\lambda \Big(T\Big(X_1, \ldots, \Big(\frac{\partial}{\partial x^\lambda}\Big)_x, \ldots, X_{s-1}\Big)\Big)^v.$$

In particular, if T is the identity tensor of type $(1, 1)$, then we obtain the geodesic flow vector field at (x, u), $\zeta_{(x,u)} = \sum u^\lambda (\partial/\partial x^\lambda)^h_{(x,u)}$, and the canonical vertical vector at (x, u), $\mathcal{U}_{(x,u)} = \sum u^\lambda (\partial/\partial x^\lambda)^v_{(x,u)}$.

Moreover $h\{T(X_1, \ldots, u, \ldots, u, \ldots, X_{s-1})\}$ and $v\{T(X_1, \ldots, u, \ldots, u, \ldots, X_{s-1})\}$ are introduced in a similar way.

The bracket operation of vector fields on the tangent bundle is given by

$$[X^h, Y^h]_{(x,u)} = [X, Y]^h_{(x,u)} - v\{R(X_x, Y_x)u\}, \tag{2.8}$$
$$[X^h, Y^v]_{(x,u)} = (\nabla_X Y)^v_{(x,u)}, \tag{2.9}$$
$$[X^v, Y^v]_{(x,u)} = 0 \tag{2.10}$$

for all vector fields X and Y on M.

Besides vertical and horizontal lifts of vector fields, there are many other special vector fields on the tangent bundle of a manifold obtained by some "lifting" technics of tensor fields on the base manifold. When the base manifold is Riemannian, these special vector fields can be expressed by means of some vertical and horizontal vector fields. We give here two of such special vector fields (see [31] for definitions):

(a) The complete lift X^c to TM of a vector field X on M is expressed as

$$X^c_{(x,u)} = X^h_{(x,u)} + v\{\nabla_u X\} \tag{2.11}$$

for all $(x, u) \in TM$.

(b) The vector field ιP on TM defined from a $(1,1)$-tensor field P on M and expressed as

$$(\iota P)_{(x,u)} = v\{P(u)\} \tag{2.12}$$

for all $(x,u) \in TM$. If P is the field I of identity endomorphisms, then $\iota I = \mathcal{U}$.

It is easy to see that $X^c = X^h + \iota(\nabla X)$.

3. g-natural metrics

In this section, we recall some basic facts about g-natural metrics on tangent bundles and their induced metrics on unit tangent bundles. For more elaborate expositions, we refer to [1], [9], [10], [24] for the construction of g-natural metrics on tangent bundles and the basic properties and [3], [4], [5], [6], [7] for g-natural metrics on unit tangent bundles.

3.1. g-natural metrics on tangent bundles. As a Riemannian manifold, the tangent bundle TM of a Riemannian manifold (M,g) were classically endowed with the well-known Sasaki metric g^s, which is defined by

$$g^s(X^h,Y^h) = g^s(X^v,Y^v) = g(X,Y), \quad g^s(X^h,Y^v) = 0$$

for all $X,Y \in \mathfrak{X}(M)$ (see [16], [26]).

Other (classes of) metrics have been then considered and the more general class is that of g-natural metrics which encompasses almost all the metrics previously considered. As their name suggests, those metrics arise from a very 'natural' construction starting from a Riemannian metric g over M. For more details about the concept of naturality and related notions, we refer to [21]. Such metrics are characterized as follows:

Proposition 3.1 ([10]). *For any g-natural metric G on the tangent bundle TM of a Riemannian manifold (M,g) there exist six smooth functions $\alpha_i,\beta_i\colon \mathbb{R}^+ \to \mathbb{R}$, $i=1,2,3$, such that*

$$\begin{aligned} G_{(x,u)}(X^h,Y^h) &= (\alpha_1+\alpha_3)(r^2)g_x(X,Y) + (\beta_1+\beta_3)(r^2)g_x(X,u)g_x(Y,u),\\ G_{(x,u)}(X^h,Y^v) &= G_{(x,u)}(X^v,Y^h)\\ &= \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u),\\ G_{(x,u)}(X^v,Y^v) &= \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,u)g_x(Y,u) \end{aligned} \tag{3.1}$$

for every $u,X,Y \in M_x$, where $r^2 = g_x(u,u)$.

Put $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$, $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t)$ and $\phi(t) = \phi_1(t) \times (\phi_1 + \phi_3)(t) - \phi_2^2(t)$ for all $t \in \mathbb{R}^+$. It it easily seen (see [9]) that G is

▷ non-degenerate if and only if

$$\alpha(t) \neq 0, \quad \phi(t) \neq 0 \quad \text{for all } t \in \mathbb{R}^+;$$

▷ Riemannian if and only if

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0 \quad \text{for all } t \in \mathbb{R}^+.$$

As said before, the wide class of g-natural metrics includes several well known metrics (Riemannian and pseudo-Riemannian) on TM. In particular:

▷ *Sasaki metric* g^s is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$ in (3.1).

▷ *Kaluza-Klein metrics* (in the sense of [11]) are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$ in (3.1) (see Remark 3.1 below).

▷ *Metrics of Kaluza-Klein type* are defined by the geometric condition of orthogonality between horizontal and vertical distributions, see [13], [14]. Thus, a g-natural metric G is of Kaluza-Klein type if $\alpha_2 = \beta_2 = 0$ in (3.1).

Remark 3.1. The terminology "Kaluza-Klein metric" in the framework of tangent bundles originates from the paper [11], in which the authors referred to the paper of Wood, see [30]. To know how the authors of [11] derived this terminology from [30], we contacted Loubeau, one of the authors, who confessed that the use of the terminology is decidedly a bit excessive and that one might use, instead, the terminology "Generalized Kaluza-Klein metric". Indeed, Wood defined the concept of Kaluza-Klein metric on principal bundles and then on their associated bundles, see [30]. When we consider the tangent bundles as associated bundles, then we find that Kaluza-Klein metrics, in the sense of Wood, are those characterized by the following properties:

(1) the projection $p\colon TM \to M$ is a Riemannian submersion;
(2) horizontal and vertical distributions are orthogonal;
(3) the metric on the fibers is induced by an arbitrary fiber metric.

Kaluza-Klein metrics on tangent bundles, as defined in [11], are rather g-natural metrics which preserve condition 2 and generalize condition 1 to the case when p is a conformal submersion. So we can define Kaluza-Klein metrics on tangent bundles, in the sense of [11], as metrics characterized by the following properties:

(1) the metric is g-natural;
(2) the projection $p\colon TM \to M$ is a conformal submersion;
(3) horizontal and vertical distributions are orthogonal.

Currently, the terminology "Kaluza-Klein metric" for tangent bundles is commonly used by geometers in the sense of [11], and so it is appropriate to adopt it.

3.2. g-natural metrics on unit tangent bundles. The *unit tangent bundle* over a Riemannian manifold (M,g) is the hypersurface of TM, given by

$$T_1M = \{(x,u) \in TM\,;\ g_x(u,u) = 1\}.$$

We will denote by $p_1\colon T_1M \to M$ the bundle projection. The tangent space of T_1M at a point $(x,u) \in T_1M$ is given by

$$(T_1M)_{(x,u)} = \{X^h + Y^v\,;\ X \in M_x,\ Y \in \{u\}^\perp \subset M_x\}. \tag{3.2}$$

By definition, *g-natural metrics on the unit tangent bundle* are the metrics induced on the hypersurface T_1M by the corresponding g-natural metrics on TM. As proved in [6] for the Riemannian case, and extended to pseudo-Riemannian settings in [12], if a g-natural metric $\widetilde{G}$ on T_1M is induced from a g-natural metric G on TM given by (3.1), then $\widetilde{G}$ is completely determined by the values of four real constants

$$a := \alpha_1(1), \quad b := \alpha_2(1), \quad c := \alpha_3(1), \quad d := (\beta_1 + \beta_3)(1),$$

i.e., by virtue of (3.2), $\widetilde{G}$ is completely determined by (1.1).

By simple calculation, using the Schmidt's orthonormalization process, it is easy to check that the vector field on TM defined by

$$N_{(x,u)} = \frac{1}{\sqrt{\varphi\phi}}[-bh\{u\} + \varphi v\{u\}] \tag{3.3}$$

for all $(x,u) \in TM$, is normal to $(T_1M, \widetilde{G})$ and unitary at any point of T_1M, where $\varphi := a + c + d$ and $\phi = \phi(1) = a\varphi - b^2$.

For $(x,u) \in T_1M$ and $X \in M_x$, since the horizontal lift $X^h_{(x,u)}$ is tangent to T_1M, we can talk about the *horizontal lift* to T_1M of a vector tangent to M at a point of T_1M. Similarly, we define the *horizontal lift of a vector field* $X \in \mathfrak{X}(M)$ as the restriction to T_1M of the horizontal lift of X to TM.

On the other hand, for $(x,u) \in T_1M$ and $X \in M_x$, $X^v_{(x,u)}$ is not necessarily tangent to T_1M. We define the *tangential lift* X^t with respect to G, of X to (x,u) as the tangential projection of the vertical lift of X to (x,u) with respect to N, that is,

$$X^t_{(x,u)} = X^v_{(x,u)} - g(u,X)v\{u\} + \frac{b}{\varphi}g(u,X)h\{u\}. \tag{3.4}$$

If X is orthogonal to u, then $X^t_{(x,u)} = X^v_{(x,u)}$. For a vector field X on M, we define it *tangential lift* to T_1M accordingly.

Let T be a tensor field of type $(1,s)$ on M. If $X_1, X_2, \dots, X_{s-1} \in M_x$ and $(x,u) \in T_1M$, we define $t\{T(X_1, \dots, u, \dots, X_{s-1})\}$ as the tangential projection of $v\{T(X_1, \dots, u, \dots, X_{s-1})\}$ on $(T_1M)_{(x,u)}$ with respect to N, given by the formula

$$t\{T(X_1, \dots, u, \dots, X_{s-1})\} = \sum u^\lambda \Big(T\Big(X_1, \dots, \Big(\frac{\partial}{\partial x^\lambda}\Big)_x, \dots, X_{s-1}\Big)\Big)^t.$$

Moreover, $t\{T(X_1, \dots, u, \dots, u, \dots, X_{s-1})\}$ is introduced in a similar way.

It is then easy to see that the tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form $X^h_{(x,u)}$ and $Y^t_{(x,u)}$, where $X, Y \in M_x$. It follows then, from (1.1) and (3.4), that g-natural metrics on T_1M admit the following explicit description.

Proposition 3.2. *Let (M,g) be a Riemannian manifold. For every pseudo-Riemannian metric $\widetilde{G}$ on T_1M induced from a g-natural G on TM there exist four constants a, b, c and d, satisfying the inequalities*

$$a \neq 0, \quad \alpha := a(a+c) - b^2 \neq 0, \quad \varphi := a + c + d \neq 0,$$

such that

$$\begin{aligned}
\widetilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)g_x(X,Y) + dg_x(X,u)g(Y,u), \\
\widetilde{G}_{(x,u)}(X^h, Y^t) &= bg_x(X,Y), \\
\widetilde{G}_{(x,u)}(X^t, Y^t) &= ag_x(X,Y) - \frac{\phi}{a+c+d} g_x(X,u)g_x(Y,u)
\end{aligned} \tag{3.5}$$

for all $(x,u) \in T_1M$, and $X, Y \in M_x$. Furthermore, $\widetilde{G}$ is Riemannian if $a > 0$, $\alpha > 0$ and $\varphi > 0$.

In particular, the *Sasaki metric* on T_1M corresponds to the case, where $a = 1$ and $b = c = d = 0$; *Kaluza-Klein metrics* are obtained when $b = d = 0$; *metrics of Kaluza-Klein type* are given by the case $b = 0$.

4. Some special vector fields on unit tangent bundles

4.1. Tangential components on T_1M of vector fields on TM. For any vector field Z on TM we define the tangential component $\tan\{Z\}$ of its restriction $Z|_{T_1M}$ to T_1M with respect to G, by $\tan\{Z\} := Z|_{T_1M} - G(Z|_{T_1M}, N)N$, obtaining a vector field on T_1M. Horizontal and tangential lifts to T_1M of vector fields on M are examples of such construction. Another example is the restriction to T_1M of the geodesic vector field ζ on TM, which is always tangent to T_1M. It defines the so-called *geodesic flow vector field on T_1M*, see [27], which we denote also by ζ. Other interesting examples are the following:

▷ The tangential component with respect to G of the complete lift X^c of X, which we call the *complete lift* to T_1M of X, and denote by $X^{\bar{c}}$. It is given by

$$X^{\bar{c}}_{(x,u)} = X^h_{(x,u)} + t\{\nabla_u X\} \tag{4.1}$$

for all $(x,u) \in T_1M$.

▷ The tangential component with respect to G of the vector field ιP on TM, which we denote by $\tilde{\iota}P$. It is given by

$$(\tilde{\iota}P)_{(x,u)} = t\{P(u)\} \tag{4.2}$$

for all $(x,u) \in T_1M$.

▷ For any vector field ξ on M and any real function f defined on $\mathbb{R}^+$, we define a vector field $*\xi_f$ on TM in the following manner: $*\xi_f := f(r^2)\xi^v + \xi^\flat \mathcal{U} + *C(\xi)$, where r^2 is the squared norm function on TM, $\xi^\flat$ is the 1-form on M dual to ξ with respect to g, $C(\xi)$ is the $(1,1)$-tensor field on M defined by $g(C(\xi)Y,Z) = -g(Y, \nabla_Z \xi)$ for all vector fields Y and Z on M and the operator $*$ acts on $(1,1)$-tensor fields on M as $*P(x,u) := h\{P(u)\}$ for all $(x,u) \in TM$. Explicitly, for $(x,u) \in TM$,

$$*\xi_f(x,u) := f(\|u\|^2)\xi^v(x,u) + g(\xi_x, u)v\{u\} + h\{C(\xi)(u)\}. \tag{4.3}$$

It is easy to see that if we restrict ourselves to T_1M, then the first summand in (4.3) depends only on $f(1)$. On the other hand, if we take the tangential component of the vector field $*\xi_f(x,u)$, restricted to T_1M, then we can assume that the second summand in (4.3) vanishes identically. So, to define such type of vector fields, on T_1M, it suffices to choose a real constant λ and a vector field ξ on M. In this case, the tangential component $\bar{*}\xi_\lambda$ of the restriction of $*\xi_\lambda$ to T_1M is defined by

$$\bar{*}\xi_\lambda(x,u) := \lambda \xi^t(x,u) + h\{C(\xi)(u)\} \quad \text{for all } (x,u) \in T_1M. \tag{4.4}$$

Lemma 4.1. *If $\xi \in \mathfrak{X}(M)$ is parallel, then $C(\xi) = 0$ and $\bar{*}\xi_\lambda = \lambda\xi^t$.*

4.2. Fiber-preserving vector fields on T_1M. A vector field V on T_1M is called *fiber-preserving* if its flow consists of local fiber preserving transformations, i.e. local diffeomorphisms on T_1M preserving fibers, i.e. if $[V,W]$ is vertical for any vertical vector field W on T_1M.

For any vector field V on T_1M, there is a vector field $\overline{V}$ on TM which extends V. For a local coordinates system $(U, x^1, \ldots, x^n)$ of M, $\overline{V}$ can be expressed in the induced coordinate system of TM as

$$\overline{V}|_{p^{-1}(U)} = \sum_i A^i \Big(\frac{\partial}{\partial x^i}\Big)^h + \sum_i B^i \Big(\frac{\partial}{\partial x^i}\Big)^v,$$

where A^i and B^i are smooth functions on $p^{-1}(U)$. Since V is tangent to T_1M at any point of T_1M, we should have $\sum_{i,j} g_{ij}(x) B^i(x,u) u^j = 0$ for all $(x,u) \in p_1^{-1}(U) = p^{-1}(U) \cap T_1M$ expressed locally as $u = \sum_i u^i \partial/\partial x^i$. If we denote the restrictions to $p_1^{-1}(U)$ of A^i and B^i by the same notation, we can write

$$V|_{p_1^{-1}(U)} = \sum_i A^i \Big(\frac{\partial}{\partial x^i}\Big)^h + \sum_i B^i \Big(\frac{\partial}{\partial x^i}\Big)^t. \tag{4.5}$$

We have then:

Lemma 4.2. *A vector field V on T_1M is fiber preserving if and only if $h\{[V, X^t]\} = 0$ for all $X \in \mathfrak{X}(M)$, where h stands for the horizontal component and X^t is the tangential lift of X with respect to any Riemannian g-natural metric of Kaluza-Klein type.*

P r o o f. The necessary condition is obvious since any tangential lift with respect to any Riemannian g-natural metric of Kaluza-Klein type is a vertical vector field on T_1M. Conversely, any vertical vector field W on T_1M can be expressed locally as $W = \sum_i W^i (\partial/\partial x^i)^t$, where each W^i is a C^∞-function on the coordinate neighborhood in T_1M. The local expression of $[V, W]$ is $\sum_i V(W^i)(\partial/\partial x^i)^t + \sum_i W^i [V, (\partial/\partial x^i)^t]$, which is clearly tangential and thus vertical. $\square$

As a corollary, we get:

Lemma 4.3. *Let $\widetilde{G}$ be a pseudo-Riemannian g-natural metric of Kaluza-Klein type. A vector field V on T_1M is fiber preserving if and only if its horizontal component is a horizontal lift of a vector field on M.*

P r o o f. Taking into account the local expression (4.5) of V and using Lemma 5.2 below (with $b = 0$), we obtain $h\{[V, X^t]_{(x,u)}\} = -\sum_i X^t_{(x,u)}(A^i)(\partial/\partial x^i)^h_{(x,u)}$. Then $h\{[V, X^t]_{(x,u)}\} = 0$ if and only if

$$X^t_{(x,u)}(A^i) = 0 \tag{4.6}$$

for all $i = 1, \dots, n$, $(x,u) \in T_1M$ and $X \in M_x$. It follows that for $X \perp u$ we have $X^t_{(x,u)}(A^i) = 0$, i.e., $(dA^i)_{(x,u)}(X^t_{(x,u)}) = 0$ for all $X \perp u$. If we denote by A^i_x the restriction of A^i to the fiber $S_xM := T_1M \cap M_x$, we then have

$$(dA^i_x)_{(x,u)}(X^t_{(x,u)}) = (dA^i)_{(x,u)}(X^t_{(x,u)}) = 0 \tag{4.7}$$

for all $X \in M_x$ such that $g(X,u) = 0$. But dA^i_x is a linear form defined on the tangent space $(S_xM)_{(x,u)} = \{X^t_{(x,u)}/X \in M_x, g(X,u) = 0\}$. Thus, by (4.7) it follows that $(dA^i_x)_{(x,u)}$ vanishes identically on $(S_xM)_{(x,u)}$. We conclude that the restriction A^i_x is constant on S_xM. Therefore, for any $x \in M$, there is a C^∞-function ξ^i on M such that

$$A^i(x,u) = \xi^i(x) \tag{4.8}$$

for all $(x,u) \in T_1M$. $\square$

4.2.1. An extension to TM of fiber-preserving vector fields on T_1M. In this subsection, we restrict ourselves to Kaluza-Klein metrics on T_1M. Let $\widetilde{G}$ be a Kaluza-Klein metric on T_1M, i.e. with $b = d = 0$, and consider the metric G on TM extending $\widetilde{G}$, given by (3.1) with $\alpha_1 = a$, $\alpha_3 = c$ and $\alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$. To extend fiber-preserving vector fields on T_1M to vector fields on TM, we use the technics from [22].

Let Z be a fiber-preserving vector field on T_1M. Then it is known that Z is projectable to a vector field $\underline{Z}$ on M, i.e. such that $(dp_1)_u(Z_u) = \underline{Z}_x$ for all $x \in M$ and $u \in S_xM$. For all $r > 0$, let us define the immersions $j_r\colon T_1M \to TM$ and $j_0\colon M \to TM$, respectively, by $j_r(u) = ru$ for all $u \in T_1M$, and $j_0(x) = 0_x$ for all $x \in M$, where 0_x denotes the zero vector in M_x. We define a vector field $\overline{\overline{Z}}$ on TM extending Z (see [22]) by

$$\overline{\overline{Z}}_{ru} := \begin{cases} (dj_r)_u(Z_u) & \text{for } r > 0, \\ (dj_0)_x(\underline{Z}_x) & \text{for } r = 0 \end{cases}$$

for all $x \in M$ and $u \in S_xM$. We denote by $\overline{Z}$ the restriction of $\overline{\overline{Z}}$ to $TM \setminus \sigma_0$, which is clearly a vector field on $TM \setminus \sigma_0$, where $\sigma_0 := j_0(M)$ is the zero section of TM. For the horizontal and tangential lifts to T_1M of vector fields on M, the preceding extensions become as follows:

Lemma 4.4. *Let $X, Y \in \mathfrak{X}(M)$. Then*
(a) $\underline{X^h} = X$ *and* $\underline{X^t} = 0$;
(b) $\overline{\overline{X^h}} = X^h$;

(c) $\overline{\overline{X^t}}_{\restriction \sigma_0} = 0$ *and* $\overline{X^t} = r.X^t$, *where the quantity* X^t *on the right hand side of the second identity is the tangential lift with respect to the corresponding tangent bundle, i.e.* $X^t_{ru} := [X - r^{-2} g(X,u)u]^v_{ru}$ *for all* $u \in T_1M$ *and* $r > 0$.

We have then the following:

Lemma 4.5. *Let* Z, X *and* Y *fiber-preserving vector fields on* T_1M, *then*

$$(\mathcal{L}_{\overline{Z}} G(\overline{X}, \overline{Y}))_{ru} = (1 - r^2)(a + c)(\mathcal{L}_{\underline{Z}} g(\underline{X}, \underline{Y}))_x + r^2 (\mathcal{L}_Z \widetilde{G}(X, Y))_u \tag{4.9}$$

for all $x \in M$, $u \in S_xM$ *and* $r > 0$.

Proof. Putting $X = X_1^h + X_2^t$ and $Y = Y_1^h + Y_2^t$ and using Lemma 4.4, we have for all $x \in M$, $u \in S_xM$ and $r > 0$

$$\begin{aligned} G_{ru}(\overline{X}, \overline{Y}) &= (a + c) g_x(X_1, Y_1) + r^2 a \Big(g_x(X_2, Y_2) - \frac{1}{r^2} g_x(X_2, u) g_x(Y_2, u) \Big) \\ &= (1 - r^2)(a + c) g_x(\underline{X}, \underline{Y}) + r^2 \widetilde{G}_u(X, Y), \end{aligned}$$

since $\underline{X} = X_1$ and $\underline{Y} = Y_1$. We deduce then that

$$\begin{aligned} \overline{Z}_{ru}(G(\overline{X}, \overline{Y})) &= (1 - r^2)(a + c) \underline{Z}_x(g(\underline{X}, \underline{Y})) + r^2 Z_u(\widetilde{G}(X, Y)), \\ G_{ru}(\overline{[Z, X]}, \overline{Y}) &= (1 - r^2)(a + c) g_x([\underline{Z}, \underline{X}], \underline{Y}) + r^2 \widetilde{G}_u([Z, X], Y), \\ G_{ru}(\overline{X}, \overline{[Z, Y]}) &= (1 - r^2)(a + c) g_x(\underline{X}, [\underline{Z}, \underline{Y}]) + r^2 \widetilde{G}_u(X, [Z, Y]). \end{aligned}$$

The identity (4.9) follows then from the three preceding formulas. □

5. Some operations on vector fields on unit tangent bundles

Lemma 5.1 ([2]). *For all* $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ *and* $(x, u) \in T_1M$, *we have*

(i) $X^h(u^i) = -\sum_{j,k} \Gamma^i_{jk} X^j u^k$ *and*

$$X^t(u^i) = X^i - g(X, u)u^i - \frac{b}{\varphi} g(X, u) \sum_{j,k} \Gamma^i_{jk} u^j u^k \quad \textit{for all } i = 1, \ldots, n;$$

(ii) $X^h(f \circ p_1) = X(f) \circ p_1$;

(iii) $X^t(f \circ p_1) = \dfrac{b}{\varphi} g(X, u) u(f)$;

(iv) $X^h_{(x,u)}(g(Y, .)) = g(\nabla_{X_x} Y, u)$;

(v) $X^t_{(x,u)}(g(Y, .)) = g(X_x, Y_x) - g(X_x, u) g(Y_x, u) + \dfrac{b}{\varphi} g(X_x, u) g(\nabla_u Y, u)$.

Lemma 5.2. *For all $X, Y \in \mathfrak{X}(M)$ and $(x,u) \in T_1M$, we have*

$$\text{(i)} \qquad [X^h, Y^h]_{(x,u)} = [X,Y]^h_{(x,u)} - t\{R(X_x, Y_x)u\},$$

$$\text{(ii)} \qquad \begin{aligned}[X^h, Y^t]_{(x,u)} &= \frac{-b}{\varphi} g(Y_x, u) h\{\nabla_u X\} + (\nabla_{X_x} Y)^t_{(x,u)} \\ &\quad - \frac{b}{\varphi} g(Y_x, u) t\{R(X_x, u)u\},\end{aligned}$$

$$\text{(iii)} \qquad \begin{aligned}[X^t, Y^t]_{(x,u)} &= \frac{b}{\varphi}[g(Y_x, u)X^h_{(x,u)} - g(X_x, u)Y^h_{(x,u)}] \\ &\quad + \frac{b}{\varphi}[g(X_x, u)t\{\nabla_u Y\} - g(Y_x, u)t\{\nabla_u X\}] \\ &\quad + \Big(\frac{b^2 d}{\varphi\alpha} - 1\Big)[g(Y_x, u)X^t_{(x,u)} - g(X_x, u)Y^t_{(x,u)}].\end{aligned}$$

P r o o f. Follows easily from (2.8)–(2.10), using (3.4) and Lemma 5.1. □

Lemma 5.3. *For all $\xi \in \mathfrak{X}(M)$, $(x,u) \in TM$ and $X, Y \in M_x$, we have*

$$\begin{aligned}&(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^h, Y^h) \\ &\quad = b\Big\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) - \frac{2b}{\varphi} g(\xi_x, u) g(R(X,u)u, Y)\Big\} \\ &\qquad + d\{g(\xi_x, X)g(Y,u) + g(\xi_x, Y)g(X,u) - 2g(\xi_x, u)g(X,u)g(Y,u)\}, \\ &(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^h, Y^t) \\ &\quad = a\{g(\nabla_X \xi, Y) - g(Y,u)g(\nabla_X \xi, u)\} \\ &\qquad + \frac{b}{\varphi}\{bg(Y,u)g(\nabla_u \xi, Y) - g(\xi_x, u)g(R(X,u)u, Y)\} \\ &\qquad + \frac{bd}{\alpha\varphi}(\alpha - b^2)\{g(\xi_x, X)g(Y,u) - g(\xi_x, u)g(X,Y)\}, \\ &(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^t, Y^t) \\ &\quad = \frac{ab}{\varphi}\{g(X,u)g(\nabla_u \xi, Y) + g(Y,u)g(\nabla_u \xi, X) - 2g(X,u)g(Y,u)g(\nabla_u \xi, u)\} \\ &\qquad - \frac{b^2\phi}{\alpha\varphi}\{g(X,\xi_x)g(Y,u)g(Y,\xi_x)g(X,u) - 2g(X,Y)g(\xi_x,u)\} \\ &\qquad - 2ag(\xi_x, u)\{g(X,Y) - g(X,u)g(Y,u)\}.\end{aligned}$$

P r o o f. Extending the vectors X, Y to vector fields on M, which we denote by $\overline{X}$, $\overline{Y}$, we have by definition of the Lie derivative

$$\text{(5.1)} \qquad \begin{aligned}(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^h, Y^h) &= \xi^t_{(x,u)}(\widetilde{G}(\overline{X}^h, \overline{Y}^h)) - \widetilde{G}(X^h, [\xi^t, \overline{Y}^h]_{(x,u)}) \\ &\quad - \widetilde{G}(Y^h, [\xi^t, \overline{X}^h]_{(x,u)}).\end{aligned}$$

Using (3.5) and (iii) and (iv) of Lemma 5.1, we obtain

$$
\begin{aligned}
(5.2)\qquad \xi^t_{(x,u)}(\widetilde{G}(\overline{X}^h,\overline{Y}^h)) &= \xi^t_{(x,u)}((a+c)g(\overline{X},\overline{Y})\circ p_1 + dg(\overline{X},.)g(\overline{Y},.)) \\
&= (a+c)\xi^t_{(x,u)}(g(\overline{X},\overline{Y})\circ p_1) \\
&\quad + d[g(X,u)\xi^t_{(x,u)}(g(\overline{Y},.)) + g(Y,u)\xi^t_{(x,u)}(g(\overline{X},.))] \\
&= \frac{(a+c)b}{\varphi} g(\xi_x,u)u(g(\overline{X},\overline{Y})) \\
&\quad + d\Big[g(X,u)g(\xi_x,Y) + g(Y,u)g(\xi_x,X) - 2g(\xi_x,u)g(X,u)g(Y,u) \\
&\qquad + \frac{b}{\varphi}g(\xi_x,u)(g(X,u)g(\nabla_u\overline{Y},u) + g(Y,u)g(\nabla_u\overline{X},u))\Big].
\end{aligned}
$$

On the other hand, using (ii) of Lemma 5.2 and (3.5), we have

$$
\begin{aligned}
(5.3)\qquad \widetilde{G}(X^h,[\xi^t,\overline{Y}^h]_{(x,u)}) &= \frac{b}{\varphi}g(\xi_x,u)\widetilde{G}(X^h,h\{\nabla_u Y\}) - \widetilde{G}(X^h,t\{\nabla_Y\xi\}) \\
&\quad + \frac{b}{\varphi}g(\xi_x,u)\widetilde{G}(X^h,t\{R(Y,u)u\}) \\
&= -bg(X,\nabla_Y\xi) + \frac{b}{\varphi}g(\xi_x,u) \\
&\quad \times [(a+c)g(X,\nabla_u Y) + dg(X,u)g(\nabla_u Y,u) + bg(X,R(Y,u)u)],
\end{aligned}
$$

and

$$
\begin{aligned}
(5.4)\qquad \widetilde{G}(Y^h,[\xi^t,\overline{X}^h]_{(x,u)}) &= -bg(Y,\nabla_X\xi) \\
&\quad + \frac{b}{\varphi}g(\xi_x,u)[(a+c)g(Y,\nabla_u X) + dg(Y,u)g(\nabla_u X,u) + bg(Y,R(X,u)u)].
\end{aligned}
$$

Substituting from (5.2)–(5.4) into (5.1), we obtain the first identity of the lemma. The other two identities follow in the same way using (3.5) and the appropriate formulas from Lemmas 5.1 and 5.2. □

Now, using the same arguments, we get the following two lemmas:

Lemma 5.4. *For all $\xi \in \mathfrak{X}(M)$, $(x,u) \in TM$ and $X, Y \in M_x$, we have*

$$
\begin{aligned}
(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^h,Y^h) &= (a+c)\{g(\nabla_X\xi,Y) + g(\nabla_Y\xi,X)\} \\
&\quad - bg(R(\xi_x,X)Y + R(\xi_x,Y)X,u) + d\{g(\nabla_X\xi,u)g(Y,u) + g(\nabla_Y\xi,u)g(X,u)\},
\end{aligned}
$$

$$(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^h, Y^t) = ag(R(\xi_x, X)u, Y) + bg(\nabla_X \xi, Y)$$
$$+ \frac{b}{\varphi} g(Y, u)\{(a + c)g(\nabla_u \xi, X) + dg(X, u)g(\nabla_u \xi, u) + bg(R(\xi_x, u)u, X)\},$$
$$(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^t, Y^t) = \frac{b^2}{\varphi}\{g(X, u)g(\nabla_u \xi, Y) + g(Y, u)g(\nabla_u \xi, X)\}$$
$$+ \frac{ab}{\varphi}\{g(X, u)g(R(\xi_x, u)u, Y) + g(Y, u)g(R(\xi_x, u)u, X)\}.$$

Lemma 5.5. *For all $\xi \in \mathfrak{X}(M)$, $(x, u) \in TM$ and $X, Y \in M_x$, we have*

$$(\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^h, Y^h) = (a + c)\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\}$$
$$+ b\{g(\nabla^2\xi(u, X) + R(\xi_x, X)u, Y) + g(\nabla^2\xi(u, Y) + R(\xi_x, Y)u, X)\}$$
$$+ d\{[g(\nabla_u \xi, Y) + g(\nabla_Y \xi, u)]g(X, u) + [g(\nabla_u \xi, X) + g(\nabla_X \xi, u)]g(Y, u)\}$$
$$- 2g(\nabla_u \xi, u)\Big\{\frac{b^2}{\varphi} g(R(X, u)u, Y) + dg(Y, u)g(X, u)\Big\},$$
$$(\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^h, Y^t) = ag(R(\xi_x, X)u + \nabla^2\xi(u, X), Y) + b\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\}$$
$$- \frac{b}{\varphi} g(\nabla_u \xi, u)\Big\{g(R(X, u)u, Y) + \frac{d}{\alpha}(\alpha - b^2)g(X, Y)\Big\},$$
$$(\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^t, Y^t) = a\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\} + 2\Big(\frac{b^2\phi}{\alpha\varphi} - a\Big)g(\nabla_u \xi, u)g(X, Y).$$

Finally, we have

Lemma 5.6. *For all $(x, u) \in T_1M$ and $X, Y \in M_x$, we have*

$$(\mathcal{L}_\zeta \widetilde{G})_{(x,u)}(X^h, Y^h) = -2bg(R(u, X)Y, u),$$
$$(\mathcal{L}_\zeta \widetilde{G})_{(x,u)}(X^h, Y^t) = (a + c)[g(X, Y) - g(X, u)g(Y, u)] + ag(R(u, X)u, Y),$$
$$(\mathcal{L}_\zeta \widetilde{G})_{(x,u)}(X^t, Y^t) = 2b[g(X, Y) - g(X, u)g(Y, u)].$$

P r o o f. Using the local expression $\zeta = \sum_i u^i(\partial/\partial x^l)^h$ of ζ, we have

$$(5.5)\quad (\mathcal{L}_\zeta \widetilde{G})_{(x,u)}(X^h, Y^h)$$
$$= \sum_i u^i(\mathcal{L}_{(\partial/\partial x^i)^h}\widetilde{G})_{(x,u)}(X^h, Y^h) + \sum_i \widetilde{G}\Big(X^h, Y^h(u^i)\Big(\frac{\partial}{\partial x^i}\Big)^h_{(x,u)}\Big)$$
$$+ \sum_i \widetilde{G}\Big(Y^h, X^h(u^i)\Big(\frac{\partial}{\partial x^i}\Big)^h_{(x,u)}\Big).$$

Using the first identity of Lemma 5.4, we obtain

$$
\begin{aligned}
(5.6)\qquad & \sum_i u^i \mathcal{L}_{(\partial/\partial x^i)^h_{(x,u)}} \widetilde{G}(X^h, Y^h) \\
&= (a+c)\sum_i u^i \Big\{ g\Big(\nabla_X\Big(\frac{\partial}{\partial x^i}\Big), Y\Big) + g\Big(\nabla_Y\Big(\frac{\partial}{\partial x^i}\Big), X\Big)\Big\} \\
&\quad - b\sum_i u^i g\Big(R\Big(\Big(\frac{\partial}{\partial x^i}\Big)_x, X\Big)Y + R\Big(\Big(\frac{\partial}{\partial x^i}\Big)_x, Y\Big)X, u\Big) \\
&\quad + d\sum_i u^i \Big\{ g\Big(\nabla_X\Big(\frac{\partial}{\partial x^i}\Big), u\Big) g(Y,u) + g\Big(\nabla_Y\Big(\frac{\partial}{\partial x^i}\Big), u\Big) g(X,u)\Big\} \\
&= (a+c)\{g(\nabla_X U, Y) + g(\nabla_Y U, X)\} - 2bg(R(u,X)Y,u) \\
&\quad + d\{g(\nabla_X U, u)g(Y,u) + g(\nabla_Y U, u)g(X,u)\},
\end{aligned}
$$

where U is the local vector field on M given by $U := \sum_i u^i \partial/\partial x^i$. Notice that using U and the identities (i) of Lemma 5.1, we have

$$
\sum_i X^h(u^i)\frac{\partial}{\partial x^i} = -\nabla_X U, \tag{5.7}
$$

$$
\sum_i X^t(u^i)\frac{\partial}{\partial x^i} = X - g(X,u)U - \frac{b}{\varphi}\nabla_U U. \tag{5.8}
$$

Using (5.7) and (3.5), we have

$$
\begin{aligned}
(5.9)\qquad \sum_i \widetilde{G}\Big(X^h, Y^h(u^i)\Big(\frac{\partial}{\partial x^i}\Big)^h_{(x,u)}\Big) &= -\widetilde{G}(X^h, (\nabla_Y U)^h_{(x,u)}) \\
&= -(a+c)g(X, \nabla_Y U) - dg(X,u)g(\nabla_Y U), u),
\end{aligned}
$$

and

$$
\sum_i \widetilde{G}\Big(Y^h, X^h(u^i)\Big(\frac{\partial}{\partial x^i}\Big)^h_{(x,u)}\Big) = -(a+c)g(Y, \nabla_X U) - dg(Y,u)g(\nabla_X U), u). \tag{5.10}
$$

Substituting from (5.6), (5.9) and (5.10) into (5.5), we obtain the first formula of the lemma. The other two formulas can be proved similarly using the second and third identities of Lemma 5.4, (5.7), (5.9) and (3.5). $\square$

Lemma 5.7. *Let V be a fiber-preserving vector field on T_1M expressed, locally, as $V = \xi^h + \sum_i B^i(\partial/\partial x^i)^t$, where $\xi \in \mathfrak{X}(M)$. For all $(x,u) \in T_1M$, $X, Y \in M_x$, we* *have*

$$
\begin{aligned}
&(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^h)\\
&\quad=(a+c)\{g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)\}+d\{g(\nabla_X\xi,u)g(Y,u)+g(\nabla_Y\xi,u)g(X,u)\\
&\qquad+g(K(V_{(x,u)}),X)g(Y,u)+g(K(V_{(x,u)}),Y)g(X,u)\},\\
&(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^t)\\
&\quad=a\Big\{g(R(\xi_x,X)u,Y)+\sum_i B^i(x,u)g\Big(\nabla_X\frac{\partial}{\partial x^i},Y\Big)+\sum_i X^h(B^i)g\Big(\frac{\partial}{\partial x^i}(x),Y\Big)\Big\},\\
&(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^t,Y^t)\\
&\quad=a\sum_i\Big\{X^t(B^i)g\Big(\frac{\partial}{\partial x^i}(x),Y-g(Y,u)u\Big)+Y^t(B^i)g\Big(\frac{\partial}{\partial x^i}(x),X-g(X,u)u\Big)\Big\},
\end{aligned}
$$

where all the lifts are taken at (x,u) and $K\colon TTM\to TM$ is the connection map corresponding to the Levi-Civita connection ∇ of (M,g), characterized by $K(X^h)=0$ and $K(X^v)=X$ for all $X\in TM$.

P r o o f. We have

$$
\begin{aligned}
(5.11)\quad (\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^h)&=(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^h,Y^h)\\
&\quad+\sum_i B^i(x,u)(\mathcal{L}_{(\partial/\partial x^i)^t}\widetilde{G})_{(x,u)}(X^h,Y^h)\\
&\quad+\sum_i X^h(B^i)\widetilde{G}\Big(\Big(\frac{\partial}{\partial x^i}\Big)^t_{(x,u)},Y^h\Big)+\sum_i Y^h(B^i)\widetilde{G}\Big(X^h,\Big(\frac{\partial}{\partial x^i}\Big)^t_{(x,u)}\Big)\\
&=\mathcal{L}_{\xi^h_{(x,u)}}\widetilde{G}(X^h,Y^h)+\sum_i B^i(x,u)\mathcal{L}_{(\partial/\partial x^i)^t_{(x,u)}}\widetilde{G}(X^h,Y^h),
\end{aligned}
$$

since $b=0$. Using the first identity of Lemma 5.3 with $b=0$ and the fact that $\sum_{i,j}g_{ij}(x)B^i(x,u)u^j=0$, we obtain

$$
\begin{aligned}
(5.12)\quad &\sum_i B^i(x,u)(\mathcal{L}_{(\partial/\partial x^i)^t}\widetilde{G})_{(x,u)}(X^h,Y^h)\\
&\qquad=d\sum_i B^i(x,u)\Big[g\Big(\Big(\frac{\partial}{\partial x^i}\Big)_x,X\Big)g(Y,u)+g\Big(\Big(\frac{\partial}{\partial x^i}\Big)_x,Y\Big)g(X,u)\Big].
\end{aligned}
$$

On the other hand, it is easy to see that $K(X^t_{(x,u)})=X-g(X,u)u$ for all $(x,u)\in T_1M$ and $X\in M_x$. Hence $K(V_{(x,u)})=\sum_i B^i(x,u)\partial/\partial x^i(x)$ for all $(x,u)\in p_1^{-1}(U)$. We deduce then that

$$
\begin{aligned}
(5.13)\quad &\sum_i B^i(x,u)(\mathcal{L}_{(\partial/\partial x^i)^t}\widetilde{G})_{(x,u)}(X^h,Y^h)\\
&\qquad=d[g(K(V_{(x,u)}),X)g(Y,u)+g(K(V_{(x,u)}),Y)g(X,u)].
\end{aligned}
$$

Substituting from (5.13) and the first identity of Lemma 5.4 (with $b = 0$) into (5.11), we obtain the first identity of our lemma. The second and third identities follow in a similar way. □

As a corollary of Lemma 5.7, it is easy to conclude:

Lemma 5.8. *Let $\widetilde{G}$ be a Kaluza-Klein metric on T_1M, i.e. $b = d = 0$. Then V is a Killing vector field on $(T_1M, \widetilde{G})$ if and only if it is Killing on $(T_1M, \widetilde{g^s})$, where $\widetilde{g^s}$ is the Sasaki metric on T_1M.*

In the same way as before, using Lemmas 5.1 and 5.2, we have:

Lemma 5.9. *If ξ is a vector field on M, $\lambda \in \mathbb{R}$, $(x,u) \in TM$ and $X, Y \in M_x$, we have*

$$\begin{aligned}
&(\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G})_{(x,u)}(X^h, Y^h)\\
&\quad = -(a+c)g(\nabla^2\xi(X,Y) + \nabla^2\xi(Y,X), u)\\
&\qquad - d[g(X,u)g(\nabla^2\xi(Y,u) + \nabla^2\xi(u,Y), u) + g(Y,u)g(\nabla^2\xi(X,u) + \nabla^2\xi(u,X), u)]\\
&\qquad + \lambda d[g(\xi_x, X)g(Y,u) + g(\xi_x, Y)g(X,u) - 2g(\xi_x, u)g(X,u)g(Y,u)],\\
&(\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G})_{(x,u)}(X^h, Y^t)\\
&\quad = [(a+c) + a\lambda][g(\nabla_X\xi, Y) - g(Y,u)g(\nabla_X\xi, u)]\\
&\qquad + dg(X,u)[g(\nabla_u\xi, Y) - g(Y,u)g(\nabla_u\xi, u)] + ag(R(C(\xi_x)(u), X)u, Y),\\
&(\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G})_{(x,u)}(X^t, Y^t) = -2a\lambda[g(X,Y) - g(X,u)g(Y,u)]g(\xi_x, u).
\end{aligned}$$

6. Proofs of the main results

If we consider the unit tangent bundle endowed with a pseudo-Riemannian g-natural metric $\widetilde{G}$, we can give the following useful characterization of conformal vector fields:

Lemma 6.1. *A vector field V on T_1M is conformal with respect to $\widetilde{G}$, if and only if there is a smooth function $\tilde{f}$ on T_1M such that the identities*

$$\begin{aligned}
(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h, Y^h) &= 2\tilde{f}(x,u)\widetilde{G}(X^h, Y^h),\\
(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h, Y^t) &= 2\tilde{f}(x,u)\widetilde{G}(X^h, Y^t),\\
(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^t, Y^t) &= 2\tilde{f}(x,u)\widetilde{G}(X^t, Y^t)
\end{aligned} \tag{6.1}$$

hold for all $(x,u) \in T_1M$ and $X, Y \in M_x$.

P r o o f. Follows from the fact that every tangent vector to T_1M can be decomposed into a sum of horizontal and tangential lifts of vectors on M. $\square$

P r o o f of Theorem 1.1. Suppose that ξ^t is a conformal vector field, that is, there is a C^∞-function $\tilde{f}$ on T_1M such that $\mathcal{L}_{\xi^t}\widetilde{G} = 2\tilde{f}\widetilde{G}$. In particular, we have

$$(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^t, Y^t) = 2a\tilde{f}(x,u)[g(X,Y) - g(X,u)g(Y,u)] \tag{6.2}$$

for all $(x,u) \in T_1M$ and $X, Y \in M_x$. Substituting from the third equation of Lemma 5.3 into (6.2) and taking $X \perp u$ and $Y = X$ (or $X = u$ and $Y \perp u$), we obtain

$$\tilde{f}(u) = \Big[\frac{b^2\phi}{a\alpha\varphi} - 1\Big]g(\xi_x, u) \quad \text{for all } u \in M_x \tag{6.3}$$

or, respectively,

$$\frac{ab}{\varphi}g(\nabla_u\xi, Y) - \frac{b^2\phi}{\alpha\varphi}g(\xi_x, u) = 0 \quad \text{for all } Y \perp u. \tag{6.4}$$

On the other hand, we have

$$(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^h, Y^t) = 2b\tilde{f}(x,u)g(X,Y) \tag{6.5}$$

for all $(x,u) \in TM$ and $X, Y \in M_x$. Substituting from the second equation of Lemma 5.3 into (6.5) and taking $X = u$ and $Y \perp u$, we obtain

$$g(\nabla_u\xi, Y) = 0 \quad \text{for all } Y \perp u. \tag{6.6}$$

We deduce from (6.4) and (6.6) that

$$\frac{b^2\phi}{\alpha\varphi}g(\xi_x, u) = 0 \quad \text{for all } Y \perp u. \tag{6.7}$$

For $b \neq 0$, (6.7) implies that $g(\xi_x, u) = 0$ for all $Y \perp u$. Since u is arbitrary, we have $\xi_x = 0$, and hence $\xi = 0$.

For $b = 0$, (6.3) is equivalent to

$$\tilde{f}(u) = -g(\xi_x, u) \quad \text{for all } u \in M_x. \tag{6.8}$$

Since by conformality of ξ,

$$(\mathcal{L}_{\xi^t}\widetilde{G})_{(x,u)}(X^h, Y^h) = 2\tilde{f}(x,u)[(a+c)g(X,Y) + dg(X,u)g(Y,u)] \tag{6.9}$$

for all $(x,u) \in TM$ and $X, Y \in M_x$, substituting from the first equation of Lemma 5.3 into (6.9) and taking $X = Y = u$, we obtain $\tilde{f} = 0$, i.e., ξ is a Killing vector field. So by virtue of (6.8) we have $g(\xi_x, u) = 0$ for all $u \in T_1M$. We conclude that $\xi = 0$. $\square$

P r o o f of Theorem 1.2. (i) $\rightarrow$ (iii): Suppose that ξ^h is a conformal vector field, that is, there is a C^∞-function $\tilde{f}$ on T_1M such that $\mathcal{L}_{\xi^h}\widetilde{G} = 2\tilde{f}\widetilde{G}$. In particular, we have

$$(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^t, Y^t) = 2a\tilde{f}(x,u)[g(X,Y) - g(X,u)g(Y,u)] \tag{6.10}$$

for all $(x,u) \in T_1M$ and $X, Y \in M_x$. Substituting from the third equation of Lemma 5.4 into (6.10) and taking $X = Y = u$, we obtain

$$\frac{b^2}{\varphi} g(\nabla_u \xi, u) = 0. \tag{6.11}$$

Case 1: $b \neq 0$. We claim that ξ is parallel. Indeed, from the preceding equation, we have

$$g(\nabla_u \xi, u) = 0 \quad \text{for all } u \in TM. \tag{6.12}$$

Hence, by (2.3), ξ is a Killing vector field on (M, g).

On the other hand,

$$(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^h, Y^h) = 2\tilde{f}(x,u)[(a+c)g(X,Y) + dg(X,u)g(Y,u)] \tag{6.13}$$

for all $(x,u) \in TM$ and $X, Y \in M_x$. Then substituting from the first equation of Lemma 5.4 into (6.13) and taking $X = Y = u$, we obtain by virtue of (6.12)

$$\tilde{f}(u) = \frac{a+c}{\varphi} g(\nabla_u \xi, u) = 0 \quad \text{for all } u \in T_1M. \tag{6.14}$$

Hence ξ^h is a Killing vector field on $(T_1M, \widetilde{G})$.

Now, substituting again from the first equation of Lemma 5.4 into (6.13) and taking $X \perp u$ and $Y = X$, we obtain by virtue of $b \neq 0$ that $g(R(\xi_x, X)X, u) = 0$ for all $u \perp X$. Since $g(R(\xi_x, X)X, X) = 0$, we have

$$R(\xi_x, X)X = 0 \quad \text{for all } X \in M_x. \tag{6.15}$$

Since ξ^h is a Killing vector field on $(T_1M, \widetilde{G})$, we have in particular

$$(\mathcal{L}_{\xi^h}\widetilde{G})_{(x,u)}(X^h, Y^t) = 0. \tag{6.16}$$

Substituting from the second equation of Lemma 5.4 into (6.16) and taking $X = u$ and $Y \perp u$, we have by virtue of (6.15) and the inequality $b \neq 0$ that $g(\nabla_u \xi, Y) = 0$

for all $Y \perp u$, and taking into account (6.12), we deduce that $g(\nabla_u \xi, Y) = 0$ for all $Y \in M_x$. It follows that ξ is parallel.

Case 2: $b = 0$. Substituting from the first equation of Lemma 5.4 into (6.13) with $b = 0$, we have easily

$$R(\xi, .). = 0. \tag{6.17}$$

On the other hand, substituting from the third equation of Lemma 5.4 into (6.10) with $b = 0$, and taking $X \perp u$ and $Y = X$, we obtain $\tilde{f} = 0$. Now, substituting from the first equation of Lemma 5.4 into (6.13) and taking first $X \perp u$ and $Y = X$ and then $X = Y = u$, we obtain $g(\nabla_X \xi, X) = 0$ for all $X \in M_x$, i.e., ξ is a Killing vector field.

Subcase 2.1: $d = 0$, i.e. $\widetilde{G}$ is a Kaluza-Klein metric. We have proved that ξ is a Killing vector field and $R(\xi, .). = 0$ (see (5.17)).

Subcase 2.2: $d \neq 0$. Substituting again from the first equation of Lemma 5.4 into (6.13), and taking into account that $\tilde{f} = 0$, ξ is a Killing vector field and $R(\xi, .). = 0$, we get, by virtue of $d \neq 0$,

$$g(\nabla_X \xi, u) g(Y, u) + g(\nabla_Y \xi, u) g(X, u) = 0. \tag{6.18}$$

Then, putting first $X = u$ and $Y \perp u$ and then $X = Y = u$ in (6.18), we deduce that ξ is parallel.

(iii) $\to$ (ii): If ξ is parallel, then we have, in particular, $R(\xi, .). = 0$, and substituting into Lemma 5.4, we get

$$(\mathcal{L}_{\xi^h} \widetilde{G})_{(x,u)}(X^h, Y^h) = (\mathcal{L}_{\xi^h} \widetilde{G})_{(x,u)}(X^h, Y^t) = (\mathcal{L}_{\xi^h} \widetilde{G})_{(x,u)}(X^t, Y^t) = 0. \tag{6.19}$$

In a similar way, if $b = d = 0$, ξ is a Killing vector field and $R(\xi, .). = 0$, then substituting into Lemma 5.4, we get (6.19). We deduce then, from Lemma 6.1, that ξ^h is conformal with vanishing potential function, i.e. ξ^h is a Killing vector field.

(ii) $\to$ (i): is trivial. □

P r o o f of Theorem 1.3. (i) $\to$ (iii): Suppose that $\xi^{\bar{c}}$ is a conformal vector field, that is, there is a C^∞-function $\tilde{f}$ on $T_1 M$ such that $\mathcal{L}_{\xi^{\bar{c}}} \widetilde{G} = 2 \tilde{f} \widetilde{G}$. In particular, we have

$$(\mathcal{L}_{\xi^{\bar{c}}} \widetilde{G})_{(x,u)}(X^t, Y^t) = 2a \tilde{f}(x, u)[g(X, Y) - g(X, u) g(Y, u)] \tag{6.20}$$

for all $(x, u) \in T_1 M$ and $X, Y \in M_x$. Substituting from the third equation of Lemma 5.5 into (6.20) and taking $X \perp u$ and $Y = X$, we obtain

$$F(u) \|X\|^2 = g(\nabla_X \xi, X) \tag{6.21}$$

for all $X \perp u$, where F is the C^∞-function on T_1M defined by

$$F(u) = \tilde{f}(u) + \Big(1 - \frac{b^2\phi}{a\alpha\varphi}\Big) g(\nabla_u \xi, u). \tag{6.22}$$

We claim that F is constant on each fiber of T_1M. Indeed, for any $x \in M$ and $u, v \in S_xM := T_1M \cap M_x$ we have $u^\perp \cap v^\perp \neq \{0\}$, since $\dim M > 2$. Let $X \in (u^\perp \cap v^\perp) \setminus \{0\}$. By (6.21) we have $F(u) = F(v)$, which proves our claim. We deduce that there is $f \in C^\infty(M)$ such that $F = f \circ p_1$, which implies, by (6.21), that ξ is a conformal vector field on M with potential function f. In particular $g(\nabla_u\xi, u) = f(x)$ for all $x \in M$ and $u \in S_xM$. Substituting from this last formula into (6.22), we deduce that $\tilde{f}$ is constant on each fiber equal to

$$\tilde{f}(u) = \frac{b^2\phi}{a\alpha\varphi} f(x) \tag{6.23}$$

for all $x \in M$ and $u \in S_xM$.

Now, substituting from the second equation of Lemma 5.5 into

$$(\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^h, Y^t) = 2b\tilde{f}(x,u) g(X,Y) \tag{6.24}$$

and taking $X \perp u$ and $Y = X$, we obtain, by virtue of (2.4) that $X(f) = 0$ for all $X \perp u$. Since u is arbitrary, f is constant.

In the same way, substituting from the first equation of Lemma 5.5 into

$$(\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^h, Y^h) = 2\tilde{f}(x,u)[(a+c)g(X,Y) + dg(X,u)g(Y,u)] \tag{6.25}$$

and taking $X = Y = u$, we obtain, by virtue of (2.5) and the fact that f is constant,

$$\Big(1 - \frac{b^2\phi}{a\alpha\varphi}\Big) f = 0. \tag{6.26}$$

We claim that $f = 0$. Indeed, suppose that $f \neq 0$. Then by (6.26) we have $b^2\phi/a\alpha\varphi = 1$, and then we have, by virtue of (6.23), $\tilde{f} = f \circ p_1$. In particular $b \neq 0$. Substituting from the first equation of Lemma 5.5 into (6.25) and taking $X \perp u$ and $Y = X$, we obtain, by virtue of (2.5), the fact that f is constant and $b \neq 0$, the equality

$$g(R(X,u)u, X) = 0 \tag{6.27}$$

for all $X \perp u$. Substituting from the first equation of Lemma 5.5 into (6.24) and taking $X = u$ and $Y \perp u$, we obtain by virtue of (2.5), (6.27), the fact that f is constant and $b \neq 0$, the identity $\alpha - b^2 = 0$. Then $b^2\phi/a\alpha\varphi = 1$ becomes $\phi = a\varphi$, which implies that $b^2 = 0$, which is a contradiction. It follows that $\tilde{f} = f \circ p_1 = 0$, i.e. ξ and $\xi^{\bar{c}}$ are Killing vector fields.

(iii) $\rightarrow$ (ii): Suppose that ξ is a Killing vector field. Then substituting from (2.2) and (2.5) into Lemma 5.5, we get

$$(6.28)\qquad (\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^h, Y^h) = (\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^h, Y^t) = (\mathcal{L}_{\xi^{\bar{c}}}\widetilde{G})_{(x,u)}(X^t, Y^t) = 0.$$

It follows then from Lemma 6.1 that $\xi^{\bar{c}}$ is a Killing vector field.

(ii) $\rightarrow$ (i): is trivial. □

P r o o f of Theorem 1.4. (i) $\rightarrow$ (iii): Suppose that ζ is a conformal vector field on T_1M, that is, there is a C^∞-function $\tilde{f}$ on T_1M such that $\mathcal{L}_\zeta\widetilde{G} = 2\tilde{f}\widetilde{G}$. In particular, we have

$$(6.29)\qquad (\mathcal{L}_\zeta\widetilde{G})_{(x,u)}(h\{u\}, h\{u\}) = 2\varphi\tilde{f}(x,u)$$

for all $(x,u) \in T_1M$. Substituting from the first equation of Lemma 5.6 into (6.29), we deduce that $\tilde{f}$ vanishes identically, i.e. ζ is a Killing vector field on T_1M. In particular, using the third equation of Lemma 5.6, we have

$$0 = (\mathcal{L}_\zeta\widetilde{G})_{(x,u)}(X^t, X^t) = 2b\|X\|^2$$

for all $X \perp u$. We deduce that $b = 0$. On the other hand, using the Killing equation for ζ to the second equation of Lemma 5.6, we obtain

$$(6.30)\qquad R_uX = \frac{a+c}{a}X$$

for all $(x,u) \in T_1M$ and $X \in M_x$, where $R_uX = R(X,u)u$ denotes the Jacobi operator associated to u. We deduce that (M,g) has constant sectional curvature $k = (a+c)/a$.

(iii) $\rightarrow$ (ii): For $b = 0$ and (M,g) of constant sectional curvature $k = (a+c)/a$, it is easy to see, from Lemmas 6.1 and 5.6, that ζ is a Killing vector field on T_1M.

(ii) $\rightarrow$ (i): is trivial. □

P r o o f of Theorem 1.5. (i) $\rightarrow$ (iii): Let V be a fiber-preserving vector field on T_1M. By Lemma 4.3, there is a vector field ξ on M such that V is locally expressed as $V = \xi^h + \sum\limits_i B^i(\partial/\partial x^i)^t$. We shall give necessary and sufficient conditions for V

to be conformal with respect to $\widetilde{G}$. Suppose that V is conformal, then by Lemma 6.1 there is a C^∞-function $\tilde{f}$ on T_1M such that

$$
\begin{aligned}
(6.31)\qquad & (\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^h) = 2\tilde{f}(x,u)[(a+c)g(X,Y)+dg(X,u)g(y,u)],\\
& (\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^t) = 0,\\
& (\mathcal{L}_V\widetilde{G})_{(x,u)}(X^t,Y^t) = 2a\tilde{f}(x,u)[g(X,Y)-g(X,u)g(Y,u)]
\end{aligned}
$$

for all $(x,u) \in T_1M$ and $X,Y \in M_x$. Substituting from the first identity of Lemma 5.7 into the first identity of (6.31) and taking first $X = Y = u$ and then $X \perp u$ and $Y = X$, we obtain

$$(6.32)\qquad \tilde{f}(x,u) = g(\nabla_u\xi,u) \quad \text{and} \quad g(\nabla_X\xi,X) = g(\nabla_u\xi,u)\|X\|^2$$

for all $X \perp u$. We deduce that

$$(6.33)\qquad g(\nabla_X\xi,X) = g(\nabla_u\xi,u)\|X\|^2$$

for all $(x,u) \in T_1M$ and $X \in M_x$. Then, by bilinearity, we get

$$(6.34)\qquad g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X) = 2g(\nabla_u\xi,u)g(X,Y)$$

for any $(x,u) \in T_1M$ and $X,Y \in M_x$. Substituting from the first identity of Lemma 5.7 and (6.34) into the first identity of (6.31) and taking $X = u$ and $Y \perp u$, we obtain

$$(6.35)\qquad d[g(\nabla_Y\xi,u)+g(K(V_{(x,u)}),Y)] = 0$$

for all $Y \perp u$. So, we have two possibilities:

Case 1: $d \neq 0$, then $g(\nabla_Y\xi,u)+g(K(V_{(x,u)}),Y) = 0$ for all $Y \perp u$. On the other hand, using (6.34), we obtain $g(\nabla_Y\xi,u)+(\nabla_u\xi,Y) = 2g(\nabla_u\xi,u)g(u,Y) = 0$. It follows that

$$(6.36)\qquad g(K(V_{(x,u)})-\nabla_u\xi,Y) = 0$$

for all $Y \perp u$. Now, since $K(V_{(x,u)}) = \sum_i B^i(x,u)\partial/\partial x^i(x)$, we have $g(K(V_{(x,u)}),u) = \sum_i B^i(x,u)u_i = 0$, and hence

$$(6.37)\qquad g(K(V_{(x,u)})-\nabla_u\xi,u) = -g(\nabla_u\xi,u)$$

for all $(x,u) \in T_1M$. Combining (6.36) and (6.37), we obtain $K(V_{(x,u)}) = \nabla_u \xi - g(\nabla_u \xi, u)u$, and consequently

$$V_{(x,u)} = \xi^h_{(x,u)} + t\{\nabla_u \xi - g(\nabla_u \xi, u)u\} = \xi^h_{(x,u)} + t\{\nabla_u \xi\} = \xi^{\bar{c}}_{(x,u)}. \tag{6.38}$$

Case 2: $d = 0$. Substituting again from the first identity of Lemma 5.7 into the first identity of (6.31), we obtain

$$(\mathcal{L}_\xi)_x g = 2\tilde{f}(x,u) g_x \tag{6.39}$$

for all $(x,u) \in T_1M$. Then $\tilde{f}$ is constant on the fibers of T_1M, that is, there exists a C^∞-function f on M such that $\tilde{f} = f \circ p_1$ and ξ is a conformal vector field on M. We shall prove that f vanishes identically on M.

Now, let G be the metric on TM extending $\widetilde{G}$, given by (3.1) with $\alpha_1 = a$, $\alpha_3 = c$ and $\alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0$, and let $\overline{V}$ and $\overline{\overline{V}}$ denote the extensions of V to $TM \setminus \sigma_0$ and TM, respectively, as in Section 4.2. Fixing $u \in S_xM$ and taking, in (4.9), $Z = V$ and $X = Y = W^t$ for $W \in \mathfrak{X}(M)$ such that $0 \neq W_x \perp u$, and using Lemma 4.4, we obtain

$$(\mathcal{L}_{\overline{V}} G)_{ru}(W^v_{ru}, W^v_{ru}) = \mathcal{L}_{Z_u} \widetilde{G}(V^v_u, Y^v_u).$$

From the fact that V is a conformal vector field on $(T_1M, \widetilde{G})$, we deduce

$$(\mathcal{L}_{\overline{V}} G)_{ru}(W^v_{ru}, W^v_{ru}) = 2f(x) g(W_x, W_x)$$

for all $r > 0$. When $r \to 0$, we have by continuity

$$(\mathcal{L}_{\overline{\overline{V}}} G)_0(W^v_0, W^v_0) = 2f(x) g(W_x, W_x).$$

But since $\overline{\overline{V}}_0 = \xi^h_0$ and $\mathcal{L}_{\xi^h} G(W^v, W^v) = 0$, we have $f(x) g(W_x, W_x) = 0$, and consequently $f(x) = 0$. Hence f vanishes identically, and thus V and ξ are Killing vector fields.

Using the Corollary in [22], we deduce from Lemma 5.8 (since $b = d = 0$) that $V = (\xi^c + \iota P)|_{T_1M} = \xi^{\bar{c}} + \tilde{\iota} P$, with ξ a Killing vector field and P a skew-symmetric parallel $(1,1)$-tensor field on M. Note that since ξ is Killing or P is skew-symmetric, respectively, then ξ^c or ιP is tangent to T_1M at any point of T_1M.

(iii) $\to$ (ii): First, if $V = \xi^{\bar{c}}$, where ξ is a Killing vector field, then, by Theorem 1.3, $V = \xi^{\bar{c}}$ is a Killing vector field. On the other hand, suppose that $\widetilde{G}$ is a Kaluza-Klein metric, i.e. $b = d = 0$, and that $V = \xi^{\bar{c}} + \tilde{\iota} P$ with ξ a Killing vector field and P a skew-symmetric parallel $(1,1)$-tensor field on M. Then

$$B^i(x,u) = \sum_j (\xi^i_{;j} + P^i_j)(x) u^j, \tag{6.40}$$

where $\xi^i_{;j}$ are the local components of the $(1,1)$-tensor field $\nabla\xi$, i.e. $\nabla_{\partial/\partial x^j}\xi = \sum_i \xi^i_{;j}\partial/\partial x^i$, and P^i_j are the local components of P. Since ξ is a Killing vector field (and hence satisfies (2.2) with $f=0$) and $d=0$, the first identity of Lemma 5.7 becomes

$$(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^h)=0. \tag{6.41}$$

On the other hand, using Lemma 5.1, we have

$$\begin{aligned}
&\sum_i X^h(B^i)\Big(\frac{\partial}{\partial x^i}\Big)_x+\sum_i B^i(x,u)\nabla_X\frac{\partial}{\partial x^i}\\
&\quad= -\sum_{i,j,k,l}\Gamma^l_{jk}(x)X^j[\xi^i_{;l}+P^i_l](x)u^k\Big(\frac{\partial}{\partial x^i}\Big)_x+\sum_{i,k}[X(\xi^i_{;k})+X(P^i_k)]u^k\Big(\frac{\partial}{\partial x^i}\Big)_x\\
&\qquad+\sum_{i,k,l}\Gamma^l_{ij}(x)X^j[\xi^i_{;k}+P^i_k](x)\Big(\frac{\partial}{\partial x^l}\Big)_x\\
&\quad= -\sum_{i,l}\Big[\sum_{j,k}\Gamma^l_{jk}(x)X^ju^k\Big]\xi^i_{;l}(x)\Big(\frac{\partial}{\partial x^i}\Big)_x-\sum_{i,l}\Big[\sum_{j,k}\Gamma^l_{jk}(x)X^ju^k\Big]P^il(x)\Big(\frac{\partial}{\partial x^i}\Big)_x\\
&\qquad+\Big\{\sum_i X\Big(\sum_k\xi^i_{;k}u^k\Big)\Big(\frac{\partial}{\partial x^i}\Big)_x+\sum_l\Gamma^l_{ij}(x)X^j\Big(\sum_k\xi^i_{;k}(x)u^k\Big)\Big(\frac{\partial}{\partial x^l}\Big)_x\Big\}\\
&\qquad+\Big\{\sum_i X\Big(\sum_k P^i_ku^k\Big)\Big(\frac{\partial}{\partial x^i}\Big)_x+\sum_l\Gamma^l_{ij}(x)X^j\Big(\sum_k P^i_k(x)u^k\Big)\Big(\frac{\partial}{\partial x^l}\Big)_x\Big\}\\
&\quad= -\nabla_{\nabla_XU}\xi-P(\nabla_XU)+\nabla_X\nabla_U\xi+\nabla_X(P(U))\\
&\quad= \nabla^2\xi(u,X)+(\nabla_XP)(u),
\end{aligned}$$

where $X=\sum_i X^i(\partial/\partial x^i)_x$ and U is the local vector field expressed as $U := \sum_i u^i\partial/\partial x^i$. Hence the second identity of Lemma 5.7 becomes

$$(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^h,Y^t)=ag(R(\xi_x,X)u+\nabla^2\xi(u,X)+(\nabla_XP)(u),Y)=0, \tag{6.42}$$

since ξ is a Killing vector field (and so satisfies (2.5)) and P is parallel.

Finally, using again Lemma 5.1, we have

$$\begin{aligned}
\sum_i X^t(B^i)\Big(\frac{\partial}{\partial x^i}\Big)_x &= \sum_{i,j}(X^j-g(X,u)u^j)(\xi^i_{;j}(x)+P^i_j(x))\Big(\frac{\partial}{\partial x^i}\Big)_x\\
&= \nabla_X\xi+P(X)-g(X,u)(\nabla_u\xi+P(u)).
\end{aligned}$$

Hence the second identity of Lemma 5.7 becomes

$$
\begin{aligned}
(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^t,Y^t) &= g(\nabla_X\xi + P(X) - g(X,u)(\nabla_u\xi + P(u)), Y - g(Y,u)u) \\
&\quad \times g(\nabla_Y\xi + P(Y) - g(Y,u)(\nabla_u\xi + P(u)), X - g(X,u)u) \\
&= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + g(P(X),Y) + g(X,P(Y)) \\
&\quad - g(X,u)[g(\nabla_u\xi, Y) + g(\nabla_Y\xi, u) + g(P(u),Y) + g(P(Y),u)] \\
&\quad - g(Y,u)[g(\nabla_u\xi, X) + g(\nabla_X\xi, u) + g(P(u),X) + g(P(X),u)] \\
&\quad + 2g(X,u)g(Y,u)[g(\nabla_u\xi, u) + g(P(u),u)].
\end{aligned}
$$

Using the facts that ξ is a Killing vector field and that P is skew-symmetric, we obtain

$$(\mathcal{L}_V\widetilde{G})_{(x,u)}(X^t,Y^t) = 0. \tag{6.43}$$

We deduce from Lemma 6.1, using the identities (6.41)–(6.43), that V is a Killing vector field.

(ii) $\to$ (i): is trivial. □

Now to prove Theorem 1.6, we need:

Lemma 6.2. *Let $\lambda \in \mathbb{R}$ and let ξ be a vector field on M. Suppose that G is a Kaluza-Klein type metric on T_1M. Then $\bar{*}\xi_\lambda$ is a conformal vector field on $(T_1M,\widetilde{G})$ if and only if the following assertions hold:*

(i) $$
\begin{aligned}
&(a+c)g(\nabla^2\xi(X,Y) + \nabla^2\xi(Y,X), u) \\
&\quad + d[g(X,u)g(\nabla^2\xi(Y,u) + \nabla^2\xi(u,Y), u) + g(Y,u)g(\nabla^2\xi(X,u) + \nabla^2\xi(u,X), u) \\
&= \lambda[2(a+c)g(X,Y)g(\xi_x,u) + d(g(\xi_x,X)g(Y,u) + g(\xi_x,Y)g(X,u))];
\end{aligned}
$$

(ii) $$
\begin{aligned}
&R(C(\xi_x)(Y),X)Z + R(C(\xi_x)(Z),X)Y = \\
&= \Big[\frac{a+c}{a} + \lambda\Big][2g(Y,Z)\nabla_X\xi - g(\nabla_X\xi,Y)Z - g(\nabla_X\xi,Z)Y] \\
&\quad + \frac{d}{a}[g(X,Y)(\nabla_Z\xi - g(\nabla_Z\xi,u)u) + g(X,Z)(\nabla_Y\xi - g(\nabla_Y\xi,u)u)]
\end{aligned}
$$

for all $(x,u) \in T_1M$ and $X,Y,Z \in M_x$.

P r o o f. Suppose that $\bar{*}\xi_\lambda$ is a conformal vector field on $(T_1M,\widetilde{G})$, that is, there is a C^∞-function $\tilde{f}$ on T_1M such that $\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G} = 2\tilde{f}\widetilde{G}$. In particular, we have

$$(\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G})_{(x,u)}(X^t,Y^t) = 2a\tilde{f}(x,u)[g(X,Y) - g(X,u)g(Y,u)] \tag{6.44}$$

for all $(x,u) \in TM$ and $X, Y \in M_x$. Substituting from the third equation of Lemma 5.9 into (6.44), we obtain

$$\tilde{f}(x,u) = -\lambda g(\xi_x, u) \quad \text{for all } (x,u) \in T_1M. \tag{6.45}$$

It follows then that

$$(\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G})_{(x,u)}(X^h, Y^h) = -2\lambda g(\xi_x, u)[(a+c)g(X,Y) + dg(X,u)g(Y,u)] \tag{6.46}$$

for all $(x,u) \in TM$ and $X, Y \in M_x$. Substituting from the first equation of Lemma 5.9 into (6.46), we obtain the condition (i) of the theorem. On the other hand, substituting from the second equation of Lemma 5.9 into $\mathcal{L}_{\bar{*}\xi_\lambda}\widetilde{G}(X^h, Y^t) = 0$, we obtain the condition (ii) of the theorem. The converse is obvious. □

P r o o f of Theorem 1.6. Recall that since (M,g) is a space of constant sectional curvature k of dimension $n > 2$, we have

$$R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y \quad \text{for all } X, Y, Z \in \mathfrak{X}(M). \tag{6.47}$$

(i) → (iii): Suppose that $\bar{*}\xi_\lambda$ is a conformal vector field. Then the condition (ii) of Lemma 6.2 becomes, by virtue of (6.47),

$$\begin{aligned} 0 = {} & \Big[\frac{a+c}{a} + \lambda\Big][g(\nabla_X\xi, Y) - g(\nabla_X\xi, u)g(Y,u)] \\ & + \frac{d}{a}g(X,u)[g(\nabla_u\xi, Y) - g(\nabla_u\xi, u)g(Y,u)] \\ & \times k[g(X,u)g(\nabla_Y\xi, u) - g(X,Y)g(\nabla_u\xi, u)] \end{aligned} \tag{6.48}$$

for all $(x,u) \in T_1M$, and $X, Y \in M_x$. Taking in (6.48) $X = Y \perp u$ and $\|X\| = 1$, we get

$$kg(\nabla_u\xi, u) = \Big[\frac{a+c}{a} + \lambda\Big]g(\nabla_X\xi, X) \tag{6.49}$$

for all $u, X \in S_xM$, $X \perp u$. So, we have three possibilities:

Case 1: $k[(a+c)/a+\lambda] = 0$ and $\lambda \neq k-(a+c)/a$. Then by (6.49) $g(\nabla_X\xi, X) = 0$ for all $X \in TM$, i.e. ξ is a Killing vector field on (M,g). Taking in (6.48) $X = u$, we obtain

$$\Big[k - \frac{\varphi}{a} - \lambda\Big]g(\nabla_Y\xi, u) = 0 \tag{6.50}$$

for all $(x,u) \in T_1M$, and $Y \in M_x$.

On the other hand, since ξ is a Killing vector field, (i) of Lemma 6.2 becomes, by virtue of (2.2) and (6.47),

$$
\begin{aligned}
(6.51) \qquad 0 = {} & [k\varphi - \lambda d][g(\xi_x, X)g(Y,u) + g(\xi_x, Y)g(X,u)] \\
& - 2(a+c)(\lambda + k)g(X,Y)g(\xi_x, u) - 4kdg(X,u)g(Y,u)g(\xi_x,u).
\end{aligned}
$$

Let $x_0 \in M$ be such that $\xi_{x_0} \neq 0$. Taking, in (6.51), $u \perp \xi_{x_0}$, $X = u$ and $Y = \xi_{x_0}$, we get

$$(6.52) \qquad k\varphi - \lambda d = 0,$$

and (6.51) becomes

$$(6.53) \qquad 2(a+c)(\lambda + k)g(X,Y)g(\xi_x,u) + 4kdg(X,u)g(Y,u)g(\xi_x,u) = 0.$$

Taking, in (6.53), $u = \|\xi_{x_0}\|^{-1}\xi_{x_0}$, $X \perp u$ and $Y = X$, we obtain

$$(6.54) \qquad \lambda + k = 0,$$

and (6.51) becomes

$$(6.55) \qquad kdg(X,u)g(Y,u)g(\xi_x,u) = 0.$$

Taking, in (6.55), $u = \|\xi_{x_0}\|^{-1}\xi_{x_0}$ and $X = Y = u$, we obtain

$$(6.56) \qquad kd = 0.$$

If $k = 0$, then $\lambda = 0$, by (6.54). Hence, we deduce from (6.50) that $g(\nabla_Y \xi, u) = 0$ for all $(x,u) \in T_1M$, and $Y \in M_x$, i.e. ξ is parallel. It follows then from Lemma 4.1 that $\bar{*}\xi_\lambda = \lambda \xi^t = 0$, since $\lambda = 0$.

If $k \neq 0$, then $d = 0$. Hence, by (6.52), $\varphi = 0$, which is a contradiction.

Case 2: $k \neq 0$ and $\lambda \neq -(a+c)/a$. Since u is arbitrary, we deduce from (6.49) that $k = \lambda + (a+c)/a$ and that the function $X \mapsto g(\nabla_X \xi, X)$ is constant on S_xM, i.e. there is a function $f \in C^\infty(M)$ such that $g(\nabla_X \xi, X) = f\|X\|^2$ for all $X \in \mathfrak{X}(M)$. Hence, by bilinearity, we have $\mathcal{L}_\xi g(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2fg(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$, i.e. ξ is a conformal vector field on (M,g). Since $\dim M > 2$ for any $X \perp Y \in M_x$, take $u \in T_1M \cap \operatorname{span}(X,Y)^\perp$. Then we have, by (6.48), $g(\nabla_X \xi, Y) = 0$ for all $X \perp Y \in M_x$. Hence, by virtue of $g(\nabla_X \xi, X) = f\|X\|^2$, we have $\nabla_X \xi = fX$ for all $X \in \mathfrak{X}(M)$. Then $\nabla^2 \xi(X,Y) = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi = X(f)Y$

for all $X, Y \in \mathfrak{X}(M)$. Substituting from the last equality into (i) of Lemma 6.2, we obtain

$$(6.57) \quad \varphi[X(f)g(Y,u) + Y(f)g(X,u)] + 2du(f)g(X,u)g(Y,u) \\ = \lambda[2(a+c)g(X,Y)g(\xi_x,u) + d(g(\xi_x,X)g(Y,u) + g(\xi_x,Y)g(X,u))]$$

for all $(x,u) \in T_1M$, and $X, Y \in M_x$.

For $\lambda \neq 0$, we take $X \perp u$ and $Y = X$ in (6.57) to get $g(\xi_x, u) = 0$ for all $u \in S_xM$, and then for all $u \in M_x$. We deduce that $\xi = 0$.

For $\lambda = 0$, we have $k = (a+c)/a$. We take $Y = u$ and $X \perp u$ in (6.57) to get $X(f) = 0$ for all $X \in M_x$. Then by connectedness, f is constant. In this case, $g(C(\xi)(u), X) = -g(\nabla_X \xi, u) = -fg(u,X)$ for all $(x,u) \in T_1M$, and $X, Y \in M_x$. We deduce that $C(\xi)(u) = -fu$, and hence $\bar{*}\xi_0(u) = -fu^h$ for all $u \in T_1M$, i.e., up to a real factor, $\bar{*}\xi_0$ is the geodesic flow vector field on T_1M.

Case 3: $k = 0 = \lambda + (a+c)/a$. In particular we have $\lambda = -(a+c)/a \neq 0$. We can suppose that (M,g) is $\mathbb{R}^n$ equipped with its standard metric. In this case, (i) of Lemma 6.2 is equivalent to the system

$$\begin{aligned} &\frac{\partial^2 \xi^k}{\partial x^i \partial x^j} = \lambda \delta_{ij} \xi^k, \quad i \neq k,\, j \neq k; \\ &\frac{\partial^2 \xi^k}{\partial x^i \partial x^k} = \frac{\lambda d}{2\varphi} \xi^i, \quad i \neq k; \\ &(\varphi + d)\frac{\partial^2 \xi^k}{(\partial x^i)^2} = \lambda \varphi \xi^k, \end{aligned}$$

$i, j, k = 1, \ldots, n$, whose solution is $\xi = 0$. Indeed:

For $\varphi + d = 0$, we have by the third equation $\xi^k = 0$ for each k.

For $\varphi + d \neq 0$, then differentiating the first equation (with $i = j$) with respect to x^k and the second equation with respect to x^i, we have $\lambda \partial \xi^k / \partial x^k = \lambda d / 2\varphi \partial \xi^i / \partial x^i$ for all $i \neq k$. Interchanging i and k and making the sum of the two formulas, we obtain $\lambda(1 - d/2\varphi)(\partial \xi^k / \partial x^k + \partial \xi^i / \partial x^i) = 0$. But $\lambda \neq 0$ and $1 - d/2\varphi \neq 0$, since if not we would have $0 = 2\varphi - d = \varphi + (a+c)$, which is a contradiction. Then $\partial \xi^k / \partial x^k = -\partial \xi^i / \partial x^i$ for all $k \neq i$. Since i and k are arbitrary and $\dim M > 2$, then $\partial \xi^k / \partial x^k = 0$ for all k. Substituting into the third equation of the preceding system, we have $\xi^k = 0$ for all k.

(iii) $\rightarrow$ (ii): Follows immediately from Theorem 1.4.

(ii) $\rightarrow$ (i): is trivial. $\square$

Acknowledgement. The authors are very grateful to the anonymous referee for his valuable comments and suggestions improving this paper.

References

[1] *M. T. K. Abbassi*: Métriques naturelles sur le fibré tangent à une variété Riemannienne. Editions universitaires europeennes, Paris, 2012. (In French.)

[2] *M. T. K. Abbassi, N. Amri, G. Calvaruso*: Kaluza-Klein type Ricci solitons on unit tangent sphere bundles. Differ. Geom. Appl. *59* (2018), 184–203. zbl MR doi

[3] *M. T. K. Abbassi, G. Calvaruso*: g-natural contact metrics on unit tangent sphere bundles. Monaths. Math. *151* (2007), 89–109. zbl MR doi

[4] *M. T. K. Abbassi, G. Calvaruso*: g-natural metrics of constant curvature on unit tangent sphere bundles. Arch. Math., Brno *48* (2012), 81–95. zbl MR doi

[5] *M. T. K. Abbassi, O. Kowalski*: On g-natural metrics with constant scalar curvature on unit tangent sphere bundles. Topics in Almost Hermitian Geometry and Related Fields. World Scientific, Hackensack, 2005, pp. 1–29. zbl MR doi

[6] *M. T. K. Abbassi, O. Kowalski*: Naturality of homogeneous metrics on Stiefel manifolds $SO(m+1)/SO(m-1)$. Differ. Geom. Appl. *28* (2010), 131–139. zbl MR doi

[7] *M. T. K. Abbassi, O. Kowalski*: On Einstein Riemannian g-natural metrics on unit tangent sphere bundles. Ann. Global Anal. Geom. *38* (2010), 11–20. zbl MR doi

[8] *M. T. K. Abbassi, M. Sarih*: Killing vector fields on tangent bundles with Cheeger-Gromoll metric. Tsukuba J. Math. *27* (2003), 295–306. zbl MR doi

[9] *M. T. K. Abbassi, M. Sarih*: On natural metrics on tangent bundles of Riemannian manifolds. Arch. Math., Brno *41* (2005), 71–92. zbl MR

[10] *M. T. K. Abbassi, M. Sarih*: On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds. Differ. Geom. Appl. *22* (2005), 19–47. zbl MR doi

[11] *M. Benyounes, E. Loubeau, L. Todjihounde*: Harmonic maps and Kaluza-Klein metrics on spheres. Rocky Mt. J. Math. *42* (2012), 791–821. zbl MR doi

[12] *G. Calvaruso, V. Martín-Molina*: Paracontact metric structures on the unit tangent sphere bundle. Ann. Mat. Pura Appl. (4) *194* (2015), 1359–1380. zbl MR doi

[13] *G. Calvaruso, D. Perrone*: Homogeneous and H-contact unit tangent sphere bundles. J. Aust. Math. Soc. *88* (2010), 323–337. zbl MR doi

[14] *G. Calvaruso, D. Perrone*: Metrics of Kaluza-Klein type on the anti-de Sitter space $\mathbb{H}^3_1$. Math. Nachr. *287* (2014), 885–902. zbl MR doi

[15] *S. Deshmukh*: Geometry of conformal vector fields. Arab J. Math. Sci. *23* (2017), 44–73. zbl MR doi

[16] *P. Dombrowski*: On the geometry of the tangent bundle. J. Reine Angew. Math. *210* (1962), 73–88. zbl MR doi

[17] *S. Ewert-Krzemieniewski*: On Killing vector fields on a tangent bundle with g-natural metric. I. Note Mat. *34* (2014), 107–133. zbl MR doi

[18] *A. Gezer, L. Bilen*: On infinitesimal conformal transformations with respect to the Cheeger-Gromoll metric. An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. *20* (2012), 113–128. zbl MR doi

[19] *G. S. Hall*: Symmetries and Curvature Structure in General Relativity. World Scientific Lecture Notes in Physics 46, World Scientific, River Edge, 2004. zbl MR doi

[20] *S. Hedayatian, B. Bidabad*: Conformal vector fields on tangent bundle of a Riemannian manifold. Iran. J. Sci. Technol, Trans. A, Sci. *29* (2005), 531–539. zbl MR

[21] *I. Kolář, P. W. Michor, J. Slovák*: Natural Operations in Differential Geometry. Springer, Berlin, 1993. zbl MR doi

[22] *T. Konno*: Killing vector fields on tangent sphere bundles. Kodai Math. J. *21* (1998), 61–72. zbl MR doi

[23] *T. Konno*: Infinitesimal isometries on tangent sphere bundles over three-dimensional manifolds. Hiroshima Math. J. *41* (2011), 343–366. zbl MR doi

[24] *O. Kowalski, M. Sekizawa*: Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles - a classification. Bull. Tokyo Gakugei Univ., Sect. IV, Ser. Math. Nat. Sci. *40* (1988), 1–29. zbl MR

[25] *E. Peyghan, A. Heydari*: Conformal vector fields on tangent bundle of a Riemannian manifold. J. Math. Anal. Appl. *347* (2008), 136–142. zbl MR doi

[26] *S. Sasaki*: On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Math. J., II. Ser. *10* (1958), 338–354. zbl MR doi

[27] *S. Sasaki*: On the differential geometry of tangent bundles of Riemannian manifolds. II. Tohoku Math. J., II. Ser. *14* (1962), 146–155. zbl MR doi

[28] *F. M. Şimşir, C. Tezer*: Conformal vector fields with respect to the Sasaki metric tensor. J. Geom. *84* (2005), 133–151. zbl MR doi

[29] *S. Tanno*: Killing vectors and geodesic flow vectors on tangent bundles. J. Reine Angew. Math. *282* (1976), 162–171. zbl MR doi

[30] *C. M. Wood*: An existence theorem for harmonic sections. Manuscr. Math. *68* (1990), 69–75. zbl MR doi

[31] *K. Yano, S. Kobayashi*: Prolongations of tensor fields and connections to tangent bundles. I: General theory. J. Math. Soc. Japan *18* (1966), 194–210. zbl MR doi

[32] *K. Yano, S. Kobayashi*: Prolongations of tensor fields and connections to tangent bundles. II: Infinitesimal automorphisms. J. Math. Soc. Japan *18* (1966), 236–246. zbl MR doi

Authors' address: Mohamed Tahar Kadaoui Abbassi (corresponding author), Noura Amri, Laboratory of Mathematical Sciences and Applications, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdallah, B.P. 1796, Fès-Atlas, Fez, Morocco, e-mail: `mtk_abbassi@Yahoo.fr`, `amri.noura1992@gmail.com`.