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# HOMOGENIZATION OF A THREE-PHASE COMPOSITES OF DOUBLE-POROSITY TYPE

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*Cordially dedicated to Professor Mongi Mabrouk*

*Abstract.* In this work we consider a diffusion problem in a periodic composite having three phases: matrix, fibers and interphase. The heat conductivities of the medium vary periodically with a period of size $\varepsilon^\beta$ ($\varepsilon > 0$ and $\beta > 0$) in the transverse directions of the fibers. In addition, we assume that the conductivity of the interphase material and the anisotropy contrast of the material in the fibers are of the same order $\varepsilon^2$ (the so-called double-porosity type scaling) while the matrix material has a conductivity of order 1. By introducing a partial unfolding operator for anisotropic domains we identify the limit problem. In particular, we prove that the effect of the interphase properties on the homogenized models is captured only when the microstructural length scale is of order $\varepsilon^\beta$ with $0 < \beta \leqslant 1$.

*Keywords*: homogenization; three-phase composite; unfolding operator; double-porosity type

*MSC 2020*: 35B27, 35B45, 35K55, 35K65, 76S05

## 1. Introduction

It is well known that the effective mechanical and thermal properties of a combination of fibers, which have highly-anisotropic characteristics, with matrix materials of a completely different nature is strongly influenced by the presence of an inevitable interphase between these two phases (see e.g. [8] and [12]). As a consequence, the fiber-matrix interphase has a crucial impact on the behavior and properties of the fiber-reinforced composites.

In recent years, by applying numerical homogenization techniques, several studies have shown the influence of the interphase material parameters on the overall mechanical behavior of fibrous composites (see e.g. [2], [10], [11]). However, it should

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be noted that a few mathematical results have been established rigorously in the analysis of those effects.

To the author's knowledge, probably one of the first papers performing a rigorous mathematical analysis of the effects of the interphase in the overall behavior of fiber-reinforced composites was published in 2009 by the first author (see [3]). Using the two-scale convergence method, the author studied the homogenization of a heat transfer problem in a fiber-reinforced composite taking into account the combined effects of the fiber-matrix interphase together with the high anisotropy of the material in the fibers. More precisely, he considered a degenerate elliptic-parabolic equation in a composite, whose physical properties oscillate rapidly on the scale of order $\varepsilon$ ($\varepsilon$ is a small dimensionless parameter). The most interesting results concerned the critical case, where the conductivity of the interphase material and the anisotropy contrast in the fiber are of the same order $\varepsilon^2$ (the so-called double-porosity type scaling). Those results showed, in particular, that the homogenized problem is an integro-differential one displaying nonlocality along the fibers.

More recently, in 2015, the first author has also studied, using the same method applied in [3], an elastostatic problem in a periodic medium having three phases: matrix, fibers, and fiber coatings (see [4]). He particularly has shown that, as a result of the soft interphase material competing with the highly anisotropic fibers, the effective transverse traction and the longitudinal stress in the fibers account for an unconventional behavior in the homogenized model.

The purpose of the present work is twofold. First, we shall introduce the concept of partial unfolding operator which maps functions defined on domain, depending on small parameter $\varepsilon$ in some specific directions, onto functions defined on the whole fixed domain. In particular, this partial unfolding operator allows us to analyze directly some highly anisotropic fiber-reinforced composites, where the physical properties are only relevant in the transversal directions. Second, to continue the previous work [3] by investigating the homogenization of composites involving three materials, with double-porosity type scaling, taking into account the microstructural length scale. To this purpose, we shall assume that the physical properties of the composite vary periodically with a period cell of size $\varepsilon^\beta$, $\beta > 0$ in the cross section.

To end this section, let us comment on how our study is related to previous works in homogenization of three-phase composites. Although our work is one of a few papers, along with [3] and [4], that treat the homogenization of an elliptic equation involving high contrast, of double porosity type, between the coefficients; it would be interesting to compare our results with those in [3] and [4]. Especially if noted that the homogenized model derived, in the critical scale (i.e. $\beta = 1$), is the same as the one obtained in [3], Theorem 3.2 and in [4], Theorem 4.1, in the nonstationary setting and in the context of linear elasticity, respectively. Moreover, in this critical scale, the effects of the interphase on the homogenized problem are obvious. However, the novelty of the present study is the result given in Section 4, which shows clearly that the effect of the interphase properties on the homogenized limit is captured, in the subcritical scale (i.e. $\beta < 1$), via the term $|\widetilde{Y}_3|$ in (4.6); whereas, in the supercritical scale (i.e. $\beta > 1$) it is not observed.

## 2. Statement of the problem and a priori estimates

**2.1. Formulation of the problem.** Let $\widetilde{\Omega}$ be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. We consider the following cylindrical domain:

$$\Omega = \widetilde{\Omega} \times I, \quad \text{where } I = ]0,1[.$$

To introduce the reference periodic medium we shall denoted by $Y$ and $\widetilde{Y}$ the unit cube in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively (i.e. $\widetilde{Y} := ]0,1[^2$ and $Y := \widetilde{Y} \times I$). We assume that $\widetilde{Y}$ is partitioned as

$$\widetilde{Y} = \widetilde{Y}_1 \cup \widetilde{Y}_{13} \cup \widetilde{Y}_3 \cup \widetilde{Y}_{23} \cup \widetilde{Y}_2,$$

where $\widetilde{Y}_1$, $\widetilde{Y}_2$, $\widetilde{Y}_3$ are three connected open subsets such that $\widetilde{Y}_1 \cap \widetilde{Y}_2 = \emptyset$, $\partial\widetilde{Y} \cap \widetilde{Y}_3 = \emptyset$ and where $\widetilde{Y}_{\alpha 3}$, $\alpha \in \{1,2\}$ is the interface between $\widetilde{Y}_\alpha$ and $\widetilde{Y}_3$ (see Figure 1). We denote by $\chi_i$ the characteristic function of $Y_i$, $i = 1,2,3$.

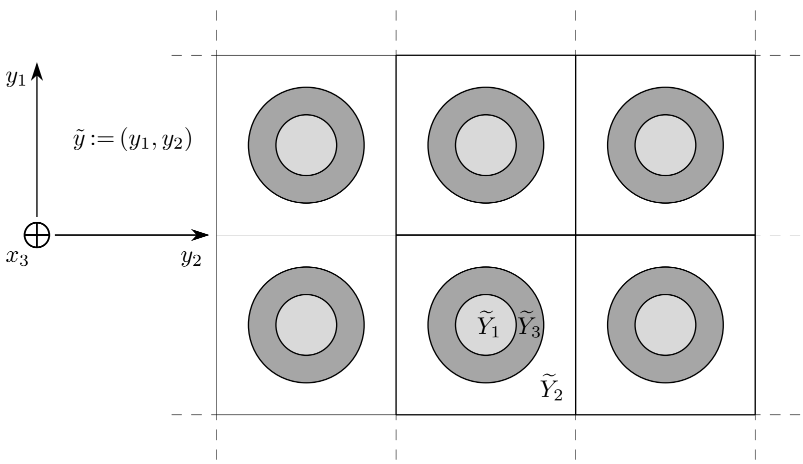


Figure 1. The transversal geometry of the medium

For $\varepsilon > 0$, $\beta > 0$, $i \in \{1,2,3\}$ and $\alpha = 1,2$, we set $\Omega_i^\varepsilon = [\varepsilon^\beta(\widetilde{Y}_i + \mathbb{Z}^2) \times I] \cap \Omega$ and $\Gamma_{\alpha 3}^\varepsilon = [\varepsilon^\beta(\widetilde{Y}_{\alpha 3} + \mathbb{Z}^2) \times I] \cap \Omega$. Thus, the boundaries $\partial\Omega_i^\varepsilon$ of $\Omega_i^\varepsilon$, $i = 1,2,3$ are given by

$$\partial\Omega_1^\varepsilon = \Gamma_{13}^\varepsilon \cup (\widetilde{\Omega}_1^\varepsilon \times \{0,1\}), \quad \partial\Omega_2^\varepsilon = \Gamma_{23}^\varepsilon \cup (\widetilde{\Omega}_2^\varepsilon \times [0,1]), \quad \partial\Omega_3^\varepsilon = \Gamma_{13}^\varepsilon \cup \Gamma_{23}^\varepsilon \cup (\widetilde{\Omega}_3^\varepsilon \times \{0,1\}).$$

The regions $\Omega_1^\varepsilon$, $\Omega_2^\varepsilon$ and $\Omega_3^\varepsilon$ represent, respectively, the fibers, the so-called matrix material and the interphase.

Now, for $i = 1, 2, 3$ let $a_i$ be the heat conductivity of the material in the $i$th phase. They are $\widetilde{Y}$-periodic functions given on the basic cell $\widetilde{Y}$ and satisfying the following assumptions:

▷ $a_i \in L^\infty(\mathbb{R}^3)$,

▷ $a_0 \leqslant a_i(\tilde{y})$ a.e $\tilde{y} \in \widetilde{Y}$ with $a_0 > 0$, independent of $\tilde{y}$.

In what follows, $x = (\widetilde{x}, x_3)$ denotes points of $\mathbb{R}^3$ and the corresponding $\varepsilon^\beta \widetilde{Y}$-periodic coefficients are then defined by

$$a_i^\varepsilon(x) = a_i\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)$$

for every $x \in \Omega_i^\varepsilon$, $i = 1, 2, 3$. Hence, the global conductivity of the material in $\Omega$ is given by

$$A^\varepsilon(x) = \sum_{i=1}^{3} \chi_i^\varepsilon(x) A_i^\varepsilon(x), \quad \chi_i^\varepsilon(x) := \chi_i\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big),$$

where

$$A_1^\varepsilon(x) := a_1^\varepsilon(x) \left(\begin{array}{c|c} \varepsilon^2 I_2 & 0 \\ \hline 0 & 1 \end{array}\right), \quad A_2^\varepsilon(x) := a_2^\varepsilon(x) I_3, \quad A_3^\varepsilon(x) := \varepsilon^2 a_3^\varepsilon(x) I_3$$

with $I_2$ and $I_3$ being the identity matrices in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Let us assume that the temperature is maintained fixed (homogeneous Dirichlet conditions) on both the lateral and bottom boundaries of $\Omega$ while the top boundary is insulted (homogeneous Neumann condition). Then the temperature filed $u^\varepsilon$ is governed by the following elliptic boundary value problem:

$$(P^\varepsilon) \begin{cases} -\mathrm{div}(A^\varepsilon \nabla u^\varepsilon) = f^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \Gamma_D := \partial\widetilde{\Omega} \times [0, 1[, \\ A^\varepsilon \nabla u^\varepsilon n = 0 & \text{on } \partial\Omega \setminus \Gamma_D, \end{cases}$$

where $f^\varepsilon \in L^2(\Omega)$ and $n$ denotes the outer normal to $\Omega$.

As an immediate consequence of the Lax-Milgram Theorem for any $\varepsilon > 0$, problem $(P^\varepsilon)$ has a unique solution $u^\varepsilon$ in

$$H_D^1(\Omega) := \{u \in H^1(\Omega),\ u = 0 \text{ on } \Gamma_D\}.$$

Let us mention that the variational formulation of problem $(P^\varepsilon)$ is:

(2.1) Find $u^\varepsilon \in H^1_D(\Omega)$ such that

$$\int_{\Omega_1^\varepsilon} a_1^\varepsilon(x) \begin{pmatrix} \varepsilon^2 \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \begin{pmatrix} \nabla_{\tilde{x}} \psi \\ \partial_{x_3} \psi \end{pmatrix} \mathrm{d}x + \int_{\Omega_2^\varepsilon} a_2^\varepsilon(x) \nabla u^\varepsilon \nabla \psi \, \mathrm{d}x + \varepsilon^2 \int_{\Omega_3^\varepsilon} a_3^\varepsilon(x) \nabla u^\varepsilon \nabla \psi \, \mathrm{d}x = \int_\Omega f^\varepsilon \psi \, \mathrm{d}x \quad \forall\, \psi \in H^1_D(\Omega).$$

Furthermore, let us emphasize that our approach to study the asymptotic behavior of the solution $u^\varepsilon$ as $\varepsilon$ passes to zero is based on a peculiar Poincaré-type inequality, involving the double-porosity type scaling, and the use of the periodic unfolding method. In particular, we prove that the limit problem depend on the value of the power scale $\varepsilon^\beta$. Moreover, in the most interesting case $\beta = 1$, we derive some corrector results. The paper is organized as follows: In Section 2 we present some crucial a priori estimates. In Section 3 we introduce the partial unfolding operator and we give some useful proprieties. Further, we obtain some convergence results. In Section 4 we summarize the main results of this work. All proofs are presented in Section 5. In Section 6 we proved some corrector results.

Throughout the paper, by $C$ we denote a constant not depending on $\varepsilon$, whose value may vary from one line to another. From a bounded sequence in $L^2$-space we can take a subsequence that converges weakly but in effect all the subsequences converge to the same limit as the limiting equations have a unique solution, so we generally ignore to mention the expression "a subsequence".

**2.2. A priori estimates.** First, we shall prove the following a priori estimates for $u^\varepsilon$ the solution of problem (2.1).

**Lemma 2.1.** *Let $u^\varepsilon$ be the solution of problem* (2.1). *Then there exists a constant $C > 0$ independent of $\varepsilon$ such that*

$$\left( \left\| \begin{pmatrix} \varepsilon \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \right\|_{L^2(\Omega_1^\varepsilon)}, \|\nabla u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)}, \|\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_3^\varepsilon)} \right) \leqslant C, \tag{2.2}$$

$$\|u^\varepsilon\|_{L^2(\Omega)} \leqslant C. \tag{2.3}$$

Proof. We choose $\psi = u^\varepsilon$ as a test function in (2.1) to obtain

$$\int_{\Omega_1^\varepsilon} a_1^\varepsilon \begin{pmatrix} \varepsilon \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \begin{pmatrix} \varepsilon \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \mathrm{d}x + \int_{\Omega_2^\varepsilon} a_2^\varepsilon \nabla u^\varepsilon \nabla u^\varepsilon \, \mathrm{d}x + \varepsilon^2 \int_{\Omega_3^\varepsilon} a_3^\varepsilon \nabla u^\varepsilon \nabla u^\varepsilon \, \mathrm{d}x = \int_\Omega f^\varepsilon u^\varepsilon \, \mathrm{d}x.$$

By using Young's inequality one has

$$\int_\Omega f^\varepsilon u^\varepsilon \, dx \leqslant \frac{\alpha^2}{2} \|f^\varepsilon\|^2_L(\Omega)^2 + \frac{1}{2\alpha^2} \|u^\varepsilon\|^2_{L^2(\Omega)} \quad \forall\, \alpha > 0.$$

Since $0 < a_0 \leqslant a_k(\tilde{y})$, $k = 1, 2, 3$, we have

$$\left\| \begin{pmatrix} \varepsilon \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \right\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \|\varepsilon \nabla u^\varepsilon\|^2_{L^2(\Omega_3^\varepsilon)} \leqslant \frac{\alpha^2}{2a_0} \|f^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2a_0\alpha^2} \|u^\varepsilon\|^2_{L^2(\Omega)}.$$

It is known (see [3], Lemma 2.1) that there exists $C > 0$ independent of $\varepsilon$ such that

$$\|u^\varepsilon\|^2_{L^2(\Omega)} \leqslant C(\|\delta_{x_3} u^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{2\beta} \|\nabla u^\varepsilon\|^2_{L^2(\Omega_3^\varepsilon)}). \tag{2.4}$$

From (2.4) we deduce

$$\begin{aligned} &\left\| \begin{pmatrix} \varepsilon \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \right\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \|\varepsilon \nabla u^\varepsilon\|^2_{L^2(\Omega_3^\varepsilon)} \\ &\qquad \leqslant \frac{C\alpha^2}{2a_0} + \frac{C}{2a_0\alpha^2} (\|\partial_{x_3} u^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{2\beta} \|\nabla u^\varepsilon\|^2_{L^2(\Omega_3^\varepsilon)}). \end{aligned}$$

We take $\alpha^2 = C/2a_0$ and notice that in the case $\beta \geqslant 1$ we have $\varepsilon^{2\beta} \leqslant \varepsilon^2$, thus

$$\left\| \begin{pmatrix} \varepsilon \nabla_{\tilde{x}} u^\varepsilon \\ \partial_{x_3} u^\varepsilon \end{pmatrix} \right\|^2_{L^2(\Omega_1^\varepsilon)} + \|\nabla u^\varepsilon\|^2_{L^2(\Omega_2^\varepsilon)} + \|\varepsilon \nabla u^\varepsilon\|^2_{L^2(\Omega_3^\varepsilon)} \leqslant C$$

and (2.2) follows. In view of (2.4) we deduce (2.3).

However, for the case $\beta < 1$ we need a particular scaling in $\Omega_3^\varepsilon$ of the function $f^\varepsilon$ to prove the a priori estimate. For this reason we shall assume that

$$f^\varepsilon(x) = \Big(\chi_1^\varepsilon(x) + \chi_2^\varepsilon(x) + \frac{\varepsilon}{\varepsilon^\beta} \chi_3^\varepsilon(x)\Big) f(x), \quad f \in L^2(\Omega).$$

Under this assumption and by a similar manner as above we obtain the desired estimates (2.2) and

$$\|u^\varepsilon\|^2_{L^2(\Omega)} \leqslant C + C\frac{\varepsilon^{2\beta}}{\varepsilon^2}. \tag{2.5}$$

This finishes the proof. □

By the way, let us point out that the basic assumption in the proof of (2.4) is the Dirichlet condition on the lateral and bottom boundaries of $\Omega$. Nevertheless, we may conjecture that the Poincaré's inequality (2.4) continues to hold true even if we have only the Dirichlet condition on the top or the bottom of $\Omega$. We postpone the proof of this conjecture to a subsequent work.

## 3. Partial unfolding in $L^2$-space

**3.1. The partial unfolding operator $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$.** The unfolding operator $\mathcal{T}_\varepsilon$ was introduced in 2002 by Damlamian, Cioranescu and Griso (see [5]) for $\varepsilon$-periodic domains. The disadvantage of this unfolding operator is, that in general, it does not conserve the integral, i.e.

$$\int_{\Omega\times\widetilde{Y}} \mathcal{T}_\varepsilon(\phi)(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} \neq \int_\Omega \phi(x)\,\mathrm{d}x.$$

To overcome this problem, Franců and Svansted proposed in [9] a modified operator $\mathcal{T}_\varepsilon(\phi)$ which conserves the integral. In this work we shall use this approach. Nevertheless, in our case, the material of the composite is highly heterogeneous only in the transverse directions. So, we need to define a partial unfolding operator with respect to the relevant transverse vector $\tilde{x}$, which enables us to simplify the homogenization process significantly.

The coordinates of any vector $x$ in $\Omega$ can be split into a "fast" part $x = (\tilde{x}, x_3)$ and a "slow" part $\tilde{y} := \tilde{x}/\varepsilon^\beta$. Based on the $\varepsilon^\beta$-periodicity assumption, for each $\tilde{x} \in \mathbb{R}^2$ one has the unique decomposition

$$\tilde{x} = \varepsilon^\beta\left(\left[\frac{\tilde{x}}{\varepsilon^\beta}\right]_{\widetilde{Y}} + \left\{\frac{\tilde{x}}{\varepsilon^\beta}\right\}_{\widetilde{Y}}\right),$$

where for any $z \in \mathbb{R}^2$ we denote by $[z]_{\widetilde{Y}}$ the unique integer combination such that $z - [z]_{\widetilde{Y}}$ belongs to $\widetilde{Y}$. The following notations are also used:

$$\begin{aligned}
&\Xi_\varepsilon = \{\xi \in \mathbb{Z}^2,\ Y_\xi^\varepsilon = \varepsilon^\beta(\xi + \widetilde{Y}) \times I \subset \Omega\},\\
&\widehat{\Omega}_\varepsilon = \text{interior}\left\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon^\beta(\xi + \overline{\widetilde{Y}}) \times I\right\},\\
&\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon.
\end{aligned} \tag{3.1}$$

The set $\widehat{\Omega}_\varepsilon$ is the largest union of $\varepsilon^\beta(\xi + \overline{\widetilde{Y}}) \times I$ cells included in $\Omega$, while $\Lambda_\varepsilon$ is the subset of $\Omega$ containing the parts from $\varepsilon^\beta(\xi + \overline{\widetilde{Y}}) \times I$ cells intersecting the boundary $\partial\Omega$. To define the partial unfolding operator with respect to the transverse vector $\tilde{x}$, we introduce the following two-scale mapping $t_\varepsilon\colon\ \Omega \times \widetilde{Y} \to \Omega$:

$$t_\varepsilon(x,\tilde{y}) = \begin{cases} \left(\varepsilon^\beta\left[\dfrac{\tilde{x}}{\varepsilon^\beta}\right]_{\widetilde{Y}} + \varepsilon^\beta\tilde{y}, x_3\right) & \text{if } (x,\tilde{y}) \in \widehat{\Omega}_\varepsilon \times \widetilde{Y},\\ x & \text{if } (x,\tilde{y}) \in \Lambda_\varepsilon \times \widetilde{Y}. \end{cases}$$

Now, we shall define the partial unfolding operator in the $\tilde{x}$-directions and of power $\beta$, which will be denoted $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$, that maps measurable functions defined in $\Omega$ into measurable functions in $\Omega \times \widetilde{Y}$.

**Definition 3.1.** Let $\varepsilon, \beta > 0$. For any function $\phi$ Lebesgue-measurable on $\Omega$, the partial unfolding operator $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$ is defined by

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)(x,\tilde{y}) = \phi(t_\varepsilon(x,\tilde{y})) \quad \text{for } (x,\tilde{y}) \in \Omega \times \widetilde{Y}. \tag{3.2}$$

We start by summarizing the most important properties of this unfolding operator $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$. The proofs of the majority of these results follow easily from the above definition and we shall omit them.

### 3.2. Fundamental properties of $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$.

**Proposition 3.2.**

(1) *The operator $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$ conserves the integral: for all $\phi \in L^1(\Omega)$ we have*

$$\int_{\Omega\times\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} = \int_\Omega \phi(x)\,\mathrm{d}x.$$

*Moreover, if $\phi \in L^2(\Omega)$, one has*

$$\|\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)\|_{L^2(\Omega\times\widetilde{Y})} = \|\phi\|_{L^2(\Omega)}.$$

(2) *The operator $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$ is linear and continuous from $L^2(\Omega)$ to $L^2(\Omega\times\widetilde{Y})$ and*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi\phi) = \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi) \quad \forall\, \psi,\phi \in L^2(\Omega).$$

(3)

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)\Big(x, \Big\{\frac{\tilde{x}}{\varepsilon^\beta}\Big\}\Big) = \psi(x) \quad \forall \psi \in L^2(\Omega).$$

### 3.3. Asymptotic properties of $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$.

**Proposition 3.3.**

(1) *Let $w_\varepsilon$ be a bounded sequence in $L^2(\Omega)$. Then there exists $w \in L^2(\Omega\times\widetilde{Y})$ such that*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \rightharpoonup w \quad \text{weakly in } L^2(\Omega\times\widetilde{Y}).$$

(2) *For $w \in L^2(\Omega)$ we have $\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w) \to w$ strongly in $L^2(\Omega\times\widetilde{Y})$.*

(3) *Let $w_\varepsilon$ be a sequence in $L^2(\Omega)$ such that $w_\varepsilon \to w$ strongly in $L^2(\Omega)$. Then*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \to w \quad \text{strongly in } L^2(\Omega\times\widetilde{Y}).$$

(4) *If* $w_\varepsilon \in L^2(\Omega)$ *satisfies* $\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \rightharpoonup \widehat{w}$ *weakly in* $L^2(\Omega \times \widetilde{Y})$, *then*

$$w_\varepsilon \rightharpoonup w = \int_{\widetilde{Y}} \widehat{w} \, \mathrm{d}\tilde{y} \quad \textit{weakly in } L^2(\Omega).$$

(5) *Let* $\{w_\varepsilon\} \subset L^2(\Omega)$ *such that* $\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \rightharpoonup u$, *weakly in* $L^2(\Omega \times \widetilde{Y})$, *and for every* $\{v_\varepsilon\} \subset L^2(\Omega)$ *with* $\mathcal{T}_{\tilde{x},\varepsilon^\beta}(v_\varepsilon) \to v$, *strongly in* $L^2(\Omega \times \widetilde{Y})$. *Then for all* $\phi \in \mathcal{D}(\Omega)$ *we have*

$$\int_\Omega w_\varepsilon(x) v_\varepsilon(x) \phi(x) \, \mathrm{d}x \to \int_\Omega \int_{\widetilde{Y}} u(x,\tilde{y}) v(x,\tilde{y}) \phi(x) \, \mathrm{d}x \, \mathrm{d}\tilde{y}.$$

Now, to achieve the asymptotic analysis of the solution of problem (2.1) we need the following technical lemma:

**Lemma 3.4.** *Let* $u^\varepsilon$ *be a bounded sequence in* $L^2(\Omega)$. *Then*

$$\int_\Omega u^\varepsilon(x) \chi_i^\varepsilon(x) \, \mathrm{d}x = \int_{\Omega \times \widetilde{Y}_i} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)(x,\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} + O(\varepsilon^\beta).$$

P r o o f. Firstly, we recall that $\Lambda_\varepsilon$ is the subset of $\Omega$ containing the parts from $\varepsilon^\beta(\xi + \overline{\widetilde{Y}}) \times I$ cells intersecting the boundary $\partial\Omega$. Since the number of cells in the boundary is of order $\varepsilon^{\beta - 2\beta}$ (see [7], page 35), we have $|\Lambda_\varepsilon| = O(\varepsilon^\beta)$. Secondly, we note that

$$\begin{aligned}
\int_\Omega u^\varepsilon(x) \chi_i^\varepsilon(x) \, \mathrm{d}x &= \int_{\Omega \times \widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)(x,\tilde{y}) \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\chi_i^\varepsilon)(x,\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} \\
&= \int_{\widehat{\Omega}_\varepsilon \times \widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)(x,\tilde{y}) \chi_i(\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} + \int_{\Lambda_\varepsilon \times \widetilde{Y}} u^\varepsilon(x) \chi_i^\varepsilon(x) \, \mathrm{d}x \, \mathrm{d}\tilde{y} \\
&= \int_{\Omega \times \widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)(x,\tilde{y}) \chi_i(\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} - \int_{\Lambda_\varepsilon \times \widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)(x,\tilde{y}) \chi_i(\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} \\
&\quad + \int_{\Lambda_\varepsilon \times \widetilde{Y}} u^\varepsilon(x) \chi_i^\varepsilon(x) \, \mathrm{d}x \, \mathrm{d}\tilde{y} \\
&= \int_{\Omega \times \widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)(x,\tilde{y}) \chi_i(\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} - \int_{\Lambda_\varepsilon \times \widetilde{Y}} u^\varepsilon(x) \chi_i(\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} \\
&\quad + \int_{\Lambda_\varepsilon} u^\varepsilon(x) \chi_i^\varepsilon(x) \, \mathrm{d}x.
\end{aligned}$$

On the other hand, $u^\varepsilon$ is bounded in $L^2(\Omega)$ and $\int_{\Lambda_\varepsilon} \chi_i^\varepsilon(x) \, \mathrm{d}x = O(\varepsilon^\beta)$, so we deduce that

$$-\int_{\Lambda_\varepsilon \times \widetilde{Y}} u^\varepsilon(x) \chi_i(\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y} + \int_{\Lambda_\varepsilon} u^\varepsilon(x) \chi_i^\varepsilon(x) \, \mathrm{d}x = O(\varepsilon^\beta),$$

which finishes the proof. □

**Corollary 3.5.** *Let $\psi \in L^2(\widetilde{Y})$ be a $\widetilde{Y}$-periodic function. Then for all $(x,\tilde{y}) \in \Omega \times \widetilde{Y}$ one has*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi_\varepsilon)(x,\tilde{y}) = \begin{cases} \psi(\tilde{y}) & \text{if } (x,\tilde{y}) \in \widehat{\Omega}_\varepsilon \times \widetilde{Y}, \\ \psi\Big(\dfrac{\tilde{x}}{\varepsilon^\beta}\Big) & \text{if } (x,\tilde{y}) \in \Lambda_\varepsilon \times \widetilde{Y}, \end{cases} \tag{3.3}$$

*and*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi_\varepsilon) \to \psi \quad \text{in } L^2(\Omega \times \widetilde{Y}). \tag{3.4}$$

P r o o f. From the definition of the partial unfolding operator we derive (3.3). The convergence in (3.4) follows by similar argument as in the proof of Lemma 3.4. □

**Remark 3.6.** For $\phi \in (L^2(\Omega))^N$, $\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)$ is defined componentwise:

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)(x,\tilde{y}) := \begin{pmatrix} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi_1)(x,\tilde{y}) \\ \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi_2)(x,\tilde{y}) \\ \vdots \\ \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi_N)(x,\tilde{y}) \end{pmatrix} \quad \forall\, (x,\tilde{y}) \in \Omega \times \widetilde{Y}.$$

**3.4. Partial unfolding of gradients.** Due to Definition 3.1 for any $w \in H^1(\Omega)$ we have

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla w) = \begin{cases} \begin{pmatrix} \dfrac{1}{\varepsilon^\beta} \nabla_{\tilde{y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(w) \\ \partial_{x_3} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(w) \end{pmatrix} & \text{on } \widehat{\Omega}_\varepsilon \times \widetilde{Y}, \\ \nabla w & \text{on } \Lambda_\varepsilon \times \widetilde{Y}. \end{cases} \tag{3.5}$$

In particular, one has obviously the following commutative property between $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$ and $\partial_{x_3}$:

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta} \circ \partial_{x_3} = \partial_{x_3} \circ \mathcal{T}_{\tilde{x},\varepsilon^\beta}. \tag{3.6}$$

**Theorem 3.7.** *Let $\{w_\varepsilon\}$ be a sequence in $H^1(\Omega)$ which converges weakly to $w$. Then there exists a subsequence still denoted $\{w_\varepsilon\}$ and there exists $\widehat{w}$ in $L^2(\Omega; H^1_{\rm per}(\widetilde{Y}))$ such that*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla w_\varepsilon) \rightharpoonup \begin{pmatrix} \nabla_{\tilde{x}} w + \nabla_{\tilde{y}} \widehat{w} \\ \partial_{x_3} w \end{pmatrix} \quad \text{in } L^2(\Omega \times \widetilde{Y})^3.$$

*Moreover,*

$$\mathcal{M}_{\widetilde{Y}}(\widehat{w}) := \int_{\widetilde{Y}} \widehat{w}\, \mathrm{d}\tilde{y} = 0.$$

*The space $H^1_{\rm per}(\widetilde{Y})$ is the closure with respect to the $H^1$-norm, of the space $\mathcal{C}^\infty_{\rm per}(\widetilde{Y})$ of infinitely differentiable $\widetilde{Y}$-periodic functions.*

P r o o f. The proof of this theorem follows along the same general lines as the proof of Theorem 3.5 in [6] and we shall omit it. □

**Corollary 3.8.** *Let* $\{w_\varepsilon\}$ *be a sequence in* $H^1(\Omega)$ *such that* $\{w_\varepsilon\}$ *is a bounded sequence in* $L^2(\Omega)$*. Moreover, we assume that*

$$\varepsilon\|\nabla_{\tilde{x}} w_\varepsilon\|_{L^2(\Omega)} \leqslant C, \quad \|\partial_{x_3} w_\varepsilon\|_{L^2(\Omega)} \leqslant C.$$

*Then there exists a subsequence* $\{w_\varepsilon\}$ *(still denoted* $w_\varepsilon$*) and*

$$w \in L^2(\Omega; H^1_{\text{per}}(\widetilde{Y})) \cap L^2(\widetilde{\Omega}; H^1(I))$$

*such that*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \rightharpoonup w(x,\tilde{y}),$$
$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3} w_\varepsilon) \rightharpoonup \partial_{x_3} w(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y}).$$

*Moreover,*

$$\varepsilon\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} w_\varepsilon) \rightharpoonup 0 \quad \text{and} \quad \varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} w_\varepsilon) \rightharpoonup \nabla_{\tilde{y}} w(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y})$$

*when* $\beta < 1$ *and* $\beta \geqslant 1$*, respectively.*

P r o o f. Since $\|w_\varepsilon\|_{L^2(\Omega)} \leqslant C$ and in view of Proposition 3.3, there exists a subsequence $\{w_\varepsilon\}$ and $w \in L^2(\Omega \times \widetilde{Y})$ such that

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \rightharpoonup w(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y}).$$

It follows from $\|\partial_{x_3} w_\varepsilon\|_{L^2(\Omega)} \leqslant C$ that there exists $u \in L^2(\Omega \times \widetilde{Y})$ such that

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3} w_\varepsilon) \rightharpoonup u(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y}).$$

From (3.6) we deduce that

$$w \in L^2(\Omega; H^1_{\text{per}}(\widetilde{Y})) \cap L^2(\widetilde{\Omega}; H^1(I)) \quad \text{and} \quad u(x,\tilde{y}) = \partial_{x_3} w(x,\tilde{y}).$$

Now, to derive the last convergence we note that there exists $\zeta \in L^2(\Omega \times \widetilde{Y})$ such that

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\varepsilon\nabla_{\tilde{x}} w_\varepsilon) \rightharpoonup \zeta(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y}),$$

since

$$\varepsilon\|\nabla_{\tilde{x}} w_\varepsilon\|_{L^2(\Omega)} \leqslant C.$$

In order to identify $\zeta$ we choose the test function $\psi(x) = \varphi(x)\psi(\tilde{x}/\varepsilon^\beta)$, where $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{\text{per}}(\widetilde{Y})$.

Since we have the conservation of the integral,

$$\int_{\Omega} \varepsilon \nabla_{\tilde{x}} u^{\varepsilon}(x)\varphi(x)\psi^{\varepsilon}(\tilde{x})\,\mathrm{d}x = \int_{\Omega\times\widetilde{Y}} \varepsilon \mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(\nabla_{\tilde{x}} u^{\varepsilon})\mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(\varphi)\mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(\psi^{\varepsilon})\,\mathrm{d}x\,\mathrm{d}\tilde{y}$$
$$\to \int_{\Omega\times\widetilde{Y}} \zeta(x,\tilde{y})\varphi(x)\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

After integration by parts, we get

$$\int_{\Omega} \varepsilon \nabla_{\tilde{x}} u^{\varepsilon}(x)\varphi(x)\psi^{\varepsilon}(\tilde{x})\,\mathrm{d}x = -\int_{\Omega} u^{\varepsilon}(x)\Big[\varepsilon \mathrm{div}_{\tilde{x}}\varphi(x)\psi^{\varepsilon}(\tilde{x}) + \frac{\varepsilon}{\varepsilon^{\beta}}\varphi(x)\mathrm{div}_{\tilde{y}}\psi^{\varepsilon}(\tilde{x})\Big]\,\mathrm{d}x.$$

Then, if $\beta < 1$ and $\varepsilon$ tends to zero, we obtain

$$0 = \int_{\Omega\times\widetilde{Y}} \zeta(x,\tilde{y})\varphi(x)\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

Thus

$$\zeta = 0 \quad \text{and} \quad \varepsilon \mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(\nabla_{\tilde{x}} w_{\varepsilon}) \rightharpoonup 0.$$

Finally, if $\beta \geqslant 1$, then

$$\varepsilon^{\beta}\|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \leqslant \varepsilon\|\nabla w_{\varepsilon}\|_{L^2(\Omega)} \leqslant C,$$

moreover,

$$\varepsilon^{\beta}\mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(\nabla w_{\varepsilon}) = \nabla_{\tilde{y}}(\mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(w)).$$

By a similar argument as above, one has

$$\varepsilon^{\beta}\mathcal{T}_{\tilde{x},\varepsilon^{\beta}}(\nabla_{\tilde{x}} w_{\varepsilon}) \rightharpoonup \nabla_{\tilde{y}} w(x,\tilde{y}) \quad \text{in } L^2(\Omega\times\widetilde{Y}).$$

This completes the proof. □

**3.5. The partial averaging operator.** We shall now introduce the following partial averaging operator.

**Definition 3.9.** The partial averaging operator $\mathcal{U}_{\tilde{x},\varepsilon^{\beta}}\colon L^2(\Omega\times\widetilde{Y}) \mapsto L^2(\Omega)$ is defined as

$$\mathcal{U}_{\tilde{x},\varepsilon^{\beta}}(\phi)(x) = \begin{cases} \displaystyle\int_{\widetilde{Y}} \phi\Big(\Big(\varepsilon^{\beta}\Big[\frac{\tilde{x}}{\varepsilon^{\beta}}\Big]_{\widetilde{Y}} + \varepsilon^{\beta}\tilde{z}, x_3\Big), \Big\{\frac{\tilde{x}}{\varepsilon^{\beta}}\Big\}_{\widetilde{Y}}\Big)\,\mathrm{d}\tilde{z} & \text{for } x \in \widehat{\Omega}_{\varepsilon}, \\ \displaystyle\int_{\widetilde{Y}} \phi(x,\tilde{y})\,\mathrm{d}\tilde{y} & \text{for } x \in \Lambda_{\varepsilon}. \end{cases}$$

Let us first note that we have the following duality-like formula between $\mathcal{U}_{\tilde{x},\varepsilon^\beta}$ and $\mathcal{T}_{\tilde{x},\varepsilon^\beta}$. Indeed, for any $\psi \in L^2(\Omega)$, $\phi \in L^2(\Omega \times \widetilde{Y})$ one has

$$\int_{\Omega\times\widetilde{Y}} \phi(x,\tilde{y})\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} = \int_\Omega \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)(x)\psi(x)\,\mathrm{d}x. \tag{3.7}$$

Using successively the change of variables $\tilde{x} = \varepsilon^\beta\xi + \varepsilon^\beta\tilde{z}$ and $\tilde{x} = \varepsilon^\beta\xi + \varepsilon^\beta\tilde{y}$ one gets

$$\begin{aligned}
&\int_{\Omega\times\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)(x,\tilde{y})\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\
&\quad= \int_{\widehat{\Omega}_\varepsilon\times\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)(x,\tilde{y})\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} + \int_{\Lambda_\varepsilon\times\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)(x,\tilde{y})\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\
&\quad= \sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon^\beta(\xi+\widetilde{Y})\times I\times\widetilde{Y}} \psi(\varepsilon^\beta\xi+\varepsilon^\beta\tilde{y},x_3)\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} + \int_{\Lambda_\varepsilon\times\widetilde{Y}} \psi(x)\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\
&\quad= \varepsilon^{2\beta}\sum_{\xi\in\Xi_\varepsilon}\int_{\widetilde{Y}\times I\times\widetilde{Y}} \psi(\varepsilon^\beta\xi+\varepsilon^\beta\tilde{y},x_3)\phi((\varepsilon^\beta\xi+\varepsilon^\beta\tilde{y},x_3),\tilde{z})\,\mathrm{d}x_3\,\mathrm{d}\tilde{z}\,\mathrm{d}\tilde{y}\\
&\qquad+ \int_{\Lambda_\varepsilon\times\widetilde{Y}} \psi(x)\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\
&\quad= \sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+\widetilde{Y})\times I\times\widetilde{Y}} \psi(x)\phi\Big(\Big(\varepsilon\Big[\frac{\tilde{x}}{\varepsilon}\Big]_{\widetilde{Y}}+\varepsilon\tilde{z},x_3\Big),\Big\{\frac{\tilde{x}}{\varepsilon}\Big\}_{\widetilde{Y}}\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{z}\\
&\qquad+ \int_{\Lambda_\varepsilon\times\widetilde{Y}} \psi(x)\phi(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\
&\quad= \int_{\widehat{\Omega}_\varepsilon} \psi(x)\Big(\int_{\widetilde{Y}}\phi\Big(\Big(\varepsilon^\beta\Big[\frac{\tilde{x}}{\varepsilon^\beta}\Big]_{\widetilde{Y}}+\varepsilon^\beta\tilde{z},x_3\Big),\Big\{\frac{\tilde{x}}{\varepsilon}\Big\}_{\widetilde{Y}}\Big)\,\mathrm{d}\tilde{z}\Big)\,\mathrm{d}x\\
&\qquad gr + \int_{\Lambda_\varepsilon}\psi(x)\Big(\int_{\widetilde{Y}}\phi(x,\tilde{y})\,\mathrm{d}\tilde{y}\Big)\,\mathrm{d}x = \int_\Omega \psi(x)\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)(x)\,\mathrm{d}x.
\end{aligned}$$

This establishes the above formula (3.7).

**Proposition 3.10.** *The operator $\mathcal{U}_{\tilde{x},\varepsilon^\beta}$ is linear and continuous from $L^2(\Omega\times\widetilde{Y})$ to $L^2(\Omega)$ and satisfies the following properties:*

(1) $\|\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)\|_{L^2(\Omega)} \leqslant \|\phi\|_{L^2(\Omega\times\widetilde{Y})}$.

(2) $\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi))(x) = \phi(x)\ \forall\, x\in\Omega$.

(3)

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi))(x,\tilde{y}) = \begin{cases}\displaystyle\int_{\widetilde{Y}}\phi\Big(\Big(\varepsilon^\beta\Big[\frac{\tilde{x}}{\varepsilon^\beta}\Big]_{\widetilde{Y}}+\varepsilon^\beta\tilde{z},x_3\Big),\tilde{y}\Big)\,\mathrm{d}\tilde{z} & \textit{if } (x,\tilde{y})\in\widehat{\Omega}_\varepsilon\times\widetilde{Y},\\ \displaystyle\int_{\widetilde{Y}}\phi(x,\tilde{y})\,\mathrm{d}\tilde{y} & \textit{if } (x,\tilde{y})\in\Lambda_\varepsilon\times\widetilde{Y}.\end{cases}$$

(4) *Let* $(w_\varepsilon)$ *be a sequence in* $L^2(\Omega \times \widetilde{Y})$ *such that* $w_\varepsilon \to w$ *strongly in* $L^2(\Omega \times \widetilde{Y})$. *Then*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathcal{U}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon)) \to w \text{ strongly in } L^2(\Omega \times \widetilde{Y}).$$

(5) *Let* $(w_\varepsilon)$ *be a sequence in* $L^2(\Omega)$. *Then*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \longrightarrow \widehat{w} \text{ in } L^2(\Omega \times \widetilde{Y}) \Leftrightarrow w_\varepsilon - \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\widehat{w}) \longrightarrow 0 \text{ in } L^2(\Omega).$$

Proof. For item (1) let $\psi \in L^2(\Omega)$, $\phi \in L^2(\Omega \times \widetilde{Y})$. One has

$$\int_{\Omega \times \widetilde{Y}} \phi(x,\tilde{y}) \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi)(x,\tilde{y}) \,\mathrm{d}x \,\mathrm{d}\tilde{y} = \int_\Omega \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)(x)\psi(x)\,\mathrm{d}x.$$

Now, we take $\psi(x) = \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)(x)$, then by using *Hölder* inequality we obtain

$$\begin{aligned}
\left|\int_\Omega (\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi))^2(x)\,\mathrm{d}x\right| &= \left|\int_{\Omega\times\widetilde{Y}} \phi(x,\tilde{y})\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi))(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\right|,\\
\|\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)\|^2_{L^2(\Omega)} &\leqslant \|\phi\|_{L^2(\Omega\times\widetilde{Y})}\|\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi))\|_{L^2(\Omega\times\widetilde{Y})}\\
&= \|\phi\|_{L^2(\Omega\times\widetilde{Y})}\|\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)\|_{L^2(\Omega)},\\
\|\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\phi)\|_{L^2(\Omega)} &\leqslant \|\phi\|_{L^2(\Omega\times\widetilde{Y})}.
\end{aligned}$$

For item (2) we observe that

$$\begin{aligned}
\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi))(x) &= \begin{cases} \displaystyle\int_{\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)\Big(\Big(\varepsilon^\beta\Big[\frac{\tilde{x}}{\varepsilon^\beta}\Big]_{\widetilde{Y}} + \varepsilon^\beta\tilde{z}, x_3\Big), \Big\{\frac{\tilde{x}}{\varepsilon^\beta}\Big\}_{\widetilde{Y}}\Big)\,\mathrm{d}\tilde{z} & \text{if } x \in \widehat{\Omega}_\varepsilon,\\ \displaystyle\int_{\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi)(x,\tilde{y})\,\mathrm{d}\tilde{y} & \text{if } x \in \Lambda_\varepsilon, \end{cases}\\
&= \begin{cases} \displaystyle\int_{\widetilde{Y}} \phi\Big(\varepsilon^\beta\Big[\frac{\tilde{x}}{\varepsilon^\beta}\Big]_{\widetilde{Y}} + \varepsilon^\beta\Big\{\frac{\tilde{x}}{\varepsilon^\beta}\Big\}_{\widetilde{Y}}, x_3\Big)\,\mathrm{d}\tilde{z} & \text{if } x \in \widehat{\Omega}_\varepsilon,\\ \displaystyle\int_{\widetilde{Y}} \phi(x)\,\mathrm{d}\tilde{y} & \text{if } x \in \Lambda_\varepsilon, \end{cases}
\end{aligned}$$

hence, $\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi))(x) = \phi(x)\ \forall\, x \in \Omega$.

Item (3) is a direct consequence of the definition of the partial operator. Item (4) follows from Proposition 3.3. Finally, let us prove the "if" part of item (5). First, from Proposition 3.3 we deduce

$$\begin{aligned}
\|w_\varepsilon - \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\hat{w})\|_{L^2(\Omega)} &= \|\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon)) - \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\hat{w})\|_{L^2(\Omega)}\\
&= \|\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) - \hat{w})\|_{L^2(\Omega)}\\
&\leqslant \|\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) - \hat{w}\|_{L^2(\Omega\times\widetilde{Y})} \to 0.
\end{aligned}$$

Conversely, one has

$$w_\varepsilon - \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\hat{w}) \to 0 \quad \text{in } L^2(\Omega),$$

then

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon - \mathcal{U}_{\tilde{x},\varepsilon^\beta}(\hat{w})) \to 0 \quad \text{in } L^2(\Omega \times \widetilde{Y})$$

and consequently

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(w_\varepsilon) \to \hat{w} \quad \text{in } L^2(\Omega \times \widetilde{Y}).$$

□

**Remark 3.11.** An important feature of this partial averaging operator is that we have for any $\phi \in L^2(\Omega)$ the following identity: $\mathcal{U}_{\tilde{x},\varepsilon^\beta}(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi))(x) = \phi(x)$ which holds true not only in $\widehat{\Omega}_\varepsilon$ but for every $x \in \Omega$.

In the following section we state the main results of this paper.

## 4. Main results

The asymptotic behavior of the solution $u^\varepsilon$ is characterized by a critical power of the scale $\varepsilon^\beta$, namely $\beta = 1$. In order to describe the main results of the present work it is convenient to split them up into three different scales.

- ▷ Supercritical scale: $\beta > 1$.
- ▷ Critical scale: $\beta = 1$.
- ▷ Subcritical scale: $\beta < 1$.

First, let us introduce function

$$s_2 = (s_2^1, s_2^2, s_2^3) \in (H^1_{\text{per}}(\widetilde{Y}))^3,$$

where for $k = 1, 2, 3$, $s_2^k$ is the unique solution of the so-called cellular problem

$$\begin{aligned} &-\operatorname{div}_{\tilde{y}}[a_2(\tilde{y})[\nabla_{\tilde{y}} s_2^k(\tilde{y}) + e_k]] = 0 \quad \text{in } \widetilde{Y}_2,\\ &a_2(\tilde{y})[\nabla_{\tilde{y}} s_2^k(\tilde{y}) + e_k] n_2(\tilde{y}) = 0 \quad \text{on } \widetilde{Y}_{23},\\ &a_2(\tilde{y}) \nabla_{\tilde{y}} s_2^k n_2(\tilde{y})_{|\partial \widetilde{Y}_2 \cap \partial \widetilde{Y}} \quad \widetilde{Y}\text{-periodic.} \end{aligned} \tag{4.1}$$

Now, we define the following homogenized matrix:

$$A_{kj}^{2,\text{hom}} = \int_{\widetilde{Y}_2} a_2(\tilde{y})[e_k + \nabla_{\tilde{y}} s_2^k(\tilde{y})][e_j + \nabla_{\tilde{y}} s_2^j(\tilde{y})]\, \mathrm{d}\tilde{y}, \quad k, j = 1, 2, 3.$$

*1. Supercritical scale*: $\beta > 1$.

**Theorem 4.1.** *Let* $u^\varepsilon$ *be the solution of* (2.1)*. Then, there exists*

$$(u, \tilde{v}_2) \in H^1_D(\Omega) \times L^2(\Omega; H^1_{\text{per}}(\widetilde{Y}_2)/\mathbb{R})$$

*such that*

$$u^\varepsilon(x) \rightharpoonup u(x) \quad \textit{in } L^2(\Omega)$$

*and* $(u, \tilde{v}_2)$ *is the unique solution of the homogenized problem*

$$\begin{aligned}
(4.2) \qquad & -\operatorname{div}_x(A^{2,\text{hom}}\nabla u) - a^{1,\text{hom}}\partial^2_{x_3} u = f \quad \textit{in } \Omega,\\
& \partial_{x_3} u(\tilde{x}, 1) = 0, \quad \tilde{x} \in \widetilde{\Omega},\\
& a_2(\tilde{y})(\nabla u(\tilde{x}, 1) + \nabla_{\tilde{y}} \tilde{v}_2((\tilde{x}, 1), \tilde{y})) = 0, \quad \tilde{x} \in \widetilde{\Omega}, \text{ a.e. } \tilde{y} \in \widetilde{Y}_2,\\
& a^{1,\text{hom}} = \int_{\widetilde{Y}_1} a_1(\tilde{y})\,\mathrm{d}\tilde{y},\\
& \tilde{v}_2(x, \tilde{y}) = \sum_{k=1}^{3} \partial_{x_k} u(x) s^k_2(\tilde{y}).
\end{aligned}$$

*2. Critical scale*: $\beta = 1$.

**Theorem 4.2.** *Let* $u^\varepsilon$ *be the solution of* (2.1)*. Then there exists*

$$(u_2, \tilde{v}_1, \tilde{v}_2) \in H^1_D(\Omega) \times [L^2(\Omega; H^1(\widetilde{Y}_1)/\mathbb{R}) \cap L^2(\widetilde{\Omega}; H^1(I))] \times L^2(\Omega; H^1_{\text{per}}(\widetilde{Y}_2)/\mathbb{R}),$$
$$w_3 \in L^2(\Omega; H^1(\widetilde{Y}_3)/\mathbb{R})$$

*such that*

$$u^\varepsilon \rightharpoonup \int_{\widetilde{Y}_1} \tilde{v}_1(x, \tilde{y})\,\mathrm{d}\tilde{y} + (1 - |\widetilde{Y}_1|)u_2(x) + \int_{\widetilde{Y}_3} w_3(x, \tilde{y})\,\mathrm{d}\tilde{y} \quad \textit{weakly in } L^2(\Omega)$$

*and* $(u_2, \tilde{v}_1, \tilde{v}_2, w_3)$ *is the unique solution of the two-scale homogenized problem*

$$\begin{aligned}
(4.3) \qquad & -\operatorname{div}_x(A^{2,\text{hom}}\nabla u_2) - \int_{\widetilde{Y}_{23}} a_3(\tilde{y})\nabla_{\tilde{y}} w_3(x, \tilde{y}) n_3(\tilde{y})\,\mathrm{d}s(\tilde{y}) = |\widetilde{Y}_2| f \quad \textit{in } \Omega,\\
& a_2(\tilde{y})(\nabla u_2(\tilde{x}, 1) + \nabla_{\tilde{y}} \tilde{v}_2((\tilde{x}, 1), \tilde{y})) = 0 \quad \tilde{x} \in \widetilde{\Omega}, \text{ a.e. } \tilde{y} \in \widetilde{Y}_2,\\
& \tilde{v}_2(x, \tilde{y}) = \sum_{k=1}^{3} \partial_{x_k} u_2(x) s^k_2(\tilde{y})
\end{aligned}$$

$$
(4.4) \qquad \begin{aligned} &-\operatorname{div}_{\tilde{y}}[a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})] - a_1(\tilde{y})\partial^2_{x_3}\tilde{v}_1(x,\tilde{y}) = f(x) \quad \text{in } \widetilde{Y}_1,\\ &a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})n_1(\tilde{y}) = a_3(\tilde{y})\nabla_{\tilde{y}}w_3(x,\tilde{y})n_1(\tilde{y}) \quad \text{on } \widetilde{Y}_{13},\\ &\partial_{x_3}\tilde{v}_1(\tilde{x},1,\tilde{y}) = 0, \quad \tilde{x}\in\widetilde{\Omega}, \text{ a.e. } \tilde{y}\in\widetilde{Y}_2 \end{aligned}
$$

$$
(4.5) \qquad \begin{aligned} &-\operatorname{div}_{\tilde{y}}[a_3(\tilde{y})\nabla_{\tilde{y}}w_3(x,\tilde{y})] = f(x) \quad \text{in } \widetilde{Y}_3,\\ &w_3(x,\tilde{y}) = 0 \quad \text{on } \widetilde{Y}_{23},\\ &w_3(x,\tilde{y}) = \tilde{v}_1(x,\tilde{y}) - u_2(x) \quad \text{on } \widetilde{Y}_{13}. \end{aligned}
$$

*3. Subcritical scale*: $\beta < 1$.

**Theorem 4.3.** *Let $u^\varepsilon$ be the solution of* (2.1). *Then there exists*

$$
(u_2,\tilde{v}_1,\tilde{v}_2) \in H^1_D(\Omega)\times[L^2(\Omega;H^1(\widetilde{Y}_1)/\mathbb{R})\cap L^2(\widetilde{\Omega};H^1(I))]\times L^2(\Omega;H^1_{\mathrm{per}}(\widetilde{Y}_2)/\mathbb{R})
$$

*such that*

$$
\begin{aligned} &u^\varepsilon(x)\chi_1\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big) \rightharpoonup \int_{\widetilde{Y}_1}\tilde{v}_1(x,\tilde{y})\,\mathrm{d}\tilde{y} \quad \text{weakly in } L^2(\Omega),\\ &u^\varepsilon(x)\chi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big) \rightharpoonup |\widetilde{Y}_2|u_2(x) \quad \text{weakly in } L^2(\Omega) \end{aligned}
$$

*and $(u_2,\tilde{v}_1,\tilde{v}_2)$ is the unique solution of the two-scale homogenized problem*

$$
(4.6) \qquad \begin{aligned} &-\operatorname{div}_x(A^{2,\mathrm{hom}}\nabla u_2) - \partial^2_{x_3}\int_{\widetilde{Y}_1} a_1(\tilde{y})\tilde{v}_1(x,\tilde{y})\,\mathrm{d}\tilde{y} = (1-|\widetilde{Y}_3|)f(x) \quad \text{in } \Omega,\\ &a_2(\tilde{y})(\nabla u_2(\tilde{x},1) + \nabla_{\tilde{y}}\tilde{v}_2((\tilde{x},1),\tilde{y})) = 0 \quad \tilde{x}\in\widetilde{\Omega}, \text{ a.e. } \tilde{y}\in\widetilde{Y}_2,\\ &\tilde{v}_2(x,\tilde{y}) = \sum_{k=1}^{3}\partial_{x_k}u_2(x)s_2^k(\tilde{y}), \end{aligned}
$$

$$
(4.7) \qquad \begin{aligned} &-a_1(\tilde{y})\partial^2_{x_3}\tilde{v}_1(x,\tilde{y}) = f(x) \quad \text{in } \Omega, \text{ a.e. } \tilde{y}\in\widetilde{Y}_1,\\ &\partial_{x_3}\tilde{v}_1(\tilde{x},1,\tilde{y}) = 0, \quad \tilde{x}\in\widetilde{\Omega}, \text{ a.e. } \tilde{y}\in\widetilde{Y}_1. \end{aligned}
$$

## 5. Proofs of the main results

In this section, we shall prove the main results stated in Section 4. Let us now begin with:

**5.1. Supercritical scale.** Before proving theorem 4.1, we first establish the following lemma which gives some convergence results.

**Lemma 5.1.** *Let $u^\varepsilon$ be the solution of problem* (2.1). *Then there exist $u \in H^1_D(\Omega)$ and $\tilde{v}_2 \in L^2(\Omega; H^1_{\rm per}(\widetilde{Y}_2)/\mathbb{R})$ such that:*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon) \rightharpoonup u(x) \quad \text{in } L^2(\Omega \times \widetilde{Y}), \tag{5.1}$$

$$u^\varepsilon \rightharpoonup u(x) \quad \text{in } L^2(\Omega), \tag{5.2}$$

$$\varepsilon^\beta \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} u^\varepsilon_{|\Omega_1^\varepsilon}) \to 0 \quad \text{in } L^2(\Omega \times \widetilde{Y}_1), \tag{5.3}$$

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla u^\varepsilon_{|\Omega_2^\varepsilon}) \rightharpoonup \begin{pmatrix} \nabla_{\tilde{x}} u(x) + \nabla_{\tilde{y}} \tilde{v}_2(x,\tilde{y}) \\ \partial_{x_3} u(x) \end{pmatrix} \quad \text{in } L^2(\Omega \times \widetilde{Y}_2), \tag{5.4}$$

$$\varepsilon^\beta \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla u^\varepsilon{}_{|\Omega_3^\varepsilon}) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } L^2(\Omega \times \widetilde{Y}_3), \tag{5.5}$$

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3} u^\varepsilon_{|\Omega_1^\varepsilon}) \rightharpoonup \partial_{x_3} u(x) \quad \text{in } L^2(\Omega \times \widetilde{Y}_1). \tag{5.6}$$

P r o o f. First, we note that

$$\varepsilon^\beta \nabla_{\tilde{x}} u^\varepsilon(x)_{|\Omega_1^\varepsilon} = \frac{\varepsilon^\beta}{\varepsilon} \varepsilon \nabla_{\tilde{x}} u^\varepsilon(x)_{|\Omega_1^\varepsilon},$$

using the boundedness of $\varepsilon \|\nabla_{\tilde{x}} u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leqslant C$, we deduce that $\varepsilon^\beta \nabla_{\tilde{x}} u^\varepsilon(x)_{|\Omega_1^\varepsilon}$ converges strongly to zero in $L^2(\Omega)$. Thus $\varepsilon^\beta \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} u^\varepsilon{}_{|\Omega_1^\varepsilon}) \to 0$ in $L^2(\Omega \times \widetilde{Y}_1)$. From

$$\varepsilon^\beta \|\nabla_{\tilde{x}} u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leqslant \varepsilon \|\nabla_{\tilde{x}} u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leqslant C$$

and Corollary 3.8 there exists

$$\tilde{v}_1 \in L^2(\Omega; H^1(\widetilde{Y}_1))$$

such that

$$\varepsilon^\beta \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} u^\varepsilon_{|\Omega_1^\varepsilon}) \rightharpoonup \nabla_{\tilde{y}} \tilde{v}_1(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y}_1).$$

Thus $\nabla_{\tilde{y}} \tilde{v}_1(x,\tilde{y}) = 0$, then $\tilde{v}_1(x,\tilde{y}) = u_1(x)$. As before, we deduce moreover the existence of

$$w \in L^2(\Omega; H^1(\widetilde{Y}_3))$$

such that

$$\varepsilon^\beta \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} u^\varepsilon{}_{|\Omega_3^\varepsilon}) \rightharpoonup \nabla_{\tilde{y}} w(x,\tilde{y}) \quad \text{in } L^2(\Omega \times \widetilde{Y}_3),$$

thus $\nabla_{\tilde{y}} w(x,\tilde{y}) = 0$. This proves (5.3)–(5.5).

On the other hand, for all $\phi^\varepsilon(x) = \varphi(x)\psi^\varepsilon(\tilde{x})$, where $\varphi \in \mathcal{D}(\Omega)$, $\psi \in H^1_{\rm per}(\widetilde{Y})$ and by using Proposition 3.2 one has

$$\begin{aligned} \int_\Omega \varepsilon^\beta \nabla_{\tilde{x}} u^\varepsilon(x) \varphi \psi^\varepsilon(\tilde{x}) \, \mathrm{d}x \\ = \int_{\Omega \times \widetilde{Y}} \varepsilon^\beta \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}} u^\varepsilon) \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\varphi)(x,\tilde{y}) \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi^\varepsilon)(x,\tilde{y}) \, \mathrm{d}x \, \mathrm{d}\tilde{y}. \end{aligned} \tag{5.7}$$

It follows from Corollary 3.5 and Corollary 3.8 that

$$\int_{\Omega\times\widetilde{Y}_1} \nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})\varphi(x)\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} + \int_{\Omega\times\widetilde{Y}_3} \nabla_{\tilde{y}}w(x,\tilde{y})\varphi(x)\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} = 0. \tag{5.8}$$

By integration by parts, we deduce

$$\begin{aligned}\int_\Omega \varepsilon^\beta\nabla_{\tilde{x}}u^\varepsilon(x)\varphi(x)\psi^\varepsilon(\tilde{x})\,\mathrm{d}x &= -\int_\Omega u^\varepsilon(x)[\varepsilon^\beta \mathrm{div}_{\tilde{x}}\varphi(x)\psi^\varepsilon(\tilde{x}) + \varphi(x)\mathrm{div}_{\tilde{y}}\psi^\varepsilon(\tilde{x})]\,\mathrm{d}x\\ &= -\int_{\Omega\times\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon)[\varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathrm{div}_{\tilde{x}}\varphi)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\psi^\varepsilon) + \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\varphi)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathrm{div}_{\tilde{y}}\psi^\varepsilon)]\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &=: I_1^\varepsilon + I_2^\varepsilon.\end{aligned}$$

Then passing to the limit and using the fact that $\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\mathrm{div}_{\tilde{y}}\psi^\varepsilon) \to \mathrm{div}_{\tilde{y}}\psi(\tilde{y})$ in $L^2(\Omega\times\widetilde{Y})$ and integrating by parts, we obtain:

$$\begin{aligned} I_1^\varepsilon &\to 0,\\ I_2^\varepsilon \to I_2 = &-\int_{\Omega\times\widetilde{Y}_{13}} u_1(x)\varphi(x)\psi(\tilde{y})n_1(\tilde{y})\,\mathrm{d}s(\tilde{y})\,\mathrm{d}x\\ &+\int_{\Omega\times\widetilde{Y}_{23}} u_2(x)\varphi(x)\psi(\tilde{y})n_2(\tilde{y})\,\mathrm{d}s(\tilde{y})\,\mathrm{d}x\\ &-\int_{\Omega\times(\partial\widetilde{Y}_2\cap\partial\widetilde{Y})} u_2(x)\varphi(x)\psi(\tilde{y})n_2(\tilde{y})\,\mathrm{d}s(\tilde{y})\,\mathrm{d}x\\ &+\int_{\Omega\times\widetilde{Y}_{13}} w(x)\varphi(x)\psi(\tilde{y})n_1(\tilde{y})\,\mathrm{d}s(\tilde{y})\,\mathrm{d}x\\ &-\int_{\Omega\times\widetilde{Y}_{23}} w(x)\varphi(x)\psi(\tilde{y})n_2(\tilde{y})\,\mathrm{d}s(\tilde{y})\,\mathrm{d}x.\end{aligned} \tag{5.9}$$

Finally, we set $w_3 = w - u_2$, we deduce

$$\begin{aligned} w_3(x,\tilde{y}) &= 0, & \tilde{y}&\in\widetilde{Y}_{23},\\ w_3(x,\tilde{y}) &= u_1(x) - u_2(x), & \tilde{y}&\in\widetilde{Y}_{13}.\end{aligned}$$

Since $\nabla_{\tilde{y}}w(x,\tilde{y}) = 0$, $\nabla_{\tilde{y}}w_3(x,\tilde{y}) = 0$ and $w_3(x,\tilde{y})$ does not depend on $\tilde{y}$ ($\widetilde{Y}_3$ is connected), as a consequence $w_3(x,\tilde{y}) = 0$ and $u_1(x) = u_2(x) = u(x)$. Convergences (5.1), (5.4) and (5.6) are proved from Corollary 3.5 and Theorem 3.7, from Proposition 3.3 we get immediately (5.2). □

Proof of Theorem 4.1. Let $\mathcal{D}^1_D(\bar{\Omega}) := \{\phi\in\mathcal{D}(\bar{\Omega}),\ \phi = 0 \text{ on } \Gamma_D\}$.
We choose as test function $\psi^\varepsilon(x) = \varphi(x) + \varepsilon^\beta\phi_2(x)\Psi_2(\tilde{x}/\varepsilon^\beta)$ in (2.1), where

$$\phi_2\in\mathcal{D}(\Omega),\quad \varphi\in\mathcal{D}^1_D(\bar{\Omega}),\quad \Psi_2\in H^1_{\mathrm{per}}(\widetilde{Y}).$$

We split the integral over $\Omega$ into integrals over $\Omega_i^\varepsilon$, $i = 1, 2, 3$, and from Lemma 3.4, we obtain:

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\nabla\psi^\varepsilon(x)\,\mathrm{d}x =: I_{11}^\varepsilon + I_{12}^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + O(\varepsilon^\beta),$$

where

$$\begin{aligned}
I_{11}^\varepsilon = \int_{\Omega\times\widetilde{Y}_1} & \mathcal{T}_{\tilde{x},\varepsilon^\beta}(a_1^\varepsilon)\varepsilon\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}}u^\varepsilon)\ \varepsilon\Big(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}}\varphi) + \varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}}\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}\Big(\Psi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)\Big) \\
&+ \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}\Big(\nabla_{\tilde{y}}\Psi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y}, \\
I_{12}^\varepsilon = \int_{\Omega\times\widetilde{Y}_1} & \mathcal{T}_{\tilde{x},\varepsilon^\beta}(a_1^\varepsilon)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3}u^\varepsilon)(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3}\varphi) + \varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3}\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\Psi_2))\,\mathrm{d}x\,\mathrm{d}\tilde{y}, \\
I_2^\varepsilon = \int_{\Omega\times\widetilde{Y}_2} & \mathcal{T}_{\tilde{x},\varepsilon^\beta}(a_2^\varepsilon)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla u^\varepsilon)\Big(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla\varphi) + \varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}\Big(\Psi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)\Big) \\
&+ \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}\Big(\nabla_{\tilde{y}}\Psi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y}, \\
I_3^\varepsilon = \int_{\Omega\times\widetilde{Y}_3} & \mathcal{T}_{\tilde{x},\varepsilon^\beta}(a_3^\varepsilon)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\varepsilon\nabla u^\varepsilon)\varepsilon\Big(\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla\varphi) + \varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\Psi_2) \\
&+ \mathcal{T}_{\tilde{x},\varepsilon^\beta}(\phi_2)\mathcal{T}_{\tilde{x},\varepsilon^\beta}\Big(\nabla_{\tilde{y}}\Psi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y}.
\end{aligned}$$

Now, we can pass to the limit in each term. For $I_{11}^\varepsilon$, we use the boundedness of $\varepsilon\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}}u^\varepsilon(x))$, Proposition 3.3, Proposition 3.2 and Corollary 3.5 and we obtain

$$I_{11}^\varepsilon \to 0, \quad I_3^\varepsilon \to 0.$$

For $I_{12}^\varepsilon$, in view of (5.6), Proposition 3.3, Proposition 3.2 and Corollary 3.5 , we have

$$I_{12}^\varepsilon \to \int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}u(x)\partial_{x_3}\varphi(x)\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

And for $I_2^\varepsilon$ by using Theorem 3.7 and (5.4) we derive

$$I_2^\varepsilon \to \int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla\varphi(x) + \phi_2(x)\nabla_{\tilde{y}}\Psi_2(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

Thus, we obtain the following variational problem:

$$\begin{aligned}
(5.10)\quad & \int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}u(x)\partial_{x_3}\varphi(x)\,\mathrm{d}x\,\mathrm{d}\tilde{y} \\
&\quad + \int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla\varphi(x) + \phi_2(x)\nabla_{\tilde{y}}\Psi_2(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y} \\
&\quad = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x.
\end{aligned}$$

By the Lax-Milgram Theorem, the above equation has a unique solution in

$$H_D^1(\Omega) \times L^2(\Omega; H^1_{\mathrm{per}}(\widetilde{Y}_2)/\mathbb{R}).$$

We are now ready to give the homogenized version of the above limit problem. Firstly, we take $\varphi = 0$ in (5.10) and integrating by parts we obtain the following system:

$$\begin{aligned}
&-\operatorname{div}_{\tilde y}[a_2(\tilde y)(\nabla u(x) + \nabla_{\tilde y}\tilde v_2(x,\tilde y))] = 0 && \text{in } \widetilde{Y}_2,\\
&a_2(\tilde y)(\nabla u(x) + \nabla_{\tilde y}\tilde v_2(x,\tilde y))n_2(\tilde y) = 0 && \text{on } \widetilde{Y}_{23},\\
&a_2(\tilde y)(\nabla u(x) + \nabla_{\tilde y}\tilde v_2(x,\tilde y))n_2(\tilde y)_{|\partial\widetilde{Y}_2\cap\partial\widetilde{Y}} && \widetilde{Y}\text{-periodic}.
\end{aligned}$$

Secondly, we take $\psi_2 = 0$ in (5.10) and integrating by parts, we get for an arbitrary $\varphi \in H_D^1(\Omega)$:

$$\begin{aligned}
&-\operatorname{div}_x\left(\int_{\widetilde{Y}_2} a_2(\tilde y)(\nabla u + \nabla_{\tilde y}\tilde v_2)\,\mathrm{d}\tilde y\right) - \int_{\widetilde{Y}_1} a_1(\tilde y)\partial^2_{x_3}u(x)\,\mathrm{d}\tilde y = f(x),\\
&\partial_{x_3}u(\tilde x, 1) = 0, \quad \tilde x \in \widetilde{\Omega},\\
&a_2(\tilde y)(\nabla u(\tilde x,1) + \nabla_{\tilde y}\tilde v_2((x,1),\tilde y)) = 0, \quad \tilde x \in \widetilde{\Omega},\ \text{a.e. } \tilde y \in \widetilde{Y}_2.
\end{aligned}$$

Finally, we decouple the macroscopic and microscopic variables $(x, \tilde y)$ by taking $\tilde v_2(x,\tilde y) = \sum_{k=1}^{3} \partial_{x_k}u(x)s_2^k(\tilde y)$, where $s_2^k$ is defined by (4.1).

$$\begin{aligned}
-\operatorname{div}_x&\left(\int_{\widetilde{Y}_2} a_2(\tilde y)(\nabla u + \sum_{k=1}^{3}\partial_{x_k}u(x)\nabla_{\tilde y}s_2^k(\tilde y))\,\mathrm{d}\tilde y\right) - \int_{\widetilde{Y}_1} a_1(\tilde y)\partial^2_{x_3}u(x)\,\mathrm{d}\tilde y\\
&= -\operatorname{div}_x\left(\int_{\widetilde{Y}_2} a_2(\tilde y)\sum_{k=1}^{3}\partial_{x_k}u(e_k + \nabla_{\tilde y}s_2^k(\tilde y))\,\mathrm{d}\tilde y\right) - \int_{\widetilde{Y}_1} a_1(\tilde y)\partial^2_{x_3}u(x)\,\mathrm{d}\tilde y = f(x).
\end{aligned}$$

As a consequence we deduce the homogenized problem (4.2). □

**5.2. Critical scale.** In this subsection, we focus on the case $\beta = 1$ which gives the most interesting result. We start with the following lemma.

**Lemma 5.2.** *Let $u^\varepsilon$ be the solution of problem* (2.1), *then, up to a subsequence, there exist*

$$\begin{aligned}
&u_2 \in H_D^1(\Omega), \quad \tilde v_1 \in L^2(\Omega; H^1(\widetilde{Y}_1)/\mathbb{R}) \cap L^2(\widetilde{\Omega}; H^1(I),\\
&\tilde v_2 \in L^2(\Omega; H^1_{\mathrm{per}}(\widetilde{Y}_2)/\mathbb{R}) \cap L^2(\widetilde{\Omega}; H^1(I))\\
&\tilde v_3 \in L^2(\Omega; H^1(\widetilde{Y}_3)/\mathbb{R})
\end{aligned}$$

*such that:*

$$(5.11)\quad \mathcal{T}_{\tilde{x},\varepsilon}(u^\varepsilon) \rightharpoonup \chi_1(\tilde{y})\tilde{v}_1(x,\tilde{y}) + \chi_2(\tilde{y})u_2(x) + \chi_3(\tilde{y})\tilde{v}_3(x,\tilde{y}) \quad \textit{in } L^2(\Omega\times\widetilde{Y}),$$

$$(5.12)\quad u^\varepsilon \rightharpoonup \int_{\widetilde{Y}_1} \tilde{v}_1(x,\tilde{y})\,\mathrm{d}\tilde{y} + |\widetilde{Y}_2|u_2(x) + \int_{\widetilde{Y}_3} \tilde{v}_3(x,\tilde{y})\,\mathrm{d}\tilde{y} \quad \textit{in } L^2(\Omega),$$

$$(5.13)\quad \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}} u^\varepsilon_{|\Omega_1^\varepsilon}) \rightharpoonup \nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y}) \quad \textit{in } L^2(\Omega\times\widetilde{Y}_1),$$

$$(5.14)\quad \mathcal{T}_{\tilde{x},\varepsilon}(\nabla u^\varepsilon{}_{|\Omega_2^\varepsilon}) \rightharpoonup \begin{pmatrix} \nabla_{\tilde{x}} u_2(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}) \\ \partial_{x_3} u_2(x) \end{pmatrix} \quad \textit{in } L^2(\Omega\times\widetilde{Y}_2),$$

$$(5.15)\quad \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla u^\varepsilon_{|\Omega_3^\varepsilon}) \rightharpoonup (\,\nabla_{\tilde{y}}\tilde{v}_3(x,\tilde{y})\ 0\,) \quad \textit{in } L^2(\Omega\times\widetilde{Y}_3),$$

$$(5.16)\quad \mathcal{T}_{\tilde{x},\varepsilon}(\partial_{x_3} u^\varepsilon_{|\Omega_1^\varepsilon}) \rightharpoonup \partial_{x_3}\tilde{v}_1(x,\tilde{y}) \quad \textit{in } L^2(\Omega\times\widetilde{Y}_1).$$

P r o o f. The above convergence results are straightforward from Lemma 2.1, Corollary 3.8, and Theorem 3.7. □

P r o o f of Theorem 4.2. Note that for all $\phi^\varepsilon(x) = \varphi(x)\psi^\varepsilon(\tilde{x})$, where $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{\mathrm{per}}(\widetilde{Y})$, one has

$$(5.17)\quad \int_\Omega \varepsilon\nabla_{\tilde{x}} u^\varepsilon \varphi\psi^\varepsilon(\tilde{x})\,\mathrm{d}x$$
$$= \int_{\Omega\times\widetilde{Y}} \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}} u^\varepsilon)(x,\tilde{y})\mathcal{T}_{\tilde{x},\varepsilon}(\varphi)(x,\tilde{y})\mathcal{T}_{\tilde{x},\varepsilon}(\psi^\varepsilon)(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}$$
$$\to \int_{\Omega\times\widetilde{Y}_1} \nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})\varphi(x)\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} + \int_{\Omega\times\widetilde{Y}_3} \nabla_{\tilde{y}}\tilde{v}_3(x,\tilde{y})\varphi(x)\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

Integration by parts yields the following:

$$\int_\Omega \varepsilon\nabla_{\tilde{x}} u^\varepsilon(x)\varphi(x)\psi^\varepsilon(\tilde{x})\,\mathrm{d}x = -\int_\Omega u^\varepsilon(x)[\varepsilon\mathrm{div}_{\tilde{x}}\varphi(x)\psi^\varepsilon(\tilde{x}) + \varphi(x)\mathrm{div}_{\tilde{y}}\psi^\varepsilon(\tilde{x})]\,\mathrm{d}x$$
$$= -\int_{\Omega\times\widetilde{Y}} \mathcal{T}_{\tilde{x},\varepsilon}(u^\varepsilon)[\varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\mathrm{div}_{\tilde{x}}\varphi)\mathcal{T}_{\tilde{x},\varepsilon}(\psi^\varepsilon) + \mathcal{T}_{\tilde{x},\varepsilon}(\varphi)\mathcal{T}_{\tilde{x},\varepsilon}(\mathrm{div}_{\tilde{y}}\psi^\varepsilon)]\,\mathrm{d}x\,\mathrm{d}\tilde{y} =: I_1^\varepsilon + I_2^\varepsilon.$$

Then passing to the limit as $\varepsilon$ tends to zero, we have

$$I_1^\varepsilon \to 0,$$
$$I_2^\varepsilon \to I_2 = -\int_{\Omega\times\widetilde{Y}_1} \tilde{v}_1(x,\tilde{y})\varphi(x)\mathrm{div}_{\tilde{y}}\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} - \int_{\Omega\times\widetilde{Y}_2} u_2(x)\varphi(x)\mathrm{div}_{\tilde{y}}\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}$$
$$- \int_{\Omega\times\widetilde{Y}_3} \tilde{v}_3(x,\tilde{y})\varphi(x)\mathrm{div}_{\tilde{y}}\psi(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

We put $w_3 = \tilde{v}_3 - u_2$, thus we deduce

$$\begin{aligned} w_3(x,\tilde{y}) &= 0 && \text{on } \widetilde{Y}_{23}, \\ w_3(x,\tilde{y}) &= \tilde{v}_1(x,\tilde{y}) - u_2(x) && \text{on } \widetilde{Y}_{13}. \end{aligned}$$

The limit problem is then derived by taking as test function in (2.1)

$$\psi^\varepsilon(x) = \varphi(x) + \varepsilon\phi_2(x)\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big) + \phi(x)\Psi\Big(\frac{\tilde{x}}{\varepsilon}\Big)$$

and

$$\Psi(\tilde{y}) = \begin{cases} \Psi_\alpha(\tilde{y}) & \text{for } \tilde{y} \in \widetilde{Y}_\alpha,\ \alpha = 1.3, \\ 0 & \text{for } \tilde{y} \in \widetilde{Y}_2, \end{cases}$$

where $\phi_2, \phi \in \mathcal{D}(\Omega)$, $\varphi \in \mathcal{D}^1_D(\bar{\Omega})$, $\Psi_\alpha, \Psi_2 \in H^1_{\rm per}(\widetilde{Y})$. Similarly, by using the same lines as in the proof of Theorem 4.1 we have first

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\nabla\psi^\varepsilon(x)\,\mathrm{d}x =: I^\varepsilon_{11} + I^\varepsilon_{12} + I^\varepsilon_2 + I^\varepsilon_3 + O(\varepsilon^\beta),$$

where

$$\begin{aligned}
I^\varepsilon_{11} = \int_{\Omega\times\widetilde{Y}_1} & \mathcal{T}_{\tilde{x},\varepsilon}(a^\varepsilon_1)\varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}}u^\varepsilon)\Big(\varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}}\varphi) + \varepsilon^2\mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}}\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big) \\
& + \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\nabla_{\tilde{y}}\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big) + \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}}\phi)\mathcal{T}_{\tilde{x},\varepsilon}(\Psi_1) \\
& + \mathcal{T}_{\tilde{x},\varepsilon}(\phi)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\nabla_{\tilde{y}}\Psi_1\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y} \\
I^\varepsilon_{12} = \int_{\Omega\times\widetilde{Y}_1} & \mathcal{T}_{\tilde{x},\varepsilon}(a^\varepsilon_1)\mathcal{T}_{\tilde{x},\varepsilon}(\partial_{x_3}u^\varepsilon)\Big(\mathcal{T}_{\tilde{x},\varepsilon}(\partial_{x_3}\varphi) + \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\partial_{x_3}\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big) \\
& + \mathcal{T}_{\tilde{x},\varepsilon}(\partial_{x_3}\phi)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\Psi_1\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y} \\
I^\varepsilon_2 = \int_{\Omega\times\widetilde{Y}_2} & \mathcal{T}_{\tilde{x},\varepsilon}(a^\varepsilon_2)\mathcal{T}_{\tilde{x},\varepsilon}(\nabla u^\varepsilon)\Big(\mathcal{T}_{\tilde{x},\varepsilon}(\nabla\varphi) + \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big) \\
& + \mathcal{T}_{\tilde{x},\varepsilon}(\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\nabla_{\tilde{y}}\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y} \\
I^\varepsilon_3 = \int_{\Omega\times\widetilde{Y}_3} & \mathcal{T}_{\tilde{x},\varepsilon}(a^\varepsilon_3)\varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla u^\varepsilon)\Big(\varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla\varphi) + \varepsilon^2\mathcal{T}_{\tilde{x},\varepsilon}(\nabla\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}(\Psi_2) \\
& + \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\phi_2)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\nabla_{\tilde{y}}\Psi_2\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big) + \varepsilon\mathcal{T}_{\tilde{x},\varepsilon}(\nabla\phi)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\Psi_3\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big) \\
& + \mathcal{T}_{\tilde{x},\varepsilon}(\phi)\mathcal{T}_{\tilde{x},\varepsilon}\Big(\nabla_{\tilde{y}}\Psi_3\Big(\frac{\tilde{x}}{\varepsilon}\Big)\Big)\Big)\,\mathrm{d}x\,\mathrm{d}\tilde{y}.
\end{aligned}$$

Now, passing to the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned}
I^\varepsilon_{11} &\to \int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})\phi(x)\nabla_{\tilde{y}}\Psi_1(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}, \\
I^\varepsilon_{12} &\to \int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}\tilde{v}_1(x,\tilde{y})(\partial_{x_3}\varphi(x) + \partial_{x_3}\phi(x)\Psi_1(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y},
\end{aligned}$$

$$I_2^\varepsilon \to \int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u_2(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla\varphi(x) + \phi_2\nabla_{\tilde{y}}\Psi_2(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y},$$
$$I_3^\varepsilon \to \int_{\Omega\times\widetilde{Y}_3} a_3(\tilde{y})\nabla_{\tilde{y}} w_3(x,\tilde{y})\phi(x)\nabla_{\tilde{y}}\Psi_3(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.$$

We deduce the following homogenized limit problem:

$$\begin{aligned}(5.18)\quad &\int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})\phi(x)\nabla_{\tilde{y}}\Psi_1(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &+\int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}\tilde{v}_1(x,\tilde{y})(\partial_{x_3}\varphi(x) + \partial_{x_3}\phi(x)\Psi_1(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &+\int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u_2(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla\varphi(x) + \phi_2(x)\nabla_{\tilde{y}}\Psi_2(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &+\int_{\Omega\times\widetilde{Y}_3} a_3(\tilde{y})\nabla_{\tilde{y}} w_3(x,\tilde{y})\phi(x)\nabla_{\tilde{y}}\Psi_3(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &=\int_{\Omega\times\widetilde{Y}_1} f(x)\phi(x)\Psi_1(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y} + \int_\Omega f(x)\phi(x)\,\mathrm{d}x\\ &+\int_{\Omega\times\widetilde{Y}_3} f(x)\phi(x)\Psi_3(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.\end{aligned}$$

By the Lax-Milgram Theorem, there exists a unique solution

$$(u_2,\tilde{v}_1,\tilde{v}_2) \in H_D^1(\Omega)\times[L^2(\Omega; H^1(\widetilde{Y}_1)/\mathbb{R})\cap L^2(\widetilde{\Omega}; H^1(I))]\times L^2(\Omega; H^1_{\mathrm{per}}(\widetilde{Y}_2)/\mathbb{R}),$$
$$w_3 \in L^2(\Omega; (H^1(\widetilde{Y}_3)/\mathbb{R})).$$

Now, we derive the two-scale homogenized problem. Firstly, we take $\phi = 0$ and $\psi_2 = \psi_1 = 0$ in (5.18). After integration by parts, we deduce

$$\begin{aligned}(5.19)\quad &-\mathrm{div}_{\tilde{y}}[a_3(\tilde{y})\nabla_{\tilde{y}} w_3(x,\tilde{y})] = f(x), \quad \text{a.e. } \tilde{y}\in\widetilde{Y}_3,\ x\in\Omega,\\ &a_3(\tilde{y})\nabla_{\tilde{y}} w_3(x,\tilde{y})n_3(\tilde{y}) = 0, \quad \text{a.e. } \tilde{y}\in\partial\widetilde{Y}_3,\ x\in\Omega.\end{aligned}$$

Secondly, let $\phi = 0$ and $\psi_2 = 0$ in (5.18), integrating by parts and using (5.19), we get

$$\begin{aligned}(5.20)\quad &-\mathrm{div}_{\tilde{y}}[a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})] - a_1(\tilde{y})\partial^2_{x_3}\tilde{v}_1(x,\tilde{y}) = f(x) \quad \text{a.e. } \tilde{y}\in\widetilde{Y}_1,\ x\in\Omega,\\ &a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})n_1(\tilde{y}) = -a_3(\tilde{y})\nabla_{\tilde{y}} w_3(x,\tilde{y})n_3(\tilde{y}) \quad \text{a.e. } \tilde{y}\in\widetilde{Y}_{13},\ x\in\Omega.\end{aligned}$$

Thirdly, let $\psi_2 = \psi_1 = \psi_3 = 0$ in (5.18) and integrating by parts,

$$-\int_{\widetilde{Y}_2}\mathrm{div}_x[a_2(\tilde{y})(\nabla u_2 + \nabla_{\tilde{y}}\tilde{v}_2)]\,\mathrm{d}\tilde{y} - \int_{\widetilde{Y}_1} a_1(\tilde{y})\partial^2_{x_3}\tilde{v}_1(x,\tilde{y})\,\mathrm{d}\tilde{y} = \int_{\widetilde{Y}} f(x)\,\mathrm{d}\tilde{y} \quad \text{in } \Omega,$$
$$a_2(\tilde{y})(\nabla u_2(\tilde{x},1) + \nabla_{\tilde{y}}\tilde{v}_2((\tilde{x},1),\tilde{y})) = 0 \quad \tilde{x}\in\widetilde{\Omega},\ \text{a.e. } \tilde{y}\in\widetilde{Y}_2,$$
$$\partial_{x_3}\tilde{v}_1(\tilde{x},1,\tilde{y}) = 0 \quad \text{in } \widetilde{\Omega}.$$

In (5.20) and (5.19) we take, respectively, the average over $\widetilde{Y}_1$ and $\widetilde{Y}_3$ and we get

$$-\int_{\widetilde{Y}_1} \operatorname{div}_{\tilde{y}}[a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})]\,\mathrm{d}\tilde{y} - \int_{\widetilde{Y}_1} a_1(\tilde{y})\partial^2_{x_3}\tilde{v}_1(x,\tilde{y})\,\mathrm{d}\tilde{y} = \int_{\widetilde{Y}_1} f(x)\,\mathrm{d}\tilde{y}$$

and

$$-\int_{\widetilde{Y}_3} \operatorname{div}_{\tilde{y}}[a_3(\tilde{y})\nabla_{\tilde{y}}w_3(x,\tilde{y})]\,\mathrm{d}\tilde{y} = \int_{\widetilde{Y}_3} f(x)\,\mathrm{d}\tilde{y}.$$

Integrating by parts, we deduce

$$-\operatorname{div}_x\Bigl(\int_{\widetilde{Y}_2} a_2(\tilde{y})(\nabla u_2+\nabla_{\tilde{y}}\tilde{v}_2)\,\mathrm{d}\tilde{y}\Bigr) - \int_{\widetilde{Y}_{23}} a_3(\tilde{y})\nabla_{\tilde{y}}w_3(x,\tilde{y})n_3(\tilde{y})\,\mathrm{d}s(\tilde{y}) = \int_{\widetilde{Y}_2} f(x)\,\mathrm{d}\tilde{y}.$$

Finally, we put $\tilde{v}_2(x,\tilde{y}) = \sum_{k=1}^{3} \partial_{x_k}u_2(x)s_2^k(\tilde{y})$, where $s_2^k$ is defined by (4.1), then we deduce (4.3). □

**5.3. Subcritical scale.** In this section, we investigate the subcritical scale. We first prove the following lemma.

**Lemma 5.3.** *Let $u^\varepsilon$ be the solution of* (2.1). *Then there exist*

$$u_2 \in H^1_D(\Omega), \quad \tilde{v}_1 \in L^2(\Omega; H^1(\widetilde{Y}_1)/\mathbb{R}) \cap L^2(\widetilde{\Omega}; H^1(I))$$

*and $\tilde{v}_2 \in L^2(\Omega; H^1_{\mathrm{per}}(\widetilde{Y}_2)/\mathbb{R})$, such that*

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla u^\varepsilon_{|\Omega_2^\varepsilon}) \rightharpoonup \begin{pmatrix} \nabla_{\tilde{x}}u_2(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}) \\ \partial_{x_3}u_2(x) \end{pmatrix} \quad \text{in } L^2(\Omega\times\widetilde{Y}_2), \tag{5.21}$$

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\partial_{x_3}u^\varepsilon_{|\Omega_1^\varepsilon}) \rightharpoonup \partial_{x_3}\tilde{v}_1(x,\tilde{y}) \quad \text{in } L^2(\Omega\times\widetilde{Y}_1). \tag{5.22}$$

**Remark 5.4.** Let us mention that our a priori estimates are not enough to deduce the behavior of $\varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla_{\tilde{x}}u^\varepsilon_{|\Omega_1^\varepsilon})$ and $\varepsilon^\beta\mathcal{T}_{\tilde{x},\varepsilon^\beta}(\nabla u^\varepsilon{}_{|\Omega_3^\varepsilon})$.

P r o o f of Lemma 5.3. To prove the lemma, we shall give some a priori estimates for the sequence $\{u^\varepsilon\}$ in $\Omega_1^\varepsilon$ and $\Omega_2^\varepsilon$.

Since $\widetilde{\Omega}_2^\varepsilon$ is connected and $u^\varepsilon = 0$ on $\partial\widetilde{\Omega}_2^\varepsilon \cap \partial\widetilde{\Omega}$, we have

$$\|u^\varepsilon\|_{L^2(\widetilde{\Omega}_2^\varepsilon)} \leqslant C\|\nabla u^\varepsilon\|_{L^2(\widetilde{\Omega}_2^\varepsilon)},$$

by applying [1], Lemma A.4. Now, by using integration over the interval $I$ with respect to the $x_3$ variable we obtain

$$\|u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leqslant C\|\nabla u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)}. \tag{5.23}$$

Moreover, as in the proof of [3], Lemma 2.1 one obtains

$$\|u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leqslant C\|\partial_{x_3} u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}. \tag{5.24}$$

From estimates (2.2), (5.23)–(5.24) one can deduce that

$$u^\varepsilon(x)\chi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big) \quad \text{and} \quad u^\varepsilon(x)\chi_1\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)$$

are bounded in $L^2(\Omega)$, then $(u_2, \tilde{v}_1) \in H^1_D(\Omega) \times [L^2(\Omega; H^1(\widetilde{Y}_1)/\mathbb{R}) \cap L^2(\widetilde{\Omega}; H^1(I))]$ such that

$$\mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon_{|\Omega_1^\varepsilon}) \rightharpoonup \tilde{v}_1(x,\tilde{y}), \quad \mathcal{T}_{\tilde{x},\varepsilon^\beta}(u^\varepsilon_{|\Omega_2^\varepsilon}) \rightharpoonup u_2(x)$$

and

$$u^\varepsilon(x)\chi_1\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big) \rightharpoonup \int_{\widetilde{Y}_1} \tilde{v}_1(x,\tilde{y})\,\mathrm{d}\tilde{y}, \quad u^\varepsilon(x)\chi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big) \rightharpoonup |\widetilde{Y}_2| u_2(x) \quad \text{weakly in } L^2(\Omega).$$

We apply Corollary 3.8 and Theorem 3.7 and get immediately convergences (5.21)–(5.22). □

Thanks to the convergence results presented in Lemma 5.3, we can now give the proof of the homogenized result stated in Theorem 4.3.

*Proof* of Theorem 4.3. In this case, we take as test function in (2.1)

$$\psi^\varepsilon(x) = \varphi(x) + \varepsilon^\beta \phi_2(x)\Psi_2\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big) + \phi_1(x)\Psi_1\Big(\frac{\tilde{x}}{\varepsilon^\beta}\Big)$$

such that $\psi_1(\tilde{y}) = 0$ for all $\tilde{y} \in \widetilde{Y}_2$, where $\phi_2, \phi_1 \in \mathcal{D}(\Omega)$, $\varphi \in \mathcal{D}^1_D(\bar{\Omega})$ and $\Psi_1, \Psi_2 \in H^1_{\rm per}(\widetilde{Y})$.

By using the same technical argument as in the proof of the previous theorem, one gets

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\nabla\psi^\varepsilon(x)\,\mathrm{d}x =: I^\varepsilon_{11} + I^\varepsilon_{12} + I^\varepsilon_2 + I^\varepsilon_3 + O(\varepsilon^\beta).$$

Passing to the limit we obtain

$$\begin{aligned}
&I^\varepsilon_{11} \to 0, \quad I^\varepsilon_3 \to 0,\\
&I^\varepsilon_{12} \to \int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}\tilde{v}_1(x,\tilde{y})(\partial_{x_3}\varphi(x) + \partial_{x_3}\phi_1(x)\Psi_1(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y},\\
&I^\varepsilon_2 \to \int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u_2(x) + \nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla\varphi(x) + \phi_2(x)\nabla_{\tilde{y}}\Psi_2(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}.
\end{aligned}$$

Then we derive the following limit problem:

$$\begin{aligned}(5.25)\quad &\int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}\tilde{v}_1(x,\tilde{y})(\partial_{x_3}\varphi(x)+\partial_{x_3}\phi_1(x)\Psi_1(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &\quad+\int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u_2(x)+\nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla\varphi(x)+\phi_2(x)\nabla_{\tilde{y}}\Psi_2(\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &\quad=\int_{\Omega\times\widetilde{Y}_1} f(x)\phi_1(x)\Psi_1(\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}+\int_\Omega f(x)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Based on the Lax-Milgram Theorem, this equation has a unique solution. To derive the two-scale homogenized problem (4.3) we take first $\varphi=0$ and $\psi_2=0$ in (5.25) and after integration by parts we obtain (4.7). Then we take $\psi_2=\psi_1=0$ in (5.25) to get (4.6). □

## 6. Corrector results

We are going to construct some corrector results by using the partial averaging operator $\mathcal{U}_{\tilde{x},\varepsilon^\beta}$. Let us deal only with the critical case in this section. The other cases are quite similar and follow the same pattern.

**Theorem 6.1.** *Using the hypotheses of Theorem* 4.2*, we have*

$$\begin{aligned}&\varepsilon\nabla_{\tilde{x}}u^\varepsilon{}_{|\Omega_1^\varepsilon}-\mathcal{U}_{\tilde{x},\varepsilon}(\nabla_{\tilde{y}}\tilde{v}_1)\to 0 && \text{in } L^2(\Omega),\\ &\partial_{x_3}u^\varepsilon_{|\Omega_1^\varepsilon}-\mathcal{U}_{\tilde{x},\varepsilon}(\partial_{x_3}\tilde{v}_1)\to 0 && \text{in } L^2(\Omega),\\ &\nabla u^\varepsilon_{|\Omega_2^\varepsilon}-\nabla_x u_2-\mathcal{U}_{\tilde{x},\varepsilon}(\nabla_{\tilde{y}}\tilde{v}_2)\to 0 && \text{in } L^2(\Omega),\\ &\varepsilon\nabla u^\varepsilon{}_{|\Omega_3^\varepsilon}-\mathcal{U}_{\tilde{x},\varepsilon}(\nabla_{\tilde{y}}w_3)\to 0 && \text{in } L^2(\Omega).\end{aligned}$$

We start with the so-called energy-convergence. Under the hypotheses of Theorem 4.2 and (5.18) and using the same arguments as in [4], we obtain easily the following lemma.

**Lemma 6.2.**

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\nabla u^\varepsilon(x)\,\mathrm{d}x &= \int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})\nabla_{\tilde{y}}\tilde{v}_1(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &+\int_{\Omega\times\widetilde{Y}_1} a_1(\tilde{y})\partial_{x_3}\tilde{v}_1(x,\tilde{y})\partial_{x_3}\tilde{v}_1(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &+\int_{\Omega\times\widetilde{Y}_2} a_2(\tilde{y})(\nabla u_2(x)+\nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))(\nabla u_2(x)+\nabla_{\tilde{y}}\tilde{v}_2(x,\tilde{y}))\,\mathrm{d}x\,\mathrm{d}\tilde{y}\\ &+\int_{\Omega\times\widetilde{Y}_3} a_3(\tilde{y})\nabla_{\tilde{y}}w_3(x,\tilde{y})\nabla_{\tilde{y}}w_3(x,\tilde{y})\,\mathrm{d}x\,\mathrm{d}\tilde{y}.\end{aligned}$$

Using the same arguments as in Theorem 4.2, we get the following corollary.

**Corollary 6.3.**

$$\begin{aligned}
&\varepsilon \mathcal{T}_{\tilde{x},\varepsilon}(\nabla_{\tilde{x}} u^{\varepsilon}_{|\Omega_1^{\varepsilon}}) \to \nabla_{\tilde{y}} \tilde{v}_1(x,\tilde{y}) && \textit{in } L^2(\Omega \times \widetilde{Y}_1),\\
&\mathcal{T}_{\tilde{x},\varepsilon}(\nabla u^{\varepsilon}_{|\Omega_2^{\varepsilon}}) \to \begin{pmatrix} \nabla_{\tilde{x}} u_2(x) + \nabla_{\tilde{y}} \tilde{v}_2(x,\tilde{y}) \\ \partial_{x_3} u_2(x) \end{pmatrix} && \textit{in } L^2(\Omega \times \widetilde{Y}_2),\\
&\varepsilon \mathcal{T}_{\tilde{x},\varepsilon}(\nabla u^{\varepsilon}{}_{|\Omega_3^{\varepsilon}}) \to \begin{pmatrix} \nabla_{\tilde{y}} w_3(x,\tilde{y}) \\ 0 \end{pmatrix} && \textit{in } L^2(\Omega \times \widetilde{Y}_3),\\
&\mathcal{T}_{\tilde{x},\varepsilon}(\partial_{x_3} u^{\varepsilon}_{|\Omega_1^{\varepsilon}}) \to \partial_{x_3} \tilde{v}_1(x,\tilde{y}) && \textit{in } L^2(\Omega \times \widetilde{Y_1}).
\end{aligned}$$

P r o o f of Theorem 6.1. By using Lemma 6.2, Corollary 6.3 and Proposition 3.10 we deduce Theorem 6.1. □

## 7. Conclusion

In this work, we succeeded to provide the solutions of an homogenization problem for a three-phase composite material: fibres, interphase and matrix. The conductivity matrix $A^{\varepsilon}$ of the medium varies periodically with a period of size $\varepsilon^{\beta}$ ($\varepsilon > 0$ and $\beta > 0$) in the transverse directions of the fibers and having different scaling in different phases and directions. It is equal to $\varepsilon^2$ (the so-called the double-porosity type scaling) in both the interphase and the direction of the fibers and no scaling in both the transverse directions of the fibers and the matrix. Recall that for composite materials with highly contrasting parameters of order $\varepsilon^2$, the homogenized limit may exhibit a nonlocal effect which is demonstrated rigorously in [3], [4]. Our first objective was to study the effects of weakly conducting interphase on the homogenized model for this double-porosity type composites. We have showed that the result depends on the magnitude of the exponent $\beta$, being lower, equal or greater than 1. More precisely, the effects of the interphase properties on the homogenized models are captured only when the microstural length scale is of order $\varepsilon^{\beta}$ with $0 < \beta \leqslant 1$. Thus, the result is physically very interesting, especially in the case $\beta > 1$, where the interphase insures a perfect conductor between the fibers and the matrix at the macroscopic scale.

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