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ASYMPTOTIC LOWER BOUNDS FOR EIGENVALUES OF THE STEKLOV EIGENVALUE PROBLEM WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, using a new correction to the Crouzeix-Raviart finite element eigenvalue approximations, we obtain asymptotic lower bounds of eigenvalues for the Steklov eigenvalue problem with variable coefficients on d-dimensional domains ($d = 2, 3$). In addition, we prove that the corrected eigenvalues converge to the exact ones from below. The new result removes the conditions of eigenfunction being singular and eigenvalue being large enough, which are usually required in the existing arguments about asymptotic lower bounds. Further, we prove that the corrected eigenvalues still maintain the same convergence order as uncorrected eigenvalues. Finally, numerical experiments validate our theoretical results.

Keywords: correction; Steklov eigenvalue problem; Crouzeix-Raviart finite element; asymptotic lower bounds; convergence order

MSC 2020: 65N25, 65N30

1. Introduction

It is an important topic to obtain upper and lower bounds for eigenvalues. As we all know, thanks to the minimum-maximum principle, it is easy to obtain guaranteed upper bounds of eigenvalues by conforming finite element methods. Naturally, attention has been paid to finding lower bounds of eigenvalues by nonconforming finite elements, such as the rotated bilinear ($Q_1^{\rm rot}$) finite element [21], [23], [15], [19], the extension of $Q_1^{\rm rot}$ finite element [21], [18], [19], [16], the enriched Crouzeix-Raviart (ECR) finite element [15], [16], [19], [22], [25], the Wilson finite element [21], [36],

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the Morley element [7], [16], [32], etc. Especially, a lot of work has been done on asymptotic lower bounds for eigenvalues based on the Crouzeix-Raviart (CR) finite element approximations (see, e.g., [2], [23], [34], [16], [30], [19], [31] and the citations therein). Asymptotic lower bounds require that the mesh size is small enough. Recently, finding guaranteed lower bounds has become an attractive topic, which has no requirement for the mesh size. There are also some researches on finding guaranteed lower bounds for eigenvalues based on the CR finite element (see, e.g., [8], [24], [17], [29], [35]).

In this paper, we discuss asymptotic lower bounds for eigenvalues of the Steklov eigenvalue problem with variable coefficients

$$\begin{cases} -\mathrm{div}(\alpha\nabla u) + \beta u = 0 & \text{in } \Omega, \\ \alpha\dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded polygonal or polyhedral domain and $\partial u/\partial\nu$ is the outward normal derivative on $\partial\Omega$. Symbols ∇ and div denote the gradient and the divergence operators, respectively, $\beta = \beta(x) \in L^\infty(\Omega)$ has a positive lower bound, $\alpha = \alpha(x) \in W^{1,\infty}(\Omega)$, and $\alpha_0 \leqslant \alpha(x)$ for a constant $\alpha_0 > 0$.

Among the above references for the Steklov eigenvalue problem with constant coefficients, [19], [31], [22] and [35] discuss asymptotic and guaranteed lower eigenvalue bounds, respectively. The paper [19] states that the CR finite element produces asymptotic lower bounds for eigenvalues in the case of a singular eigenfunction. And [19], [31] also prove that property of asymptotic lower bounds in the case of a non-singular eigenfunction but under an additional condition that the eigenvalue is large enough. The paper [35] obtains guaranteed lower bounds for eigenvalue by correcting the CR finite element eigenvalue approximations, but convergence order of the corrected eigenvalues cannot achieve that of the uncorrected eigenvalues.

Based on the above work, we further discuss asymptotic lower bounds of eigenvalues for the Steklov eigenvalue problem with variable coefficients. We introduce a new correction formula (3.5) to the CR finite element eigenvalue approximations λ_h and obtain the corrected eigenvalues λ_h^c. Our work has the following features:

(1) For shape-regular meshes including quasi-uniform meshes and adaptive meshes with local refinement, when mesh diameter h is sufficiently small and $\|u - u_h\| \geqslant Ch^{1+\varepsilon_0}$ (for some $\varepsilon_0 > 0$), we prove the conclusion

$$\lambda \geqslant \lambda_h^c$$

in Theorem 3.1, which shows that the corrected eigenvalues are asymptotic lower bounds of the exact ones. The new result removes the conditions of eigenfunction being singular and eigenvalue being large enough (see Section 3 for details), which are usually required in the existing arguments about asymptotic lower bounds.

(2) The result in Theorem 3.2 implies that the corrected eigenvalues converge to the exact ones without the loss of convergence order, i.e., convergence order of the corrected eigenvalues is still the same as that of the uncorrected eigenvalues.

(3) For d-dimensional domains ($d = 2, 3$), we implement numerical experiments in Section 4. Numerical results coincide in the theoretical analysis. We are particularly pleased that the correction takes very little time.

In [19], [22], it has been obtained that the ECR finite element can produce asymptotic lower bounds of eigenvalues for the Steklov eigenvalue problem with a constant coefficient whether the eigenfunctions are singular or not. However, up to now, it has not been proved that the ECR element produces asymptotic lower eigenvalue bounds for the Steklov eigenvalue problem with variable coefficients. It should be pointed out that the correction method and theoretical analysis in this paper are also valid for the ECR finite element (see Remark 3.3 in Section 3).

As for the basic theory of the finite element and spectral approximation, we refer to [3], [4], [26], [6], [1], and [5]. Throughout this paper, C denotes a generic positive constant independent of the mesh size, which may not be the same at each occurrence.

2. Preliminary

Let $H^m(\Omega)$ denote the Sobolev space with the real order m on Ω. Let $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ be the norm and seminorm on $H^m(\Omega)$, respectively, and $H^0(\Omega) = L^2(\Omega)$. Furthermore, $H^m(\partial\Omega)$ denotes the Sobolev space with the real order m on $\partial\Omega$, $\|\cdot\|_{m,\partial\Omega}$ is the norm on $H^m(\partial\Omega)$ and $H^0(\partial\Omega) = L^2(\partial\Omega)$.

The weak form of (1.1) can be written as: find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ with $\|u\|_{0,\partial\Omega} = 1$ such that

$$a(u, v) = \lambda b(u, v) \quad \forall\, v \in H^1(\Omega), \tag{2.1}$$

where

$$a(u, v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + \beta uv)\, \mathrm{d}x, \tag{2.2}$$

$$b(u, v) = \int_{\partial\Omega} uv\, \mathrm{d}s. \tag{2.3}$$

Let $\pi_h = \{\kappa\}$ be a regular partition of Ω with the mesh diameter $h = \max\{h_\kappa\}$, where h_κ is the diameter of element κ. Let ε_h be the set of $(d-1)$-dimensional faces of π_h. We denote by $|\kappa|$ the measure of the element κ.

We consider the CR finite element space V_h, proposed by Crouzeix and Raviart [12], as

$$V_h = \{v \in L^2(\Omega)\colon\ v|_\kappa \in P_1(\kappa),\ v \text{ is continuous at the barycenters of the } (d-1)\text{-dimensional faces of } \kappa \text{ for all } \kappa \in \pi_h\}.$$

Put $\|v\|_h = \Big(\sum\limits_{\kappa\in\pi_h} \|v\|_{1,\kappa}^2\Big)^{1/2}$, then $\|v\|_h$ means the norm on V_h. To construct the CR finite element approximation of (2.1) means to find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $\|u_h\|_{0,\partial\Omega} = 1$ such that

$$a_h(u_h, v) = \lambda_h b(u_h, v) \quad \forall\, v \in V_h, \tag{2.4}$$

where

$$a_h(u_h, v) = \sum_{\kappa\in\pi_h} \int_\kappa (\alpha \nabla u_h \cdot \nabla v + \beta u_h v)\,\mathrm{d}x. \tag{2.5}$$

From Theorem 4 in [27] and Remark 2.1 in [14], we have the following regularity result.

Regularity: Assume that φ is the solution of the source problem associated with (2.1) with the right-hand side f.

▷ In the case of $\Omega \subset \mathbb{R}^2$, if $f \in L^2(\partial\Omega)$, there exist constants r and $C_r > 0$ such that $\varphi \in H^{1+r}(\Omega)$ and

$$\|\varphi\|_{1+r/2} \leqslant C_r \|f\|_{0,\partial\Omega};$$

if $f \in H^{1/2}(\partial\Omega)$, then $\varphi \in H^{1+r}(\Omega)$ and

$$\|\varphi\|_{1+r} \leqslant C_r \|f\|_{1/2,\partial\Omega},$$

here $r = 1$ when the largest inner angle θ of Ω satisfies $\theta < \pi$, and $r < \pi/\theta$ which can be arbitrarily close to π/θ when $\theta > \pi$.

▷ In the case of $\Omega \subset \mathbb{R}^3$, if $f \in L^2(\partial\Omega)$, there exist constants $r \in (0, \frac{1}{2})$ and $C_r > 0$ such that $\varphi \in H^{1+r}(\Omega)$ and

$$\|\varphi\|_{1+r} \leqslant C_r \|f\|_{0,\partial\Omega},$$

where C_r is the regularity constant independent of f.

Lemma 2.1. *Let (λ_h, u_h) be the jth eigenpair of (2.4) and λ be the jth eigenvalue of (2.1). Then there exists an eigenfunction u corresponding to λ and when $u \in$*

$H^{1+t}(\Omega)$ and h is sufficiently small, it holds

$$\|u_h - u\|_h \leqslant Ch^t, \tag{2.6}$$

$$|\lambda_h - \lambda| \leqslant Ch^{2t}, \tag{2.7}$$

$$\|u - u_h\|_{0,\partial\Omega} \leqslant Ch^s \|u - u_h\|_h, \tag{2.8}$$

where $r \leqslant t \leqslant 1$, $s = r/2$ if $\Omega \subset \mathbb{R}^2$ and $s = r$ if $\Omega \subset \mathbb{R}^3$. The eigenfunction u is called singular when $t < 1$.

Proof. By using standard arguments in nonconforming finite element error estimates, (2.6) and (2.7) can be proved directly. The two estimates have also been given by Theorem 2.2 in [13] and (2.6) in [14]. Now we prove (2.8).

In order to prove the error estimate, we need to define the solution operator $A\colon\, L^2(\partial\Omega) \to H^1(\Omega)$ associated with the source problem of (2.1) by

$$a(Af, v) = b(f, v) \quad \forall\, v \in H^1(\Omega)$$

and the operator $T\colon\, L^2(\partial\Omega) \to L^2(\partial\Omega)$ by

$$Tf = (Af)',$$

where the prime denotes the restriction to $\partial\Omega$, namely $Tf = Af|_{\partial\Omega}$.

Analogously, we can define the discrete versions A_h and T_h corresponding to A and T, respectively. Define $A_h\colon\, L^2(\partial\Omega) \to V_h$ by

$$a_h(A_h f, v) = b(f, v) \quad \forall\, v \in V_h$$

and the operator $T_h\colon\, L^2(\partial\Omega) \to L^2(\partial\Omega)$ by

$$T_h f = (A_h f)'.$$

By the Nitsche technique (see also (2.13) in [30]), we derive

$$\|Tu - T_h u\|_{0,\partial\Omega} = \|Au - A_h u\|_{0,\partial\Omega} \leqslant Ch^s \|Au - A_h u\|_h.$$

From (2.7) and (2.8) in [34] (see also Lemma 3.1 in [19]), we have

$$\|u - u_h\|_h = \lambda \|Au - A_h u\|_h + R,$$
$$\|u - u_h\|_{0,\partial\Omega} \leqslant C\|Tu - T_h u\|_{0,\partial\Omega},$$

where $|R| \leqslant C\|Tu - T_h u\|_{0,\partial\Omega}$.

The estimate (2.8) is a direct consequence of the above three relations. $\square$

Define the Crouzeix-Raviart interpolation operator $I_h\colon\ H^1(\Omega) \to V_h$ by

$$\int_e I_h u \,\mathrm{d}s = \int_e u \,\mathrm{d}s \quad \forall\, e \in \varepsilon_h, \quad u \in H^1(\Omega). \tag{2.9}$$

Note that the interpolation operator I_h has an important orthogonality property (see the equality (2.9) in [2]): for each element $\kappa \in \pi_h$, it is

$$\int_\kappa \nabla(u - I_h u) \cdot \nabla v_h \,\mathrm{d}x = \int_{\partial\kappa} (u - I_h u) \nabla v_h \cdot \nu \,\mathrm{d}s = 0 \quad \forall\, v_h \in V_h. \tag{2.10}$$

The estimation of constants in the Poincaré and trace inequalities is a concern of academe (see, e.g., [28], [9], [7], [10], [20], [35], [24] and therein). From Theorem 4.2 in [24], we have following Lemma 2.2.

Lemma 2.2. *For any element κ, the conclusion*

$$\|u - I_h u\|_{0,\kappa} \leqslant C_{h_\kappa} |u - I_h u|_{1,\kappa} \quad \forall\, u \in H^1(\kappa), \tag{2.11}$$

is valid where

▷ $C_{h_\kappa} = 0.1893 h_\kappa$ *for a triangle element κ in $\mathbb{R}^2$,*
▷ $C_{h_\kappa} = 0.3804 h_\kappa$ *for a tetrahedron element κ in $\mathbb{R}^3$.*

Consider any element κ with the vertices $P_1, P_2, \ldots, P_{d+1}$. The edge/face opposite to the vertex P_{d+1} is denoted by e. The measure of e is $|e|$. Let H_κ be the height of element κ with respect to e. It is easy to see that

$$H_\kappa = \frac{d|\kappa|}{|e|}.$$

Thanks to Lemma 2 of [7] and Theorem 3.3 of [35], we have following Lemma 2.3.

Lemma 2.3. *For a given element κ, it is*

$$\|u - I_h u\|_{0,e} \leqslant C_{h_e} |u - I_h u|_{1,\kappa} \quad \forall\, u \in H^1(\kappa), \tag{2.12}$$

where

▷ $C_{h_e} = 0.6711 \dfrac{h_\kappa}{\sqrt{H_\kappa}}$ *for a triangle element κ in $\mathbb{R}^2$,*
▷ $C_{h_e} = 1.0932 \dfrac{h_\kappa}{\sqrt{H_\kappa}}$ *for a tetrahedron element κ in $\mathbb{R}^3$.*

Proof. The proof can be found in Theorem 3.3 of [35]. For convenience of reading in case of $d = 3$, we present the proof here.

For any $v \in H^1(\kappa)$ and any point $x = (x_1, x_2, x_3)$ in κ, we have

$$\int_\kappa ((x_1, x_2, x_3) - P_4) \cdot \nabla(v^2)\,\mathrm{d}x = \int_{\partial\kappa} ((x_1, x_2, x_3) - P_4) \cdot \mathbf{n} v^2\,\mathrm{d}s - \int_\kappa 3v^2\,\mathrm{d}x \tag{2.13}$$

from the Green formula. We deduce

$$\begin{aligned} &((x_1, x_2, x_3) - P_4) \cdot \mathbf{n} \\ &\quad = \begin{cases} 0 & \text{for any } x \text{ on the faces } P_1P_2P_4,\ P_1P_3P_4, \text{ and } P_2P_3P_4, \\ \dfrac{3|\kappa|}{|e|} & \text{for any } x \text{ on the face } P_1P_2P_3. \end{cases} \end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13), we obtain

$$\begin{aligned} \frac{3|\kappa|}{|e|} \int_e v^2\,\mathrm{d}s &= \int_\kappa 3v^2\,\mathrm{d}x + \int_\kappa ((x_1, x_2, x_3) - P_4) \cdot \nabla(v^2)\,\mathrm{d}x \\ &\leqslant 3\int_\kappa v^2\,\mathrm{d}x + \int_\kappa |(x_1, x_2, x_3) - P_4||\nabla(v^2)|\,\mathrm{d}x \\ &\leqslant 3\int_\kappa v^2\,\mathrm{d}x + 2h_\kappa \int_\kappa |v||\nabla v|\,\mathrm{d}x \\ &\leqslant 3\|v\|_{0,\kappa}^2 + 2h_\kappa \|v\|_{0,\kappa}\|\nabla v\|_{0,\kappa}. \end{aligned} \tag{2.15}$$

Taking $v = u - I_h u$ and applying the estimate (2.11), we deduce

$$\|u - I_h u\|_{0,e}^2 \leqslant \frac{|e|}{3|\kappa|}(3C_{h_\kappa}^2 + 2h_\kappa C_{h_\kappa})|u - I_h u|_{1,\kappa}^2,$$

which implies that (2.12) is valid when $\Omega \subset \mathbb{R}^3$. □

3. The asymptotic lower bounds property of corrected eigenvalues

For the problem (1.1), thanks to the minimum-maximum principle, it is easy to obtain guaranteed upper bounds for eigenvalues by conforming finite element methods. From [19], we know that the CR finite element method gives asymptotic lower bounds for eigenvalues when the corresponding eigenfunctions are singular or the eigenvalues are large enough. In this section, we introduce a correction for eigenvalues of the problem (1.1) and we prove that the corrected eigenvalues converge to the exact ones from below. The conclusion holds without the conditions that eigenfunction is singular and eigenvalue is large enough. First we prove the following inequality (3.1) and Lemma 3.1.

Using (2.10), we have

$$\int_\kappa \nabla(u-I_hu)\cdot\nabla(u-I_hu)\,\mathrm{d}x = \int_\kappa \nabla(u-I_hu)\cdot\nabla(u-u_h)\,\mathrm{d}x \leqslant |u-I_hu|_{1,\kappa}|u-u_h|_{1,\kappa},$$

so then

$$|u-I_hu|_{1,\kappa} \leqslant |u-u_h|_{1,\kappa}. \tag{3.1}$$

The identity in following Lemma 3.1 is an equivalent form of the identity (4.1) in [19], which is a generalization of the identities (2.12) in [2] and (2.3) in [36].

Lemma 3.1. *Let* (λ, u) *and* (λ_h, u_h) *be eigenpairs of* (2.1) *and* (2.4), *respectively. Then the following identity is valid:*

$$\begin{aligned}\lambda-\lambda_h &= a_h(u-u_h,u-u_h)-\lambda_h b(u-u_h,u-u_h)\\ &\quad -2a_h(I_hu-u,u_h)-2\lambda_h b(u-I_hu,u_h).\end{aligned} \tag{3.2}$$

P r o o f. From $\|u\|_{0,\partial\Omega}=1=\|u_h\|_{0,\partial\Omega}$, we get

$$a_h(u,u)=\lambda,\ \ a_h(u_h,u_h)=\lambda_h.$$

Therefore,

$$\begin{aligned}\lambda-\lambda_h &= a_h(u,u)+a_h(u_h,u_h)-2a_h(u_h,u_h)\\ &= a_h(u,u)+a_h(u_h,u_h)-2a_h(u,u_h)+2a_h(u-u_h,u_h)\\ &= a_h(u-u_h,u-u_h)+2a_h(u-u_h,u_h).\end{aligned} \tag{3.3}$$

From $b(I_hu-u_h,u_h)=b(I_hu-u,u_h)+b(u-u_h,u_h-\frac{1}{2}u+\frac{1}{2}u)$, we obtain

$$\lambda_h b(I_hu-u_h,u_h)=\lambda_h b(I_hu-u,u_h)-\tfrac{1}{2}\lambda_h b(u-u_h,u-u_h),$$

which together with (2.4) yields

$$\begin{aligned}a_h(u-u_h,u_h) &= a_h(u-I_hu,u_h)+a_h(I_hu-u_h,u_h)\\ &= a_h(u-I_hu,u_h)+\lambda_h b(I_hu-u_h,u_h)\\ &= a_h(u-I_hu,u_h)+\lambda_h b(I_hu-u,u_h)-\tfrac{1}{2}\lambda_h b(u-u_h,u-u_h).\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), we get (3.2). □

Now we give the correction formula (3.5). In addition, we prove that the correction provides asymptotic lower bounds for eigenvalues of the problem (2.1).

Denote by I_0 the piecewise constant interpolation operator on Ω. Let (λ, u) be an eigenpair of (2.1) and (λ_h, u_h) be the corresponding CR finite element approximations. We introduce the following formula to correct the CR finite element approximations λ_h:

$$\lambda_h^c = \frac{\lambda_h}{1 + M/\lambda_h}, \tag{3.5}$$

where

$$M = \frac{\delta}{\alpha_0} \sum_{\kappa\in\pi_h} (\|(\alpha - I_0\alpha)\nabla u_h\|_{0,\kappa} + C_{h_\kappa}\|\beta u_h\|_{0,\kappa})^2 \tag{3.6}$$

and $\delta > 1$ is any given constant.

By the interpolation error estimate, we know that

$$\|\alpha - I_0\alpha\|_{0,\infty,\kappa} \leqslant Ch_\kappa\|\alpha\|_{1,\infty,\kappa}. \tag{3.7}$$

Noting that $C_{h_\kappa} = 0.1893h_\kappa$, we derive

$$0 \leqslant M \leqslant Ch^2. \tag{3.8}$$

In practical computation, we cannot guarantee that λ_h are lower bounds of λ if we are not sure that the eigenfunctions are singular or the eigenvalues are large enough. Now we prove that the corrected eigenvalues λ_h^c are asymptotic lower bounds of the exact ones, which holds without the conditions of singularity and large eigenvalues.

Theorem 3.1. *Let λ_h^c be a corrected eigenvalue obtained by* (3.5). *Assume that the conditions of Lemma* 2.1 *and $\|u - u_h\|_h \geqslant Ch^{1+\varepsilon_0}$ $(\varepsilon_0 = \min\{\frac{1}{4}, \frac{1}{2}r\})$ hold. Then, if h is sufficiently small, we have*

$$\lambda \geqslant \lambda_h^c. \tag{3.9}$$

Proof. We now estimate each of the four terms on the right-hand side of (3.2). Since $\alpha \geqslant \alpha_0$, the first term

$$a_h(u - u_h, u - u_h) \geqslant \sum_{\kappa\in\pi_h} \left(\alpha_0|u - u_h|_{1,\kappa}^2 + \int_\kappa \beta(u - u_h)^2\,\mathrm{d}x\right). \tag{3.10}$$

From (2.8), the second term

$$\lambda_h b(u-u_h,u-u_h)=\lambda_h\|u-u_h\|_{0,\partial\Omega}^2\leqslant Ch^{2s}\|u-u_h\|_h^2. \tag{3.11}$$

Now we estimate the third term. From (2.10), we have

$$\begin{aligned} a_h(I_hu-u,u_h)=&\sum_{\kappa\in\pi_h}\int_\kappa((\alpha-I_0\alpha)\nabla(I_hu-u)\cdot\nabla u_h\\ &+I_0\alpha\nabla(I_hu-u)\cdot\nabla u_h+\beta(I_hu-u)u_h)\,\mathrm{d}x\\ =&\sum_{\kappa\in\pi_h}\int_\kappa((\alpha-I_0\alpha)\nabla(I_hu-u)\cdot\nabla u_h+\beta(I_hu-u)u_h)\,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Applying Schwarz's inequality and (2.11) to the above equality and combining it with (3.1), we deduce that

$$\begin{aligned} a_h(I_hu-u,u_h)&\leqslant\sum_{\kappa\in\pi_h}(|u-I_hu|_{1,\kappa}\|(\alpha-I_0\alpha)\nabla u_h\|_{0,\kappa}+\|u-I_hu\|_{0,\kappa}\|\beta u_h\|_{0,\kappa})\\ &\leqslant\sum_{\kappa\in\pi_h}|u-I_hu|_{1,\kappa}(\|(\alpha-I_0\alpha)\nabla u_h\|_{0,\kappa}+C_{h_\kappa}\|\beta u_h\|_{0,\kappa})\\ &\leqslant\sum_{\kappa\in\pi_h}|u-u_h|_{1,\kappa}(\|(\alpha-I_0\alpha)\nabla u_h\|_{0,\kappa}+C_{h_\kappa}\|\beta u_h\|_{0,\kappa}), \end{aligned}$$

which together with Young's inequality yields

$$\begin{aligned} 2a_h(I_hu-u,u_h)\leqslant&\frac{\alpha_0}{\delta}\sum_{\kappa\in\pi_h}|u-u_h|_{1,\kappa}^2\\ &+\frac{\delta}{\alpha_0}\sum_{\kappa\in\pi_h}(\|(\alpha-I_0\alpha)\nabla u_h\|_{0,\kappa}+C_{h_\kappa}\|\beta u_h\|_{0,\kappa})^2. \end{aligned} \tag{3.13}$$

It remains to estimate the last term. For the later proof, we introduce the piecewise constant interpolation operator I_0^b on $\partial\Omega$. Using (2.9), Schwarz's inequality, (2.12), $\|u-I_0^bu\|_{0,e}\leqslant Ch^{\min\{1,1/2+r\}}\|u\|_{H^{1/2+r}(e)}$, the trace inequality and (3.1), we get

$$\begin{aligned} b(u-I_hu,u)&=\sum_{e\in\varepsilon_h\cap\partial\Omega}\int_e((u-I_hu)(u-I_0^bu)+(u-I_hu)I_0^bu)\,\mathrm{d}s\\ &\leqslant\sum_{e\in\varepsilon_h\cap\partial\Omega}\|u-I_hu\|_{0,e}\|u-I_0^bu\|_{0,e}\\ &\leqslant Ch^{\min\{1,1/2+r\}}\Bigg(\sum_{\substack{\kappa\in\pi_h,\\ e\in\partial\kappa\cap\partial\Omega}}C_{h_e}^2|u-I_hu|_{1,\kappa}^2\Bigg)^{1/2}\\ &\leqslant Ch^{1+2\varepsilon_0}\Bigg(\sum_{\kappa\in\pi_h}|u-u_h|_{1,\kappa}^2\Bigg)^{1/2}, \end{aligned}$$

where $\varepsilon_0 = \min\{\frac{1}{4}, \frac{1}{2}r\}$. Combining it with $\|u - u_h\|_h \geqslant Ch^{1+\varepsilon_0}$, we derive that

$$b(u - I_h u, u) \leqslant Ch^{\varepsilon_0}\|u - u_h\|_h^2. \tag{3.14}$$

From Schwarz's inequality, (2.12), (2.8) and (3.1), we conclude that

$$\begin{aligned} b(u - I_h u, u_h - u) &\leqslant \sum_{e \in \varepsilon_h \cap \partial\Omega} \|u - I_h u\|_{0,e}\|u_h - u\|_{0,e} \\ &\leqslant Ch^{1/2}\Big(\sum_{\kappa \in \pi_h} |u - I_h u|_{1,\kappa}^2\Big)^{1/2} h^s \|u_h - u\|_h \\ &\leqslant Ch^{1/2+s}\|u_h - u\|_h^2. \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we deduce

$$2\lambda_h b(u - I_h u, u_h) \leqslant Ch^{\varepsilon_0}\|u_h - u\|_h^2. \tag{3.16}$$

Substituting (3.10), (3.11), (3.13), (3.16) into (3.2), we obtain

$$\begin{aligned} \lambda - \lambda_h \geqslant{}& \Big(1 - \frac{1}{\delta}\Big)\alpha_0 \sum_{\kappa \in \pi_h} |u - u_h|_{1,\kappa}^2 \\ &+ \sum_{\kappa \in \pi_h} \int_\kappa \beta(u - u_h)^2 \,\mathrm{d}x - Ch^{2s}\|u - u_h\|_h^2 \\ &- \frac{\delta}{\alpha_0} \sum_{\kappa \in \pi_h} (\|(\alpha - I_0\alpha)\nabla u_h\|_{0,\kappa} + C_{h_\kappa}\|\beta u_h\|_{0,\kappa})^2 - Ch^{\varepsilon_0}\|u_h - u\|_h^2. \end{aligned} \tag{3.17}$$

From the definition of M, we have

$$\begin{aligned} \Big(1 + \frac{1}{\lambda_h}M\Big)\lambda - \lambda_h \geqslant{}& \Big(1 - \frac{1}{\delta}\Big)\alpha_0 \sum_{\kappa \in \pi_h} |u - u_h|_{1,\kappa}^2 + \sum_{\kappa \in \pi_h} \int_\kappa \beta(u - u_h)^2 \,\mathrm{d}x \\ &- Ch^{2s}\|u - u_h\|_h^2 - Ch^{\varepsilon_0}\|u_h - u\|_h^2 - \frac{\lambda_h - \lambda}{\lambda_h}M. \end{aligned} \tag{3.18}$$

It is obvious that, when h is sufficiently small, the third and the fourth terms on the right-hand side of (3.18) are infinitesimals of higher order compared with the sum of the first two terms. From (3.8), (2.7) and $\|u - u_h\|_h \geqslant Ch^{1+\varepsilon_0}$, we get

$$\Big|\frac{\lambda_h - \lambda}{\lambda_h}M\Big| \leqslant Ch^{2+2t} \leqslant Ch^t h^{2+t} \leqslant Ch^t\|u - u_h\|_h^2,$$

which is a quantity of high order. Hence, the sign of the right-hand side of (3.18) is determined by the sum of the first two terms, i.e.

$$\Big(1 + \frac{1}{\lambda_h}M\Big)\lambda - \lambda_h \geqslant 0.$$

From (3.5), we know that (3.9) is valid. The proof is completed. □

R e m a r k 3.1. A condition on the lower bound for $\|u - u_h\|_h$ is necessary in Theorem 3.1, otherwise the proof does not work. The condition $\|u - u_h\|_h \geqslant Ch^r$ has been used in Theorem 2.3 of [2]. It is valid on quasi-uniform meshes but not on adaptive meshes with local refinement. Therefore, in order to make the conclusion of Theorem 3.1 hold on shape-regular meshes including quasi-uniform meshes and adaptive meshes with local refinement, we use the condition $\|u - u_h\|_h \geqslant Ch^{1+\varepsilon_0}$ ($\varepsilon_0 = \min\{\frac{1}{4}, \frac{1}{2}r\}$) rather than $\|u - u_h\|_h \geqslant Ch^r$. There are some papers on the adaptive algorithm that have discussed the rationality of this type of hypothesis (see, e.g., (2.33) and Remark 2.1 in [33]).

The following theorem shows that λ_h^c converge to λ and maintain the same convergence order as λ_h.

Theorem 3.2. *Let* (λ, u) *and* (λ_h, u_h) *be eigenpairs of* (2.1) *and* (2.4), *respectively. If* λ_h^c *is a corrected eigenvalue obtained by* (3.5), *then we have*

$$\lambda - \lambda_h^c = \lambda - \lambda_h + \frac{\lambda_h M}{\lambda_h + M}, \tag{3.19}$$

where $|M| \leqslant Ch^2$.

P r o o f. From (3.6), we have

$$\lambda - \lambda_h^c = \lambda - \lambda_h + \lambda_h - \frac{\lambda_h}{1 + M/\lambda_h} = \lambda - \lambda_h + \frac{\lambda_h M}{\lambda_h + M}.$$

The proof is completed. □

R e m a r k 3.2. From (3.17) and the definition of M, we can also get another correction formula

$$\lambda_h^N = \lambda_h - M.$$

Here λ_h^N is still an asymptotic lower bound for the eigenvalue. In addition, it can be obtained directly that

$$\lambda - \lambda_h^N = \lambda - \lambda_h + M,$$

which indicates that λ_h^N converge to λ and maintain the same convergence order as λ_h. However, from the inequality

$$(\lambda - \lambda_h^c) - (\lambda - \lambda_h^N) = \frac{\lambda_h M}{\lambda_h + M} - M = -\frac{M^2}{\lambda_h + M} \leqslant 0,$$

we know that the error of λ_h^N is larger than λ_h^c when the mesh size is small enough.

R e m a r k 3.3 (The correction to the ECR finite element eigenvalue approximations). Let (λ_h, u_h) be an approximate eigenpair of (2.1) obtained by the ECR element and $\beta \in W^{1,\infty}(\Omega)$. Assume that the condition $\|u - u_h\|_h \geqslant Ch^{1+\varepsilon_0}$ holds. Following closely the arguments used to prove Theorem 3.1, if h is sufficiently small, we can also obtain a similar correction

$$\lambda_h^c = \frac{\lambda_h}{1 + \delta\lambda_h^{-1}\alpha_0^{-1}\sum_{\kappa\in\pi_h}\|(\alpha - I_o\alpha)\nabla u_h\|_{0,\kappa}^2}$$

such that

$$\lambda \geqslant \lambda_h^c$$

and λ_h^c maintain the same convergence order as λ_h.

Actually, throughout the proof, we just need to replace the second term in (3.12) with the term

$$\begin{aligned}\sum_{\kappa\in\pi_h}\int_\kappa \beta(u - I_h u)u_h \,\mathrm{d}x &= \sum_{\kappa\in\pi_h}\int_\kappa (u - I_h u)(\beta u_h - I_0(\beta u_h))\,\mathrm{d}x \\ &\leqslant C\sum_{\kappa\in\pi_h} h_\kappa^2 |u - I_h u|_{1,\kappa}\|\beta u_h\|_{1,\kappa},\end{aligned}$$

which is a quantity of high order. Then we can obtain the desired.

4. Numerical experiments

In this section, to validate the theoretical results of this paper, we apply the correction (3.5) to (1.1) on the domain Ω. The discrete eigenvalue problems are solved in MATLAB 2018b on an Lenovo IdeaPad PC with 1.8 GHz CPU and 8 GB RAM. Our code was compiled under the iFEM package [11]. In order to investigate the error, we use the approximate eigenvalues given by extrapolation method as the reference value. The following notations are adopted in tables and figures.

h_0: The diameter of Ω.
h: The diameter of meshes.
λ_j: The jth eigenvalue of (2.1).
$\lambda_{j,h}$: The jth eigenvalue of (2.4) computed by the CR finite element.
$\lambda_{j,h}^c$: The approximation obtained by correcting $\lambda_{j,h}$.
$t(s)$: The CPU time to compute eigenvalues on the finest meshes.

4.1. Numerical results on $\Omega \subset \mathbb{R}^2$. In this subsection, we present two numerical examples. The first is $\alpha = \beta = 1$, the second is $\alpha = 10\sin^2(x_1 + x_2) + \frac{1}{6}$ and

$\beta = \mathrm{e}^{(x_1-1/2)(x_2-1/2)}$. We compute on the unit square $(0,1)^2$ ($h_0 = \sqrt{2}$), on the L-shaped domain $(-1,1)^2 \setminus ([0,1) \times (-1,0])$ ($h_0 = 2\sqrt{2}$) and on the regular hexagon with the side length of 1 ($h_0 = 2$); for convenience, we refer to the domains as **S**, **L** and **H**, respectively.

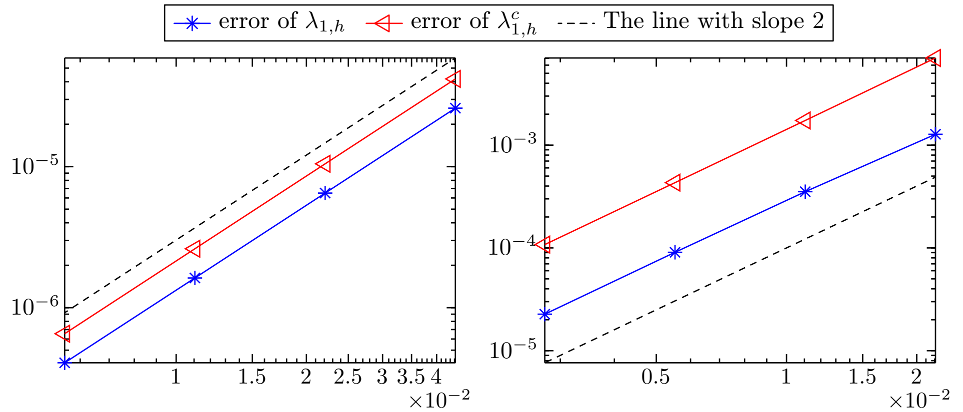

Figure 1. The error curves of the first eigenvalues on the unit square: $\alpha = \beta = 1$ (left) and $\alpha = 10\sin^2(x_1+x_2) + \frac{1}{6}$, $\beta = \mathrm{e}^{(x_1-1/2)(x_2-1/2)}$ (right). Vertical axis: The relative error of eigenvalue. Horizontal axis: The diameter of meshes.

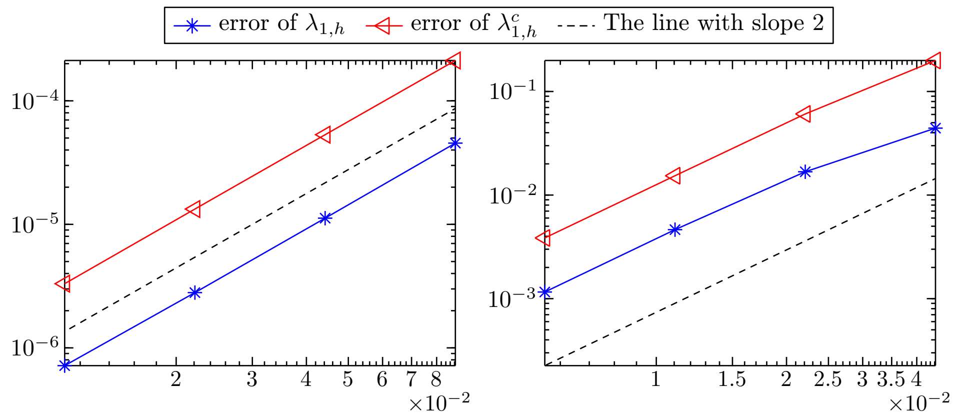

Figure 2. The error curves of the first eigenvalues on the L-shaped domain: $\alpha = \beta = 1$ (left) and $\alpha = 10\sin^2(x_1+x_2) + \frac{1}{6}$, $\beta = \mathrm{e}^{(x_1-1/2)(x_2-1/2)}$ (right). Vertical axis: The relative error of eigenvalue. Horizontal axis: The diameter of meshes.

In order to obtain asymptotic lower bounds for the problem (1.1), we use (3.5) to correct $\lambda_{1,h}$. The error curves are depicted in Figures 1–3. New approximate eigenvalues $\lambda^c_{1,h}$ are listed in Tables 1 and 2. From Figures 1–3 we can see that on each domain, the error curves of $\lambda^c_{1,h}$ and $\lambda_{1,h}$ are almost parallel to the line with slope 2, which indicates that $\lambda^c_{1,h}$ and $\lambda_{1,h}$ have the same and optimal convergence order $\mathcal{O}(h^2)$. This result coincides with the conclusion of Theorem 3.2. In addition, we can assume that the eigenfunctions corresponding to λ_1 are smooth. From Tables 1 and 2, on the one hand, we see that $\lambda_{1,h}$ converge to λ_1 from above and the corrected eigenvalues $\lambda_{1,h}^c$ converge to λ_1 from below, which indicates that the correction (3.5) can provide lower bounds for eigenvalues even though eigenfunctions are smooth. This result coincides with the conclusion of Theorem 3.1. On the other hand, on each domain, the CPU time to compute $\lambda_{1,h}^c$ is almost the same as that of $\lambda_{1,h}$, which tells us that the correction takes very little time.

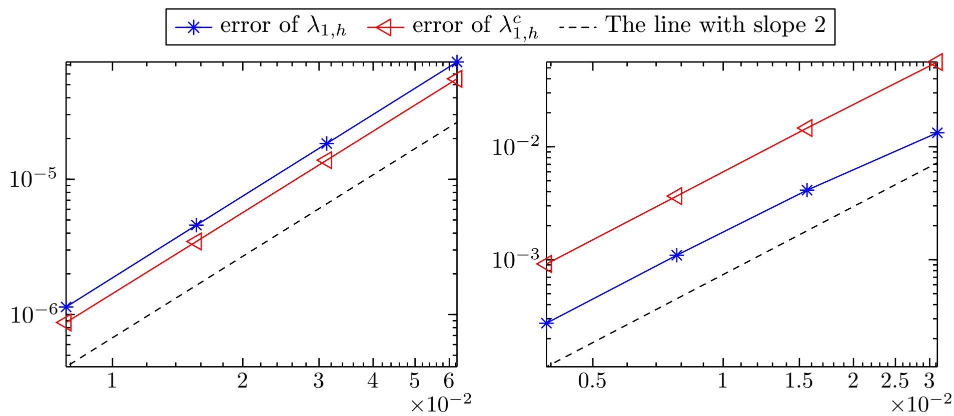

Figure 3. The error curves of the first eigenvalues on the regular hexagon: $\alpha = \beta = 1$ (left) and $\alpha = 10\sin^2(x_1 + x_2) + \frac{1}{6}$, $\beta = \mathrm{e}^{(x_1-1/2)(x_2-1/2)}$ (right). Vertical axis: The relative error of eigenvalue. Horizontal axis: The diameter of meshes.

Domain	**S**		**L**		**H**	
h	$\lambda_{1,h}$	$\lambda_{1,h}^c$	$\lambda_{1,h}$	$\lambda_{1,h}^c$	$\lambda_{1,h}$	$\lambda_{1,h}^c$
$\frac{h_0}{32}$	0.24008533	0.24006902	0.34143156	0.34134357	0.39334226	0.39329159
$\frac{h_0}{64}$	0.24008065	0.24007657	0.34141986	0.34139787	0.39332055	0.39330788
$\frac{h_0}{128}$	0.24007948	0.24007846	0.34141699	0.34141149	0.39331513	0.39331196
$\frac{h_0}{256}$	0.24007918	0.24007893	0.34141628	0.34141490	0.39331377	0.39331298
$\frac{h_0}{512}$	0.24007911	0.24007905	0.34141610	0.34141576	0.39331344	0.39331324
$t(s)$	31.10	31.20	22.74	22.81	25.34	25.41
Trend	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Table 1. The uncorrected eigenvalues and the corrected eigenvalues on $\Omega \subset \mathbb{R}^2$: $\delta = \frac{100}{99}$, $\alpha = \beta = 1$.

Domain	**S**		**L**		**H**	
h	$\lambda_{1,h}$	$\lambda_{1,h}^c$	$\lambda_{1,h}$	$\lambda_{1,h}^c$	$\lambda_{1,h}$	$\lambda_{1,h}^c$
$\frac{h_0}{32}$	0.24696	0.23963	0.53724	0.27358	0.56181	0.45333
$\frac{h_0}{64}$	0.24645	0.24441	0.51661	0.39617	0.55267	0.51468
$\frac{h_0}{128}$	0.24623	0.24571	0.50301	0.46469	0.54766	0.53740
$\frac{h_0}{256}$	0.24616	0.24603	0.49700	0.48710	0.54600	0.54341
$\frac{h_0}{512}$	0.24614	0.24611	0.49528	0.49280	0.54556	0.54491
$t(s)$	37.71	40.05	28.06	29.74	28.39	30.07
Trend	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Table 2. The uncorrected eigenvalues and the corrected eigenvalues on $\Omega \subset \mathbb{R}^2$: $\delta = \frac{100}{99}$, $\alpha = 10\sin^2(x_1 + x_2) + \frac{1}{6}$, $\beta = \mathrm{e}^{(x_1-1/2)(x_2-1/2)}$.

Domain	**C**			**F**			
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{5,h}$	h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{3,h}$
0.6124	0.162344	1.11356	1.56489	0.8660	0.268747	0.54947	0.72763
0.3062	0.162226	1.14537	1.65619	0.4330	0.268359	0.56641	0.73377
0.1531	0.162196	1.15272	1.68222	0.2165	0.268268	0.57235	0.73615
0.0765	0.162189	1.15448	1.68924	0.1083	0.268247	0.57441	0.73687
Trend	$\searrow$	$\nearrow$	$\nearrow$	–	$\searrow$	$\nearrow$	$\nearrow$

Table 3. The CR finite element eigenvalue approximations on $\Omega \subset \mathbb{R}^3$: $\delta = \frac{100}{99}$.

Domain	**C**			**F**	
h	$\lambda_{1,h}$	$\lambda_{1,h}^c$	h	$\lambda_{1,h}$	$\lambda_{1,h}^c$
0.6124	0.162344	0.156854	0.8660	0.268747	0.244062
0.3062	0.162226	0.160802	0.4330	0.268359	0.261752
0.1531	0.162196	0.161837	0.2165	0.268268	0.266587
0.0765	0.162189	0.162099	0.1083	0.268247	0.267824
$t(s)$	150.07	150.23	–	223.12	223.26
Trend	$\searrow$	$\nearrow$	–	$\searrow$	$\nearrow$

Table 4. The uncorrected eigenvalues and the corrected eigenvalues on $\Omega \subset \mathbb{R}^3$: $\delta = \frac{100}{99}$.

4.2. Numerical results on $\Omega \subset \mathbb{R}^3$. In this subsection, we select $\alpha = \beta = 1$. We compute in the cube $(0,1)^3$ and the Fichera corner domain $(-1,1)^3 \setminus (-1,0]^3$. For convenience, we denote the domains by **C** and **F**, respectively. The quasi-uniform mesh samples of the cube and the Fichera corner domain are depicted in Figure 4. In the two domains, we compute the first three eigenvalues using the CR finite element and list the results in Table 3. In the cube, λ_2 and λ_5 are the eigenvalues with multiplicity of 3. Corrected eigenvalues $\lambda^c_{1,h}$ are listed in Table 4. Error curves are depicted in Figure 5.

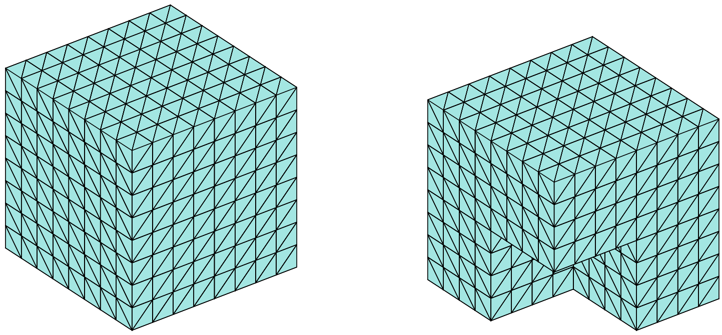

Figure 4. The quasi-uniform mesh samples of the cube (left) and the Fichera corner domain (right).

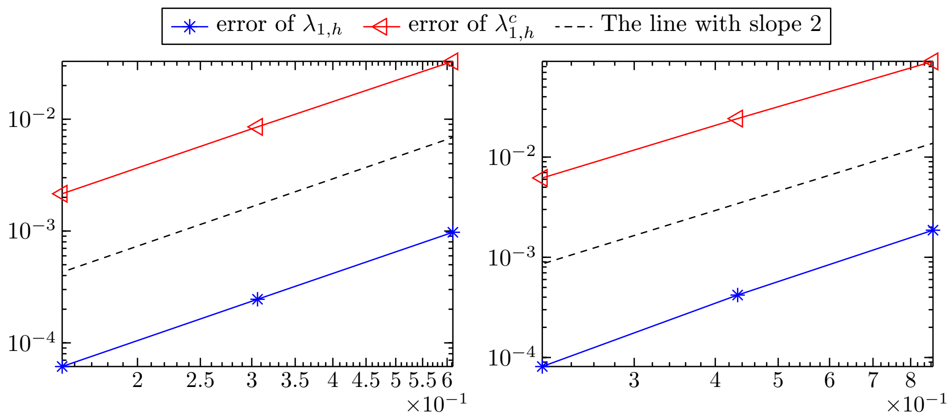

Figure 5. The error curves of the first eigenvalues in the cube (left) and the Fichera corner domain (right). Vertical axis: The relative error of eigenvalue. Horizontal axis: The diameter of meshes.

From Figure 5, we see that the error curves of $\lambda^c_{1,h}$ and $\lambda_{1,h}$ are parallel to the line with slope 2, which indicates that $\lambda^c_{1,h}$ and $\lambda_{1,h}$ have the same and optimal convergence order $\mathcal{O}(h^2)$. Also we assume that the eigenfunctions corresponding to λ_1 are smooth. From Table 3, we see that on each domain, $\lambda_{1,h}$ converge to λ_1 from above. This shows that the CR finite element eigenvalue approximations may not be lower bounds of exact eigenvalues in the case of smooth eigenfunctions. From Table 4, we see that the corrected eigenvalues $\lambda^c_{1,h}$ converge to λ_1, which indicates that the correction (3.5) provides lower bounds for eigenvalues even though the eigenfunctions are smooth. The numerical results on three dimensional domains coincide with the conclusions of Theorem 3.1 and Theorem 3.2.

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