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Hongzhao Lin; Zhongming Teng

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REDUCING SUBSPACES OF TOEPLITZ OPERATORS ON DIRICHLET TYPE SPACES OF THE BIDISK

Hongzhao Lin, Zhongming Teng, Fujian

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Abstract. The reducing subspaces of Toeplitz operators $T_{z_1^N \bar{z}_2^M}$ on Dirichlet type spaces of the $\mathcal{D}_\alpha(\mathbb{D}^2)$ are described, which extends the results for the corresponding operators on Bergman spaces of the bidisk.

Keywords: reducing subspace; Toeplitz operator; Dirichlet type space; bidisk

MSC 2020: 47B35

1. Introduction

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ denote the set of nonnegative integers. Let $\mathbb{D}$ be the open unit disk of complex plane $\mathbb{C}$ and $\mathbb{D}^2 = \{(z_1, z_2);\ z_1 \in \mathbb{D},\ z_2 \in \mathbb{D}\}$ is called the *bidisk*. We say that a function $f\colon \mathbb{D}^2 \to \mathbb{C}$ is holomorphic if it is holomorphic in each variable separately. Each holomorphic function f on the bidisk can be represented as

$$f(z, w) = \sum_{i,j \in \mathbb{N}} a_{i,j} z_1^i z_2^j$$

with $(z, w) \in \mathbb{D}^2$ and $a_{i,j} \in \mathbb{C}$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, the Dirichlet type space of the bidisk $\mathcal{D}_\alpha(\mathbb{D}^2)$ consisting of all holomorphic functions f on the bidisk satisfying

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{D}^2)} = \sum_{i,j \in \mathbb{N}} |a_{i,j}|^2 (1+i)^{\alpha_1} (1+j)^{\alpha_2} < \infty.$$

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Assume that $\mathcal{D}_\alpha(\mathbb{D}^2)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \sum_{i,j\in\mathbb{N}} a_{i,j}\overline{b_{i,j}}(1+i)^{\alpha_1}(1+j)^{\alpha_2},$$

where $f = \sum\limits_{i,j\in\mathbb{N}} a_{i,j}z_1^i z_2^j$ and $g = \sum\limits_{i,j\in\mathbb{N}} b_{i,j}z_1^i z_2^j$. Given $z = (z_1, z_2) \in \mathbb{D}^2$, each point evaluation $\lambda_z^\alpha(f) = f(z)$ is a bounded linear functional on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Hence, for each $z \in \mathbb{D}^2$, there exists a unique reproducing kernel $K_z(w) \in \mathcal{D}_\alpha(\mathbb{D}^2)$ with $w = (w_1, w_2) \in \mathbb{D}^2$ such that

$$f(z) = \langle f(w), K_z(w)\rangle \quad \forall\, f \in \mathcal{D}_\alpha(\mathbb{D}^2).$$

Actually, it can be calculated that

$$K_z(w) = \sum_{i,j\geqslant 0} \frac{w_1^i w_2^j \overline{z}_1^i \overline{z}_2^j}{(1+i)^{\alpha_1}(1+j)^{\alpha_2}}.$$

One can see [6] for more details about Dirichlet type space $\mathcal{D}_\alpha(\mathbb{D}^2)$. Throughout this paper, we denote $\gamma_{\alpha_1,i} = \sqrt{(1+i)^{\alpha_1}}$ and $\gamma_{\alpha_2,j} = \sqrt{(1+j)^{\alpha_2}}$. It follows that $\|z_1^i z_2^j\|_{\mathcal{D}_\alpha(\mathbb{D}^2)} = \gamma_{\alpha_1,i}\gamma_{\alpha_2,j}$. For simplicity, we denote $\|z_1^i z_2^j\|_{\mathcal{D}_\alpha(\mathbb{D}^2)}$ by $\|z_1^i z_2^j\|$.

It is easy to see that $\mathcal{D}_{(0,0)}(\mathbb{D}^2)$ is the Hardy space over the bidisk $H^2(\mathbb{D}^2)$ and $\mathcal{D}_{(-1,-1)}(\mathbb{D}^2)$ is the Bergman space over the bidisk $A^2(\mathbb{D}^2)$. In this paper, we only deal with $\mathcal{D}_\alpha(\mathbb{D}^2)$ satisfying $\alpha_1\alpha_2 \neq 0$.

Given a holomorphic function f on the bidisk $\mathbb{D}^2$, if $hf \in \mathcal{D}_\alpha(\mathbb{D}^2)$ for any $h \in \mathcal{D}_\alpha(\mathbb{D}^2)$, we define $T_f\colon\ \mathcal{D}_\alpha(\mathbb{D}^2) \to \mathcal{D}_\alpha(\mathbb{D}^2)$ by

$$T_f(h) = fh \quad \forall\, h \in \mathcal{D}_\alpha(\mathbb{D}^2).$$

Let N, M be integers larger than 1 with $N \neq M$; it is easy to check that $T_{z_1^N}$ (or $T_{\overline{z}_2^M}$) is a bounded linear operator on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Note that

$$\|T_{z_1^N\overline{z}_2^M}\| = \|T_{z_1^N}T_{\overline{z}_2^M}\| \leqslant \|T_{z_1^N}\|\|T_{\overline{z}_2^M}\|,$$

where $T_{z_1^N\overline{z}_2^M}$ are bounded linear operators on $\mathcal{D}_\alpha(\mathbb{D}^2)$.

Suppose that $\mathfrak{M}$ is a closed subspace of Hilbert space $\mathcal{H}$. Recall that $\mathfrak{M}$ is a reducing subspace of the operator T if $T(\mathfrak{M}) \subseteq \mathfrak{M}$ and $T^*(\mathfrak{M}) \subseteq \mathfrak{M}$. A reducing subspace $\mathfrak{M}$ is said to be minimal if there are none nontrivial reducing subspaces of T contained in $\mathfrak{M}$.

Stessin and Zhu in [10] completely characterized the reducing subspaces of the power of scalar weighted unilateral shifts. As an consequence, they gave the description of the reducing subspaces of T_{z^N} on the Bergman space and Dirichlet space of the unit disk. For more general symbols, the reducing subspaces of the Toeplitz operators with finite Blaschke product were well studied (see [4], [5], [12] for example). Lu, Shi and Zhou extended the result in [10] to Bergman space with several variables. They characterized the reducing subspaces of $T_{z_1^N}$, $T_{z_1^N z_2^N}$ and $T_{z_1^N z_2^M}$ on the weighted Bergman space over the bidisk and polydisk (see [8], [9], [11]). However, we knew little about the reducing subspaces of Toeplitz operators with non-analytic symbols. On the weighted Bergman space over the bidisk, Lu and his students identified reducing subspaces of $T_{z_1^N \bar{z}_2^M}$ in [1] and $T_{z_1^N+\alpha\bar{z}_2^M}$ in [2], respectively. Recently, Gu in [3] extended the results about $T_{z_1^N+\alpha\bar{z}_2^M}$ to the weighted Hardy space case.

The author in [7] has described the reducing subspaces of Toeplitz operators $T_{z_1^N}$ (or $T_{z_2^N}$), $T_{z_1^N z_2^N}$ and $T_{z_1^N z_2^M}$ on Dirichlet type spaces of the bidisk $\mathcal{D}_\alpha(\mathbb{D}^2)$. Motivated by the above work, we will investigate the reducing subspaces of Toeplitz operators $T_{z_1^N \bar{z}_2^M}$ on Dirichlet type spaces of the bidisk, which generalizes the results in [1]. We characterize the reducing subspaces of $T_{z_1^N \bar{z}_2^M}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1| = |\alpha_2|$ in Section 2 and $|\alpha_1| \neq |\alpha_2|$ in Section 3, respectively.

Throughout this paper, we denote $T = T_{z_1^N \bar{z}_2^M}$ and $[f]$ be the reducing subspace of T generated by $f \in \mathcal{D}_\alpha(\mathbb{D}^2)$. By a direct computation for $k, l, h \in \mathbb{N}$ we have

$$T^h(z_1^k z_2^l) = \begin{cases} \dfrac{\gamma_{\alpha_2,l}^2}{\gamma_{\alpha_2,l-hM}^2} z_1^{k+hN} z_2^{l-hM}, & l \geqslant hM, \\ 0, & \text{else} \end{cases}$$

and

$$T^{*h}(z_1^k z_2^l) = \begin{cases} \dfrac{\gamma_{\alpha_1,k}^2}{\gamma_{\alpha_1,k-hN}^2} z_1^{k-hN} z_2^{l+hM}, & k \geqslant hN, \\ 0, & \text{else.} \end{cases}$$

2. The case of Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1| = |\alpha_2|$

In this section, we will characterize reducing subspace of T on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1| = |\alpha_2|$. The following lemma is easy but useful.

Lemma 2.1. *Suppose* $|\alpha_1| = |\alpha_2|$ *and*

$$f(x) = \left(\frac{a-x}{b-x}\right)^{\alpha_2} \left(\frac{c+x}{d+x}\right)^{\alpha_1}$$

with $a, b, c, d \in \mathbb{R}$. *If* $f(0) = f(\lambda_1) = f(\lambda_2)$, *where nonzero* $\lambda_1, \lambda_2 \in \mathbb{R}$ *with* $\lambda_1 \neq \lambda_2$, *then* $a = b$ *and* $c = d$.

P r o o f. First suppose $\alpha_1 = \alpha_2$. Let $f_1 = (a-x)(c+x)$ and $f_2 = (b-x)(d+x)$, then we have

$$f(0) = \frac{f_1^{\alpha_2}(0)}{f_2^{\alpha_2}(0)}, \quad f(\lambda_1) = \frac{f_1^{\alpha_2}(\lambda_1)}{f_2^{\alpha_2}(\lambda_1)}, \quad f(\lambda_2) = \frac{f_1^{\alpha_2}(\lambda_2)}{f_2^{\alpha_2}(\lambda_2)}.$$

By the assumption, it follows that

$$f_1(0) = f_2(0)\frac{f_1(0)}{f_2(0)}, \quad f_1(\lambda_1) = f_2(\lambda_1)\frac{f_1(0)}{f_2(0)}, \quad f_1(\lambda_2) = f_2(\lambda_2)\frac{f_1(0)}{f_2(0)}.$$

Since f_1 and f_2 are both quadratic polynomials, it follows that $f_1(x) = f_2(x)$. Therefore, $a = b$ and $c = d$.

Now suppose $\alpha_1 = -\alpha_2$. Then

$$f(x) = \Big(\frac{a-x}{b-x}\Big)^{\alpha_2}\Big(\frac{d+x}{c+x}\Big)^{\alpha_2}.$$

By the discussion above, we have $a = b$ and $c = d$. Thus, the desired result is proved. $\square$

Observe that $\mathbb{N}^2 = \mathbb{N}\times\mathbb{N} = \bigcup\limits_{i=0}^{5} E_i$. It follows that

$$\mathcal{D}_\alpha(\mathbb{D}^2) = \bigoplus_{i=0}^{5} \overline{\operatorname{span}}\{z_1^k z_2^l;\, (k,l)\in E_i\} := \bigoplus_{i=0}^{5} \mathfrak{M}_i,$$

where

$$\begin{aligned}
E_0 &= \{(k,l)\in\mathbb{N}^2\colon\ 0\leqslant k<N,\ 0\leqslant l<M\},\\
E_1 &= \{(k,l)\in\mathbb{N}^2\colon\ k\geqslant 2N\},\\
E_2 &= \{(k,l)\in\mathbb{N}^2\colon\ 0\leqslant k<2N,\ l\geqslant 2M\},\\
E_3 &= \{(k,l)\in\mathbb{N}^2\colon\ N\leqslant k<2N,\ M\leqslant l<2M\},\\
E_4 &= \{(k,l)\in\mathbb{N}^2\colon\ 0\leqslant k<N,\ M\leqslant l<2M\},\\
E_5 &= \{(k,l)\in\mathbb{N}^2\colon\ N\leqslant k<2N,\ 0\leqslant l<M\}.
\end{aligned}$$

Letting

$$f(x) = \Big(\frac{(1+l)/M - x}{(1+q)/M - x}\Big)^{\alpha_2}\Big(\frac{(1+p)/N + x}{(1+k)/N + x}\Big)^{\alpha_1},$$

we define two equivalences on E_4 and E_5, respectively, by

(i) for $(p,q),(k,l)\in E_4$, $(p,q)\sim_1(k,l)$ if and only if $f(0)=f(1)$, which is equivalent to

$$\frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+N}}{\gamma^2_{\alpha_2,l-M}\gamma^2_{\alpha_1,k}} = \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}}.$$

(ii) for $(p,q),(k,l) \in E_5$, $(p,q) \sim_2 (k,l)$ if and only if $f(0) = f(-1)$, which is equivalent to

$$\frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k-N}}{\gamma^2_{\alpha_2,l+M}\gamma^2_{\alpha_1,k}} = \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p-N}}{\gamma^2_{\alpha_2,q+M}\gamma^2_{\alpha_1,p}}.$$

It is easy to check the following statements:

(1) $(p,q) \in E_4$ if and only if $(p+N, q-M) \in E_5$,

(2) for $(p,q),(k,l) \in E_4$, $(p,q) \sim_1 (k,l)$ if and only if $(p+N, q-M) \sim_2 (k+N, l-M)$,

(3) for $(p,q),(k,l) \in E_5$, $(p,q) \sim_2 (k,l)$ if and only if $(p-N, q+M) \sim_1 (k-N, l+M)$.

It is easy to see that $\mathfrak{M}_0$ is a reducing subspace of T. Next, we will study the orthogonal decomposition of $z_1^k z_2^l$ with respect to $\mathfrak{M}$, where $\mathfrak{M} \subset \mathcal{D}_\alpha(\mathbb{D}^2)$ and $\mathfrak{M} \perp \mathfrak{M}_0$.

Lemma 2.2. *Suppose $\mathfrak{M}$ is a reducing subspace of T and $\mathfrak{M} \perp \mathfrak{M}_0$. Let $P_{\mathfrak{M}}$ be the orthogonal projection from $\mathcal{D}_\alpha(\mathbb{D}^2)$ to $\mathfrak{M}$. Then the following statements hold.*

(1) *If $(k,l) \in E_1 \cup E_2 \cup E_3$, then $P_{\mathfrak{M}} z_1^k z_2^l = \lambda z_1^k z_2^l$, where $\lambda = 0$ or 1.*

(2) *If $(k,l) \in E_4$, then $P_{\mathfrak{M}} z_1^k z_2^l \in \mathfrak{M}_4$.*

(3) *If $(k,l) \in E_5$, then $P_{\mathfrak{M}} z_1^k z_2^l \in \mathfrak{M}_5$.*

Proof. Note that

$$T^{*h}T^h(z_1^k z_2^l) = \frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+hN}}{\gamma^2_{\alpha_2,l-hM}\gamma^2_{\alpha_1,k}} z_1^k z_2^l \quad \forall\, l \geqslant hM.$$

It follows that

$$\begin{aligned}\langle P_{\mathfrak{M}} T^{*h}T^h(z_1^k z_2^l), z_1^p z_2^q\rangle &= \Big\langle P_{\mathfrak{M}} \frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+hN}}{\gamma^2_{\alpha_2,l-hM}\gamma^2_{\alpha_1,k}} z_1^k z_2^l, z_1^p z_2^q\Big\rangle \\ &= \frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+hN}}{\gamma^2_{\alpha_2,l-hM}\gamma^2_{\alpha_1,k}} \langle P_{\mathfrak{M}} z_1^k z_2^l, z_1^p z_2^q\rangle \quad \forall\, l \geqslant hM.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle T^{*h}T^h P_{\mathfrak{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle &= \langle P_{\mathfrak{M}}(z_1^k z_2^l), T^{*h}T^h(z_1^p z_2^q)\rangle \\ &= \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+hN}}{\gamma^2_{\alpha_2,q-hM}\gamma^2_{\alpha_1,p}} \langle P_{\mathfrak{M}} z_1^k z_2^l, z_1^p z_2^q\rangle \quad \forall\, q \geqslant hM.\end{aligned}$$

Since $\mathfrak{M}$ is a reducing subspace of T, the operators T^{*h} and T^h commute with $P_{\mathfrak{M}}$. If $\langle P_{\mathfrak{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle \neq 0$ for $l \geqslant hM$, $q \geqslant hM$ we have

$$\frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+hN}}{\gamma^2_{\alpha_2,l-hM}\gamma^2_{\alpha_1,k}} = \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+hN}}{\gamma^2_{\alpha_2,q-hM}\gamma^2_{\alpha_1,p}}, \tag{2.1}$$

which is equivalent to

$$\frac{(1+l)^{\alpha_2}(1+p)^{\alpha_1}}{(1+q)^{\alpha_2}(1+k)^{\alpha_1}} = \frac{(1+l-hM)^{\alpha_2}(1+p+hN)^{\alpha_1}}{(1+q-hM)^{\alpha_2}(1+k+hN)^{\alpha_1}}. \tag{2.2}$$

(1) If $(k,l) \in E_1 \cup E_2 \cup E_3$, we only need to show that the equation (2.2) holds if and only if $p = k$ and $q = l$.

(i) If $(k,l) \in E_2$, then $l \geqslant 2M$. By the assumption, $T^{*h}T^h$ commutes with $P_{\mathfrak{M}}$. Then the equations (2.1) and (2.2) show that

$$f(0) = f(1) = f(2),$$

where

$$f(x) = \Big(\frac{(1+l)/M - x}{(1+q)/M - x}\Big)^{\alpha_2} \Big(\frac{(1+p)/N + x}{(1+k)/N + x}\Big)^{\alpha_1}$$

with $|\alpha_1| = |\alpha_2|$. By Lemma 2.1, we get

$$\frac{1+l}{M} = \frac{1+q}{M}, \quad \frac{1+p}{N} = \frac{1+k}{N},$$

which is equivalent to $p = k$ and $q = l$.

(ii) If $(k,l) \in E_1$, then $k \geqslant 2N$. By the assumption, T^hT^{*h} commutes with $P_{\mathfrak{M}}$. Then a detailed computation like equations (2.1) and (2.2) show that

$$f(0) = f(-1) = f(-2),$$

which leads to $p = k$ and $q = l$ by Lemma 2.1.

(iii) If $(k,l) \in E_3$, then $M \leqslant l < 2M$ and $N \leqslant k < 2N$. We consider that T^*T and TT^* both commute with $P_{\mathfrak{M}}$. Then a detailed computation shows that

$$f(0) = f(-1) = f(1),$$

which also leads to $p = k$ and $q = l$ by Lemma 2.1. Therefore, the statement (1) holds.

(2) If $(k,l) \in E_4$, the statement (2) holds by showing $P_{\mathfrak{M}} z_1^k z_2^l \perp \mathfrak{M}_i$ where $i = 1,2,3,5$, which is implied by the fact that for $(n,m) \in \bigcup_{i=1}^{3} E_i$,

$$\begin{aligned} \langle P_{\mathfrak{M}}(z_1^k z_2^l), z_1^n z_2^m \rangle &= \langle z_1^k z_2^l, P_{\mathfrak{M}}(z_1^n z_2^m) \rangle \\ &= \bar{\lambda} \langle z_1^k z_2^l, z_1^n z_2^m \rangle \quad \text{(by statement (1))} \\ &= 0 \end{aligned}$$

and for $(n,m)\in E_5$,

$$\begin{aligned}\langle P_{\mathfrak{M}}(z_1^kz_2^l), z_1^nz_2^m\rangle &= \frac{\gamma^2_{\alpha_2,l-M}\gamma^2_{\alpha_1,k}}{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+N}}\langle P_{\mathfrak{M}}T^*Tz_1^kz_2^l, z_1^nz_2^m\rangle\\ &= \frac{\gamma^2_{\alpha_2,l-M}\gamma^2_{\alpha_1,k}}{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+N}}\langle T^*P_{\mathfrak{M}}Tz_1^kz_2^l, z_1^nz_2^m\rangle\\ &= \frac{\gamma^2_{\alpha_2,l-M}\gamma^2_{\alpha_1,k}}{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+N}}\langle P_{\mathfrak{M}}Tz_1^kz_2^l, Tz_1^nz_2^m\rangle \quad (\text{since } Tz_1^nz_2^m=0)\\ &= 0.\end{aligned}$$

(3) Replacing T^*T by TT^* in the case (2), we can get the statement (3) with a similar argument. □

By Lemma 2.2, the structure of the reducing subspaces on $\bigoplus_{i=0}^{3}\mathfrak{M}_i$ is relatively clear. However, we still know little about the structure of the reducing subspaces on $\mathfrak{M}_4$ or $\mathfrak{M}_5$. In order to describe it, we introduce some notations. Given $(n,m)\in E_4$, define

$$P_{n,m}\colon\ \mathcal{D}_\alpha(\mathbb{D}^2)\to\mathfrak{M}_{n,m}$$

as the orthogonal projection, where $\mathfrak{M}_{n,m}=\text{span}\{z_1^pz_2^q\colon\ (p,q)\sim_1(n,m), (p,q)\in E_4\}$.

Similarly given $(n,m)\in E_5$, we can define the orthogonal projection

$$Q_{n,m}\colon\ \mathcal{D}_\alpha(\mathbb{D}^2)\to\mathfrak{M}_{n,m},$$

where $\mathfrak{M}_{n,m}=\text{span}\{z_1^pz_2^q\colon\ (p,q)\sim_2(n,m), (p,q)\in E_5\}$.

For $f\in\mathcal{D}_\alpha(\mathbb{D}^2)$, note that $T^*P_{n,m}f=0$, $T^2P_{n,m}f=0$ and

$$T^*TP_{n,m}f=\frac{\gamma^2_{\alpha_2,m}\gamma^2_{\alpha_1,n+N}}{\gamma^2_{\alpha_2,m-M}\gamma^2_{\alpha_1,n}}P_{n,m}f,$$

and we have

$$[P_{n,m}f]=\text{span}\{P_{n,m}f, TP_{n,m}f\}. \tag{2.3}$$

Similarly, we have

$$[Q_{n,m}f]=\text{span}\{Q_{n,m}f, T^*Q_{n,m}f\}. \tag{2.4}$$

Lemma 2.3. *Let $\mathfrak{M}\perp\mathfrak{M}_0$ be the reducing subspace of T and $(n,m)\in E_4$. Then the following statements hold.*

(1) *$P_{n,m}P_{\mathfrak{M}} = P_{\mathfrak{M}}P_{n,m}$ and $Q_{n+N,m-M}P_{\mathfrak{M}} = P_{\mathfrak{M}}Q_{n+N,m-M}$. Thus if $f \in \mathfrak{M}$, then $[P_{n,m}f] \subseteq \mathfrak{M}$ and $[Q_{n+N,m-M}f] \subseteq \mathfrak{M}$.*

(2) *If $f_1, f_2 \in P_{n,m}\mathfrak{M}$ and $f_1 \perp f_2$, then $[f_1] \perp [f_2]$.*

(3) *If $f \in \mathfrak{M}$, then $P_{n,m}T^*f = T^*Q_{n+N,m-M}f$ and $TP_{n,m}f = Q_{n+N,m-M}Tf$.*

(4) *If $f \in \mathfrak{M}$, then $[P_{n,m}f] = [Q_{n+N,m-M}Tf]$ and $[Q_{n+N,m-M}f] = [P_{n,m}T^*f]$.*

(5) *$P_{n,m}\mathfrak{M} \oplus Q_{n+N,m-M}\mathfrak{M} \subseteq \mathfrak{M}$ is a reducing subspace of T.*

Proof. By Lemma 2.2, we have $P_{\mathfrak{M}}z_1^k z_2^l \in E_4$ if $(k,l) \in E_4$ and $P_M z_1^k z_2^l \in E_4^{\perp}$ if $(k,l) \notin E_4$, which implies that

$$P_{\mathfrak{M}}P_{n,m} = P_{n,m}P_{\mathfrak{M}}.$$

Thus, $P_{n,m}f \in \mathfrak{M}$. It follows that $[P_{n,m}f] \subseteq \mathfrak{M}$. Similarly, we get $Q_{n+N,m-M}P_{\mathfrak{M}} = P_{\mathfrak{M}}Q_{n+N,m-M}$ and $[Q_{n+N,m-M}f] \subseteq \mathfrak{M}$. So, statement (1) holds.

By equation (2.3), we have $[f_i] = \operatorname{span}\{f_i, Tf_i\}$ since $f_i \in P_{n,m}\mathfrak{M}$ for $i = 1$ or 2. Note that since $Tf_i \in \mathfrak{M}_5$ if $f_i \in \mathfrak{M}_4$

$$Tf_i \perp f_j \tag{2.5}$$

for $i, j = 1$ or 2. Also we get

$$Tf_i \perp Tf_j \tag{2.6}$$

by the fact that

$$\langle Tf_1, Tf_2\rangle = \langle T^*Tf_1, f_2\rangle = \frac{\gamma^2_{\alpha_2,m}\gamma^2_{\alpha_1,n+N}}{\gamma^2_{\alpha_2,m-M}\gamma^2_{\alpha_1,n}}\langle f_1, f_2\rangle = 0.$$

Then statement (2) holds by equations (2.5) and (2.6).

Write $f = \sum\limits_{i,j\in\mathbb{N}} a_{i,j}z_1^i z_2^j \in \mathfrak{M}$. Recall that

$$Tz_1^k z_2^l = \frac{\gamma^2_{\alpha_2,l}}{\gamma^2_{\alpha_2,l-M}} z_1^{k+N} z_2^{l-M}.$$

Then $TP_{n,m}f = Q_{n+N,m-M}Tf$ holds since

$$TP_{n,m}f = T\sum_{(i,j)\sim_1(n,m)} a_{i,j}z_1^i z_2^j = \sum_{(i,j)\sim_1(n,m)} a_{i,j}\frac{\gamma^2_{\alpha_2,j}}{\gamma^2_{\alpha_2,j-M}} z_1^{i+N} z_2^{j-M}$$

and

$$\begin{aligned} Q_{n+N,m-M}Tf &= Q_{n+N,m-M}\sum_{i,j\in\mathbb{N}} a_{i,j}\frac{\gamma^2_{\alpha_2,j}}{\gamma^2_{\alpha_2,j-M}} z_1^{i+N} z_2^{j-M} \\ &= \sum_{(i+N,j-M)\sim_2(n+N,m-M)} a_{i,j}\frac{\gamma^2_{\alpha_2,j}}{\gamma^2_{\alpha_2,j-M}} z_1^{i+N} z_2^{j-M} \\ &= \sum_{(i,j)\sim_1(n,m)} a_{i,j}\frac{\gamma^2_{\alpha_2,j}}{\gamma^2_{\alpha_2,j-M}} z_1^{i+N} z_2^{j-M}. \end{aligned}$$

We may prove the second half of the statement (3) in a similar way.

By equations (2.3), (2.4), statement (3) and

$$T^*TP_{n,m}f = \frac{\gamma^2_{\alpha_2,m}\gamma^2_{\alpha_1,n+N}}{\gamma^2_{\alpha_2,m-M}\gamma^2_{\alpha_1,n}}P_{n,m}f,$$

we have

$$\begin{aligned} [Q_{n+N,m-M}Tf] &= \operatorname{span}\{Q_{n+N,m-M}Tf, T^*Q_{n+N,m-M}Tf\} \\ &= \operatorname{span}\{TP_{n,m}f, T^*TP_{n,m}f\} \\ &= \operatorname{span}\{TP_{n,m}f, P_{n,m}f\} = [P_{n,m}f] \end{aligned}$$

and

$$\begin{aligned} [P_{n,m}T^*f] &= \operatorname{span}\{P_{n,m}T^*f, TP_{n,m}T^*f\} \\ &= \operatorname{span}\{T^*Q_{n+N,m-M}f, TT^*Q_{n+N,m-M}f\} \\ &= \operatorname{span}\{T^*Q_{n+N,m-M}f, Q_{n+N,m-M}f\} = [Q_{n+N,m-M}f]. \end{aligned}$$

Thus, statement (4) holds.

By statement (1), we obtain $P_{n,m}\mathfrak{M}\oplus Q_{n+N,m-M}\mathfrak{M}\subseteq\mathfrak{M}$. Noticing that $TQ_{n+N,m-M}\mathfrak{M}=\{0\}$, $T^*P_{n,m}\mathfrak{M}=\{0\}$ and statement (3), it follows that statement (5) holds since

$$\begin{aligned} T(P_{n,m}\mathfrak{M}\oplus Q_{n+N,m-M}\mathfrak{M}) &= TP_{n,m}\mathfrak{M}\oplus TQ_{n+N,m-M}\mathfrak{M} = TP_{n,m}\mathfrak{M} \\ &= Q_{n+N,m-M}T\mathfrak{M}\subseteq Q_{n+N,m-M}\mathfrak{M} \\ &\subseteq P_{n,m}\mathfrak{M}\oplus Q_{n+N,m-M}\mathfrak{M} \end{aligned}$$

and

$$\begin{aligned} T^*(P_{n,m}\mathfrak{M}\oplus Q_{n+N,m-M}\mathfrak{M}) &= T^*P_{n,m}\mathfrak{M}\oplus T^*Q_{n+N,m-M}\mathfrak{M} = T^*Q_{n+N,m-M}\mathfrak{M} \\ &= P_{n,m}T^*\mathfrak{M}\subseteq P_{n,m}\mathfrak{M}\subseteq P_{n,m}\mathfrak{M}\oplus Q_{n+N,m-M}\mathfrak{M}. \end{aligned}$$

□

Theorem 2.4. *Let $\mathfrak{M} \perp \mathfrak{M}_0$ be the reducing subspace of T on the bidisk. Then $\mathfrak{M} = M_1 \oplus M_2$, where*

(1) $M_1 = \bigoplus\limits_{(p,q)\in\Lambda} [z_1^p z_2^q]$ *with* $\Lambda = \{(p,q) \in E_1 \cup E_2 \cup E_3 \colon\ z_1^p z_2^q \in \mathfrak{M}\}$,

(2) M_2 *is a direct sum of minimal reducing subspace* $[f]$ *with* $f \in P_{n,m}\mathfrak{M}$ *for some* $(n,m) \in E_4$.

P r o o f. Firstly, we claim that $\mathfrak{M} = M_1 \oplus \bigoplus\limits_{(n,m)\in E} H_{n,m}$, where E is the partition of E_4 by the equivalence $\sim_1$ and $H_{n,m} = P_{n,m}\mathfrak{M} \oplus Q_{n+N,m-M}\mathfrak{M}$.

By Lemma 2.2, statement (1) for each $(p,q) \in \Lambda$ we have that $z_1^p z_2^q \in \mathfrak{M}$ and $[z_1^p z_2^q] \subseteq \mathfrak{M}$ is a minimal reducing subspace of T. Note that $\bigoplus\limits_{(n,m)\in E} H_{n,m} \subseteq \mathfrak{M}$ by Lemma 2.3, statement (5), it follows that $M_1 \cup \bigoplus\limits_{(n,m)\in E} H_{n,m} \subseteq \mathfrak{M}$.

For each $g \in \mathfrak{M}$, write $g = g_1 + g_2$ with

$$g_1 = \sum_{(p,q)\in E_1\cup E_2\cup E_3} a_{p,q} z_1^p z_2^q \quad \text{and} \quad g_2 = \sum_{(p,q)\in E_4\cup E_5} a_{p,q} z_1^p z_2^q.$$

Lemma 2.2, statement (1) shows that $g_1 \in \mathfrak{M}$, which implies that $g_2 = g - g_1 \in \mathfrak{M}$. It follows that $g_2 = \sum\limits_{(n,m)\in E} P_{n,m} g_2 + Q_{n+N,m-M} g_2 \in \bigoplus\limits_{n,m\in E} H_{n,m}$. Therefore, $\mathfrak{M} \subseteq \mathfrak{M}_1 \oplus \bigoplus\limits_{(n,m)\in E} H_{n,m}$. So we have $\mathfrak{M} = M_1 \oplus \bigoplus\limits_{(n,m)\in E} H_{n,m}$.

To complete the proof, we only need to show that each $H_{n,m}$ is the direct sum of minimal reducing subspaces as $[f] = \operatorname{span}\{f, Tf\}$ with $f \in P_{n,m}\mathfrak{M}$.

Suppose $P_{n,m}\mathfrak{M} \neq \emptyset$. Take $0 \neq f_1 \in P_{n,m}\mathfrak{M}$, then $[f_1] = \operatorname{span}\{f_1, Tf_1\} \subseteq H_{n,m}$. If $P_{n,m}\mathfrak{M} \ominus \mathbb{C}f_1 \neq \emptyset$, take $0 \neq f_2 \in P_{n,m}\mathfrak{M} \ominus \mathbb{C}f_1$. Then $[f_2] = \operatorname{span}\{f_2, Tf_2\} \subseteq H_{n,m} \ominus [f_1]$. If $[f_1] \oplus [f_2] \neq H_{n,m}$, we continue this process. This process will stop in finite steps, since the dimension of $H_{n,m}$ is finite. The proof is complete. $\square$

Remark 2.5. If $\mathfrak{M}$ is a reducing subspace generated by $g = g_1 + g_2$, then by Theorem 2.4 $[g] = [g_1] \oplus [g_2] = [g_1] \oplus [P_{n,m}g, Q_{n+N,m-M}g]$. In fact, since $[P_{n,m}g] = \operatorname{span}\{P_{n,m}g, TP_{n,m}g\}$, by Lemma 2.3 we have

$$\begin{aligned}
[P_{n,m}g, Q_{n+N,m-M}g] &= [P_{n,m}g, T^*Q_{n+N,m-M}g] = [P_{n,m}g, P_{n,m}T^*g] \\
&= \operatorname{span}\{P_{n,m}g, TP_{n,m}g, P_{n,m}T^*g, TP_{n,m}T^*g\} \\
&= \operatorname{span}\{P_{n,m}g, Q_{n,m}Tg, P_{n,m}T^*g, TT^*Q_{n,m}g\} \\
&= \operatorname{span}\{P_{n,m}g, P_{n,m}T^*g\} \oplus \operatorname{span}\{Q_{n,m}Tg, Q_{n,m}g\}.
\end{aligned}$$

Albaseer, Shi and Lu in [1] completely describe all the reducing subspaces of $T_{z_1^N \overline{z}_2^M}$ on the common Bergman space of the bidisk. Comparing with the results in [1], Theorem 2.4 implies that $T_{z_1^N \overline{z}_2^M}$ shares the same structure of reducing subspaces on each Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1| = |\alpha_2|$, which extend the result of [1]. In other words, the structure of reducing subspaces of $T_{z_1^N \overline{z}_2^M}$ on $\mathcal{D}_\alpha(\mathbb{D}^2)$ is independent of the weight α whenever $|\alpha_1| = |\alpha_2|$.

3. The case on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1| \neq |\alpha_2|$

In this section, we will study the reducing subspace of $T_{z_1^N \overline{z}_2^M}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1| \neq |\alpha_2|$. Generally, we follow the main idea in Section 2, but it is slightly more complicated. As an analog to Lemma 2.1, we have the next lemma.

Lemma 3.1. *Suppose $\beta = |\alpha_1| + |\alpha_2|$ and*

$$f(x) = \Big(\frac{a-x}{b-x}\Big)^{\alpha_2}\Big(\frac{c+x}{d+x}\Big)^{\alpha_1}$$

with $a, b, c, d > 0$. If $f(0) = f(\lambda_1) = \ldots = f(\lambda_n)$, where $\lambda_i \neq 0$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and $n \geqslant \beta$, then $a = b$ and $c = d$.

Proof. First suppose $\alpha_1, \alpha_2 > 0$. Let $f_1 = (a-x)^{\alpha_2}(c+x)^{\alpha_1}$ and $f_2 = (b-x)^{\alpha_2}(d+x)^{\alpha_1}$, then we have

$$f(x) = \frac{f_1(x)}{f_2(x)} \quad \text{and} \quad f(0) = \frac{f_1(0)}{f_2(0)}, \quad f(\lambda_i) = \frac{f_1(\lambda_i)}{f_2(\lambda_i)} \quad \text{for } i = 1, 2, \ldots, n.$$

By the assumption, it follows that

$$f_1(0) = f_2(0)\frac{f_1(0)}{f_2(0)}, \quad f_1(\lambda_i) = f_2(\lambda_i)\frac{f_1(0)}{f_2(0)} \quad \text{for } i = 1, 2, \ldots, n.$$

Since f_1 and f_2 are both polynomials with degree $\beta = |\alpha_1| + |\alpha_2|$, it follows that $f_1(x) = f_2(x)$. Therefore, $a = b$ and $c = d$.

Now suppose $\alpha_1\alpha_2 < 0$. Without loss of generality, we may assume $\alpha_1 > 0$ and $\alpha_2 < 0$. Then

$$f(x) = \Big(\frac{b-x}{a-x}\Big)^{-\alpha_2}\Big(\frac{c+x}{d+x}\Big)^{\alpha_1}.$$

By similar discussion, we have $a = b$ and $c = d$. Thus, the desired result is obtained. $\square$

Let i, j be positive integers, observe that $\mathbb{N}^2 = E_0 \cup E_1 \cup E_2 \cup E_2' \bigcup_{3 \leqslant i+j \leqslant \beta+1} E_{i,j}$, it follows that

$$\mathcal{D}_\alpha(\mathbb{D}^2) = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_2' \bigoplus_{i+j=3}^{\beta+1} \operatorname{span}\{z_1^k z_2^l;\ (k,l) \in E_{i,j}\},$$

where

$$
\begin{aligned}
E_0 &= \{(k,l)\in\mathbb{N}^2\colon\ 0\leqslant k<N,\, 0\leqslant l<M\},\\
E_1 &= \{(k,l)\in\mathbb{N}^2\colon\ k\geqslant \beta N\},\\
E_2 &= \{(k,l)\in\mathbb{N}^2\colon\ 0\leqslant k<\beta N,\, l\geqslant \beta M\},\\
E_{i,j} &= \{(k,l)\in\mathbb{N}^2\colon\ (i-1)N\leqslant k<iN,\, (j-1)M\leqslant l<jM\} \text{ with } 1\leqslant j\leqslant i\leqslant \beta,\\
E_2' &= \mathbb{N}^2-\bigcup_{i=1}^{2}E_i-\bigcup_{3\leqslant i+j\leqslant \beta+1}E_{i,j}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{M}_0 &= \operatorname{span}\{z_1^kz_2^l\colon\ (k,l)\in E_0\},\\
\mathfrak{M}_1 &= \operatorname{span}\{z_1^kz_2^l\colon\ (k,l)\in E_1\},\\
\mathfrak{M}_2 &= \operatorname{span}\{z_1^kz_2^l\colon\ (k,l)\in E_2\},\\
\mathfrak{M}_2' &= \operatorname{span}\{z_1^kz_2^l\colon\ (k,l)\in E_2'\},\\
\mathfrak{M}_{i,j} &= \operatorname{span}\{z_1^kz_2^l\colon\ (k,l)\in E_{i,j}\}.
\end{aligned}
$$

Letting

$$
f(x)=\Big(\frac{(1+l)/M-x}{(1+q)/M-x}\Big)^{\alpha_2}\Big(\frac{(1+p)/N+x}{(1+k)/N+x}\Big)^{\alpha_1},
$$

we defined equivalence on $E_{i,j}$. For $(p,q),(k,l)\in E_{i,j}$,

(1) if $j>1$, $(p,q)\sim_{i,j}(k,l)$ if and only if $f(0)=f(1)$, which is equivalent to

$$
\frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k+N}}{\gamma^2_{\alpha_2,l-M}\gamma^2_{\alpha_1,k}}=\frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}};
$$

(2) if $j=1$, $(p,q)\sim_{i,1}(k,l)$ if and only if $f(0)=f(-1)$, which is equivalent to

$$
\frac{\gamma^2_{\alpha_2,l}\gamma^2_{\alpha_1,k-N}}{\gamma^2_{\alpha_2,l+M}\gamma^2_{\alpha_1,k}}=\frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p-N}}{\gamma^2_{\alpha_2,q+M}\gamma^2_{\alpha_1,p}}.
$$

It is easy to see that $\mathfrak{M}_0$ is a reducing subspace of T. Next, we study the orthogonal decomposition of $z_1^kz_2^l$ with respect to $\mathfrak{M}$, where $\mathfrak{M}\subset\mathcal{D}_\alpha(\mathbb{D}^2)$ and $\mathfrak{M}\perp\mathfrak{M}_0$.

Lemma 3.2. *Suppose $\mathfrak{M}$ is a reducing subspace of T and $\mathfrak{M}\perp\mathfrak{M}_0$. Let $P_{\mathfrak{M}}$ be the orthogonal projection from $\mathcal{D}_\alpha(\mathbb{D}^2)$ to $\mathfrak{M}$. Then the following statements hold.*

(1) *If $(k,l)\in E_1\cup E_2\cup E_2'$, then $P_{\mathfrak{M}}z_1^kz_2^l=\lambda z_1^kz_2^l$, where $\lambda=0$ or 1.*

(2) *If $(k,l)\in E_{i,j}$, then $P_{\mathfrak{M}}z_1^kz_2^l\in\operatorname{span}\{z_1^nz_2^m,(n,m)\in E_{i,j}\}$.*

Proof. Note that $T^{*h}T^h$ commutes with $P_{\mathfrak{M}}$ for positive integer h. If

$$\langle P_{\mathfrak{M}}(z_1^k z_2^l), z_1^p z_2^q\rangle \neq 0,$$

the same argument in Lemma 2.2 and equation (2.2) shows that for $l \geqslant hM$, $q \geqslant hM$ we get

$$\frac{(1+l)^{\alpha_2}(1+p)^{\alpha_1}}{(1+q)^{\alpha_2}(1+k)^{\alpha_1}} = \frac{(1+l-hM)^{\alpha_2}(1+p+hN)^{\alpha_1}}{(1+q-hM)^{\alpha_2}(1+k+hN)^{\alpha_1}}. \tag{3.1}$$

(1) If $(k,l) \in E_1 \cup E_2 \cup E_2'$, we only need to show that the equation (3.1) holds if and only if $p = k$ and $q = l$.

(i) If $(k,l) \in E_2$, then $l \geqslant \beta M$ with $\beta = |\alpha_1| + |\alpha_2|$. By the assumption, $T^{*h}T^h$ commutes with $P_{\mathfrak{M}}$. Then the equation (3.1) implies

$$f(0) = f(1) = \ldots = f(\beta),$$

where

$$f(x) = \Big(\frac{(1+l)/M - x}{(1+q)/M - x}\Big)^{\alpha_2}\Big(\frac{(1+p)/N + x}{(1+k)/N + x}\Big)^{\alpha_1}$$

with $|\alpha_1| \neq |\alpha_2|$. By Lemma 3.1, we get

$$\frac{1+l}{M} = \frac{1+q}{M}, \quad \frac{1+p}{N} = \frac{1+k}{N},$$

which is equivalent to $p = k$ and $q = l$.

(ii) If $(k,l) \in E_1$, then $k \geqslant \beta N$ with $\beta = |\alpha_1| + |\alpha_2|$. By the assumption, T^hT^{*h} also commutes with $P_{\mathfrak{M}}$. Then a detailed computation shows that

$$f(0) = f(-1) = \ldots = f(-\beta),$$

which also leads to $p = k$ and $q = l$ by Lemma 3.1.

(iii) If $(k,l) \in E_2'$, then (k,l) will belong to some $E_{i,j} = \{(p,q)\colon (i-1)N \leqslant p < iN,\ (j-1)M \leqslant q < jM\}$ with $j > i$. We consider $T^{*k}T^k$ and T^lT^{*l} for $1 \leqslant k < i$, $1 \leqslant l < j$ all commute with $P_{\mathfrak{M}}$. Then a detailed computation shows that

$$f(-(j-1)) = \ldots = f(-1) = f(0) = f(1) = \ldots = f(i-1).$$

This also leads to $p = k$ and $q = l$ by Lemma 3.1 since $i + j \geqslant \beta + 2$. Therefore, the statement (1) holds.

(2) We only show the case of $(k,l) \in E_{2,1}$ holds and the other case can be proved by the same way. For statement (2), it is sufficient to show that $P_{\mathfrak{M}} z_1^k z_2^l \perp \operatorname{span}\{z_1^n z_2^m\colon (n,m) \notin E_{2,1}\}$. For $(n,m) \in E_1 \cup E_2 \cup E_2'$, statement (1) shows that $P_{\mathfrak{M}} z_1^k z_2^l \perp \operatorname{span}\{z_1^n z_2^m\colon (n,m) \in E_1 \cup E_2 \cup E_2'\}$. Note that for $(n,m) \in E_{i',j'}$ with $(i',j') \neq (i,j)$, there exists some integer h satisfying one of the following:
   (a) $T^{*h}T^h z_1^n z_2^m \neq 0$ and $T^{*h}T^h z_1^k z_2^l = 0$;
   (b) $T^h T^{*h} z_1^n z_2^m \neq 0$ and $T^h T^{*h} z_1^k z_2^l = 0$.

Without loss of generality, we assume (a) holds. Then

$$\langle P_{\mathfrak{M}}(z_1^k z_2^l), T^{*h}T^h z_1^n z_2^m\rangle = \langle T^{*h}T^h P_{\mathfrak{M}}(z_1^k z_2^l), z_1^n z_2^m\rangle = \langle P_{\mathfrak{M}} T^{*h}T^h (z_1^k z_2^l), z_1^n z_2^m\rangle = 0.$$

However, a direct computation shows

$$\langle P_{\mathfrak{M}}(z_1^k z_2^l), T^{*h}T^h z_1^n z_2^m\rangle = \frac{\gamma^2_{\alpha_2,m}\gamma^2_{\alpha_1,n+hN}}{\gamma^2_{\alpha_2,m-hM}\gamma^2_{\alpha_1,n}}\langle P_{\mathfrak{M}} z_1^k z_2^l, z_1^n z_2^m\rangle.$$

Thus

$$\langle P_{\mathfrak{M}} z_1^k z_2^l, z_1^n z_2^m\rangle = 0.$$

That is, $P_{\mathfrak{M}} z_1^k z_2^l \perp z_1^n z_2^m$. This completes the proof. □

Besides the above lemma, we need further study of the structure of the reducing subspaces on $\mathfrak{M}_{i,j}$. Given $(n,m) \in E_{i,j}$, we can define the orthogonal projection

$$P^{i,j}_{n,m}\colon \mathcal{D}_\alpha(\mathbb{D}^2) \to \operatorname{span}\{z_1^p z_2^q\colon (p,q) \sim_{i,j} (n,m), (p,q) \in E_{i,j}\}.$$

For $f \in \mathcal{D}_\alpha(\mathbb{D}^2)$ and $P^{i,j}_{n,m} f \neq 0$, the minimal reducing subspace of T containing $P^{1,j}_{n,m} f$ can be represented as

$$\begin{aligned}[P^{1,j}_{n,m} f] &= \operatorname{span}\{T^{*j_1}T^{j_2} P^{1,j}_{n,m} f, j_1, j_2 = 0,1\ldots\} = \operatorname{span}\{T^{j_2-j_1} P^{1,j}_{n,m} f, j_1, j_2 = 0,1\ldots\}\\ &= \operatorname{span}\{P^{1,j}_{n,m} f, T P^{1,j}_{n,m} f, \ldots, T^{j-1} P^{1,j}_{n,m} f\},\end{aligned}$$

since $T^* P^{1,j}_{n,m} f = 0$ and $T^j P^{1,j}_{n,m} f = 0$. Moreover, we have

$$[P^{2,j}_{n,m} f] = \operatorname{span}\{T^* P^{2,j}_{n,m} f, P^{2,j}_{n,m} f, T P^{2,j}_{n,m} f, \ldots, T^{j-1} P^{2,j}_{n,m} f\}$$

and inductively

$$[P^{i,j}_{n,m} f] = \operatorname{span}\{T^{*k} P^{i,j}_{n,m} f, T^l P^{i,j}_{n,m} f,\ 1 \leqslant k \leqslant i-1,\ 0 \leqslant l \leqslant j-1\}. \tag{3.2}$$

Lemma 3.3. *Let $\mathfrak{M} \perp \mathfrak{M}_0$ be the reducing subspace of T and $(n,m) \in E_{i,j}$. Then the following statements hold.*

(1) *If $f \in \mathfrak{M}$, then $[P^{i,j}_{n,m} f] \subseteq \mathfrak{M}$.*

(2) *If $f_1, f_2 \in P^{i,j}_{n,m}\mathfrak{M}$ and $f_1 \perp f_2$, then $[f_1] \perp [f_2]$.*

(3) *If $f \in \mathfrak{M}$, then $P^{i,j}_{n,m} T^* f = T^* P^{i+1,j-1}_{n+N,m-M} f$ and $T P^{i,j}_{n,m} f = P^{i+1,j-1}_{n+N,m-M} T f$.*

(4) *If $f \in \mathfrak{M}$, then $[P^{i,j}_{n,m} f] = [P^{i+1,j-1}_{n+N,m-M} T f]$ and $[P^{i+1,j-1}_{n+N,m-M} f] = [P^{i,j}_{n,m} T^* f]$.*

(5) *$\bigoplus\limits_{k=0}^{i+j-2} P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M} \subseteq \mathfrak{M}$ is a reducing subspace of T.*

Proof. (1) By Lemma 3.2, we have

$$P_{\mathfrak{M}} z_1^k z_2^l \in \operatorname{span}\{z_1^p z_2^q, (p,q) \in E_{i,j}\} \quad \text{for } (k,l) \in E_{i,j}$$

and

$$P_{\mathfrak{M}} z_1^k z_2^l \perp \operatorname{span}\{z_1^p z_2^q, (p,q) \in E_{i,j}\} \quad \text{for } (k,l) \notin E_{i,j}.$$

It means that $P_{\mathfrak{M}} P^{i,j}_{n,m} = P^{i,j}_{n,m} P_{\mathfrak{M}}$, which implies statement (1).

(2) Note that $T^*Tf = cf$ for some nonzero constant c. By the assumption for $k_1, k_2 \in \mathbb{N}$ we have

$$\langle T^{k_1} f_1, T^{*k_2} f_2\rangle = 0, \quad \langle T^{k_1} f_1, T^{k_2} f_2\rangle = 0, \quad \langle T^{*k_1} f_1, T^{k_2} f_2\rangle = 0.$$

By equation (3.2), statement (2) holds.

(3) Write $f = \sum\limits_{(p,q)\in\mathbb{N}^2} a_{p,q} z_1^p z_2^q \in \mathfrak{M}$. Recall that since

$$T z_1^p z_2^q = \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}} z_1^{p+N} z_2^{q-M},$$

then $T P^{i,j}_{n,m} f = Q^{i+1,j-1}_{n+N,m-M} T f$ holds since

$$T P^{i,j}_{n,m} f = T \sum_{(p,q)\sim_{i,j}(n,m)} a_{p,q} z_1^p z_2^q = \sum_{(p,q)\sim_{i,j}(n,m)} a_{p,q} \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}} z_1^{p+N} z_2^{q-M}$$

and

$$\begin{aligned} Q^{i+1,j-1}_{n+N,m-M} T f &= Q^{i+1,j-1}_{n+N,m-M} \sum_{(p,q)\in\mathbb{N}^2} a_{p,q} \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}} z_1^{p+N} z_2^{q-M} \\ &= \sum_{(p+N,q-M)\sim_{i+1,j-1}(n+N,m-M)} a_{p,q} \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}} z_1^{p+N} z_2^{q-M} \\ &= \sum_{(p,q)\sim_{i,j}(n,m)} a_{p,q} \frac{\gamma^2_{\alpha_2,q}\gamma^2_{\alpha_1,p+N}}{\gamma^2_{\alpha_2,q-M}\gamma^2_{\alpha_1,p}} z_1^{p+N} z_2^{q-M}. \end{aligned}$$

We may prove the second half of the statement (3) in a similar way.

(4) By (3.2), statement (3) and $T^*TP^{i,j}_{n,m}f = cP_{n,m}f$ for some nonzero constant c, we have

$$
\begin{aligned}
&[P^{i+1,j-1}_{n+N,m-M}Tf]\\
&\qquad = \operatorname{span}\{T^{*k}P^{i+1,j-1}_{n+N,m-M}Tf, T^lP^{i+1,j-1}_{n+N,m-M}Tf,\ 1 \leqslant k \leqslant i,\ 0 \leqslant l \leqslant j-2\}\\
&\qquad = \operatorname{span}\{T^{*k}TP^{i,j}_{n,m}f, T^{l+1}P^{i,j}_{n,m}f,\ 1 \leqslant k \leqslant i,\ 0 \leqslant l \leqslant j-2\}\\
&\qquad = \operatorname{span}\{T^{*(k-1)}P^{i,j}_{n,m}f, T^{l+1}P^{i,j}_{n,m}f,\ 1 \leqslant k \leqslant i,\ 0 \leqslant l \leqslant j-2\}\\
&\qquad = \operatorname{span}\{T^{*k}P^{i,j}_{n,m}f, T^lP^{i,j}_{n,m}f,\ 1 \leqslant k \leqslant i-1,\ 0 \leqslant l \leqslant j-1\} = [P^{i,j}_{n,m}f].
\end{aligned}
$$

A similar argument shows that

$$
\begin{aligned}
&[P^{i,j}_{n,m}T^*f]\\
&\quad = \operatorname{span}\{T^{*k}P^{i,j}_{n,m}T^*f, T^lP^{i,j}_{n,m}T^*f,\ 0 \leqslant k \leqslant i-1,\ 1 \leqslant l \leqslant j-1\}\\
&\quad = \operatorname{span}\{T^{*(k+1)}P^{i+1,j-1}_{n+N,m-M}f, T^lT^*P^{i+1,j-1}_{n+N,m-M}f,\ 0 \leqslant k \leqslant i-1,\ 1 \leqslant l \leqslant j-1\}\\
&\quad = \operatorname{span}\{T^{*(k+1)}P^{i+1,j-1}_{n+N,m-M}f, T^{l-1}P^{i+1,j-1}_{n+N,m-M}f,\ 0 \leqslant k \leqslant i-1,\ 1 \leqslant l \leqslant j-1\}\\
&\quad = \operatorname{span}\{T^{*k}P^{i+1,j-1}_{n+N,m-M}f, T^lP^{i+1,j-1}_{n+N,m-M}f,\ 1 \leqslant k \leqslant i,\ 0 \leqslant l \leqslant j-2\} = [P^{i+1,j-1}_{n+N,m-M}f].
\end{aligned}
$$

Thus, statement (4) holds.

(5) By statement (1), we obtain $\bigoplus_{k=0}^{i+j-2} P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M} \subseteq \mathfrak{M}$. Notice that $TP^{i+j-1,1}_{n,m}\mathfrak{M} = \{0\}$ and $T^*P^{1,i+j-1}_{n,m}\mathfrak{M} = \{0\}$, by statements (3) and (4), it follows that statement (5) holds since

$$
\begin{aligned}
T\left(\bigoplus_{k=0}^{i+j-2} P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M}\right) &\subseteq \bigoplus_{k=0}^{i+j-3} P^{k+2,i+j-k-2}_{n+(k+1)N,m-(k+1)M}\mathfrak{M}\\
&\subseteq \bigoplus_{k=-1}^{i+j-3} P^{k+2,i+j-k-2}_{n+(k+1)N,m-(k+1)M}\mathfrak{M} = \bigoplus_{k=0}^{i+j-2} P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M}
\end{aligned}
$$

and

$$
\begin{aligned}
T^*\left(\bigoplus_{k=0}^{i+j-2} P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M}\right) &\subseteq \bigoplus_{k=1}^{i+j-2} P^{k,i+j-k}_{n+(k-1)N,m-(k-1)M}\mathfrak{M}\\
&\subseteq \bigoplus_{k=1}^{i+j-1} P^{k,i+j-k}_{n+(k-1)N,m-(k-1)M}\mathfrak{M} = \bigoplus_{k=0}^{i+j-2} P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M}.
\end{aligned}
$$

□

Remark. In the proof of statement (5) in Lemma 3.3, we also get

$$[P^{k+1,i+j-k-1}_{n+kN,m-kM}\mathfrak{M}] = [P^{l+1,i+j-l-1}_{n+lN,m-lM}\mathfrak{M}], \quad 0 \leqslant k,l \leqslant i+j-2. \tag{3.3}$$

Next we describe the structure of the reducing subspace of T.

Theorem 3.4. *Let $\mathfrak{M} \perp \mathfrak{M}_0$ be the reducing subspace of T on the bidisk. Then $\mathfrak{M} = M_1 \oplus M_2$, where*

(1) $M_1 = \bigoplus_{(p,q)\in\Lambda} [z_1^p z_2^q]$ *with* $\Lambda = \{(p,q) \in E_1 \cup E_2 \cup E_2' \colon\ z_1^p z_2^q \in \mathfrak{M}\}$,

(2) *M_2 is a direct sum of minimal reducing subspace $[f]$ with $f \in P^{i,j}_{n,m}\mathfrak{M}$ for some $(n,m) \in E_{i,j}$.*

Proof. Firstly, we claim that $\mathfrak{M} = M_1 \oplus \bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m}\mathfrak{M}$.

By Lemma 3.2, statement (1), for each $(p,q) \in \Lambda$ we have that $z_1^p z_2^q \in \mathfrak{M}$ and that $[z_1^p z_2^q] \subseteq \mathfrak{M}$ is a minimal reducing subspace of T. Noting that $\bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m}\mathfrak{M} \subseteq \mathfrak{M}$ by Lemma 3.3, statement (5), it follows that $\mathfrak{M}_1 \oplus \bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m}\mathfrak{M} \subseteq \mathfrak{M}$.

For each $g \in \mathfrak{M}$, write $g = g_1 + g_2$ with

$$g_1 = \sum_{(p,q)\in E_1\cup E_2\cup E_2'} a_{p,q} z_1^p z_2^q \quad \text{and} \quad g_2 = \sum_{(p,q)\in E_{i,j}} a_{p,q} z_1^p z_2^q.$$

Lemma 3.2, statement (1) shows that $g_1 \in \mathfrak{M}$, which implies that $g_2 = g - g_1 \in \mathfrak{M}$. It follows that

$$g_2 = \bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m} g_2 \in \bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m}\mathfrak{M}.$$

Therefore, $\mathfrak{M} \subseteq M_1 \oplus \bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m}\mathfrak{M}$. So we have $\mathfrak{M} = M_1 \oplus \bigoplus_{\substack{(n,m)\in E_{i,j}\\ 3\leqslant i+j\leqslant \beta+1}} P^{i,j}_{n,m}\mathfrak{M}$.

To complete the proof, we only need to show that each $\bigoplus_{\substack{(n,m)\in E_{i,j}\\ i+j=t}} P^{i,j}_{n,m}\mathfrak{M}$ is the direct sum of minimal reducing subspaces as $[f]$ with $f \in P^{i,j}_{n,m}\mathfrak{M}$.

Suppose $P^{i,j}_{n,m}\mathfrak{M} \neq \emptyset$ with $3 \leqslant i+j \leqslant \beta+1$ and $(n,m) \in E_{i,j}$. Take $0 \neq f_1 \in P^{i,j}_{n,m}\mathfrak{M}$. Then by equation (3.2)

$$[f_1] = \operatorname{span}\{T^{*(i-1)} f_1, \ldots, f_1, T f_1, \ldots, T^{j-1} f_1\} \subseteq \bigoplus_{\substack{(n,m)\in E_{i,j}\\ i+j=t}} P^{i,j}_{n,m}\mathfrak{M}.$$

If $P_{n,m}^{i,j}\mathfrak{M}\ominus\mathbb{C}f_1\neq\emptyset$, take $0\neq f_2\in P_{n,m}^{i,j}\mathfrak{M}\ominus\mathbb{C}f_1$. Then

$$[f_2]=\text{span}\{T^{*(i-1)}f_1,\ldots,f_1,Tf_1,\ldots,T^{j-1}f_1\}\subseteq\bigoplus_{\substack{(n,m)\in E_{i,j}\\ i+j=t}}P_{n,m}^{i,j}\mathfrak{M}\ominus[f_1].$$

If $P_{n,m}^{i,j}\mathfrak{M}\ominus\mathbb{C}f_1\ominus\mathbb{C}f_2\neq\emptyset$, we continue this process. This process will stop in finite steps, since the dimension of every $P_{n,m}^{i,j}\mathfrak{M}$ is finite. The proof is complete. $\square$

At the end of the paper, we will give an example of the reducing subspaces of $T=T_{z_1^N\bar{z}_2^M}$ on Dirichlet type spaces $\mathcal{D}_\alpha(\mathbb{D}^2)$ with $|\alpha_1|\neq|\alpha_2|$.

Example 3.5. Suppose $\alpha=(\alpha_1,\alpha_2)=(2,1)$. Let

$$f=1+z_1^4z_2^5+z_1^4z_2^{15}+z_1^9z_2^{11}+z_1^{11}z_2^{12}+z_1^{40}z_2^{50}+z_1^{50}z_2^{40},$$

and $[f]$ be the reducing subspace of $T_{z_1^{10}\bar{z}_2^{10}}$ generated by f on $\mathcal{D}_\alpha(\mathbb{D}^2)$. Then

$$[f]=[f_1]\oplus[f_2]\oplus[f_3]\oplus[f_4],$$

where

$$\begin{aligned}
&[f_1]=[1+z_1^4z_2^5]=\mathbb{C}(1+z_1^4z_2^5);\\
&[f_2]=[z_1^4z_2^{15}+z_1^9z_2^{11}]=\text{span}\{z_1^4z_2^{15}+z_1^9z_2^{11},\tfrac{8}{3}z_1^{14}z_2^5+6z_1^{19}z_2\};\\
&[f_3]=[z_1^{11}z_2^{12}]=\text{span}\{z_1z_2^{22},z_1^{11}z_2^{12},z_1^{21}z_2^2\};\\
&[f_4]=[z_1^{40}z_2^{50}]\\
&\quad=\text{span}\{z_2^{90},z_1^{10}z_2^{80},z_1^{20}z_2^{70},z_1^{30}z_2^{60},z_1^{40}z_2^{50},z_1^{50}z_2^{40},z_1^{60}z_2^{30},z_1^{70}z_2^{20},z_1^{80}z_2^{10},z_1^{90}\}.
\end{aligned}$$

Proof. Since $f_1=1+z_1^4z_2^5\in\mathfrak{M}_0$, $[f_1]=\mathbb{C}(1+z_1^4z_2^5)\subseteq[f]$ is a minimal reducing subspace of $T_{z_1^{10}\bar{z}_2^{10}}$. Thus $[f]\ominus[f_1]\perp\mathfrak{M}_0$ and $[f]\ominus[f_1]$ is a reducing subspace of $T_{z_1^{10}\bar{z}_2^{10}}$. Noting that $(40,50),(50,40)\in E_1\cup E_2\cup E_2'$, Theorem 3.4 shows that $[f_4],[f_5]\subseteq[f]$, where $f_5=z_1^{50}z_2^{40}$. Since $Tf_4=f_5$, it follows that

$$\begin{aligned}
&[f_4]=[f_5]=[z_1^{40}z_2^{50}]\\
&\quad=\text{span}\{z_2^{90},z_1^{10}z_2^{80},z_1^{20}z_2^{70},z_1^{30}z_2^{60},z_1^{40}z_2^{50},z_1^{50}z_2^{40},z_1^{60}z_2^{30},z_1^{70}z_2^{20},z_1^{80}z_2^{10},z_1^{90}\}.
\end{aligned}$$

Noting that $(4,15),(9,11)\in E_{1,2}$ and $(11,12)\in E_{2,2}$. A direct computation shows that $(4,15)\sim_{1,2}(9,11)$ and $Tf_2=\frac{8}{3}z_1^{14}z_2^5+6z_1^{19}z_2$. Lemma 3.3, statement (1) implies that $f_2=P_{4,15}^{1,2}f$ and $z_1^{11}z_2^{12}=P_{11,12}^{2,2}f$ are in $[f]$. By equation (3.2), $[f_2]=\text{span}\{f_2,Tf_2\}$ and

$$[f_3]=\text{span}\{T^*z_1^{11}z_2^{12},z_1^{11}z_2^{12},Tz_1^{11}z_2^{12}\}=\text{span}\{z_1z_2^{22},z_1^{11}z_2^{12},z_1^{21}z_2^2\}.$$

Therefore, we get the desired result by Theorem 3.4. $\square$

References

[1] *M. Albaseer, Y. Lu, Y. Shi*: Reducing subspaces for a class of Toeplitz operators on the Bergman space of the bidisk. Bull. Korean Math. Soc. *52* (2015), 1649–1660. zbl MR doi

[2] *J. Deng, Y. Lu, Y. Shi*: Reducing subspaces for a class of non-analytic Toeplitz operators on the bidisk. J. Math. Anal. Appl. *445* (2017), 784–796. zbl MR doi

[3] *C. Gu*: Reducing subspaces of non-analytic Toeplitz operators on the weighted Hardy and Dirichlet spaces of the bidisk. J. Math. Anal. Appl. *459* (2018), 980–996. zbl MR doi

[4] *K. Guo, H. Huang*: On multiplication operators on the Bergman space: Similarity, unitary equivalence and reducing subspaces. J. Oper. Theory *65* (2011), 355–378. zbl MR

[5] *K. Guo, S. Sun, D. Zheng, C. Zhong*: Multiplication operators on the Bergman space via the Hardy space of the bidisk. J. Reine Angew. Math. *628* (2009), 129–168. zbl MR doi

[6] *D. Jupiter, D. Redett*: Multipliers on Dirichlet type spaces. Acta Sci. Math. *72* (2006), 179–203. zbl MR

[7] *H. Lin*: Reducing subspaces of Toeplitz operators on the Dirichlet type spaces of the bidisk. Turk. J. Math. *42* (2018), 227–242. zbl MR doi

[8] *Y. Lu, X. Zhou*: Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk. J. Math. Soc. Japan *62* (2010), 745–765. zbl MR doi

[9] *Y. Shi, Y. Lu*: Reducing subspaces for Toeplitz operators on the polydisk. Bull. Korean Math. Soc. *50* (2013), 687–696. zbl MR doi

[10] *M. Stessin, K. Zhu*: Reducing subspaces of weighted shift operators. Proc. Am. Math. Soc. *130* (2002), 2631–2639. zbl MR doi

[11] *X. Zhou, Y. Shi, Y. Lu*: Invariant subspaces and reducing subspaces of weighted Bergman space over polydisc. Sci. Sin., Math. *41* (2011), 427–438. doi

[12] *K. Zhu*: Reducing subspaces for a class of multiplication operators. J. Lond. Math. Soc., II. Ser. *62* (2000), 553–568. zbl MR doi

Authors' address: Hongzhao Lin (corresponding author), Zhongming Teng, College of Computer and Information Sciences, Fujian Agriculture and Forestry University, Fujian 350002, P. R. China, e-mail: `hzl_076@163.com`, `peter979@163.com`.