Youness Hadder

The kh-socle of a commutative semisimple Banach algebra

Mathematica Bohemica, Vol. 145 (2020), No. 4, 387–399

Persistent URL: `http://dml.cz/dmlcz/148431`

THE KH-SOCLE OF A COMMUTATIVE SEMISIMPLE BANACH ALGEBRA

Youness Hadder, Fèz

Received August 26, 2018. Published online November 26, 2019.
Communicated by Vladimír Müller and Dagmar Medková

Abstract. Let $\mathcal{A}$ be a commutative complex semisimple Banach algebra. Denote by $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ the kernel of the hull of the socle of $\mathcal{A}$. In this work we give some new characterizations of this ideal in terms of minimal idempotents in $\mathcal{A}$. This allows us to show that a "result" from Riesz theory in commutative Banach algebras is not true.

Keywords: commutative Banach algebra; socle; kh-socle; inessential element

MSC 2010: 46J05, 46J20, 47A10

1. Introduction

For a semisimple complex algebra $\mathcal{A}$ we let $\mathrm{soc}(\mathcal{A})$ be its socle and denote by $\mathcal{I}_m(\mathcal{A})$ the set of minimal idempotents in $\mathcal{A}$. An element $u \in \mathcal{A}$ is said to be a quasi-Fredholm element of $\mathcal{A}$ if u is quasi-invertible modulo $\mathrm{soc}(\mathcal{A})$ in $\mathcal{A}$ (see [8]). We denote the set of all quasi-Fredholm elements of $\mathcal{A}$ by $q\text{-}\mathfrak{F}_r(\mathcal{A})$. If $\mathcal{A}$ is unital then we define an element $u \in \mathcal{A}$ to be a Fredholm element of $\mathcal{A}$ if u is invertible modulo $\mathrm{soc}(\mathcal{A})$ in $\mathcal{A}$ (see [8]). We denote the set of all Fredholm elements of $\mathcal{A}$ by $\mathfrak{F}_r(\mathcal{A})$. Now, let $\Pi_{\mathcal{A}}$ denote the set of all primitive ideals of $\mathcal{A}$. Let Φ be a subset of $\Pi_{\mathcal{A}}$ and S a subset of $\mathcal{A}$. The kernel of Φ in $\mathcal{A}$ is denoted by $k_{\mathcal{A}}(\Phi)$ and the hull of S in $\Pi_{\mathcal{A}}$ is denoted by $h_{\mathcal{A}}(S)$ (see [12]). Then the kernel of the hull of $\mathrm{soc}(\mathcal{A})$ in $\mathcal{A}$, which is the intersection of all primitive ideals of $\mathcal{A}$ containing $\mathrm{soc}(\mathcal{A})$, is simply denoted by $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ and is referred to as the kh-socle in $\mathcal{A}$ for short (see [14]). Assume now that $\mathcal{A}$ is a Banach algebra. By [12], Theorem 2.2.6, it is easy to see that $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ is the set of all elements $u \in \mathcal{A}$ such that $u + \overline{\mathrm{soc}(\mathcal{A})} \in \mathrm{rad}(\mathcal{A}/\overline{\mathrm{soc}(\mathcal{A})})$. We recall that an element of $\mathcal{A}$ is called inessential if its spectrum is either finite or a sequence converging to zero, and an ideal is inessential if all its elements are

DOI: 10.21136/MB.2019.0106-18

inessential (see [5]). For example it is well known that $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ is inessential (see [7], Theorem 3.4). If moreover $\mathcal{A}$ is commutative then $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ is constituted exactly by the elements $u \in \mathcal{A}$ such that $u + \overline{\mathrm{soc}(\mathcal{A})}$ is quasi-nilpotent in $\mathcal{A}/\overline{\mathrm{soc}(\mathcal{A})}$, that is to say,

$$\mathrm{kh}(\mathrm{soc}(\mathcal{A})) = \Big\{u \in \mathcal{A}\colon\ \forall \lambda \in \mathbb{C}^*,\ \frac{1}{\lambda} u \in q\text{-}\mathfrak{F}_r(\mathcal{A})\Big\}. \tag{1.1}$$

For the fundamental properties of this ideal the reader is referred to [6], [7], [13].

Finally, note that an element a in a semisimple complex Banach algebra $\mathcal{A}$ is compact if the operator ${}_aT_a$ defined by ${}_aT_a(x) = axa$ for all $x \in \mathcal{A}$ is compact on $\mathcal{A}$ (see [2]). The set of all compact elements of $\mathcal{A}$ is denoted by $\mathcal{K}(\mathcal{A})$. In particular, we say that $\mathcal{A}$ is compact if $\mathcal{A} = \mathcal{K}(\mathcal{A})$. Using [6], Theorem 2.1 we see that $\overline{\mathrm{soc}(\mathcal{A})} \subseteq \mathcal{K}(\mathcal{A}) \subseteq \mathrm{kh}(\mathrm{soc}(\mathcal{A}))$.

In what follows we consider a commutative semisimple complex Banach algebra $\mathcal{A}$. We will use Φ_A to denote the maximal ideal space of $\mathcal{A}$ and we let $\mathcal{A}_{\mathrm{c}} = C_0(\Phi_{\mathcal{A}})$ (the commutative algebra of complex-valued continuous functions which vanish at infinity on Φ_A). Moreover, if S is any subset of $\mathcal{A}$, then we let $\widehat{S} = \{\widehat{x}\colon\ x \in S\}$, where $\widehat{x}$ is the Gelfand transform of x. In this paper we will study the truth of the equality $\widehat{\mathrm{kh}(\mathrm{soc}(\mathcal{A}))} = \mathrm{kh}(\mathrm{soc}(\mathcal{A}_{\mathrm{c}}))$. To do this we will give some new characterizations of the kh-socle in terms of minimal idempotents. The motivation for this problem is as follows: In [10] we encountered the problem of whether a commutative semisimple complex Banach algebra $\mathcal{A}$ has the property that $\mathcal{K}(\mathcal{A}) = \overline{\mathrm{soc}(\mathcal{A})}$. The following affirmative answer appears as the main theorem in [3]:

(a) If $\mathcal{A}$ is a commutative semisimple unital complex Banach algebra then $\mathcal{K}(\mathcal{A}) = \overline{\mathrm{soc}(\mathcal{A})}$.

However, the argument given is not rigorous. To justify this we must, in our opinion, also examine the veracity of the following statement which appears, without proof, as Theorem 7.2 (iii) in [13]:

(b) If $\mathcal{A}$ is a commutative unital complex Banach algebra and $\mathcal{F}$ is an ideal of algebraic elements of $\mathcal{A}$ then $\overline{\overline{\mathcal{F}}} = \widehat{\mathrm{kh}(\mathcal{F})}$.

This statement turns out to be false. We therefore amend the "result" (a) and a particular case of (b) by proving that they hold under an additional hypothesis (see Section 4).

With this context in mind let us explain the organization of the paper. In Sections 2 and 3 we characterize the kh-socle in terms of minimal idempotents. Some informative results about a commutative complex unital semisimple Banach algebra $\mathcal{A}$ which satisfies $\widehat{\mathrm{kh}(\mathrm{soc}(\mathcal{A}))} = \mathrm{kh}(\mathrm{soc}(\mathcal{A}_{\mathrm{c}}))$ are given in Section 4. Finally in Section 5 we will give a counter-example to the statements (a) and (b).

2. The study of case where $\mathcal{A} = \mathcal{C}_0(\mathcal{K})$

We now place ourselves in a particular commutative context. We consider a locally compact Hausdorff space $\mathcal{K}$ and set $\mathcal{A} = \mathcal{A}_{\mathrm{c}} = \mathcal{C}_0(\mathcal{K})$, the commutative algebra of complex-valued continuous functions which vanish at infinity on $\mathcal{K}$. Suppose that the set $\mathrm{iso}(\mathcal{K})$ of all isolated points of $\mathcal{K}$ is not empty. Let $\mathrm{acc}(\mathcal{K})$ be the set of all accumulation points of $\mathcal{K}$.

The main result of this section is

Proposition 2.1. *We have*

$$\mathrm{kh}(\mathrm{soc}(\mathcal{A}_{\mathrm{c}})) = \Big\{u \in \mathcal{A}_{\mathrm{c}}\colon\ u = \sum_{k=1}^{\infty} \lambda_k e_k,\ (e_k)_{k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}_{\mathrm{c}}),\ (\lambda_k)_{k\geqslant 1} \subset \mathbb{C}\Big\}.$$

To show this result we need the following terminology and lemmas.

Given a subset F of a set X, the complement of F in X is denoted by F^{c}. Also, we denote by $\mathcal{P}_{\mathrm{f}}(X)$ and $\mathcal{P}_{\mathrm{d}}(X)$ the set of all finite subsets of X and the set of all countable subsets of X, respectively.

From [8], page 94 and using Urysohn's lemma we have the following immediate result.

Lemma 2.2. *If $u \in \mathcal{A}_{\mathrm{c}}$ then:*

(i) $u \in \mathcal{I}_m(\mathcal{A}_{\mathrm{c}}) \Leftrightarrow \exists!\, x \in \mathrm{iso}(\mathcal{K})$ *such that* $u = \delta_x$, *with* $\delta_x\colon \mathcal{K} \to \mathbb{C}$, $\delta_x(x) = 1$ *and* $\delta_x(\{x\}^{\mathrm{c}}) = \{0\}$;

(ii) $u \in \mathrm{soc}(\mathcal{A}_{\mathrm{c}}) \Leftrightarrow \exists\, F \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\mathcal{K}))$ *such that* $Z_{\mathcal{K}}(u) = F^{\mathrm{c}}$, *where* $Z_{\mathcal{K}}(u) = \{x \in \mathcal{K}\colon\ u(x) = 0\}$.

From this we deduce the following well-known result:

$$\mathrm{soc}(\mathcal{A}_{\mathrm{c}}) = \Big\{u \in \mathcal{A}_{\mathrm{c}}\colon\ u = \sum_{k=1}^{n} \lambda_k e_k,\ n \in \mathbb{N}^*,\ (e_k)_{n\geqslant k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}_{\mathrm{c}}),\ (\lambda_k)_{n\geqslant k\geqslant 1} \subset \mathbb{C}\Big\}.$$

We can easily verify the following facts:

Remark 2.3.

(i) $\mathcal{I}_m(\mathcal{A}_{\mathrm{c}})$ is orthogonal;

(ii) $\mathrm{iso}(\mathcal{K})$ and $\mathcal{I}_m(\mathcal{A}_{\mathrm{c}})$ are equipotent;

(iii) $\mathcal{I}_m(\mathcal{A}_{\mathrm{c}})$ is closed;

(iv) $\mathrm{soc}(\mathcal{C}_0(\mathcal{K})) \neq \{0\} \Leftrightarrow \mathrm{iso}(\mathcal{K}) \neq \emptyset$;

(v) $h(\mathrm{soc}(\mathcal{A}_{\mathrm{c}})) = \mathrm{acc}(\Phi_{\mathcal{A}_{\mathrm{c}}})$.

Again by [8], page 94 we deduce immediately the following characterization of the Fredholm elements of $\mathcal{A}_{\mathrm{c}}$ in terms of $\mathrm{iso}(\mathcal{K})$.

Lemma 2.4. *If $\mathcal{K}$ is compact and $u \in \mathcal{A}_{\mathrm{c}}$ then the following conditions are equivalent:*

(i) $u \in \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$;

(ii) $u(x) \neq 0$ *for all* $x \in \mathrm{acc}(\mathcal{K})$;

(iii) $Z_{\mathcal{K}}(u) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\mathcal{K}))$.

Now we will generalize this result to the case where $\mathcal{K}$ is only a locally compact Hausdorff space. To do this we will use the following:

If $\mathcal{K}$ is a locally compact non-compact Hausdorff space, let the set $\widetilde{\mathcal{K}} = \mathcal{K} \cup \{\infty\}$ be its one-point compactification. If $u \in \mathcal{A}_{\mathrm{c}}$, set the mapping $\widetilde{u}\colon\ \widetilde{\mathcal{K}} \to \mathbb{C}$, $\widetilde{u}(\infty) := 0$, $\widetilde{u}(x) := u(x)$ for each $x \in \mathcal{K}$. Then $\widetilde{u} \in \mathcal{C}(\widetilde{K})$. Let $\mathcal{A}$ be a non unital commutative semisimple complex Banach algebra and $\mathcal{A}^{\sharp}$ its unitization. Note that the mapping $u \mapsto \widetilde{u}$ of $\mathcal{A}_{\mathrm{c}}$ into $\mathcal{C}(\widetilde{K})$ is an isomorphism isometric between two Banach algebras. Now, we consider the mapping

$$T\colon\ \mathcal{A}_{\mathrm{c}}^{\sharp} \to \mathcal{C}(\widetilde{K}), \quad T(\lambda + u) := \lambda + \widetilde{u}$$

for each $\lambda \in \mathbb{C}$ and $u \in \mathcal{A}_{\mathrm{c}}$; where $\lambda + \widetilde{u}\colon\ x' \mapsto \lambda + \widetilde{u}(x')$ for each $x' \in \widetilde{\mathcal{K}}$. Then T is an isomorphism. Indeed, it suffices to prove that T is surjective. To this end, let $h \in \mathcal{C}(\widetilde{K})$. We put $\lambda := h(\infty)$ and $u(x) := h(x) - \lambda$ for each $x \in \mathcal{K}$. We can verify that $u \in \mathcal{A}_{\mathrm{c}}$ and $T(\lambda + u) = h$.

Lemma 2.5. *If $\mathcal{K}$ is only a locally compact Hausdorff space and $u \in \mathcal{A}_{\mathrm{c}}$ then the following statements are equivalent:*

(i) $u \in q\text{-}\mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$;

(ii) $\widetilde{u}(x') \neq 1$ *for all* $x' \in \mathrm{acc}(\widetilde{\mathcal{K}})$;

(iii) $Z_{\mathcal{K}}(1 - u) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\mathcal{K}))$, *where* $Z_{\mathcal{K}}(1 - u) = \{x \in \mathcal{K}\colon\ u(x) = 1\}$.

Proof. First, using [11], Corollary 3.5 and the fact that T is an isomorphism, we may therefore infer that T preserves the Fredholm elements in the both directions, that is, for each $h \in \mathcal{A}_{\mathrm{c}}^{\sharp}$, we have:

$$h \in \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}}^{\sharp}) \Leftrightarrow T(h) \in \mathfrak{F}_r(\mathcal{C}(\widetilde{\mathcal{K}})).$$

From this remark we have the following equivalences

(i) $\Leftrightarrow$ (ii): By Lemma 2.4 and the fact that, with $u \in \mathcal{A}$, $u \in q\text{-}\mathfrak{F}_r(\mathcal{A}) \Leftrightarrow 1-u \in \mathfrak{F}_r(\mathcal{A}^\sharp)$.

(i) $\Leftrightarrow$ (iii): This follows using similar reasoning since $\mathrm{iso}(\mathcal{K}) = \mathrm{iso}(\widetilde{K})$. $\square$

Now, we can also characterize the elements of $\mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c}))$ in terms of $\mathrm{iso}(\mathcal{K})$.

Lemma 2.6. *If $u \in \mathcal{A}_\mathrm{c}$ then the following properties are equivalent:*

(i) $u \in \mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c}))$;

(ii) $\widetilde{u} = 0$ *on* $\mathrm{acc}(\widetilde{K})$;

(iii) $Z_\mathcal{K}(u)^\mathrm{c} \in \mathcal{P}_\mathrm{d}(\mathrm{iso}(\mathcal{K}))$.

Proof. (i) $\Rightarrow$ (iii): Fix $u \in \mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c}))$. Then for all $\lambda \in \mathrm{sp}(u) \setminus \{0\}$, there exists a finite subset F_λ of $\mathrm{iso}(\mathcal{K})$ such that $u(x)/\lambda \neq 1$ for each $x \in F_\lambda^\mathrm{c}$ by Lemma 2.5. Let $D = \bigcup\{F_\lambda \colon \lambda \in \mathrm{sp}_{\mathcal{A}_\mathrm{c}}(u) \setminus \{0\}\}$ which is a countable subset of $\mathrm{iso}(\mathcal{K})$ since u is an inessential element. Moreover, $u(x) = 0$ for all $x \in D^\mathrm{c}$.

(iii) $\Rightarrow$ (ii): Is trivial.

(ii) $\Rightarrow$ (i): Suppose that $\widetilde{u} = 0$ on $\mathrm{acc}(\widetilde{K})$. Then for every $\lambda \in \mathbb{C}^*$, $\widetilde{u}(x')/\lambda \neq 1$ for every $x' \in \mathrm{acc}(\widetilde{K})$. Hence, by Lemma 2.5, $u/\lambda \in q\text{-}\mathfrak{F}_r(\mathcal{A}_\mathrm{c})$ for every $\lambda \in \mathbb{C}^*$. Therefore $u \in \mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c}))$ by (1.1). $\square$

Finally, we give the proof of Proposition 2.1.

Proof. Fix $u \in \mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c}))$. By Lemma 2.6, $Z_\mathcal{K}(u)^\mathrm{c}$ is a countable subset of $\mathcal{P}_\mathrm{d}(\mathrm{iso}(\mathcal{K}))$. Set $Z_\mathcal{K}(u)^\mathrm{c} = \{x_1, x_2, \ldots\}$ and $\lambda_n = u(x_n)$. It follows that, with $e_k = \delta_{x_k}$, $u = \sum\limits_{k=1}^{\infty} \lambda_k e_k$ since any element u of $\mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c}))$ has the property that $\mathrm{sp}_{\mathcal{A}_\mathrm{c}}(u) \setminus \{0\}$ is finite or an infinite sequence converging to zero by [7], Theorem 3.4. $\square$

From this we obtain the following well-known result:

$$\mathrm{kh}(\mathrm{soc}(\mathcal{A}_\mathrm{c})) = \overline{\mathrm{soc}(\mathcal{A}_\mathrm{c})} = \mathcal{K}(\mathcal{A}_\mathrm{c}).$$

Remark 2.7. According to [13], Corollary 7.4, if $\mathcal{A}$ is a function algebra, then $\overline{\mathrm{soc}(\mathcal{A})} = \mathcal{K}(\mathcal{A})$. Also this result is true for any $\mathcal{C}^*$-algebra (see [9], C*.2.4). In Sections 4 and 5 we will discuss the status of this equality in an abstract commutative context.

3. The study of "general" case

Throughout this paragraph the letter $\mathcal{A}$ denotes a complex semisimple commutative Banach algebra. Let $\mathcal{A}_{\rm c} = \mathcal{C}_0(\Phi_{\mathcal{A}})$. Suppose that $\operatorname{iso}(\Phi_{\mathcal{A}})$ is not empty. The main result of this section is

Proposition 3.1.

$$\operatorname{kh}(\operatorname{soc}(\mathcal{A})) = \Big\{u \in \mathcal{A}\colon\ \widehat{u} = \sum_{k=1}^{\infty} \lambda_k \widehat{e_k},\ (e_k)_{k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}),\ (\lambda_k)_{k\geqslant 1} \subset \mathbb{C}\Big\}.$$

To show this result we need the following facts which are very similar to those obtained in Section 2.

The idea of the next lemma comes from [1], Remark 5.26.

Lemma 3.2. *We have the following statements:*

(i) *If $e \in \mathcal{I}_m(\mathcal{A})$ then there is only one element $\phi \in \operatorname{iso}(\Phi_{\mathcal{A}})$ such that $\hat{e} = \delta_\phi$;*

(ii) *If $\phi \in \operatorname{iso}(\Phi_{\mathcal{A}})$ then $\mathcal{I}_m(\mathcal{A})$ contains only one element e such that $\hat{e} = \delta_\phi$.*

P r o o f. (i) Fix $e \in \mathcal{I}_m(\mathcal{A})$. Suppose that there exists $\phi \neq \varphi$ in $\Phi_{\mathcal{A}}$ such that $\phi(e) = 1 = \varphi(e)$. Let $a \in \mathcal{A}$ be such that $\phi(a) \neq \varphi(a)$. There exists $\lambda \in \mathbb{C}$ with $ae = \lambda e$ since e is a minimal idempotent in $\mathcal{A}$. So $\phi(a) = \lambda = \varphi(a)$, a contradiction. Thus there is only one element $\phi \in \operatorname{iso}(\Phi_{\mathcal{A}})$ which checks $\phi(e) = 1$ and therefore $\hat{e} = \delta_\phi$.

(ii) Fix $\phi \in \operatorname{iso}(\Phi_{\mathcal{A}})$. Then, from [12], Theorem 3.6.3 we can find a nonzero idempotent e in $\mathcal{A}$ such that $\{\phi\} = \{\psi\colon\ \psi(e) = 1\}$. Hence $\hat{e} = \delta_\phi$. This implies that for each $a \in \mathcal{A}$, $ae = \phi(a)e$ since $\mathcal{A}$ is semisimple. From this we may infer that $e \in \mathcal{I}_m(\mathcal{A})$. □

We can reproduce the following known result concerning the Riesz theory in complex semisimple commutative Banach algebras.

Proposition 3.3. *The following equalities are equivalent:*

(i) *$h_{\mathcal{A}}(\operatorname{soc}(\mathcal{A})) = \emptyset$;*

(ii) *$\operatorname{acc}(\Phi_{\mathcal{A}}) = \emptyset$;*

(iii) *$\mathcal{A} = \operatorname{kh}(\operatorname{soc}(\mathcal{A}))$.*

P r o o f. (i) $\Rightarrow$ (ii): Assume that there exists $\phi \in \Phi_{\mathcal{A}}$ such that $\phi = 0$ on $\operatorname{soc}(\mathcal{A})$. Then ϕ is not in $\operatorname{iso}(\Phi_{\mathcal{A}})$ by (ii) of Lemma 3.2.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are trivial.

(ii) $\Rightarrow$ (i): If $\phi \in \operatorname{acc}(\Phi_{\mathcal{A}})$ then $\phi = 0$ on $\operatorname{soc}(\mathcal{A})$ by (i) of Lemma 3.2. Therefore $\phi \in h_{\mathcal{A}}(\operatorname{soc}(\mathcal{A}))$. □

The following remark is similar to Remark 2.3.

Remark 3.4.

(i) $\mathcal{I}_m(\mathcal{A})$ is orthogonal and closed by (i) of Lemma 3.2 and (iii) of Remark 2.3;

(ii) The sets $\mathcal{I}_m(\mathcal{A})$, $\operatorname{iso}(\Phi_{\mathcal{A}})$, $\mathcal{I}_m(\mathcal{C}_0(\Phi_{\mathcal{A}}))$ and $\operatorname{iso}(\Phi_{\mathcal{C}_0(\Phi_{\mathcal{A}})})$ are in bijection;

(iii) $\operatorname{soc}(\mathcal{A}) \neq \{0\} \Leftrightarrow \operatorname{iso}(\Phi_{\mathcal{A}}) \neq \emptyset$.

By using Remark 3.2 and Lemma 2.2 we can deduce the following information.

Remark 3.5.

(i) $\widehat{\mathcal{I}_m(\mathcal{A})} = \mathcal{I}_m(\mathcal{A}_{\mathrm{c}})$ and $\widehat{\operatorname{soc}(\mathcal{A})} = \operatorname{soc}(\mathcal{A}_{\mathrm{c}})$;

(ii) $h_{\mathcal{A}}(\operatorname{soc}(\mathcal{A})) = \operatorname{acc}(\Phi_{\mathcal{A}})$;

(iii) $s \in \operatorname{soc}(\mathcal{A}) \Leftrightarrow Z_{\Phi_{\mathcal{A}}}(\hat{s})^{\mathrm{c}} \in \mathcal{P}_{\mathrm{f}}(\operatorname{iso}(\Phi_{\mathcal{A}}))$.

Then, using the semisimplicity of $\mathcal{A}$, we deduce the following well-known result:

$$\operatorname{soc}(\mathcal{A}) = \left\{u \in \mathcal{A}\colon\ u = \sum_{k=1}^{n} \lambda_k e_k,\ n \in \mathbb{N}^*,\ (e_k)_{n\geqslant k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}),\ (\lambda_k)_{n\geqslant k\geqslant 1} \subset \mathbb{C}\right\}.$$

Next we will generalize the results in Lemmas 2.4 and 2.5 to the abstract commutative case. To do this we will use the celebrated identification $\mathcal{L}$ that is defined as follows:

Let $\mathcal{A}$ be a non unital commutative semisimple complex Banach algebra and $\mathcal{A}^\sharp$ its unitization. Set

$$\varphi_\infty\colon\ \mathcal{A}^\sharp \to \mathbb{C};\quad \lambda + u \mapsto \lambda.$$

And for every $\varphi \in \Phi_{\mathcal{A}}$ we put

$$\widetilde{\varphi}\colon\ \mathcal{A}^\sharp \to \mathbb{C};\quad \lambda + u \mapsto \lambda + \varphi(u),$$

and

$$\mathcal{L}\colon\ \Phi_{\mathcal{A}} \to \Phi_{\mathcal{A}^\sharp} \setminus \{\varphi_\infty\};\quad \varphi \mapsto \widetilde{\varphi}.$$

Then

$$\Phi_{\mathcal{A}^\sharp} = \mathcal{L}(\Phi_{\mathcal{A}}) \cup \{\varphi_\infty\}.$$

We can also check that $\mathcal{L}$ is a homeomorphism for the Gelfand topologies.

Lemma 3.6. *Let $u \in \mathcal{A}$.*

(a) *Suppose that $\mathcal{A}$ is unital. Then the following relations are equivalent:*

  (i) *$u \in \mathfrak{F}_r(\mathcal{A})$;*

  (ii) *$\varphi(u) \neq 0$ for all $\varphi \in \operatorname{acc}(\Phi_{\mathcal{A}})$;*

  (iii) *$Z_{\Phi_{\mathcal{A}}}(\widehat{u}) \in \mathcal{P}_{\mathrm{f}}(\operatorname{iso}(\Phi_{\mathcal{A}}))$.*

(b) *Suppose that $\mathcal{A}$ is not unital. Then the following relations are equivalent:*

(i) $u \in q\text{-}\mathfrak{F}_r(\mathcal{A})$;

(ii) $Z_{\Phi_{\mathcal{A}}}(\widehat{1-u}) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\Phi_{\mathcal{A}}))$;

(iii) $\varphi'(u) \neq 1$ *for each* $\varphi' \in \mathrm{acc}(\Phi_{\mathcal{A}^\sharp})$.

P r o o f. (a) Suppose that $\mathcal{A}$ is unital.

(iii) $\Rightarrow$ (i): We assume that $Z_{\Phi_{\mathcal{A}}}(\widehat{u}) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\Phi_{\mathcal{A}}))$. If $u \notin \mathfrak{F}_r(\mathcal{A})$, then there exists an element ϕ_0 in $\Phi_{\mathcal{A}_0}$ such that $\phi_0 \circ \pi(u) = 0$; where $\mathcal{A}_0 = \mathcal{A}/\overline{\mathrm{soc}(\mathcal{A})}$ and π is the canonical quotient map of $\mathcal{A}$ onto $\mathcal{A}_0$. Note that $\phi_0 \circ \pi \in h(\mathrm{soc}(\mathcal{A}))$. Hence $\phi_0 \circ \pi \in \mathrm{acc}(\Phi_{\mathcal{A}})$ by (ii) of Lemma 3.5, so that $\widehat{u}(\phi_0 \circ \pi) \neq 0$ which is a contradiction.

(i) $\Rightarrow$ (ii): It suffices to use also (ii) of Lemma 3.5.

(ii) $\Rightarrow$ (iii): It follows from Lemma 2.4.

(b) Suppose that $\mathcal{A}$ is not unital.

(i) $\Rightarrow$ (ii): Assume that $u \in q\text{-}\mathfrak{F}_r(\mathcal{A})$. Then $1-u \in \mathfrak{F}_r(\mathcal{A}^\sharp)$ so that $Z_{\Phi_{\mathcal{A}^\sharp}}(\widehat{1-u}) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\Phi_{\mathcal{A}^\sharp}))$ by (a). Thus $Z_{\Phi_{\mathcal{A}}}(\widehat{1-u}) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\Phi_{\mathcal{A}}))$ by virtue of the continuity of $\mathcal{L}\colon \Phi_A \to \Phi_{\mathcal{A}^\sharp} \setminus \{\varphi_\infty\}$.

(ii) $\Rightarrow$ (i): Suppose that $Z_{\Phi_{\mathcal{A}}}(\widehat{1-u}) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\Phi_{\mathcal{A}}))$. Set $Z_{\Phi_{\mathcal{A}}}(\widehat{1-u}) = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Since $\mathcal{L}$ is open, we have $\widetilde{\varphi_k} \in \mathrm{iso}(\Phi_{\mathcal{A}^\sharp} \setminus \{\varphi_\infty\})$ for all $1 \leqslant k \leqslant n$. Hence $Z_{\Phi_{\mathcal{A}^\sharp}}(\widehat{1-u}) = \{\widetilde{\varphi_1}, \ldots, \widetilde{\varphi_n}\} \in \mathcal{P}(\mathrm{iso}(\Phi_{\mathcal{A}^\sharp} \setminus \{\varphi_\infty\}))$. Thus $Z_{\Phi_{\mathcal{A}^\sharp}}(\widehat{1-u}) \in \mathcal{P}_{\mathrm{f}}(\mathrm{iso}(\Phi_{\mathcal{A}^\sharp}))$. This implies that $1-u \in \mathfrak{F}_r(\mathcal{A}^\sharp)$ which is equivalent to the fact that $u \in q\text{-}\mathfrak{F}_r(\mathcal{A})$.

(i) $\Leftrightarrow$ (iii): The forward implication follows from part (i) $\Rightarrow$ (ii) of (b) and the reverse implication can be obtained using part (ii) $\Rightarrow$ (i) in (a) together with the fact that $u \in q\text{-}\mathfrak{F}_r(\mathcal{A}) \Leftrightarrow 1-u \in \mathfrak{F}_r(\mathcal{A}^\sharp)$. $\square$

We give some information about the relationship between $\mathfrak{F}_r(\mathcal{A})$ and $\mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$.

Proposition 3.7. *If $\mathcal{A}$ is unital then the following assertions hold:*

(i) $u \in \mathfrak{F}_r(\mathcal{A}) \Leftrightarrow \widehat{u} \in \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$, *where* $u \in \mathcal{A}$;

(ii) $\widehat{\mathfrak{F}_r(\mathcal{A})} \subseteq \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$;

(iii) $\widehat{\mathfrak{F}_r(\mathcal{A})} = \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}}) \Leftrightarrow \widehat{\mathcal{A}} = \mathcal{A}_{\mathrm{c}}$.

P r o o f. (i) and (ii) follow from part (a) of Lemma 3.6 and Lemma 2.4.

(iii) The reverse implication is trivial by (i). Now, suppose that $\widehat{\mathfrak{F}_r(\mathcal{A})} = \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$. If f is an element in $\mathcal{A}_{\mathrm{c}}$ then there exists $\lambda \in \mathbb{C}$ such that $\lambda - f$ is an invertible element in $\mathcal{A}_{\mathrm{c}}$. Hence, $\lambda - f \in \mathfrak{F}_r(\mathcal{A}_{\mathrm{c}})$. Since A is unital, we have $f \in \widehat{\mathcal{A}}$. $\square$

We now conclude a generalization of Lemma 2.6 to the “abstract” commutative case.

Lemma 3.8. *Fix $u \in \mathcal{A}$. Then the following assertions are equivalent:*

(i) $u \in \operatorname{kh}(\operatorname{soc}(\mathcal{A}))$;

(ii) $\widehat{u}(\varphi') = 0$ *for each* $\varphi' \in \operatorname{acc}(\Phi_{\mathcal{A}^\sharp})$;

(iii) $Z_{\Phi_{\mathcal{A}}}(\widehat{u})^{\mathrm{c}} \in \mathcal{P}_{\mathrm{d}}(\operatorname{iso}(\Phi_{\mathcal{A}}))$.

Proof. We can see the following equivalences by applying (1.1):

(i) $\Leftrightarrow$ (ii): We use the equivalence (i) $\Leftrightarrow$ (iii) of the part (b) of Lemma 3.6.

(i) $\Leftrightarrow$ (iii): Using (i) $\Leftrightarrow$ (ii), Lemma 2.6 and the continuity of $\mathcal{L}$. $\square$

In connection with the statement (b) we give the following assertions.

Corollary 3.9.

(i) $u \in \operatorname{kh}(\operatorname{soc}(\mathcal{A})) \Leftrightarrow \widehat{u} \in \operatorname{kh}(\operatorname{soc}(\mathcal{A}_{\mathrm{c}}))$, *where* $u \in \mathcal{A}$;

(ii) $\widehat{\operatorname{kh}(\operatorname{soc}(\mathcal{A}))} \subseteq \operatorname{kh}(\operatorname{soc}(\mathcal{A}_{\mathrm{c}}))$;

(iii) $\operatorname{kh}(\operatorname{soc}(\mathcal{A})) = k(\operatorname{acc}(\Phi_{\mathcal{A}}))$.

Finally, we turn to the proof of the result in Proposition 3.1.

Proof. By Proposition 2.1 and using the fact that $\widehat{\mathcal{I}_m(\mathcal{A})} = \mathcal{I}_m(\mathcal{A}_{\mathrm{c}})$ we see that

$$\operatorname{kh}(\operatorname{soc}(\mathcal{A}_{\mathrm{c}})) = \left\{ f \in \mathcal{A}_{\mathrm{c}} \colon \ f = \sum_{k=1}^{\infty} \lambda_k \widehat{e_k},\ (e_k)_{k \geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}),\ (\lambda_k)_{k \geqslant 1} \subset \mathbb{C} \right\}.$$

Therefore applying (i) and (ii) of Corollary 3.9 we obtain our result. $\square$

Remark 3.10. Does Proposition 3.1 stay true if we delete the symbol "$\widehat{\ }$"? We show in Section 5 that the answer is negative in general.

4. Special case study with the condition that $\widehat{\operatorname{kh}(\operatorname{soc}(\mathcal{A}))} = \operatorname{kh}(\operatorname{soc}(\mathcal{A}_{\mathrm{c}}))$

Now, to examine the veracity of the statement (b) we will study a commutative complex unital semisimple Banach algebra $\mathcal{A}$ with the property that

$$\text{(TS)} \qquad \widehat{\operatorname{kh}(\operatorname{soc}(\mathcal{A}))} = \operatorname{kh}(\operatorname{soc}(\mathcal{A}_{\mathrm{c}})).$$

Then the following two results are a summary of information concerning this algebra.

For every $x \in \mathcal{A}$, we put $\|x\|_r = r_{\mathcal{A}}(x)$ which is the spectral radius of x. Throughout this section we always assume that $\operatorname{iso}(\Phi_{\mathcal{A}})$ is not empty.

Proposition 4.1. *If $\mathcal{A}$ is a commutative complex unital semisimple Banach algebra with the property* (TS) *then*

(i) $(\mathrm{kh}(\mathrm{soc}(\mathcal{A})), \|\cdot\|_r)$ *is a Banach algebra.*

(ii) $\mathrm{kh}(\mathrm{soc}(\mathcal{A})) = \mathcal{K}(\mathcal{A}) = \overline{\mathrm{soc}(\mathcal{A})}^{\|\cdot\|_r} = \overline{\mathrm{soc}(\mathcal{A})}^{\|\cdot\|}$.

P r o o f. If $\mathcal{A}$ satisfies (TS), then the Gelfand transform and its inverse are isomorphisms between two semisimple commutative Banach algebras. By [5], Corollary 4.1.9 we may therefore infer that the norms on these two algebras are equivalent. From this one readily obtains (i) and (ii). $\square$

According to the statements (a), (b) and [10], Remark 2.5, this proposition gives us a sufficient condition on a commutative complex unital semisimple Banach algebra $\mathcal{A}$ in order to have that $\mathcal{K}(\mathcal{A}) = \overline{\mathrm{soc}(\mathcal{A})}^{\|\cdot\|}$.

From what precedes we can also see the condition (TS) like this.

R e m a r k 4.2. For every commutative complex unital semisimple Banach algebra $\mathcal{A}$, the following conditions are equivalent:

(i) $\widehat{\mathrm{kh}(\mathrm{soc}(\mathcal{A}))} = \mathrm{kh}(\mathrm{soc}(\mathcal{A}_{\mathrm{c}}))$;

(ii) $(\mathrm{kh}(\mathrm{soc}(\mathcal{A})), \|\cdot\|r)$ is a Banach algebra;

(iii) for all $f \in \mathcal{A}_{\mathrm{c}}$; $f = 0$ on $\mathrm{acc}(\Phi_{\mathcal{A}})$ we have $f \in \widehat{A}$.

In this case we can conclude the following beautiful characterization of $\mathrm{kh}(\mathrm{soc}(\mathcal{A}))$.

Corollary 4.3. *If $\mathcal{A}$ is a commutative complex unital semisimple Banach algebra with the property* (TS) *then*

$$\mathrm{kh}(\mathrm{soc}(\mathcal{A})) = \left\{u \in \mathcal{A}\colon\ u = \sum_{k=1}^{\infty} \lambda_k e_k,\ (e_k)_{k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}),\ (\lambda_k)_{k\geqslant 1} \subset \mathbb{C}\right\}.$$

We can easily notice that every function algebra, within the meaning of [5], page 76, is a commutative complex unital semisimple Banach algebra verifying (TS); consequently we have

Corollary 4.4. *If $\mathcal{A}$ is a function algebra then*

$$\mathrm{kh}(\mathrm{soc}(\mathcal{A})) = \left\{u \in \mathcal{A}\colon\ u = \sum_{k=1}^{\infty} \lambda_k e_k,\ (e_k)_{k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A}),\ (\lambda_k)_{k\geqslant 1} \subset \mathbb{C}\right\}.$$

Q u e s t i o n 4.5. Let $\mathcal{A}$ be a commutative complex unital semisimple Banach algebra verifying (TS). Is it a function algebra?

5. Counter-example

Finally, in this section we give a counter-example to the statements (a) and (b).

Example 5.1. There exists a commutative complex unital semisimple Banach algebra $\mathcal{A}$ such that

(i) $\overline{\mathrm{soc}(\mathcal{A})} \neq \mathcal{K}(\mathcal{A})$;

(ii) $\widehat{\mathrm{kh}(\mathrm{soc}(\mathcal{A}))} \neq \mathrm{kh}(\mathrm{soc}(\mathcal{A}_{\mathrm{c}}))$;

(iii) there exists $u \in \mathrm{kh}(\mathrm{soc}(\mathcal{A}))$ such that there is no family $(e_k)_{k\geqslant 1} \subseteq \mathcal{I}_m(\mathcal{A})$ and $(\lambda_k)_{k\geqslant 1} \subset \mathbb{C}$ with $u = \sum\limits_{k=1}^{\infty} \lambda_k e_k$.

Proof. In [13] the author used the algebra

$$\mathcal{A} = \{x = (x_n)_{n\geqslant 1} \colon x_n \in \mathbb{C}, \forall\, n \in \mathbb{N}^*, \sup\{n|x_n|, n \in \mathbb{N}^*\} < \infty\}$$

to show the existence of a commutative complex semisimple Banach algebra such that $\overline{\mathrm{soc}(\mathcal{A})} \neq \mathrm{kh}(\mathrm{soc}(\mathcal{A}))$; precisely he noticed that $u \notin \overline{\mathrm{soc}(\mathcal{A})}$ and $u \in \mathrm{kh}(\mathrm{soc}(\mathcal{A}))$, with $u = (1/n)_{n\geqslant 1}$. We will utilize $\mathcal{A}^\sharp$ and the element u to establish the existence of an algebra satisfying (i), (ii) and (iii). First note that $\mathcal{A}^\sharp$ is a commutative complex unital semisimple Banach algebra. Then we have the following facts:

(i) We claim that u acts compactly on $\mathcal{A}^\sharp$. It suffices to prove that u is a compact element of $\mathcal{A}$. To this end, let (z_k) be any sequence bounded in norm by 1 in $\mathcal{A}$. If $z_k = (z_{k;n})_{n\geqslant 1}$ for each $k \in \mathbb{N}^*$, then for each $k, n \in \mathbb{N}^*$,

$$\frac{1}{n}|z_{k;n}| \leqslant \frac{1}{n^2}. \tag{5.1}$$

Since $|z_{k;1}| \leqslant 1$ for all $k \in \mathbb{N}^*$, there exists a subsequence $(z_{j_k^1})$ of (z_k) such that $(z_{j_k^1;1})$ converges as $k \to \infty$. Say $z_{j_k^1;1} \to \omega_1 \in \mathbb{C}$ as $k \to \infty$ and note that $|\omega_1| \leqslant 1$. But from (5.1) we also have, for all $k \in \mathbb{N}^*$,

$$\frac{1}{2}|z_{j_k^1;2}| \leqslant \frac{1}{2^2}.$$

Hence, there exists a subsequence $(z_{j_k^2})$ of $(z_{j_k^1})$ such that $(\frac{1}{2} z_{j_k^2;2})$ converges as $k \to \infty$. Say $\frac{1}{2} z_{j_k^2;2} \to 2\omega_2 \in \mathbb{C}$ as $k \to \infty$ and note that $|2\omega_2| \leqslant 1/2^2$. We continue inductively in this way. After m steps we obtain a subsequence $(z_{j_k^m})$ of $(z_{j_k^{m-1}})$ such that $z_{j_k^m;m}/m \to m\omega_m$ as $k \to \infty$ with $|m\omega_m| \leqslant 1/m^2$. From this we obtain the subsequence $(z_{j_k^k})$ of (z_k) with the following property: For each $n \in \mathbb{N}^*$, the sequence $(z_{j_k^k;n}/n)$ converges to $n\omega_n$ as $k \to \infty$. To simplify our notation, we set $j_k^k := j_k$. Let $\omega := (\omega_n)$ and notice that $\omega \in \mathcal{A}$ since $\sup\{n|\omega_n| \colon n \in \mathbb{N}^*\} \leqslant 1$.

In order to establish the claim we need only to verify that $u^2 z_{j_k} \to \omega$ as $k \to \infty$. To get a contradiction, assume that $u^2 z_{j_k} \nrightarrow \omega$ as $k \to \infty$. Then there exists an $\varepsilon > 0$ such that for every $N \in \mathbb{N}^*$, there is some $m \geqslant N$ satisfying $\|u^2 z_{j_m} - \omega\| \geqslant \varepsilon$. Consequently, there is a subsequence (z_{i_k}) of (z_{j_k}) such that $\|u^2 z_{i_k} - \omega\| \geqslant \varepsilon$ for all $k \in \mathbb{N}^*$. In particular, this implies that for each $k \in \mathbb{N}^*$ there is some $n_k \in \mathbb{N}^*$ such that

$$\frac{2}{n_k^2} \geqslant \left| \frac{1}{n_k} z_{i_k;n_k} - n_k \omega_{n_k} \right| \geqslant \varepsilon. \tag{5.2}$$

From (5.2) it follows that the set $\{n_k \colon\ k \in \mathbb{N}^*\}$ is finite. Hence, there exists an $l \in \{n_k \colon\ k \in \mathbb{N}^*\}$ and a subsequence (z_{q_k}) of (z_{i_k}) such that for all $k \in \mathbb{N}^*$

$$\left| \frac{1}{l} z_{q_k;l} - l\omega_l \right| \geqslant \varepsilon.$$

But this is absurd since the sequence (z_{q_k}/l) converges to $l\omega_l$ as $k \to \infty$. From this we may infer that $u^2 z_{j_k} \to \omega$ as $k \to \infty$. This proves our claim.

(ii) By Proposition 4.1 we have $\widehat{\mathrm{kh}(\mathrm{soc}(\mathcal{A}^\sharp))} \neq \mathrm{kh}(\mathrm{soc}((\mathcal{A}^\sharp)_{\mathrm{c}}))$.

(iii) Just take the same element $u = (1/n)_{n \geqslant 1}$ from (i). $\square$

Question 5.2. Does $\mathrm{kh}(\mathrm{soc}(\mathcal{A})) = \mathcal{K}(\mathcal{A})$ for every commutative complex unital semisimple Banach algebra $\mathcal{A}$?

However, it is well known that this equality is false in the non-commutative context (see [4]).

Acknowledgments. I am grateful to professor Nadia Boudi for helpful conversations and valuable criticism of the manuscript. We also thank the referee for careful reading of the manuscript and for several very useful suggestions, especially the version of the proof of the counterexample.

References

[1] *P. Aiena*: Fredholm and Local Spectral Theory, with Applications to Multipliers. Kluwer Academic Publishers, Dordrecht, 2004. zbl MR doi

[2] *J. C. Alexander*: Compact Banach algebras. Proc. Lond. Math. Soc., III. Ser. *18* (1968), 1–18. zbl MR doi

[3] *A. H. Al-Moajil*: The compactum of a semi-simple commutative Banach algebra. Int. J. Math. Math. Sci. *7* (1984), 821–822. zbl MR doi

[4] *G. Androulakis, T. Schlumprecht*: Strictly singular, non-compact operators exist on the space of Gowers and Maurey. J. Lond. Math. Soc., II. Ser. *64* (2001), 655–674. zbl MR doi

[5] *B. Aupetit*: A Primer on Spectral Theory. Universitext. Springer, New York, 1991. zbl MR doi

[6] *B. Aupetit, H. du T. Mouton*: Spectrum preserving linear mappings in Banach algebras. Studia Math. *109* (1994), 91–100. zbl MR doi

[7] *B. A. Barnes*: A generalized Fredholm theory for certain maps in the regular representations of an algebra. Can. J. Math. *20* (1968), 495–504. zbl MR doi

[8] *B. A. Barnes*: The Fredholm elements of a ring. Can. J. Math. *21* (1969), 84–95. zbl MR doi

[9] *B. A. Barnes, G. J. Murphy, M. R. F. Smyth, T. T. West*: Riesz and Fredholm Theory in Banach Algebras. Research Notes in Mathematics 67. Pitman Advanced Publishing Program, Boston, 1982. zbl MR

[10] *N. Boudi, Y. Hadder*: On linear maps preserving generalized invertibility and related properties. J. Math. Anal. Appl. *345* (2008), 20–25. zbl MR doi

[11] *J. Puhl*: The trace of finite and nuclear elements in Banach algebras. Czech. Math. J. *28* (1978), 656–676. zbl MR

[12] *C. E. Rickart*: General Theory of Banach Algebras. The University Series in Higher Mathematics. D. Van Nostrand, Princeton, 1960. zbl MR

[13] *M. R. F. Smyth*: Riesz theory in Banach algebras. Math. Z. *145* (1975), 145–155. zbl MR doi

[14] *X. Wang, P. Cao*: Spectral characterization of the kh-socle in Banach Jordan algebras. J. Math. Anal. Appl. *466* (2018), 567–572. zbl MR doi

Author's address: *Youness Hadder*, Regional Center for Education and Training Fèz-Meknès, Rue Hafid Ibrahim, Fèz, Morocco, e-mail: `haddfsm@hotmail.com`.