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Applications of Mathematics, Vol. 66 (2021), No. 1, 145–176

Persistent URL: `http://dml.cz/dmlcz/148514`

KINETIC BGK MODEL FOR A CROWD: CROWD CHARACTERIZED BY A STATE OF EQUILIBRIUM

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Received June 17, 2019. Published online October 21, 2020.

Abstract. This article focuses on dynamic description of the collective pedestrian motion based on the kinetic model of Bhatnagar-Gross-Krook. The proposed mathematical model is based on a tendency of pedestrians to reach a state of equilibrium within a certain time of relaxation. An approximation of the Maxwellian function representing this equilibrium state is determined. A result of the existence and uniqueness of the discrete velocity model is demonstrated. Thus, the convergence of the solution to that of the continuous BGK equation is proven. Numerical simulations are presented to validate the proposed mathematical model.

Keywords: discrete kinetic theory; crowd dynamics; BGK model; semi-Lagrangian schemes

MSC 2020: 35A01, 35A02, 97M70, 97N40

1. Introduction

Mathematical representations of crowd motion from the microscopic to macroscopic scale have been an active field of study for the last three decades. An overview of the most common models at different scales (microscopic, macroscopic, or mesoscopic) is presented in [11]. Indeed, the most popular crowd simulation models are the individual models, namely the heuristic rule-based models [35], mechanical models [18], [20], [21], and cellular automate [31], continuous models are based on fluid dynamics [19], [23], [10], and the kinetic (gas-kinetic) models are intermediate models between the two discrete and continuous models [25], [6], [7]. Handerson was the first to apply this type of "kinetic gas" model to empirical pedestrian crowd data [22], [24].

DOI: 10.21136/AM.2020.0153-19

In 2011, Bellomo et al. [3], [2], [1], [4], [5], [9] developed the kinetic approach for crowds in a recent approach called the kinetic theory for active particles. This approach considers the crowd as a complex system. The microscopic state of each particle is characterized by a geometric variable, its position $\mathbf{x} = (x, y)$ and a mechanical variable, its velocity $\mathbf{v} = (v_x, v_y)$. In addition, there is a microscopic state related to their socio-biological behavior, called activity, and denoted u. The representation of the system is defined by a distribution function denoted $f(t, \mathbf{x}, \mathbf{v}, u)$, where $f(t, \mathbf{x}, \mathbf{v}, u)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}$ represents the number of pedestrians who at the instant t are in the elementary volume $[\mathbf{x}, \mathbf{x} + \mathrm{d}\mathbf{x}] \times [\mathbf{v}, \mathbf{v} + \mathrm{d}\mathbf{v}]$ and who have activity u. Pedestrian movements are governed by the Partial Derivative Equation (PDE) of transport applied to f:

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v}, u)}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}, u) = \Gamma[f](t, \mathbf{x}), \tag{1.1}$$

where $\Gamma[f](t, \mathbf{x})$ characterizes the different interactions between pedestrians and their environment.

In our previous work [16], the PDE given in (1.1) was considered and its second member $\Gamma[f](t, \mathbf{x})$, which models the interactions between pedestrians with various obstacles, was treated from a probabilistic point of view. In this paper, we are interested in one of the simplest ways to model the term $\Gamma[f](t, \mathbf{x})$. It consists of describing a pedestrian tendency to equilibrium similar to the BGK operator which replaces the collision operator of the Boltzmann equation [16], specifically, the case where the pedestrian system is characterized by an equilibrium configuration f_e and a relaxation parameter $\tau[\varrho]$.

In an emergency evacuation case, pedestrians try to achieve a desired velocity noted $\mathbf{v}_d$ to reach a target. The symbol $\tau[\varrho]$ is the relaxation parameter which describes the adaptation of the density f to the equilibrium density $f_{\mathrm{eq}}(\mathbf{v})$. Therefore, the interactions term takes the following simple form:

$$\Gamma[f](t, \mathbf{x}) = \frac{1}{\tau[\varrho]}(f_{\mathrm{eq}}(\mathbf{v} \to \mathbf{v}_d) - f). \tag{1.2}$$

This paper develops a special theory for pedestrian motion and these interactions namely deceleration avoidance. Consequently, the BGK model (1.1), (1.2) proposed in this work does not use the assumptions of conservation of momentum and energy. Only the conservation of mass has to be verified in our study. The equilibrium state function of pedestrians is developed and based on Henderson works [22], [24]. A mathematical framework for a theoretical study of the proposed model is determined. The rest of this paper is organized as follows: Section 2 provides the discrete velocity model derived from the continuous BGK equation. This model describes the motion of pedestrians reaching an equilibrium configuration in a domain Ω. Then an approximation of the Maxwellian discrete density representing this state of equilibrium is presented. In Section 3, a result of existence and uniqueness for the discrete velocity model is demonstrated. Then we prove the convergence of this solution to a solution of the continuous BGK equation. Section 4 is devoted to numerical simulations to validate the proposed model, and to show its ability to describe the main features of the dynamics of pedestrians.

2. Mathematical model

2.1. Boltzmann equation: the BGK model for a crowd. We consider a system made of N particles (pedestrians) randomly distributed in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$.

At the instant $t = t_0$, the pedestrians are distributed in a disk D_0 of radius r_D and center $M_0(x_0, y_0)$. And the initial overall density is

$$\varrho_0 = \frac{N}{\pi r_D^2} (\mathrm{ped}/\mathrm{m}^2).$$

This group of N pedestrians is present in the domain Ω at initial time t_0. They are in a normal traffic situation, that is to say that they have the possibility of heading in all possible directions. Then, all pedestrians seek to reach a "comfortable" destination denoted $\mathbf{v}_d$ (state of equilibrium). The group of pedestrians have the tendency to reach a target located at the point $\mathbf{x}_c$ (see Figure 1).

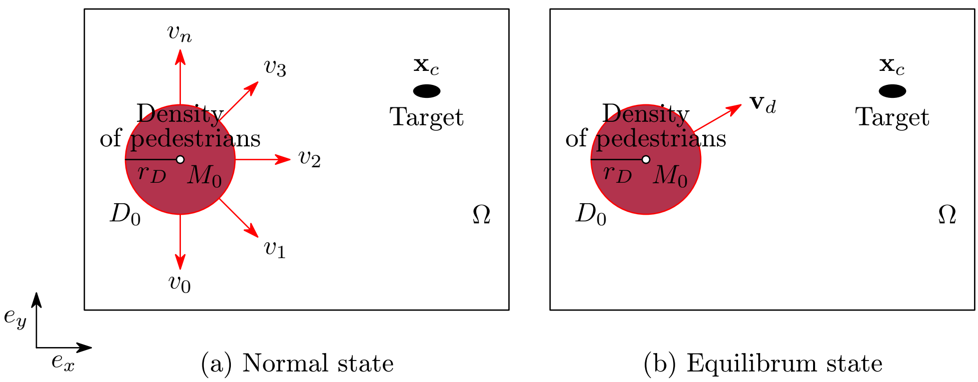


Figure 1. Density of pedestrians characterized by (a) normal state: pedestrians have the ability to direct towards n possible directions. (b) equilibrium state: all pedestrians look for a comfortable destination noted $\mathbf{v}_d$ to reach the target located at the point $\mathbf{x}_c$.

The state of the crowd is represented by the density $f(t, \mathbf{x}, \mathbf{v})$. They move with a velocity $\mathbf{v} \in D_{\mathbf{v}}$.

The average crowd quantities obtained by integrating f in the velocity space $D_{\mathbf{v}}$:

▷ density:

$$\varrho(t, \mathbf{x}) = \int_{D_{\mathbf{v}}} f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v},$$

▷ average velocity:

$$\boldsymbol{\xi}(t, \mathbf{x}) = \frac{1}{\varrho(t, \mathbf{x})} \int_{D_{\mathbf{v}}} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v},$$

▷ total energy:

$$E = \frac{1}{2} \int_{D_{\mathbf{v}}} \|\mathbf{v}\|^2 f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v}.$$

The evolution of the particle density $f(t, \mathbf{x}, \mathbf{v})$ is described by the following equation:

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\tau[\varrho]} (f_{\mathrm{eq}}(\mathbf{v}) - f). \tag{2.1}$$

Here f_{eq} denotes the equilibrium distribution function that may be parametrized by the local density ϱ and by the desired direction $\mathbf{v}_d$ towards the target, $f_{\mathrm{eq}} = f_{\mathrm{eq}}(\mathbf{v}, \mathbf{v}_d, \varrho)$, where $f_{\mathrm{eq}}(\mathbf{v}, \mathbf{v}_d, \varrho) \, \mathrm{d}\mathbf{v}$ represents the number of pedestrians who move towards desired direction $\mathbf{v}_d$. However, the normal distribution is given by $f(t, \mathbf{x}, \mathbf{v})$, where $f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{v}$ represents the number of pedestrians who at the moment t are in the elementary domain $[\mathbf{x}, \mathbf{x} + \mathrm{d}\mathbf{x}]$, and they have the ability to direct themselves towards all possible directions in $[\mathbf{v}, \mathbf{v} + \mathrm{d}\mathbf{v}]$.

The coefficient τ can depend on the density $\varrho(t, \mathbf{x})$; this term expresses that the distribution f would not go instantly to the desired velocity distribution f_{eq}, but would need some time called relaxation time τ. For reasons of simplicity, we assume that this relaxation term is a constant, i.e. $\tau[\varrho] = \tau$. According to this hypothesis, model (2.1) takes the form

$$\frac{\partial f(t, \mathbf{x}, \mathbf{v})}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) = \frac{1}{\tau} (f_{\mathrm{eq}}(\mathbf{v}) - f(t, \mathbf{x}, \mathbf{v})), \tag{2.2}$$

to which we add the following initial conditions:

$$f(t = 0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}), \quad \mathbf{x} \in \Omega, \ \mathbf{v} \in D_{\mathbf{v}}. \tag{2.3}$$

The system defined by equations (2.2), (2.3) represents the Bhatnagar-Gross-Krook (BGK) model. In this paper it describes the temporal evolution of the distribution of particles (pedestrians). This model is less expensive than the Boltzmann equation, because it is sufficient to update the macroscopic fields at each time step. On the other hand, it provides qualitatively correct solutions for macroscopic moments. These two aspects, namely the relatively low cost of calculation and the correct description of the hydrodynamic limit, explain the interest in the BGK model during the last decades.

It also shares important features with Boltzmann's original equation, such as the following conservation laws:

$$\text{conservation of mass} \quad \int_{D_{\mathbf{v}}} f_{\mathrm{eq}}(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_{D_{\mathbf{v}}} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}. \tag{2.4}$$

Thus, the BGK equation is a kinetic collision equation that takes into account only the overall effect of pedestrian interactions.

Remark 2.1. The BGK model for pedestrian motion contains corrections due to interactions such as avoidance and deceleration. Therefore, this model does not obey the conservation of momentum and energy. In our case, the only law of conservation which must be respected by the model is the conservation of the mass given by (2.4).

2.2. Maxwellian approximation: equilibrium density $f_{\mathrm{eq}}(\mathbf{v})$. In 1971 Henderson [22] suggested that the motion of people in a crowd represents a system similar to a gas molecule collection. Specifically, he suggested that the classical Maxwell-Boltzmann theory usually applied to molecular system could also describe the velocity distribution of individuals.

Henderson measured the speed distribution function for three crowd categories: a crowd of university students walking from the library to the university, a crowd of adults of all ages using a pedestrian crossing on a street, and a group of children in an outdoor playground. Analysis and estimates are made under some assumptions about the crowd, namely, the movement is defined at all times t by its position and its velocity $\mathbf{v} = (v_x, v_y)$. All the individuals in the crowd have the same mass.

The two Figures 2 (a), (b) show that Henderson's empirical results agree with the classical Maxwell-Boltzmann theory. The distribution of the v_x component of the velocity is given by the equation

$$f_{\mathrm{eq}}(v_x) = \frac{1}{N}\frac{\mathrm{d}N_{v_x}}{\mathrm{d}v_x} = \frac{1}{\sqrt{2\pi}v_m}\exp\Big(-\frac{1}{2}\frac{v_x^2}{v_m^2}\Big), \tag{2.5}$$

where

- ▷ v_m is the square root of the average module value of speed;
- ▷ N is the total number of pedestrians;
- ▷ N_{v_x} is the number of pedestrians with speed v_x.

In a similar way, for the equation of the v_y component distribution of velocity, Henderson found the result for the distribution of $\mathbf{v} = (v_x, v_y)$ given by the equation

$$f_{\mathrm{eq}}(\mathbf{v}) = \frac{1}{N}\frac{\mathrm{d}N_{\mathbf{v}}}{\mathrm{d}\mathbf{v}} = \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\mathbf{v}^2}{v_m^2}\Big), \tag{2.6}$$

where $N_{\mathbf{v}}$ is the number of pedestrians with velocity $\mathbf{v}$.

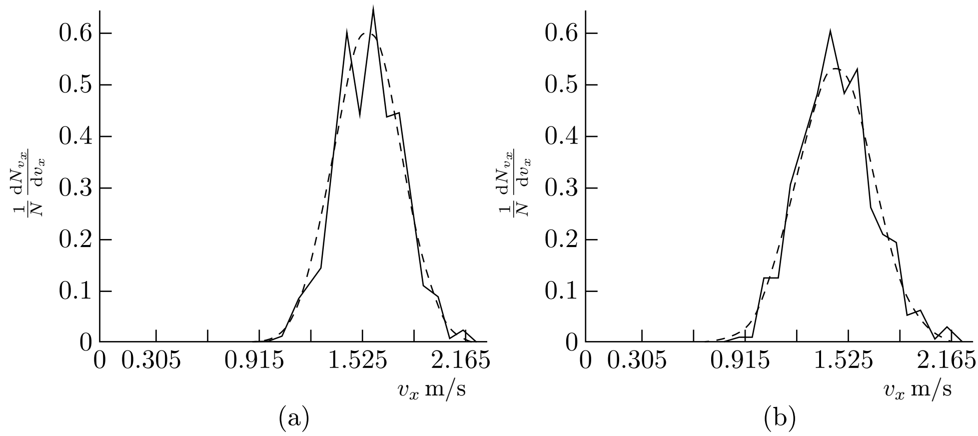


Figure 2. The density function of the first component v_x of the speed curve a represents the measured distribution and curve b represents the Maxwell-Boltzmann distribution. Two cases are studied: Case (a) for 693 students walking outside the library at the University of Sydney with $v_x = 1.53\,\mathrm{m/s}$ and $v_m = 0.201\,\mathrm{m/s}$, Case (b) for 628 pedestrians on a pedestrian crossing in Sydney with $v_x = 1.44\,\mathrm{m/s}$ and $v_m = 0.228\,\mathrm{m/s}$.

According to the above and the last remark, the density of equilibrium for pedestrians f_{eq}, which models the equilibrium state that each pedestrian wishes to achieve, is defined by

$$f_{\mathrm{eq}}(\mathbf{v}, \varrho) = \frac{\varrho(t,\mathbf{x})}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}-\mathbf{v}_d\|^2}{v_m^2}\Big), \tag{2.7}$$

where $v_m = \sqrt{\frac{1}{2}\int_{D_{\mathbf{v}}} \|\mathbf{v}\|^2 f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}}$ and $\mathbf{v}_d$ is the desired direction. The density f_{eq} can be seen as the probability density function of the Gauss probability distribution for each instant t and position $\mathbf{x}$ fixed.

Indeed, (2.7) allows all pedestrians to reach a comfortable destination defined by the direction $\mathbf{v}_d$.

The density of equilibrium that we have defined satisfies the mass conservation law defined by equation (2.4):

$$\int_{D_{\mathbf{v}}} f_{\mathrm{eq}}(\mathbf{v})\,\mathrm{d}\mathbf{v} = \int_{D_{\mathbf{v}}} \frac{\varrho(t,\mathbf{x})}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}-\mathbf{v}_d\|^2}{v_m^2}\Big)\,\mathrm{d}\mathbf{v} = \varrho(t,\mathbf{x}) = \int_{D_{\mathbf{v}}} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}.$$

2.3. Model of discrete velocities. As noticed by Keith Still in his book [38] from safety guidance guides, the crowd density, expressed as people per square meter, should not exceed four people per square meter for moving/dynamic areas (queues, slow-moving crowds) and 4.7 people per square meter for standing/static area.

That is why the number of pedestrians N is generally insufficient to justify the hypothesis of continuity of the particle distribution function $f(t, \mathbf{x}, \mathbf{v})$ with respect to velocity. Thus, for numerical simulations, a discrete velocity approximation of the BGK equation is introduced. We refer to the discrete velocity models for the Boltzmann equation developed by Rogier and Schneider [37], Buet [8], Heintz and Panferov [32], and Mieussens [29]. The proposed approximation in this work has the same conservation properties as the continuous BGK model.

Let $\mathcal{K}$ be a set of multi-indices of $\mathbb{Z}^2$, defined by $\mathcal{K} = \{\mathbf{k} = (k_1, k_2), |\mathbf{k}| \leqslant B\}$, where B is a scalar.

We define $\mathcal{V} \subset \mathbb{R}^2$, a set of N_v discrete velocities, $\mathcal{V} = \{\mathbf{v_k} = \mathbf{k}\Delta v / \mathbf{k} \in \mathcal{K}\}$, where Δv is a scalar.

The distribution f of the continuous velocities is then replaced by the N_v vector $f_{\mathcal{K}} = (f_{\mathbf{k}}(t, \mathbf{x}))_{\mathbf{k} \in \mathcal{K}}$, where each component $f_{\mathbf{k}}(t, \mathbf{x})$ is an approximation of the function $f(t, \mathbf{x}, \mathbf{v_k})$.

Thanks to the previous discretization of the velocity, we define the local density by the following equation:

$$\varrho(t, \mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{K}} f_{\mathbf{k}}(\mathbf{x}, t). \tag{2.8}$$

The discrete kinetic model associated with the equation BGK (2.2) is defined by the set of equations,

$$\frac{\partial f_{\mathbf{k}}(t, \mathbf{x})}{\partial t} + \mathbf{v_k} \cdot \nabla_{\mathbf{x}} f_{\mathbf{k}}(t, \mathbf{x}) = \frac{1}{\tau}(f_{\mathrm{eq},\mathbf{k}}(\mathbf{v_k}, \varrho) - f_{\mathbf{k}}(t, \mathbf{x})), \quad \mathbf{k} \in \mathcal{K} \text{ and } \mathbf{v_k} \in \mathcal{V}, \tag{2.9}$$

where $f_{\mathrm{eq},\mathbf{k}}$ is an approximation of the equilibrium density defined by (2.7).

The main problem is to define this approximation of the discrete density $f_{\mathrm{eq},\mathbf{k}}$ such that the property of mass conservation is satisfied. We used the natural approximation first proposed by Yang and Huang [40] and then developed by Mieussens [28]:

$$f_{\mathrm{eq},\mathbf{k}}(\mathbf{v_k}, \varrho) = f_{\mathrm{eq}}(\mathbf{v_k}, \varrho), \quad \mathbf{k} \in \mathcal{K}, \tag{2.10}$$

hence,

$$f_{\mathrm{eq},\mathbf{k}}(\mathbf{v_k}, \varrho) = \frac{\varrho(t, \mathbf{x})}{2\pi v_m^2} \exp\Big(-\frac{1}{2} \frac{\|\mathbf{v_k} - \mathbf{v}_d\|^2}{v_m^2}\Big), \tag{2.11}$$

where

$$v_m = \sqrt{\frac{1}{\varrho(t,\mathbf{x})} \sum_{\mathbf{k}\in\mathcal{K}} \|\mathbf{v_k}\|^2 f_{\mathbf{k}}(t,\mathbf{x})}.$$

The mathematical model (2.9) is then considered with an initial condition defined by:

$$f_{\mathbf{k}}(t=0,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}), \quad \mathbf{k}\in\mathcal{K},\ \mathbf{x}\in\Omega. \tag{2.12}$$

Boundary condition on $\partial\Omega$. This study focuses on the adaptation of a pedestrian equilibrium situation within the domain Ω. Their target is assumed to be inside the domain. The disk diameter $2r_D$ occupied by pedestrians is always less than the distance between the target in $\mathbf{x}_c$ and the edge of the domain $\partial\Omega$, i.e.

$$2r_D < d(\mathbf{x}_c, \partial\Omega).$$

According to this assumption, the theoretical study of our problem is extended to the whole plane $\mathbb{R}^2$ instead of Ω.

3. Theoretical study of the proposed mathematical model

Some important mathematical results concerning the BGK equation have been obtained during the last decade. For example, Perthame [33] has proved the existence and stability of a distribution solution throughout the space. This result has been extended to a bounded domain with various boundary conditions by Ringeisen [36]. More recently, Perthame and Pulvirenti [34] have proved the existence and uniqueness of a "mild" solution with weighted estimates in $\mathbb{L}^\infty$. We also mention the result of Issautier [26] who proved that the "mild" solution of Perthame and Pulvirenti is strong if some assumptions of regularity on the initial condition are made. However, it is important to note that in all these studies, the authors assume that the relaxation time is constant (i.e. $\tau = 1$).

In this paper, we are interested in the existence and uniqueness of the BGK model with a different source or a different equilibrium density to that defined in the case of fluid dynamics, namely a density that is suitable for pedestrian movement. In addition, we assume that $\tau = 1$.

It is interesting to study the convergence of such an approximation to the continuous BGK equation. We refer to Mischler's proof [30] for the convergence of a discrete velocity model for the Boltzmann equation. There are essentially two distinct points to prove:

- ▷ the existence and uniqueness of a discrete velocity model solution;
- ▷ the convergence of the discrete kinetic equation towards the continuous equation.

3.1. Existence and uniqueness of the model solution. To define a discrete velocity model of approximation given in (2.9), the following notations are considered:

$\mathcal{V}^n$ is a grid of N_n velocities defined by $\mathcal{V}^n = \{\mathbf{v}_\mathbf{k}^n = \mathbf{k}\Delta v_n/\mathbf{k} \in \mathcal{K}^n\}$, where $\mathcal{K}^n = \{\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2,\ |\mathbf{k}| \leqslant B_n\}$. In the rest of the paper, we replace $\mathcal{K}^n$ by $\mathcal{K}$.

Δv_n, B_n are two real suites, assumed such that:

$$\Delta v_n \underset{n\to\infty}{\longrightarrow} 0, \quad \Delta v_n B_n \underset{n\to\infty}{\longrightarrow} \infty.$$

The velocity's cells $\mathcal{I}_\mathbf{k}^n$ are defined by $\mathcal{I}_\mathbf{k}^n = [v_{k_1}^n, v_{k_1}^n + \frac{1}{2}\Delta v_n[\times[v_{k_2}^n, v_{k_2}^n + \frac{1}{2}\Delta v_n[$.

The discrete velocity model approximation (2.9) is then given by the following system:

$$\tag{3.1} \begin{cases} \dfrac{\partial f_\mathbf{k}^n(t,\mathbf{x})}{\partial t} + \mathbf{v}_\mathbf{k}^n \cdot \nabla_\mathbf{x} f_\mathbf{k}^n(t,\mathbf{x}) \\ \qquad\qquad = f_{\mathrm{eq},\mathbf{k}}^n(\mathbf{v}_\mathbf{k}, \varrho) - f_\mathbf{k}^n(t,\mathbf{x}))\mathcal{D}'([0,T]\times\mathbb{R}_\mathbf{x}^2), & \mathbf{k}\in\mathcal{K}, \\ f_\mathbf{k}^n(t=0,\mathbf{x}) = f_{0,\mathbf{k}}^n(\mathbf{x}), & \mathbf{k}\in\mathcal{K}, \end{cases}$$

where $f_{0,\mathbf{k}}^n$ is an approximation of the initial density f_0, and

$$\tag{3.2} \mathcal{D}'([0,T]\times\mathbb{R}_\mathbf{x}^2) = \{W\colon\ \mathcal{D}([0,T]\times\mathbb{R}_\mathbf{x}^2) \longrightarrow \mathbb{R},\ W \text{ continuous, linear}\},$$

where

$$\tag{3.3} \mathcal{D}([0,T]\times\mathbb{R}_\mathbf{x}^2) = \{f \in \mathcal{C}^\infty([0,T]\times\mathbb{R}_\mathbf{x}^2)\colon\ \operatorname{supp}(f),\ \text{compact}\}.$$

Our goal is to show the existence and uniqueness of the solution for the model given in (3.1).

We consider the characteristic curves associated with the system in (3.1), which are given by

$$\tag{3.4} \boldsymbol{\gamma}_\mathbf{k}(t) = (\gamma_{k_1}(t), \gamma_{k_2}(t)) = (x + tv_{k_1}^n, y + tv_{k_2}^n), \quad \mathbf{k} = (k_1,k_2)\in\mathcal{K}.$$

These curves are solutions of the system

$$\tag{3.5} \begin{cases} \dfrac{\mathrm{d}\boldsymbol{\gamma}_\mathbf{k}(t)}{\mathrm{d}t} = \mathbf{v}_\mathbf{k}^n, & \mathbf{k}\in\mathcal{K}, \\ \boldsymbol{\gamma}_\mathbf{k}(0) = (x,y)^\top, & \mathbf{k}\in\mathcal{K} \end{cases}$$

with $\mathbf{v}_\mathbf{k}^n = (v_{k_1}^n, v_{k_2}^n)$.

Along these curves, the solution of the system in (3.1) satisfies the following system of ordinary differential equations:

$$
\left\{
\begin{aligned}
&\frac{\mathrm{d} f^n_{\mathbf{k}}(t,\boldsymbol{\gamma}_{\mathbf{k}}(t))}{\mathrm{d}t} = f^n_{\mathrm{eq},\mathbf{k}}(\mathbf{v}_{\mathbf{k}},\varrho) - f^n_{\mathbf{k}}(t,\boldsymbol{\gamma}_{\mathbf{k}}(t)), \quad && \mathbf{k}\in\mathcal{K},\\
&f^n_{\mathbf{k}}(t=0,\mathbf{x}) = f^n_{0,\mathbf{k}}(\mathbf{x}), && \mathbf{k}\in\mathcal{K}.
\end{aligned}
\right. \tag{3.6}
$$

Let us pose $\hat{f}^n_{\mathbf{k}}(t,\mathbf{x}) = f^n_{\mathbf{k}}(t,\boldsymbol{\gamma}_{\mathbf{k}}(t))$ for $\mathbf{k}\in\mathcal{K}$, where $\hat{f}^n_{\mathbf{k}}$ is the value of f^n along these characteristic curves.

The "mild" form of system (3.6) is then obtained by integrating it along the characteristic curves (3.4). For $\mathbf{k}\in\mathcal{K}$ we get

$$
\hat{f}^n_{\mathbf{k}}(t,\mathbf{x}) = f^n_{0,\mathbf{k}} + \int_0^t (\hat{f}^n_{\mathrm{eq},\mathbf{k}}(\mathbf{v}_{\mathbf{k}},\varrho) - \hat{f}^n_{\mathbf{k}}(s,\mathbf{x}))\,\mathrm{d}s. \tag{3.7}
$$

For a given time, the following functional space is defined:

$$
\mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}}) = \Big\{ \mathbf{f}(t) = (f_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}},\ \|\mathbf{f}(t)\|_1 = \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}_{\mathbf{x}^2}} \|f_{\mathbf{k}}(t,\mathbf{x})\|\,\mathrm{d}\mathbf{x} < \infty \Big\}.
$$

For a time $T>0$, let us consider the Banach space $\mathbb{X}_T = \mathcal{C}([0,T],\mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}}))$ with the following norm: $\|\mathbf{f}\|_{\mathbb{X}_T} = \sup\limits_{t\in[0,T]} \|\mathbf{f}(t)\|_1$.

Our theoretical study is based on the following theorem:

Theorem 3.1 (Local existence)**.** *Let $f^n_0 = (f^n_{0,\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathbb{L}^\infty(\mathbb{R}^2_{\mathbf{x}}) \cap \mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}})$ with $f^n_0 \geqslant 0$. Then there is a time $T>0$ and a constant R such as for all $t<T$, problem* (3.1) *admits a unique solution $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathcal{C}([0,T],\mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}}))$, and which satisfies the following estimates:*

$$
\sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \Theta(T) \tag{3.8}
$$

and

$$
\varrho(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x}) \leqslant R. \tag{3.9}
$$

If furthermore $\sum\limits_{\mathbf{k}\in\mathcal{K}} \|f^n_{0,\mathbf{k}}\|_\infty < 1$, then the solution of the model has a physical meaning, i.e.:

$$
\varrho(t,\mathbf{x}) \leqslant 1. \tag{3.10}
$$

Since this theorem is independent of n, for sake of simplicity, the exponent n will be omitted in this section.

P r o o f of the theorem. The following function is first introduced:

$$\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) = \hat{f}_{\mathbf{k}}(t,\mathbf{x})\exp(\lambda t) \quad \text{for } \mathbf{k}\in\mathcal{K},\ \lambda>0.$$

Therefore, the system given in (3.6) is equivalent to the following system:

$$\begin{cases} \dfrac{\mathrm{d}\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x})}{\mathrm{d}t} = \lambda\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(t,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}), & \mathbf{k}\in\mathcal{K}, \\ \widehat{\psi}_{\mathbf{k}}(t=0,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}), & \mathbf{k}\in\mathcal{K}. \end{cases} \tag{3.11}$$

For all $t\in[0,T]$, we integrate the equation in (3.11) to deduce the following "mild" formulation:

$$\widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t (\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}))\,\mathrm{d}s \quad \forall\,\mathbf{k}\in\mathcal{K}.$$

Consider the following operator $\mathbf{A} = (\widehat{\mathbf{A}\psi_{\mathbf{k}}})_{\mathbf{k}\in\mathcal{K}}$:

$$\widehat{\mathbf{A}\psi_{\mathbf{k}}}(t,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t (\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}))\,\mathrm{d}s \quad \forall\,\mathbf{k}\in\mathcal{K}.$$

To show that system (3.11) has a solution, it is enough to show that the operator has a unique fixed point in the Banach space $\mathbb{X}_T$. Indeed, the set defined by

$$B_{T,a_0,\lambda,R} = \Big\{\widehat{\psi} = (\widehat{\psi})_{\mathbf{k}\in\mathcal{K}} \in \mathbb{X}_T\colon\ \widehat{\psi}_{\mathbf{k}} \geqslant 0,\ \|\widehat{\psi}\|_{\mathbb{X}_T} \leqslant a_0\|f_0\|_1, \\ \sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) \leqslant R\exp(\lambda t),\ t\in[0,T],\ \mathbf{x}\in\mathbb{R}^2_{\mathbf{x}}\Big\}$$

is introduced.

In the following, the parameters T, a_0, λ, R must be chosen carefully in order to show that the operator has a unique fixed point.

Lemma 3.1. *Let* $\widehat{\psi}\in B_{T,a_0,\lambda,R}$.

(1) *There is* λ_0 *such that for all* $\lambda\geqslant\lambda_0$ *we have* $(\widehat{A\psi})_{\mathbf{k}\in\mathcal{K}} \geqslant 0$.

(2) *Let* $\lambda=\lambda_0$. *There are two constants* R_0, T_0 *such that for all* $R\geqslant R_0$ *and* $t\in[0,T_0]$,

$$\sum_{\mathbf{k}\in\mathcal{K}} \widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x}) \leqslant R\exp(\lambda t).$$

(3) *Let* $C_1 = \sum\limits_{\mathbf{k}\in\mathcal{K}} (2\pi v_m^2)^{-1} + 1$, $\lambda = \lambda_0$ *and* $T < (C_1 + \lambda_0)^{-1} := T_1$. *Then there is a constant* $a_0 > 0$ *such that for all* $t \in [0, T]$

$$\|\widehat{A\psi}\|_{\mathbb{X}_T} \leqslant a_0 \|f_0\|_1. \tag{3.12}$$

(4) *Let* $\widehat{\psi^1}, \widehat{\psi}^2 \in B_{T,a_0,\lambda_0,R}$, $\lambda = \lambda_0$. *Then there exist* $C_2 > 0$ *such that*

$$\|\widehat{A\psi^1} - \widehat{A\psi^2}\|_{\mathbb{X}_T} \leqslant (\lambda_0 T + C_2 T + T)\|\widehat{\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T}.$$

P r o o f of the lemma.

(1) Since $f_{0,\mathbf{k}}(\mathbf{x}) \geqslant 0$, $\mathbf{k} \in \mathcal{K}$, $(\widehat{A\psi})_{\mathbf{k}\in\mathcal{K}} \geqslant 0$ if

$$(\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})) \geqslant 0, \quad \mathbf{k} \in \mathcal{K}.$$

We obtain

$$\begin{aligned}
&\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \\
&\quad = \lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \frac{\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_k(s,\mathbf{x})}{2\pi v_m^2} \exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \\
&\quad = \Big(\lambda + \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big) - 1\Big)\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \\
&\qquad + \frac{\sum_{\mathbf{l}\in\mathcal{K},\mathbf{l}\neq\mathbf{k}} \widehat{\psi}_{\mathbf{l}}(s,\mathbf{x})}{2\pi v_m^2} \exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big).
\end{aligned}$$

Since

$$\frac{\sum_{\mathbf{l}\in\mathcal{K},\mathbf{l}\neq\mathbf{k}} \widehat{\psi}_{\mathbf{l}}(s,\mathbf{x})}{2\pi v_m^2} \exp\Big(-\frac{1}{2}\frac{\|v - \mathbf{v}_d\|^2}{v_m^2}\Big) \geqslant 0,$$

we have $\lambda\widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \geqslant 0$ if

$$\Big(\lambda + \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big) - 1\Big) \geqslant 0,$$

i.e.,

$$\lambda \geqslant 1 - \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big).$$

Since

$$\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big) \leqslant 1,$$

we take

$$\lambda_0 = 1 + \frac{1}{2\pi v_m^2} > 0. \tag{3.13}$$

Then for all $\mathbf{k} \in \mathcal{K}$ $(\widehat{A\psi})_{\mathbf{k}\in\mathcal{K}} \geqslant 0$ if $\lambda \geqslant \lambda_0$.

(2) Let $\lambda = \lambda_0$ fixed. Since $\widehat{\psi} \in B_{T,a_0,\lambda,R}$, we have $\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(t,\mathbf{x}) \leqslant R\exp(\lambda t)$, hence,

$$
\begin{aligned}
&\sum_{\mathbf{k}\in\mathcal{K}} \widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x}) \\
&\quad = \sum_{\mathbf{k}\in\mathcal{K}} f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \Big(\sum_{\mathbf{k}\in\mathcal{K}} \lambda_0 \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \Big) \,\mathrm{d}s \\
&\quad = \sum_{\mathbf{k}\in\mathcal{K}} f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \Big(\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \Big(\sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} \exp\Big(-\frac{1}{2} \frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2} \Big) - 1 \Big) \\
&\qquad + \sum_{\mathbf{k}\in\mathcal{K}} \lambda_0 \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \Big) \,\mathrm{d}s \\
&\quad \leqslant \sum_{\mathbf{k}\in\mathcal{K}} f_{0,\mathbf{k}}(\mathbf{x}) + \int_0^t \Big(\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \Big(\sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} + 1 \Big) + \sum_{\mathbf{k}\in\mathcal{K}} \lambda_0 \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \Big) \,\mathrm{d}s \\
&\quad \leqslant \sum_{\mathbf{k}\in\mathcal{K}} \|f_{0,\mathbf{k}}\|_\infty + \frac{C_1 R}{\lambda_0} (\exp(\lambda_0 t) - 1) + R\exp(\lambda_0 t) - R
\end{aligned}
$$

with

$$
C_1 = \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} + 1 = \mathrm{Card}(\mathcal{K}) \frac{1}{2\pi v_m^2} + 1 > 0, \tag{3.14}
$$

where $\mathrm{Card}(\mathcal{K})$ is the cardinal number of the set $\mathcal{K}$.

Hence, $\sum_{\mathbf{k}\in\mathcal{K}} \widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x}) \leqslant R\exp(\lambda t)$ if

$$
\sum_{\mathbf{k}\in\mathcal{K}} \|f_{0,\mathbf{k}}\|_\infty + \frac{C_1 R}{\lambda_0} (\exp(\lambda t) - 1) - R \leqslant 0, \tag{3.15}
$$

hence

$$
R \geqslant R_0 = \sum_{\mathbf{k}\in\mathcal{K}} \|f_{0,\mathbf{k}}\|_\infty \tag{3.16}
$$

and

$$
t \leqslant \frac{1}{\lambda_0} \ln \Big(1 + \frac{\lambda_0}{C_1 R} \Big(R - \sum_{\mathbf{k}\in\mathcal{K}} \|f_{0,\mathbf{k}}\|_\infty \Big) \Big) =: T_0. \tag{3.17}
$$

(3) Since $\widehat{\boldsymbol{\psi}} \in B_{T,a_0,\lambda,R}$, we have $\|\widehat{\boldsymbol{\psi}}\|_{\mathbb{X}_T} \leqslant a_0 \|f_0\|_1$, moreover,

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} |\widehat{A\psi}_{\mathbf{k}}(t,\mathbf{x})| \,\mathrm{d}\mathbf{x} &= \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} \Big| f_{0,\mathbf{k}}(\mathbf{x}) \\
&\quad + \int_0^t (\lambda_0 \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})) \,\mathrm{d}s \Big| \,\mathrm{d}\mathbf{x} \\
&\leqslant \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} \Big| f_{0,\mathbf{k}}(\mathbf{x}) \Big| \,\mathrm{d}\mathbf{x} \\
&\quad + \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} \int_0^t |(\lambda_0 \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) + \widehat{\psi}_{\mathrm{eq},\mathbf{k}}(s,\mathbf{x}) - \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})) \,\mathrm{d}s| \,\mathrm{d}\mathbf{x} \\
&\leqslant \|f_0\|_1 + \int_{\mathbb{R}^2_{\mathbf{x}}} \int_0^t \Big(\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x}) \\
&\quad \times \Big(\lambda_0 + \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} \exp\Big(-\frac{1}{2} \frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2} \Big) + 1 \Big) \Big) \,\mathrm{d}s \,\mathrm{d}\mathbf{x} \\
&\leqslant \|f_0\|_1 + \int_{\mathbb{R}^2_{\mathbf{x}}} \int_0^t \Big(\sum_{\mathbf{k}\in\mathcal{K}} \widehat{\psi}_{\mathbf{k}}(s,\mathbf{x})(\lambda_0 + C_1) \Big) \,\mathrm{d}s \,\mathrm{d}\mathbf{x} \\
&\leqslant \|f_0\|_1 + (\lambda_0 + C_1) T \|\widehat{\boldsymbol{\psi}}\|_{\mathbb{X}_T} \\
&\leqslant (1 + (\lambda_0 + C_1) T a_0) \|f_0\|_1,
\end{aligned} \tag{3.18}$$

where C_1 and λ_0 are given by (3.14) and (3.13).

Therefore, the constant a_0 exists if

$$1 + (\lambda_0 + C_1) T a_0 \leqslant a_0,$$

hence, if

$$1 \leqslant (1 - (\lambda_0 + C_1) T) a_0$$

and

$$(\lambda_0 + C_1) T < 1,$$

thus, if

$$T < \frac{1}{C_1 + \lambda_0} =: T_1. \tag{3.19}$$

Then for $T < (C_1 + \lambda_0)^{-1}$ we choose a_0 such that

$$\frac{1}{1 - (\lambda_0 + C_1) T} \leqslant a_0. \tag{3.20}$$

(4) Let $\widehat{\psi^1}, \widehat{\psi^2} \in B_{T,a_0,\lambda_0,R}$,

$$\begin{aligned}
&\|\widehat{A\psi^1}(t) - \widehat{\psi^2}(t)\|_1 \\
&\quad = \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} |\widehat{A\psi^1_{\mathbf{k}}}(t,\mathbf{x}) - \widehat{A\psi^2_{\mathbf{k}}}(t,\mathbf{x})| \,\mathrm{d}\mathbf{x} \\
&\quad = \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} \Big| \int_0^t \lambda_0(\widehat{\psi^1_{\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^2_{\mathbf{k}}}(s,\mathbf{x}))\,\mathrm{d}s + \int_0^t (\widehat{\psi^1_{\mathrm{eq},\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^2_{\mathrm{eq},\mathbf{k}}}(s,\mathbf{x}))\,\mathrm{d}s \\
&\qquad + \int_0^t (\widehat{\psi^2_{\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^1_{\mathbf{k}}}(s,\mathbf{x}))\,\mathrm{d}s \Big| \,\mathrm{d}\mathbf{x} \\
&\quad \leqslant \sum_{\mathbf{k}\in\mathcal{K}} \int_{\mathbb{R}^2_{\mathbf{x}}} \int_0^t \lambda_0 |(\widehat{\psi^1_{\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^2_{\mathbf{k}}}(s,\mathbf{x}))|\,\mathrm{d}s + \int_0^t |(\widehat{\psi^1_{\mathrm{eq},\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^2_{\mathrm{eq},\mathbf{k}}}(s,\mathbf{x}))|\,\mathrm{d}s \\
&\qquad + \int_0^t |(\widehat{\psi^2_{\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^1_{\mathbf{k}}}(s,\mathbf{x}))|\,\mathrm{d}s\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

By definition of the equilibrium density, we have

$$\widehat{\psi^2_{\mathrm{eq},\mathbf{k}}}(s,\mathbf{x}) - \widehat{\psi^1_{\mathrm{eq},\mathbf{k}}}(s,\mathbf{x}) = \frac{1}{2\pi v_m^2} \exp\Big(-\frac{1}{2}\frac{\|\mathbf{v_k} - \mathbf{v}_d\|^2}{v_m^2}\Big)\Big(\sum_{\mathbf{l}\in\mathcal{K}} \widehat{\psi^1_{\mathbf{l}}}(s,\mathbf{x}) - \sum_{\mathbf{l}\in\mathcal{K}} \widehat{\psi^2_{\mathbf{l}}}(s,\mathbf{x})\Big),$$

since

$$\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v_k} - \mathbf{v}_d\|^2}{v_m^2}\Big) \leqslant 1.$$

Then

$$\sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} \exp\Big(-\frac{1}{2}\frac{\|\mathbf{v_k} - \mathbf{v}_d\|^2}{v_m^2}\Big) \leqslant \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} =: C_2.$$

Hence, we find

$$\begin{aligned}
\|\widehat{A\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T} &\leqslant \lambda_0 T \|\widehat{\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T} + C_2 T \|\widehat{\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T} + T\|\widehat{\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T} \\
&\leqslant (\lambda_0 T + C_2 T + T)\|\widehat{\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T}
\end{aligned}$$

with

$$C_2 = \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2} = \mathrm{Card}(\mathcal{K}) \frac{1}{2\pi v_m^2}, \tag{3.21}$$

which ends the proof of the lemma. □

Let us then pose

$$T = \frac{\min\{T_0; T_1\}}{2} =: T_2, \tag{3.22}$$

where T_0 and T_1 are given by (3.17) and (3.19).

According to (1), (2) and (3) of the previous lemma, for $\lambda = \lambda_0$, $R \geqslant R_0$ and $T = T_2$ given by (3.13), (3.16), (3.22), respectively, there exists a_0 satisfying (3.20), and we have:

$$\text{If } \widehat{\psi} \in B_{T,a_0,\lambda,R}, \text{ then } \widehat{A\psi} \in B_{T,a_0,\lambda,R}.$$

According to (4) of the lemma, we have

$$\|\widehat{A\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T} \leqslant (\lambda_0 T + C_2 T + T)\|\widehat{\psi^1} - \widehat{\psi^2}\|_{\mathbb{X}_T}.$$

Moreover, $(\lambda_0 T + C_2 T + T) < 1$ if

$$T < \frac{1}{\lambda_0 + C_2 + 1} =: T_3. \tag{3.23}$$

Moreover, from inequality (3.23) it follows that $T_3 < \frac{1}{2}$.

Hence, for $t \leqslant T = T_3/2$, the operator $\mathbf{A}\colon B_{T,a_0,\lambda,R} \to B_{T,a_0,\lambda,R}$ is a contraction.

Let

$$T = \min\Big\{T_2; \frac{T_3}{2}\Big\}. \tag{3.24}$$

Banach's fixed point theorem refers to the local existence of the model solution in $[0,T]$, with T given by (3.24).

From the foregoing, there exist $\lambda = \lambda_0$, R, T and a_0 such that the problem given in (3.1) has a unique positive solution $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathcal{C}([0,T], \mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}}))$ which satisfies:

$$\int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant a_0\|f_0\|_1 \exp(-\lambda t) \quad (\text{since } \|\widehat{\boldsymbol{\psi}}\|_{\mathbb{X}_T} \leqslant a_0\|f_0\|_1).$$

Hence,

$$\sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant a_0\|f_0\|_1.$$

Since $a_0\|f_0\|_1$ does not depend on n and $a_0 = a_0(T)$,

$$\sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \Theta(T),$$

where $\Theta(T) = a_0(T)\|f_0\|_1$, from where estimation (3.8) follows.

In addition, the solution satisfies

$$\varrho(t,\mathbf{x}) \leqslant R, \quad t \in [0,T],\ \mathbf{x} \in \Omega,\ R \geqslant R_0,$$

where

$$R_0 = \sum_{\mathbf{k}\in\mathcal{K}} \|f_{0,\mathbf{k}}\|_\infty.$$

Moreover, if $\sum\limits_{\mathbf{k}\in\mathcal{K}} \|f_{0,\mathbf{k}}\|_\infty < 1$, ($R_0 \leqslant 1$), then we choose $R = R_0$ such that

$$\varrho(t,\mathbf{x}) \leqslant 1 \quad \forall\, t \in [0,T],\ \forall\, \mathbf{x} \in \Omega.$$

Hence, estimation (3.10). That ends the proof of Theorem 3.1 (Local existence). □

R e m a r k 3.1. In our proof, we demonstrated first the local existence of the solution for problem (3.1) in $[0,t_1]$, where $t_1 = T = \min\{T_2; T_3/2\}$ is the existence time depending on the initial conditions $f_0 := f(t=0,\mathbf{x})$ and velocities v_m, since T_0 and T_1 depend on data v_m and T_0 depends on initial data f_0. Then through an iterative process, we can extend the solution for time greater than t_1 by successively solving the equation in (3.1) with initial conditions in $t_1 = T = T(f_0)$ until t_2, with the second existence time interval $[t_1,t_2]$ depending on data in t_1, i.e. on $f_1 := (t = t_1, \mathbf{x})$, and so on.

By concatenation, we build a maximum solution on $[0, T_{\max}[$, with $T_{\max} = \sup\limits_j\{t_j\}$. This solution belongs to $\mathcal{C}([0,T_{\max}[, \mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}}))$.

3.2. Convergence of the discrete kinetic equation towards the continuous one. According to the previous section, we have shown the existence and uniqueness of the solution of the discrete model given in (3.1): $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}} \in \mathcal{C}([0,T_{\max}[, \mathbb{L}^1(\mathbb{R}^2_{\mathbf{x}}))$, and moreover satisfies the following estimate:

$$\sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \Theta(T). \tag{3.25}$$

In order to prove the convergence of this solution, the following functions are introduced:

$$f^n(t,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathbf{k}}(t,\mathbf{x})\mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \tag{3.26}$$

$$f^n_{\mathrm{eq}}(t,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathrm{eq},\mathbf{k}}(t,\mathbf{x})\mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \tag{3.27}$$

$$C^n(\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} \mathbf{v}^n_{\mathbf{k}}\mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \tag{3.28}$$

$$f^0(0,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^{0,n}_{\mathbf{k}}(\mathbf{x})\mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \tag{3.29}$$

where $\mathbb{1}^n_{\mathbf{k}}$ is the indicator function on the velocity cells $\mathcal{I}^n_{\mathbf{k}}$.

Then the discrete model given in (3.1) can be linked to the BGK equation (2.1) by the following system:

$$\left\{\begin{aligned} &\frac{\partial f^n(t,\mathbf{x},\mathbf{v})}{\partial t} + C^n(\mathbf{v})\cdot\nabla_{\mathbf{x}} f^n(t,\mathbf{x},\mathbf{v}) \\ &\qquad = f^n_{\mathrm{eq}}(\mathbf{v}) - f^n(t,\mathbf{x},\mathbf{v}),\ \mathcal{D}'(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}), \\ &f^0(0,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^{0,n}_{\mathbf{k}}(\mathbf{x})\mathbb{1}^n_{\mathbf{k}}(\mathbf{v}), \end{aligned}\right. \tag{3.30}$$

where $\mathcal{D}'(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$, $\mathcal{D}(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$ are defined by (3.2), (3.3), respectively.

We denote then our convergence result in the following theorem:

Theorem 3.2. *Let $\mathbf{v}^n$ be such that $\Delta v_n \underset{n\to\infty}{\longrightarrow} 0$, $\Delta v_n B_n \underset{n\to\infty}{\longrightarrow}\infty$, and $C^n\colon\ \mathbb{R}^2_{\mathbf{v}}\to\mathbb{R}^2_{\mathbf{v}}$, $\mathbf{v}\mapsto C^n(\mathbf{v})$ such that*

$$C^n \text{ is locally, uniformly bounded in } (\mathbb{L}^\infty_{\mathrm{loc}}(\mathbb{R}^2_{\mathbf{v}}))^2, \tag{3.31}$$

$$C^n(\mathbf{v}) \underset{n\to\infty}{\longrightarrow} \mathbf{v} \text{ simply.} \tag{3.32}$$

Then we can extract a sub-sequence noted $(f^n)_n$ which converges weakly in $\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$ for all $T_{\max}\geqslant 0$ to a solution of equation (2.1).

P r o o f. By referring to the works of Perthame [29] and [33], the proof is divided into 4 steps:

Step 1: weak convergence of f^n. According to (3.25) and (3.26), it is obvious that f^n satisfies the estimate

$$\sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} \leqslant \Theta_1(T), \tag{3.33}$$

which implies that the following $(f^n)_n$ is equi-integrable.

According to Dunford-Pettis Theorem [14], we can extract a subsequence denoted $(f^n)_n$ which converges weakly in $\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$ towards

$$f\in\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$$

i.e.:

$$f^n \underset{n\to\infty}{\rightharpoonup} f \quad \text{in } \mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}). \tag{3.34}$$

Hence,

$$f^n \underset{n\to\infty}{\longrightarrow} f \quad \text{in } \mathcal{D}'(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}),$$

for all $T_{\max} \geqslant 0$. We thus obtain the convergence of the transport term of (3.30) towards $\partial f + \mathbf{v} \cdot \nabla f$ in $\mathcal{D}'(]0, T_{\max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}})$, i.e.

$$\partial f^n + C^n(\mathbf{v}) \cdot \nabla_{\mathbf{x}} f^n \underset{n\to\infty}{\longrightarrow} \partial f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f \quad \text{in } \mathcal{D}'(]0, T_{\max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}}). \tag{3.35}$$

Indeed, let $\varphi \in \mathcal{D}(]0, T_{\max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}})$,

$$\begin{aligned}
\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} & (\partial_t f^n + C^n(\mathbf{v}) \cdot \nabla_{\mathbf{x}} f^n)\varphi \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \\
&= -\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(\partial_t \varphi + C^n(\mathbf{v}) \cdot \nabla_{\mathbf{x}} \varphi) \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \\
&= -\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(\partial_t \varphi) \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \\
&\quad -\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(C^n(\mathbf{v}) \cdot \nabla_{\mathbf{x}} \varphi) \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v}.
\end{aligned}$$

We get

$$\begin{aligned}
\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} & f^n(C^n(\mathbf{v}) \cdot \nabla_{\mathbf{x}} \varphi) \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \\
&= \int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(C^n(\mathbf{v}) - \mathbf{v}) \cdot \nabla_{\mathbf{x}} \varphi \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \\
&\quad + \int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n \mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v}.
\end{aligned}$$

According to (3.31) and (3.32), we obtain

$$\begin{aligned}
\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} & f^n(C^n(\mathbf{v}) - \mathbf{v}) \cdot \nabla_{\mathbf{x}} \varphi \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \\
&\leqslant \|f^n\|_{\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})} \|C^n(\mathbf{v}) - \mathbf{v}\|_{\mathbb{L}^\infty(\mathbb{R}^2_{\mathbf{v}})} \|\nabla_{\mathbf{x}} \varphi\|_{\mathbb{L}^\infty(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})}.
\end{aligned}$$

From (3.32), $\|C^n(\mathbf{v}) - \mathbf{v}\|_{\mathbb{L}^\infty(\mathbb{R}^2_{\mathbf{v}})} \underset{n\to\infty}{\longrightarrow} 0$, consequently,

$$\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(C^n(\mathbf{v}) - \mathbf{v}) \cdot \nabla_{\mathbf{x}} \varphi \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.36}$$

Moreover, since $\mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi \in \mathbb{L}^\infty(]0, T_{\max}[\times \mathbb{R}^2_{\mathbf{x}} \times \mathbb{R}^2_{\mathbf{v}})$,

$$\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n \mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi \,\mathrm{d}t \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{v} \underset{n\to\infty}{\longrightarrow} \int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f \mathbf{v} \cdot \nabla_{\mathbf{x}} \varphi \,\mathrm{d}t. \tag{3.37}$$

In addition, $\partial_t\varphi \in \mathbb{L}^\infty(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$, then

$$\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n(\partial_t\varphi)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} \underset{n\to\infty}{\longrightarrow} \int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f(\partial_t\varphi)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}. \tag{3.38}$$

From (3.36), (3.37), and (3.38) we get

$$\begin{aligned}\int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} & f^n(\partial_t\varphi + C^n(\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} \\ & \underset{n\to\infty}{\longrightarrow} \int_{]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\,\mathrm{d}t\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v},\end{aligned}$$

hence, the result (3.35).

For the convergence of the non-linear part, we first have the convergence of ϱ^n, according to (3.34):

$$\int_{\mathbb{R}^2_{\mathbf{v}}} f^n(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^2_{\mathbf{v}}} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \quad \text{weakly in } \mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}),$$

i.e.

$$\varrho^n(t,\mathbf{x}) \underset{n\to\infty}{\rightharpoonup} \varrho(t,\mathbf{x}) \quad \text{in } \mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}). \tag{3.39}$$

Step 2: weak convergence of f^n_{eq}. We need the following lemma:

Lemma 3.2. *Suppose that $f^n = (f^n_{\mathbf{k}})_{\mathbf{k}\in\mathcal{K}}$ satisfies inequality (3.25). Then for all $T \geqslant 0$ there exists $C(T)$ such that*

$$\sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n_{\mathrm{eq}}(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v} \leqslant C(T). \tag{3.40}$$

Proof of Lemma 3.2. We have

$$f^n_{\mathrm{eq}}(t,\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}\in\mathcal{K}} f^n_{\mathrm{eq},\mathbf{k}}(t,\mathbf{x})\mathbb{1}^n_{\mathbf{k}}(\mathbf{v})$$

with

$$f^n_{\mathrm{eq},\mathbf{k}}(t,\mathbf{x}) = \frac{1}{2\pi v_m^2}\exp\Bigl(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d\|^2}{v_m^2}\Bigr), \quad \varrho(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} f_{\mathbf{k}}(\mathbf{x},t),$$

$$
\begin{aligned}
&\int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n_{\mathrm{eq}}(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\\
&\qquad = \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}(\mathbf{x},t) \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d\|^2}{v_m^2}\Big)\mathbb{1}^n_{\mathbf{k}}(\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\\
&\qquad = \int_{\mathbb{R}^2_{\mathbf{x}}\mathbb{R}^2_{\mathbf{v}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \int_{\mathbb{R}^2_{\mathbf{v}}} \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d\|^2}{v_m^2}\Big)\mathbb{1}^n_{\mathbf{k}}(\mathbf{v})\,\mathrm{d}\mathbf{v}\\
&\qquad = \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \int_{\mathcal{I}^n_{\mathbf{k}}} \sum_{\mathbf{k}\in\mathcal{K}} \frac{1}{2\pi v_m^2}\exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}}-\mathbf{v}_d\|^2}{v_m^2}\Big)\,\mathrm{d}\mathbf{v},
\end{aligned}
$$

from where

$$
\begin{aligned}
&\sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}} f^n_{\mathrm{eq}}(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{v}\\
&\qquad \leqslant \sup_n \sup_{[0,T]} \int_{\mathbb{R}^2_{\mathbf{x}}} \sum_{\mathbf{l}\in\mathcal{K}} f_{\mathbf{l}}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\cdot C(\mathcal{I}^n_{\mathbf{k}}) \leqslant C(T) \quad \text{according to (3.25),}
\end{aligned}
$$

and where $C(T) := \Theta(T)\cdot C(\mathcal{I}^n_{\mathbf{k}})$. □

According to this lemma, (f^n_{eq}) is weakly compact in $\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$, so we can extract a subsequence denoted (f^n_{eq}) such that $f^n_{\mathrm{eq}} \underset{n\to\infty}{\rightharpoonup} g$ in $\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$, where from steps (1) and (2), the low limit of f^n satisfies the equation

$$
\partial f + \mathbf{v}\cdot\nabla f = g - f \quad \text{in } \mathcal{D}'(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}). \tag{3.41}
$$

The next step is to show that $g = f_{\mathrm{eq}}$.

Step 3: Strong convergence of ϱ^n. According to the compactness lemma on averages obtained by Mischler [30], and that on sets of bounded velocities [17], f^n is strongly compact. Indeed, we have for any R_C (again by extracting subsequences):

$$
\int_{|\mathbf{v}|\leqslant R_C} f^n(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \underset{n\to\infty}{\longrightarrow} \int_{|\mathbf{v}|\leqslant R_C} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \tag{3.42}
$$

strongly in $\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}})$.

From the uniform estimates (3.33) we then obtain:

$$
f^n \underset{n\to\infty}{\longrightarrow} f \quad \text{in } \mathbb{L}^1(]0,T_{\max}[\times K_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})\ \forall\, K_{\mathbf{x}} \text{ compact on } \mathbb{R}^2_{\mathbf{x}}. \tag{3.43}
$$

Hence, from the above and equation (3.39)

$$
\begin{aligned}
\varrho^n(t,\mathbf{x}) &= \int_{\mathbb{R}^2_{\mathbf{v}}} f^n(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v} \underset{n\to\infty}{\longrightarrow} \varrho(t,\mathbf{x})\\
&= \int_{\mathbb{R}^2_{\mathbf{v}}} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v}, \quad \mathbb{L}^1(]0,T_{\max}[\times K_{\mathbf{x}})\ \forall\, K_{\mathbf{x}} \text{ compact.}
\end{aligned} \tag{3.44}
$$

Step 4: Passage to the limit. According to step (3),

$$\varrho^n(t,\mathbf{x}) \underset{n\to\infty}{\longrightarrow} \varrho(t,\mathbf{x}), \quad \mathbb{L}^1(]0,T_{\max}[\times K_{\mathbf{x}})\ \forall K_{\mathbf{x}} \text{ compact}, \tag{3.45}$$

then we can again extract a subsequence as

$$\varrho^n(t,\mathbf{x}) \underset{n\to\infty}{\longrightarrow} \varrho(t,\mathbf{x}) \quad \text{a.e. } [0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}. \tag{3.46}$$

On the other hand, by hypothesis we have

$$\|\mathbf{v}^n_{\mathbf{k}} - \mathbf{v}_d\| \underset{n\to\infty}{\longrightarrow} \|\mathbf{v} - \mathbf{v}_d\| \quad \forall \mathbf{k} \in \mathcal{K}. \tag{3.47}$$

Hence, from (3.46), (3.47),

$$f^n_{\text{eq}} \underset{n\to\infty}{\longrightarrow} f_{\text{eq}} \quad \text{a.e. } [0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}}. \tag{3.48}$$

Combining these results with those of step (2), we get

$$g = f_{\text{eq}}. \tag{3.49}$$

As a result, the left-hand side of equation (3.30) weakly converges to $(f_{\text{eq}} - f)$ in $\mathbb{L}^1(]0,T_{\max}[\times\mathbb{R}^2_{\mathbf{x}}\times\mathbb{R}^2_{\mathbf{v}})$. We conclude that f is a solution of the equation BGK (2.1).

4. Results and numerical simulations

4.1. Numerical method. The general idea of the semi-Lagrangian method used is to fix a grid in the velocity space and to transform the kinetic equation into a set of linear hyperbolic equations with source terms. We refer to [12], [13] for the detailed description of this numerical method. Here we recall only the basic principles.

We summarize the semi-Lagrangian numerical method used in this work as follows:

(1) The discretization of the BGK model equation in the velocity space.
(2) A splitting procedure of time between transport and relaxation operators for each of the system evolution equations that will be given in (4.1).
(3) The exact resolution of the transport equation with zero second member which means without using a spatial mesh; the initial data of this step are given by the solution of the relaxation operator.
(4) The resolution of the relaxation part on the grid with initial data defined by the value of the distribution function at the center of the cells after the transport step.

To discretize the square $[-1,1]\times[-1,1]$, a Cartesian grid $\mathcal{V}$ in $\mathbb{R}^2$ in two-dimensional velocity space, and a set $\mathcal{K}$ multi-indices of $\mathbb{Z}^2$ are introduced, such that

$$\mathcal{K} = \{(-1,0); (-1,1); (0,1); (1,1); (1,0); (1,-1); (0,-1); (-1,-1)\}.$$

In what follows, all the simulations will be made in a square space domain $\Omega = [0, 20] \times [0, 20]$.

Thanks to the discrete velocity approximation above, the continuous distribution function f is replaced by a vector of dimension 8, where each component is supposed to be an approximation of the distribution function f, i.e. $f_{\mathbf{k}}(t, \mathbf{x}) \approx f(t, \mathbf{x}, \mathbf{v}_{\mathbf{k}})$, and the original kinetic equation (1.1) is replaced by a set of 8 evolution equations for $f_{\mathbf{k}}$ of the following form:

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial f_{\mathbf{k}}(t,\mathbf{x})}{\partial t} + \mathbf{v}_{\mathbf{k}} \cdot \nabla_{\mathbf{x}} f_{\mathbf{k}}(t,\mathbf{x}) = \dfrac{1}{\tau}(f_{\mathrm{eq},\mathbf{k}}(\mathbf{v}_{\mathbf{k}}) - f_{\mathbf{k}}(t,\mathbf{x})), & \mathbf{k} \in \mathcal{K}, \\ f_{\mathbf{k}}(t=0,\mathbf{x}) = f_{0,\mathbf{k}}(\mathbf{x}), & \mathbf{k} \in \mathcal{K}. \end{cases}$$

We describe the first step of the $[t^0 \to t^1]$ method, starting from $t^0 = 0$, and then we generalize it to arbitrary time step.

With splitting, the first step is reduced to N_v linear transport equation resolution of the form:

$$\text{(4.2)} \qquad \frac{\partial f_{\mathbf{k}}(t,\mathbf{x})}{\partial t} + \mathbf{v}_{\mathbf{k}} \cdot \nabla_{\mathbf{x}} f_{\mathbf{k}}(t,\mathbf{x}) = 0, \quad \mathbf{k} \in \mathcal{K}.$$

In order to solve the transport equation without zero second member, we consider for each equation of system (4.2) the initial data defined by

$$\text{(4.3)} \qquad \widehat{f_{\mathbf{k}}}(\mathbf{x}, t=0) = f_{0,\mathbf{k}}(\mathbf{x}), \quad \mathbf{x} \in \Omega,\ \mathbf{k} \in \mathcal{K}.$$

Thanks to this reconstruction, the exact system solution (4.2) at time $t^1 = t^0 + \Delta t = \Delta t$ is given by

$$\text{(4.4)} \qquad \widehat{f^*_{\mathbf{k}}}(\mathbf{x}) = \widehat{f_{\mathbf{k}}}(\mathbf{x} - \mathbf{v}_{\mathbf{k}}\Delta t), \quad \mathbf{x} \in \Omega,\ \mathbf{k} \in \mathcal{K}.$$

To complete a step of time, we must calculate the solution of the interaction part of equation (4.1) on the points of the grid,

$$\text{(4.5)} \qquad \partial_t f_{\mathbf{k}} = \frac{1}{\tau}(f_{\mathrm{eq},\mathbf{k}} - f_{\mathbf{k}}), \quad \mathbf{k} \in \mathcal{K},$$

where the initial data is given by the resolution of the transport step to time $t^1 = t^0 + \Delta t$, $(\widehat{f^*_{\mathbf{k}}}(\mathbf{x}))_{\mathbf{k}\in\mathcal{K}}$.

To solve (4.5), we define the value of the equilibrium distribution at the instant t^1, $(f^1_{\mathrm{eq},\mathbf{k}})_{\mathbf{k}\in\mathcal{K}}$ by

$$f^1_{\mathrm{eq},\mathbf{k}}(\mathbf{v}_{\mathbf{k}}) = \frac{\varrho(t,\mathbf{x})}{2\pi v_m^2} \exp\Big(-\frac{1}{2}\frac{\|\mathbf{v}_{\mathbf{k}} - \mathbf{v}_d\|^2}{v_m^2}\Big), \quad \mathbf{k} \in \mathcal{K},$$

with

$$\varrho(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} \widehat{f}^*_{\mathbf{k}}(\mathbf{x}).$$

Finally, the solution of the relaxation equation (4.5) is given by

$$f^1_{\mathbf{k}} = \mathrm{e}^{(-\Delta t/\tau)} \widehat{f}^*_{\mathbf{k}} + (1-\mathrm{e}^{(-\Delta t/\tau)}) f^1_{\mathrm{eq},\mathbf{k}}, \quad \mathbf{k}\in\mathcal{K}. \tag{4.6}$$

Given a density value $(\widehat{f}^n_{\mathbf{k}}(\mathbf{x}))_{\mathbf{k}\in\mathcal{K}}$ at the moment t^n for $\mathbf{x}\in\Omega$, $\mathbf{k}\in\mathcal{K}$, the density at instant t^{n+1} can be calculated as follows:

$$\widehat{f}^*_{\mathbf{k}}(\mathbf{x}) = \widehat{f}^n_{\mathbf{k}}(\mathbf{x}-\mathbf{v}_{\mathbf{k}}\Delta t), \quad f^{n+1}_{\mathbf{k}} = \mathrm{e}^{(-\Delta t/\tau)} \widehat{f}^*_{\mathbf{k}} + (1-\mathrm{e}^{(-\Delta t/\tau)}) f^{n+1}_{\mathrm{eq},\mathbf{k}}.$$

The local density at the instant t^{n+1} is defined by

$$\varrho(t^{n+1},\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}} f^{n+1}_{\mathbf{k}}(\mathbf{x}).$$

By referring to [12], [39], [27], [15], this scheme is unconditionally stable; however for reasons of precision, the time step is chosen to satisfy the condition $\Delta t/\Delta x < 1$, because the maximum speed of the pedestrians is fixed at one.

In conclusion, we summarize the different steps of this numerical scheme in the following algorithm:

Algorithm 4.1

Require: $(f^0_{\mathbf{i}}(\mathbf{x}))_{\mathbf{i}\in\mathcal{K}}$: initial data.
for $m = 0\colon Nt-1$ **do**
- ▷ Resolution of the transport part given by the Nv equations of the system (4.2) with $(f^m_{\mathbf{i}})_{\mathbf{i}\in\mathcal{K}}$ the initial data, we get $(\widehat{f}^m_{\mathbf{i}})_{\mathbf{i}\in\mathcal{K}}$.
- ▷ Computation of equilibrium density $(f^m_{\mathrm{eq},\mathbf{i}})_{\mathbf{i}\in\mathcal{K}}$.
- ▷ Relaxation term resolution (4.5) with $(\widehat{f}^m_{\mathbf{i}})_{\mathbf{i}\in\mathcal{K}}$ initial data.

end for
- ▷ $(f^m_{\mathbf{i}}(\mathbf{x}))_{\mathbf{i}\in\mathcal{K}}$.

The output parameters of the algorithm are the numerical solution of the original problem defined by the two equations (4.1).

4.2. Convergence test for the semi-Lagrangian scheme. Our goal in this paper is to validate the proposed mathematical model for pedestrian motion. This choice is motivated by considering a small number of pedestrians ($N \leqslant 100$ pedestrians).

In order to analyze the convergence of the spatial discretization of the semi-Lagrangian method, we solve the kinetic equation (4.1) with initial data that is a piecewise constant function in two-dimensional space. They are concentrated densities in a center disk $\mathbf{x}_0$ with radius r_D:

$$\forall\, \mathbf{k} \in \mathcal{K},\; f_{\mathbf{k}}(t=0,\mathbf{x}) = \begin{cases} f^0_{\mathbf{k}}(\mathbf{x}), & \text{if } \|\mathbf{x}-\mathbf{x}_0\| \leqslant r_D, \\ 0. \end{cases} \tag{4.7}$$

This initial data corresponds to a small number of pedestrians $N = 100$.

The simulation is performed with a time sampling step $\Delta t = 2 \cdot 10^{-2}$. In order to estimate the accuracy of the method, we used as a reference a solution f_* computed with a space sampling step $\Delta x = \Delta y = 2^{-6} = 0.0156$, then we estimated the solution with different space sampling steps $\Delta x = (2^{n-1})^{-1}$, $n = 1, \ldots, 5$, with $\Delta x = \Delta y$.

Let us then denote by h the space sampling step, where

$$h = \Delta x = \Delta y.$$

To evaluate the convergence, we calculated the difference between the norm $\mathbb{L}^1$ of the reference solution and the density $\varrho(\mathbf{x})$ estimated at time $t = 2.5\,\mathrm{s}$ with different space steps h,

$$\mathrm{Error}(h) = \|\varrho(h, t=2.5) - \varrho_*(h, t=2.5)\|_{\mathbb{L}^1}, \tag{4.8}$$

where $\varrho_*(\cdot, t=2.5) = \sum_{\mathbf{k}\in\mathcal{K}} f_{\mathbf{k},*}(\cdot, t=2.5)$ and h is the space sampling step.

We illustrate the obtained results in Figure 3,

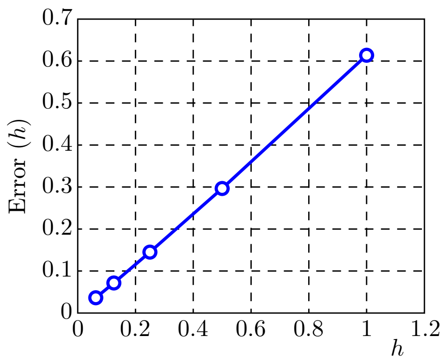


Figure 3. Precision of the schema in space with the error given by (4.8) and h being the space step.

Figure 3 shows that the norm of error $\mathbb{L}^1$ depends linearly on the space step h.

In order to demonstrate the convergence order, we compute the so called Experimental Order of Convergence (EOC):

$$\mathrm{EOC} := \log_2 \frac{\|\varrho(h, t = 2.5) - \varrho_*\|_{\mathbb{L}^1}}{\|\varrho(\frac{h}{2}, t = 2.5) - \varrho_*\|_{\mathbb{L}^1}}. \tag{4.9}$$

For considered space sampling steps $h = (2^{n-1})^{-1}$, $n = 1, \ldots, 5$, the obtained results are given in Table 1.

n the order of refinement	$h = (2^{n-1})^{-1}$	Error(h)	EOC
1	1	0.6138	$--$
2	0.5000	0.2964	1.0502
3	0.2500	0.1450	1.0315
4	0.1250	0.0715	1.0200
5	0.0625	0.0362	0.9820

Table 1. Experimental order of convergence (EOC) for the semi-Lagrangian scheme.

According to the results in Table 1, the experimental order of convergence $\mathrm{EOC} \approx 1$, showing that the method is of the first order accurate.

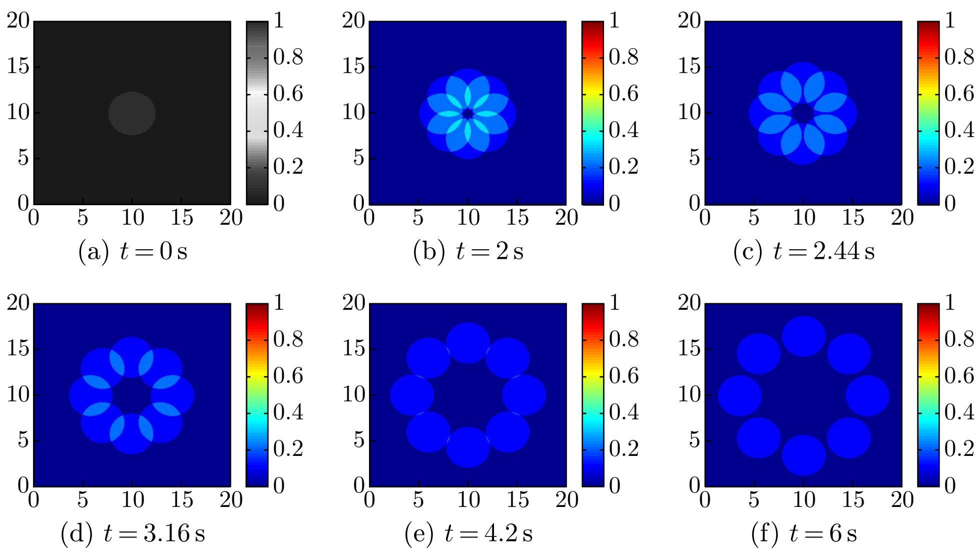


Figure 4. The evolution of the density $\varrho(x, y)$ during the instants (a) $t = 0\,\mathrm{s}$, (b) $t = 2\,\mathrm{s}$, (c) $t = 2.44\,\mathrm{s}$, (d) $t = 3.16\,\mathrm{s}$, (e) $t = 4.2\,\mathrm{s}$, (f) $t = 6\,\mathrm{s}$.

Then, to show the possible directions for pedestrians

$$\mathcal{K} = \{(-1,-1), (-1,0); (-1,1); (1,-1); (1,0); (1,1); (0,1); (0,-1)\},$$

$\mathcal{V} = \{\mathbf{v_k} = \mathbf{k}\Delta v / \mathbf{k} \in \mathcal{K},\ \Delta v = 1\}$, we represented the solution at 6 different instants, with the same initial condition defined by (4.7), and space sampling step $h = 5 \cdot 10^{-2}$ and $\Delta t = 2 \cdot 10^{-2}$; the results obtained are represented in Figure 4 a–f.

We observe that pedestrian density diffuses into space and pedestrians point to the following directions:

$$\{(\pm 1, 0); (1, \pm 1); (-1, \pm 1); (0, \pm 1)\}.$$

4.3. Movement of a group of pedestrians towards a single desired direction $\mathbf{v}_d$. In this paragraph, our aim is to show the adaptation of all pedestrians to a desired configuration, namely the movement in a desired direction $\mathbf{v}_d$. We consider the BGK model (4.1) with the same initial condition (4.7) corresponding to 100 pedestrians. The equilibrium density modeling the trend towards the desired configuration is defined as follows:

$$\text{(4.10)} \qquad \begin{cases} f_{\mathrm{eq},\mathbf{k}}(\mathbf{v_k}) = \dfrac{\varrho(t,\mathbf{x})}{2\pi v_m^2} \exp\Big(-\dfrac{1}{2}\dfrac{\|\mathbf{v}_k - \mathbf{v}_d\|^2}{v_m^2}\Big), \quad \mathbf{k} \in \mathcal{K},\ \mathbf{v}_d = [1,1]^\top, \\ \mathcal{K} = \{(-1,0); (1,-1); (1,0); (1,1); (0,1)\}, \\ \qquad\qquad\qquad \mathbf{v_k} \in \mathcal{V} = \{\mathbf{v_k} = \mathbf{k}\Delta v / \mathbf{k} \in \mathcal{K},\ \Delta v = 1\}. \end{cases}$$

For practical reasons (simplicity and validation of the BGK model to the case of a crowd), v_m is assumed to have a constant value in all numerical simulations.

We considered the same time sampling step as in the first case $\Delta t = 2 \cdot 10^{-2}$, and space sampling step $h = 5 \cdot 10^{-2}$.

The evolution of pedestrian density is then represented at 6 different instants and for two values of relaxation time $\tau = 5 \cdot 10^{-3}$ (see Figure 5) and $\tau = 5 \cdot 10^{-2}$ (see Figure 6).

At initial time $t = 0$, we have the possible directions $\{(\pm 1, 0); (1, \pm 1); (-1, \pm 1); (0, \pm 1)\}$. Then all pedestrians have the tendency to direct towards a desired direction $\mathbf{v}_d = (1,1)^\top$. On one hand, the figures show the adaptation of the desired direction $\mathbf{v}_d = (1,1)^\top$ by the pedestrian group, on the other hand, the local density reaches different values for different relaxation times, is in fact the conservation of mass. More precisely, Figure 5 shows that the time needed to direct towards $\mathbf{v}_d$ by the group of pedestrians with relaxation time $\tau = 5 \cdot 10^{-3}\,\mathrm{s}$ is less than that needed by the group with $\tau = 5 \cdot 10^{-2}\,\mathrm{s}$, see Figure 6.

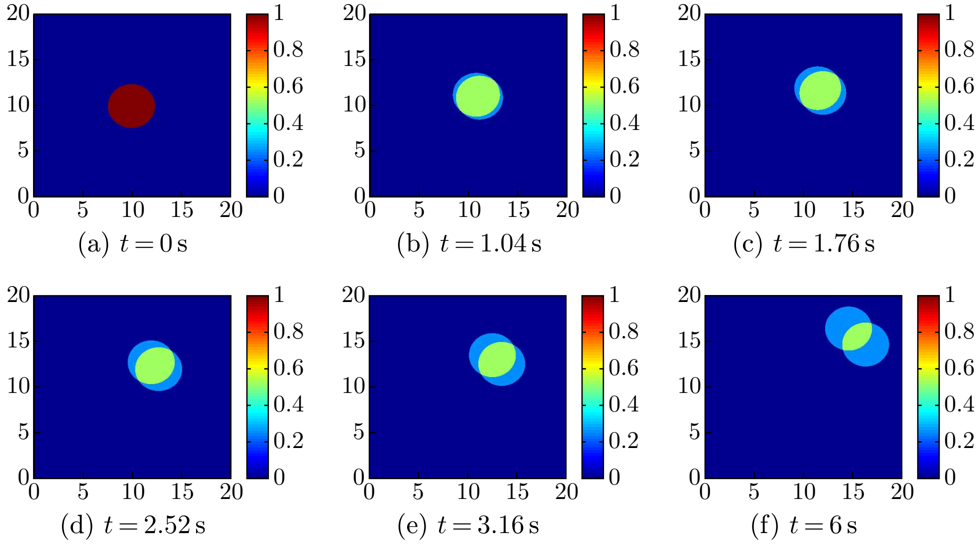


Figure 5. Evolution of the local density during the times (a) $t = 0\,\mathrm{s}$, (b) $t = 1.04\,\mathrm{s}$, (c) $t = 1.76\,\mathrm{s}$, (d) $t = 2.52\,\mathrm{s}$, (e) $t = 3.16\,\mathrm{s}$, (f) $t = 6\,\mathrm{s}$, with relaxation time $\tau = 5 \cdot 10^{-3}$.

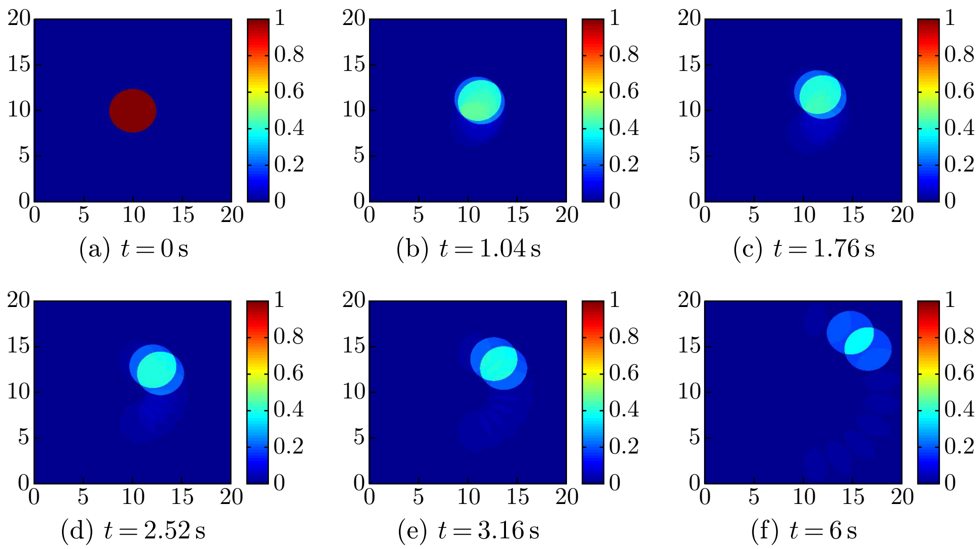


Figure 6. Evolution of the local density during the times (a) $t = 0\,\mathrm{s}$, (b) $t = 1.04\,\mathrm{s}$, (c) $t = 1.76\,\mathrm{s}$, (d) $t = 2.52\,\mathrm{s}$, (e) $t = 3.16\,\mathrm{s}$, (f) $t = 6\,\mathrm{s}$, with relaxation time $\tau = 5 \cdot 10^{-2}$.

4.4. Motion of a group of pedestrians towards 2 desired directions $\mathbf{v}_{d,1}$ and $\mathbf{v}_{d,2}$. Finally, we consider a group of pedestrians defined by the same initial data. This initial density corresponds to a number of pedestrians equal to 100 pedestrians which have the tendency to direct towards two desired directions $\mathbf{v}_{d,1} = (1,1)^\top$ and $\mathbf{v}_{d,2} = (-1,1)^\top$.

The results obtained are given in Figure 7 for $\tau = 5 \cdot 10^{-3}$ and Figure 8 for $\tau = 5 \cdot 10^{-2}$.

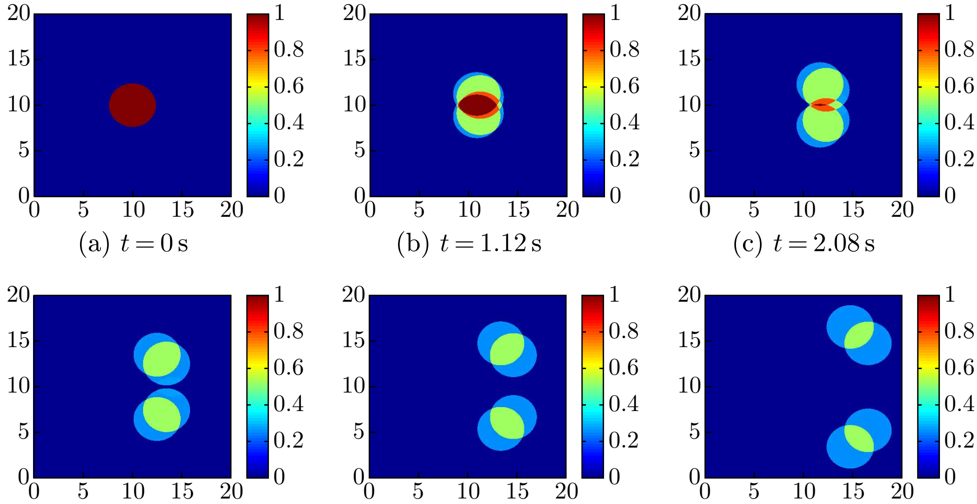
The figures show that the initial circular pedestrians group is divided into two groups in order to reach their state of equilibrium. It consists of the movement of the two groups towards the opposite corners of the domain, corresponding to two desired directions $\mathbf{v}_{d,1}$ and $\mathbf{v}_{d,2}$.

5. Conclusions and perspectives

In this paper, we developed the kinetic approach for the dynamics of a crowd, based on the BGK equation. The main aim of this paper is to adapt the BGK model in kinetic theory to the movement of a crowd. In other words, the derivation of the equilibrium function describing a pedestrian tendency to achieve a desired direction. The proposed mathematical model is capable of describing the tendency of a crowd towards a situation of equilibrium, namely the tendency towards a desired direction.

The existence and the uniqueness of the proposed discrete velocity model solution have been demonstrated by means of the Banach fixed point theorem. Thus, the convergence of this discrete model towards the continuous BGK model is proven. Numerical simulations using the semi-Lagrangian method are then performed. Two perspectives can be considered, the most important one is related to the derivation of a equilibrium function describing the panic conditions, evacuation problems or lane formation. A restrictive hypothesis concerning the constant nature of the relaxation time parameter has been made in this paper. In reality this parameter may depend on the equilibrium distribution function or on the crowd density. The second perspective is to implement this dependency in a new model.

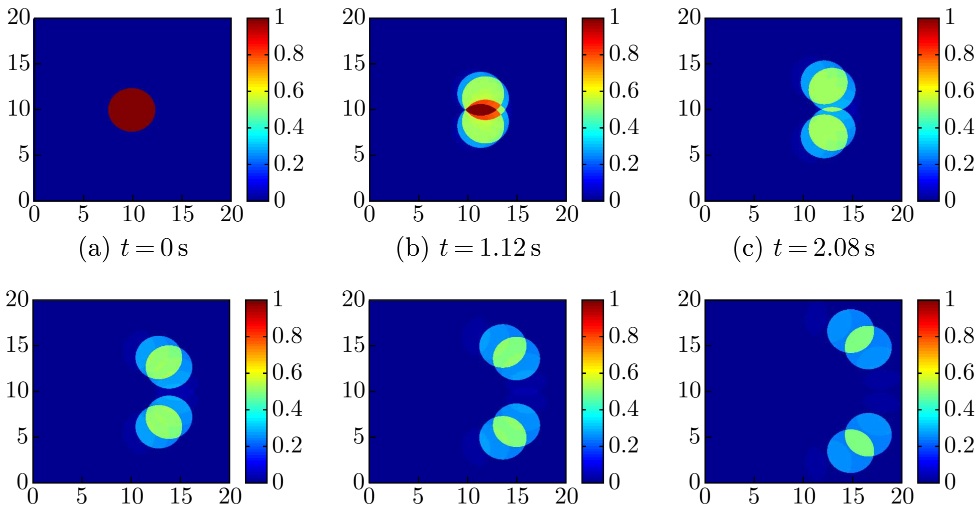
Acknowledgments. The authors are grateful to the reviewers for their insightful remarks and corrections. Their feedback has greatly influenced the quality of this paper.



(a) $t = 0\,\mathrm{s}$ (b) $t = 1.12\,\mathrm{s}$ (c) $t = 2.08\,\mathrm{s}$

(d) $t = 3.2\,\mathrm{s}$ (e) $t = 4.24\,\mathrm{s}$ (f) $t = 6\,\mathrm{s}$

Figure 7. Evolution of local density during the times (a) $t = 0\,\mathrm{s}$, (b) $t = 1.12\,\mathrm{s}$, (c) $t = 2.08\,\mathrm{s}$, (d) $t = 3.2\,\mathrm{s}$, (e) $t = 4.24\,\mathrm{s}$, (f) $t = 6\,\mathrm{s}$, with relaxation time $\tau = 5 \cdot 10^{-3}$.



(a) $t = 0\,\mathrm{s}$ (b) $t = 1.12\,\mathrm{s}$ (c) $t = 2.08\,\mathrm{s}$

(d) $t = 3.2\,\mathrm{s}$ (e) $t = 4.24\,\mathrm{s}$ (f) $t = 6\,\mathrm{s}$

Figure 8. Evolution of local density during the times (a) $t = 0\,\mathrm{s}$, (b) $t = 1.12\,\mathrm{s}$, (c) $t = 2.08\,\mathrm{s}$, (d) $t = 3.2\,\mathrm{s}$, (e) $t = 4.24\,\mathrm{s}$, (f) $t = 6\,\mathrm{s}$, with relaxation time $\tau = 5 \cdot 10^{-2}$.

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